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Equilibrium selection and stability in dynamic core-periphery models with heterogeneous preferences

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Abstract

In New Economic Geography, recent models have shown that idiosyncratic preferences of workers for locations act as a dispersion force affecting the number and stability of equilibrium population distributions. Yet those models are based on *ad hoc* deterministic adjustment procedures that have two shortcomings. Firstly, they remove the aggregate effect of idiosyncratic preferences on the collective spatial dynamics of workers, whose study would require the use of specific notions of equilibrium stability. Secondly, these adjustment dynamics lack an explicit time unit that prevents adjustment trajectories to be expressed as dynamic scenarios. Those two shortcomings strive against the use of New Economic Geography models to support policy recommendations. Starting from a classic core-periphery model of New Economic Geography, this paper proposes a novel approach to adjustment dynamics, based on stochastic migration models, by which the dynamics of the population distribution is a continuous-time Markov chain. Using a diffusion approximation, the dynamic system is reduced to a set of Itô stochastic differential equations, which is an original contribution to New Economic Geography. In those equations, deterministic and stochastic effects are still distinct at the aggregate scale, which enables to numerically compute equilibrium population distributions as well as to evaluate their stability and selection under stochastic perturbations generated by idiosyncratic preferences. Those equations also enable to complete expected adjustment trajectories with an explicit time unit and with confidence intervals, for different scenarios. Hence this paper is a substantial improvement of the capacity of New Economic Geography models to support policy recommendations.

1 Introduction

The geographic distribution of human population and economic activities is closely related to regional development (Williamson, 1965; Wheaton and Shishido, 1981; Fujita and Thisse, 2003; Desmet and Rossi-Hansberg, 2010). This is why it has become an important political issue, as exemplified by international economic integration policies such as the North American Free Trade Agreement or the European Economic Area. The distribution of human activities results from migration decisions that depend, *inter alia*, on market incentives, such as wage differentials or costs of living. Market incentives to migrate are fundamental in new economic geography, which essentially studies how trade costs interact with demand and input-output linkages in economic production (Krugman, 1991a; Venables, 1996; Fujita, 1999; Combes et al., 2008; Baldwin et al., 2005). Models of new economic geography manly use representative agents, although migrations also depend on idiosyncratic preferences for market or non-market attributes of locations (Jacobs, 1961; Hicks, 1963; Rosen, 1979; Roback, 1982; Greenwood, 1985; Glaeser, 2008). It is only recently that some researchers have started to study interactions between representative preferences, which describe market incentives to migrate, and idiosyncratic preferences, which may include non-market incentives (Tabuchi and Thisse, 2002; Murata, 2003). They showed that idiosyncratic preferences affect the number and stability of equilibrium population distributions. Yet those studies, like many models of economic geography, treat equilibrium stability locally using *ad hoc* deterministic adjustment procedures based on expected migrations. Hence they do not consider the uncertainty on adjustment dynamic trajectories that results from aggregating individual shocks generated by idiosyncratic preferences for locations. Yet this aggregate uncertainty turns the dynamic of the population distribution to a stochastic dynamical system (Longtin, 2010), in which case non-trivial effects such as noise-induced transitions (Horsthemke and Lefever, 2006) require the use specific notions of equilibrium stability (Khasminskii, 2012). From an applied perspective, those *ad hoc* adjustment dynamics also suffer from the lack of explicit time unit that prevents adjustment trajectories to be expressed as dynamic scenarios. This shortcoming strives against the use of new economic geography models to support policy recommendations.

Starting from a classic core-periphery model of new economic geography, this paper proposes a novel approach to adjustment dynamics, based on stochastic migration models, by which the dynamics of the population distribution is a continuous-time Markov chain. Using a diffusion approximation, the dynamic system is reduced to a set of Itô stochastic differential equations, which is an original contribution to new economic geography. In those equations, deterministic and stochastic effects are still distinct at the aggregate scale, which enables to numerically compute equilibrium population distributions as well as to evaluate their stability and selection under stochastic perturbations generated by idiosyncratic preferences. Those equations also enable to complete expected adjustment trajectories with an explicit time unit and confidence intervals, for different scenarios.

The remaining of this paper is structured as follows. Section 2 proposes a detailed literature review of new economic geography and probabilistic migration models, with an emphasize on how this paper bridges the gap between both disciplines. Section 3 contains the model proposed in this paper. It starts by presenting the footloose entrepreneur model with three equidistant regions, then it exposes the individual relocation dynamics of a single agent and it finishes by deriving the collective spatial dynamics of the entire population. Section 4 uses this model to numerically discuss equilibrium selection and stability in probabilistic terms. Section 5 concludes.

2 Background literature

The seminal model of new economic geography is the core-periphery model of Krugman (1991a). It shows how agglomeration and dispersion forces emerge through (horizontal) demand linkages[1] in a context of Chamberlinian monopolistic competition *à la* Dixit and Stiglitz (1977) with iceberg transport cost (for a comprehensive presentation, see Fujita et al., 1999; Baldwin et al., 2005; Combes et al., 2008). Although its original form was analytically intractable, it has been extended in many ways by a set of analytically tractable models that have collectively improved our understanding of agglomeration and dispersion forces in economic geography. For example, Baldwin (1999) considered that human capital accumulation (instead of migration) is the root of agglomeration forces. In another study, Ottaviano et al. (2002) used a quasi linear utility function with quadratic sub-utility and linear transport cost. In a related work, Pflüger (2004) also used quasi-linear utility function but a sub-utility with constant elasticity of substitution (CES). Obviously, those works depart from the original core-periphery model with Cobb-Douglas utility and CES sub-utility. Thus, looking for the slightest modifications of the original model which make it analytically tractable, Forslid and Ottaviano (2003) proposed to introduce skill heterogeneity between workers and to endow high-skill workers with a bigger interregional mobility. They naturally call this model the "footloose entrepreneur" model. Actually, Robert-Nicoud (2005) showed that all those models share some proximity to an alternative specification of monopolistic competition put forth by Flam and Helpman (1987). Further development in this encompassing perspective are provided by Ottaviano and Robert-Nicoud (2006) and Pflüger and Südekum (2008). See also the review of Redding (2013).

Most of the analytical results in new economic geography are actually based on two-regions settings (see for example Mossay, 2006). Although a better understanding of multiple-regions systems has already been called for (Krugman, 1998; Fujita et al., 1999; Neary, 2001; Ottaviano and Thisse, 2004), the consequent increase in the number of equilibria makes it even more difficult to address theoretical questions regarding the selection and stability of those equilibria (Fujita and Thisse, 2009; Behrens and Robert-Nicoud, 2011). An initial effort in developing models with multiple regions has been made by Krugman and Elizondo (1996) who proposed a three-regions model where two regions are subdivisions of a larger one. This setting has been used several times since then (Paluzie, 2001; Behrens, 2011; Commendatore et al., 2014). Another three-regions framework consists in considering equidistant regions (Fujita et al., 1999; Commendatore and Kubin, 2013). In particular, Fujita et al. (1999, chap. 6) developed the three-regions case of the classic core-periphery model and showed that for intermediate values of transport cost, both concentration and dispersion of labour are stable distributions of activities (Figure 1). Other spatial settings allowing more than three regions are the linear economy (Ago et al., 2006; Commendatore et al., 2015), the racetrack economy (Fujita et al., 1999; Castro et al., 2012; Ikeda et al., 2012; Akamatsu et al., 2012) and the hexagonal lattice (Christaller, 1933; Lösch, 1954; Ikeda et al., 2014; Ikeda and Murota, 2014). Note that some analogies can be drawn between those various spatial frameworks (Ikeda et al., 2017a,b).

It appears that new economic geography models dealing with multiple regions differ in terms of their spatial setting. However, they share two common strategies to deal with equilibrium multiplicity and the inherent problem of equilibrium selection. The first strategy is to study path-dependency effects in the adjustment dynamics of the regional economy. For example, forward-looking expectations are likely to self-reinforce to the point of influencing the selection of a long-run equilibrium. The second strategy is to select an equilibrium by introducing heterogeneity, either in the geographical assumptions, firm technologies or consumer preferences. Contributions to those two strategies are now presented.

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[1]In contrast, Krugman and Venables (1995) and Venables (1996) studied the influence of *vertical* linkages on agglomeration forces. In Puga (1999), both types of linkages are considered.
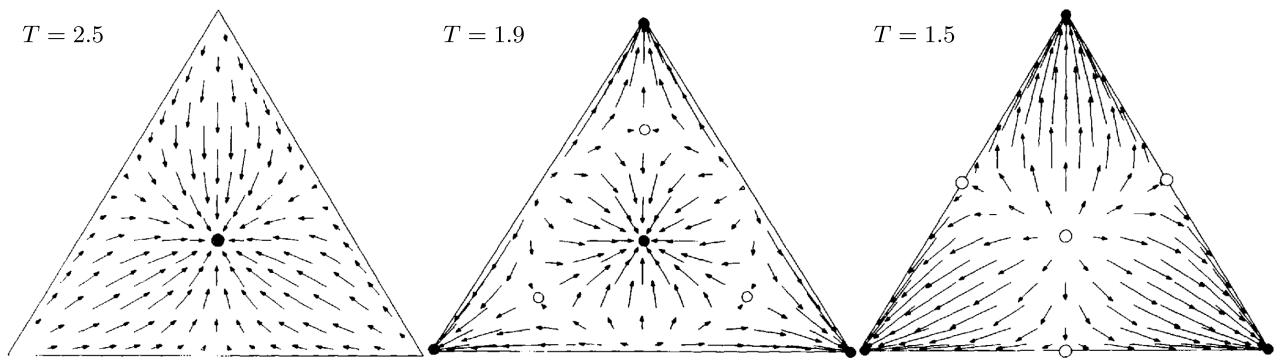
Figure 1: Core-Periphery model with three-regions (from Fujita et al., 1999, section 6.1, p.79). Simplex of the population distribution among the three regions. Each corner represents the concentration of activities in one of the three regions. Arrows depict the dynamics of the population distribution. Black dots represent stable equilibria and white dots represent the unstable ones. T is the value of transport cost (see Fujita et al., 1999, for further details).

The original adjustment dynamics of the core-periphery model of Krugman (1991a) is based on a myopic consideration of wage differentials and it is quite similar to the replicator equation of evolutionary dynamics (see or example Nowak, 2006). While this enables to study numerically the influence of initial (historical) conditions on the selection of the long-term equilibrium, the likeliness of this myopic behaviour assumption is questionable. This has pushed researchers to assume forward-looking expectations, which may generate self-fulfilling prophecies, hence refining the questions of equilibrium selection and stability in dynamic systems with multiple equilibria (Baldwin, 2001). In order to address those issues, researchers in new economic geography have started studying the adjustment dynamics using explicit differential equations under various non-myopic expectations. In a seminal effort, Krugman (1991b) and Fukao and Benabou (1993) have proposed a two-regions model described by a set of two linear differential equations including forward-looking expectations. Their setting however departs from the original core-periphery model in that agglomeration results from technological externalities and not from demand linkages. Ottaviano (1999) used an ingenious assumption on inter-temporal elasticity of substitution between consumption goods to show that Krugman (1991b) can be interpreted as a model with pecuniary agglomeration forces, hence matching the initial core-periphery model. In another study, Ottaviano et al. (2002) used a formulation with quadratic utility functions that also yields linear differential equations. As Baldwin (2001) highlighted, a severe drawback of those linear models is that they loose some important results like, for example, the possibility of simultaneously stable corner and interior solutions that is depicted on Figure 1. Matsuyama (1991) was the first to address the question of stability in a regional system described by non-linear differential equations. Like Krugman (1991b) and Fukao and Benabou (1993), it is a two-regions model where agglomeration results from technological externalities. Ottaviano (2001) used the methodology of Matsuyama (1991) in a model with pecuniary externalities, which is closer to the original core-periphery setting, and discussed the conditions under which expectations may affect the long-run equilibrium population distribution. Baldwin (2001) pushed the idea even further by coming back to the canonical formulation of the core-periphery model. He combined analytical and simulation tools to show that forward-looking expectation have no incidence on the core-periphery model when migration costs are high.

Another strategy to select a long-run equilibrium in the case of multiple equilibria is by introducing heterogeneity in the model. Firms heterogeneity has been introduced in international trade by Jean (2002) and Melitz (2003). Their contribution has been followed by several models of economic geography studying, *inter alia*, how more productive firms cluster in larger markets (Baldwin and Okubo, 2006; Nocke, 2006; Melitz and Ottaviano, 2008). Heterogeneous households' preferences for locations have been introduced in new economic geography by Tabuchi and Thisse (2002)[2]. Several papers have followed, showing how taste heterogeneity acts as a dispersion force (Mossay, 2003; Murata, 2003, 2007; Zeng, 2008; Candau and Fleurbaey, 2011). Regarding the adjustment dynamics discussed previously, those studies use *ad hoc* deterministic adjustment procedures based on expected migrations under myopic behaviours. While this improves the mathematical tractability of the models, it prevents from modelling the aggregate stochasticity generated by taste heterogeneity. This is even more damaging that aggregate stochasticity is expected to affect equilibrium selection and stability, especially in the nonlinear adjustment procedures proposed by Matsuyama (1991, 1995), Ottaviano (2001) and Baldwin (2001). For example, noise-induced transitions, which are equilibrium selections induced by stochasticity, may occur (Horsthemke and Lefever, 2006; Longtin, 2010; Khasminskii, 2012).

There is a literature on nonlinear probabilistic migratory systems that has evolved independently from new economic geography. This research field can be traced back to the entropy-based gravitational model of Wilson (1970), which shares analogies with Alonso's theory of movement (Alonso, 1978; Ledent, 1981). In particular, Moss (1979) proposed a model of migration choice that is based on the discrete choice theory and random utility framework proposed by McFadden (1974) and Manski (1977). This setting has been used in the study of population interactions in migratory systems (Miyao, 1978; Miyao and Shapiro, 1981; de Palma and Lefevre, 1983, 1985; Ginsburgh et al., 1985; Tabuchi, 1986). This idea has further been formalized by Ben-Akiva and de Palma (1986), Kanaroglou et al. (1986b,a) and Kanaroglou and Ferguson (1996) using a nested Logit structure (see Ben-Akiva and Lerman, 1985), which enables to decompose the migration decision between a decision to leave (the "*push*" factor) and a choice of destination (the "*pull*" factor). This "push and pull" model has been used by many empirical studies afterwards (Liaw and Ledent, 1987; Liaw, 1990; Anderson and Papageorgiou, 1994a,b; Newbold and Liaw, 1994; Frey et al., 1996; Pellegrini and Fotheringham, 1999; Lee and Waddell, 2010). See also the review of Pellegrini and Fotheringham (2002). Instead of starting from discrete choice theory, Weidlich and Haag (1983) and Haag (1989) developed an alternative dynamic decision theory that is inspired from stochastic physics and its social applications (the so-called *synergetics*, see Haken, 2004). In those works, individual transition rates from a location to another are mathematically expressed by an exponential of the difference between the so called "*dynamic utilities*" at destination and origin (Weidlich, 2006). Their model has been used in empirical research as well (Haag and Grützmann, 1993; Weidlich and Haag, 1988). No matter they follow from discrete choice theory or from synergetics, models from de Palma and Lefevre (1983), Ben-Akiva and de Palma (1986), Kanaroglou et al. (1986b) and Haag and Weidlich (1984) all end up with a Master equation that is hardly tractable, and then they use a deterministic approximation of the interregional dynamics. de Palma and Lefevre (1983), Ben-Akiva and de Palma (1986) and Kanaroglou et al. (1986b) rely on a deterministic approximation proposed by Kurtz (1978), whilst Haag and Weidlich (1984) use the condition of detailed balance that results from their particular exponential form of transition rates. Thus they are similar to the aforementioned literature on heterogeneity in new economic geography in that they do not provide a satisfactory understanding of the stochasticity that results from aggregated idiosyncratic preferences on the adjustment dynamics of interregional systems.

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[2]See Amiti and Pissarides (2005) and Mori and Turrini (2005) for heterogeneous skills among households. See also Nocco (2009) for a combined treatment of taste and skills heterogeneity.

This paper distinguishes from the previous literature by studying the aggregate effect of idiosyncratic preferences on the adjustment dynamics of core-periphery models. Especially, idiosyncratic preferences are added to the footloose entrepreneur model (Forslid and Ottaviano, 2003; Baldwin et al., 2005), which is taken as an illustrative case for its analytical tractability and its proximity to the original core-periphery model. The geographical setting is made up with three equidistant regions. It is done for simplicity but the proposed methodology does not impose any limit neither on the number nor on the spatial structure of regions. The dynamic assumptions are inspired by the nested Logit approach of Ben-Akiva and de Palma (1986), Kanaroglou et al. (1986b,a) and Kanaroglou and Ferguson (1996), but they are also consistent with the exponential transition rate of Weidlich and Haag (1983) and Haag (1989). An original contribution of this paper is to overcome the problem of aggregating idiosyncratic preferences by using an analytical diffusion approximation of the continuous-time Markov process (Gardiner, 1985; Allen, 2003, 2007; Fuchs, 2013). This enables to express the dynamics of the interregional population distribution by a system of nonlinear *stochastic* differential equations, hence following the nonlinear studies of Matsuyama (1991, 1995), Ottaviano (2001) and Baldwin (2001), but emphasizing the influence of taste heterogeneity (instead of forward-looking expectations) on equilibrium selection and stability.

3   A dynamic footloose entrepreneur model

Consider a population of workers spread among three regions, denoted by the subset of positive integers $\{1, 2, 3\}$. They consume an horizontally differentiated manufacturing good and a so called agricultural good, both being traded between regions. Each worker is endowed with one unit of labour, of either high-skilled or low-skilled type. High-skilled workers are employed in the manufactoral sector whilst low-skill workers are employed in both agricultural and manufactural sectors. The regional high-skill and low-skill endowments of the ith region at time t respectively write $h_i(t)$ and $l_i(t)$, such that at any time t, $h_1(t) + h_2(t) + h_3(t) = H$ and $l_1(t) + l_2(t) + l_3(t) = L$. High-skilled workers are perfectly mobile whilst low-skilled workers are immobile and evenly distributed. Thus at any time, low-skill endowment of region i is $l_i(t) = L/3$, and the distribution of high-killed workers is given by the vector $\mathbf{h}(t)$ whose components are the $h_i(t)$'s.

In this subsection, time is fixed (hence the variable t is omitted for ease of reading) and the distribution of high-skilled workers is given. One derives the clearing-market values of quantities and prices under the given distribution of high-skilled workers. Note that most of the computations are skipped. The reader is invited to consult Forslid and Ottaviano (2003) or Baldwin et al. (2005) for any detail that would remain unclear.

On the demand side, considering workers as customers, high-skilled and low-skilled workers only differ by their wage, whose regional values respectively write y_i and y_i^{L}. Their consumption preferences are represented by the utility function $u(x_i, a_i) = \mu \ln(x_i) + (1 - \mu) \ln(a_i)$ with

$$x_i = \left(\int\limits_M d_i(m)^{(\sigma-1)/\sigma} \mathrm{d}m \right)^{\sigma/(\sigma-1)}, \tag{1}$$

where $\mu \in [0, 1]$ is a constant, x_i is individual consumption of manufactures in the ith region, a_i is individual consumption of agricultural products in the ith region, M is the set of all varieties of x, $d_i(m)$ is the individual consumption of variety m in the ith region and finally, σ is both the demand-elasticity of any variety and the elasticity of substitution between any two varieties.

Turning now to the supply side, on the one hand, firms of the agricultural sector produce an homogeneous good under perfect competition and constant return to scale using low-skilled labour. Units are freely chosen so that one labour unit produces one output unit. The agricultural good is freely traded and it is chosen as the numeraire[3]. On the other hand, firms of the manufactural sector

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[3]Note that the chosen unit of the homogeneous agricultural good a_i, the assumption of perfect competition and the choice of a_i as the numeraire altogether imply $y_i^{\text{L}} = 1$, $\forall i \in \{1, 2, 3\}$. This holds as long as the agricultural good is produced in all regions, which requires $\mu < \sigma/(2\sigma - 1)$ (Forslid and Ottaviano, 2003). This assumption is assumed to hold for now on.

produce differentiated varieties, they are monopolistically competitive and employ both low-skill and high-skill labour under increasing returns to scale. More precisely, a one-to-one relationship exists between firms and variety, and the total production cost of the firm producing $x_i(m)$ units of variety m in region i is given by

$$c_i(m) = \gamma \, y_i + \eta \, x_i(m) y_i^{\text{L}} \, , \qquad (2)$$

where γ is the fixed input requirement of high-skill labour and $\eta \, x_i(m)$ is the marginal input requirement of low-skill labour. The manufactured good endures trade barriers, which are modelled as iceberg costs. Assuming that the 3 regions are equidistant, $\tau \in [1, +\infty[$ is the amount of manufactured good that has to leave the origin region in order to deliver one unit at the region of destination.

Forslid and Ottaviano (2003) have shown that maximizing consumers utility, firms profit and clearing markets yields the indirect utility of high-skilled workers in any region, given their interregional distribution. More precisely, freely indexing the three regions by i, j and k, the indirect utility in the ith region writes

$$U_i(\mathbf{h}) = \ln \left(\mu^\mu (1-\mu)^{1-\mu} \left[\frac{(\sigma-1)\gamma^{1/(1-\sigma)}}{\sigma\eta} \right]^\mu \left[h_i + \tau^{1-\sigma} \left(h_j + h_k \right) \right]^{\mu/(\sigma-1)} y_i^* \right) \, , \qquad (3)$$

where y_i^* is the local equilibrium wage of high-skilled workers, implicitly given by

$$y_i^* = \theta \left[\frac{y_i^* h_i + L/3}{h_i + \phi\left(h_j + h_k\right)} + \frac{\phi\left(y_j^* h_j + L/3\right)}{h_j + \phi\left(h_i + h_k\right)} + \frac{\phi\left(y_k^* h_k + L/3\right)}{h_k + \phi\left(h_i + h_j\right)} \right] \, , \qquad (4)$$

with $\phi = \tau^{1-\sigma}$ and $\theta = \mu/\sigma$. Since the three regions are perfectly symmetric, indirect utilities in other regions write accordingly. Thus $\{y_1^*, y_2^*, y_3^*\}$ is a system of linear equations that can be solved to obtain the equilibrium wages as functions of the spatial distribution of high-skilled workers.

3.1 Individual relocation dynamics

Consider an arbitrary distribution of high-skilled workers among the three regions. This distribution can evolve through time due to workers interregional migration, whilst prices and quantities are assumed to adjust instantaneously to those changes. In this section, a first step is made in modelling the adjustment dynamics of the distribution of high-skilled workers by considering the move of a single worker. The resulting individual relocation dynamic is a continuous-time Markov process whose transition rates follow an exponential law similar to those of Weidlich and Haag (1983); Haag (1989); Weidlich (2006).

High-skilled workers' relocation choice follows two assumptions. Firstly, it results from both economic incentives that are observable at the population scale and, idiosyncratic preferences that are non-observable and potentially non-economic. Secondly, it is origin-specific because non-observable incentives give a particular status to the location of origin. As a result, the decision is sequential: a worker starts by deciding to move or not, and then he chooses where to go, if appropriate. Under those assumptions, the relocation choice can be modelled as a nested Logit model (Ben-Akiva and Lerman, 1985), which has already been used in literature on migrations (Ben-Akiva and de Palma, 1986; Kanaroglou et al., 1986b; Haag and Grützmann, 1993).

Formally, workers' tastes for regions are described by an utility function $\widetilde{U} : \{1, 2, 3\} \mapsto \mathbb{R} : i \mapsto \widetilde{U}_i(\mathbf{h}) = U_i(\mathbf{h}) + \varepsilon$ where $U_i(\mathbf{h})$ is the (deterministic) indirect utility function given in (3), and ε is a random variable standing for idiosyncratic preferences. ε's are Gumbell distributed with zero mean, following the Logit model of discrete choice theory (McFadden, 1974; Anderson et al., 1992). Consider again the free indexing of the three regions by i, j and k, and take a worker located in region i. This worker's origin location has a complementary set $R_i = \{j, k\}$. This partition of the set of regions is the support of the nested structure of the relocation decision whose first stage, between i or R_i, is the choice to stay or to move, whilst its second stage decision, within R_i, is the potential destination choice
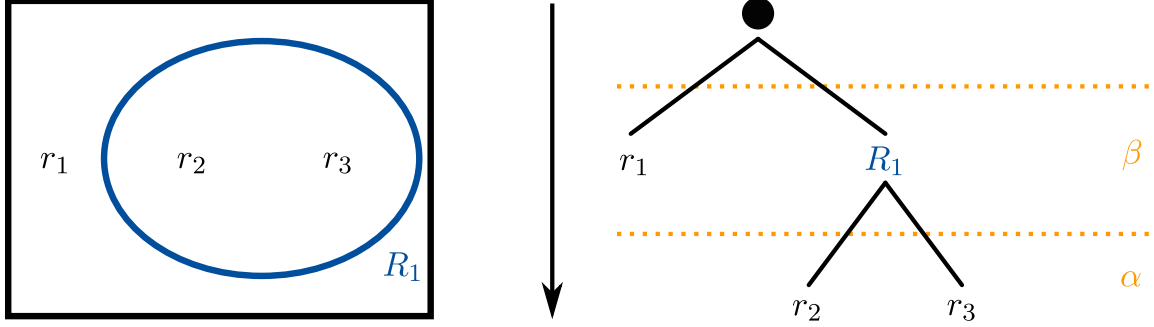
Figure 2: Nested structure of individual relocation decision. The left box shows how the current location, say r_1, of an high-skilled worker defines a partition of the set of possible locations $\{r_1, r_2, r_3\}$. This partition is the basis of a two-stage decision process that is represented by a decision tree on the right. First, the worker decides whether to leave its current location or not, and second he may choose a destination. The scale parameters of the Gumbell distributions of idiosyncratic utilities at each stage respectively write β and α.

(Figure 2). Following the classic nested Logit approach (Ben-Akiva and Lerman, 1985), the probability for an arbitrary worker to move from region i to any other destination, say j, is the product of the probability for him to choose to leave its current location $p_o(i, \mathbf{h})$ and the probability for him to choose region j for destination $p(j|i, \mathbf{h})$. Those probabilities write

$$p_o(i, \mathbf{h}) = \frac{e^{\mathbb{U}(R_i, \mathbf{h})/\beta}}{e^{U_i(\mathbf{h})/\beta} + e^{\mathbb{U}(R_i, \mathbf{h})/\beta}} \ , \tag{5}$$

$$p(j|i, \mathbf{h}) = e^{\left[U_j(\mathbf{h}) - \mathbb{U}(R_i, \mathbf{h})\right]/\alpha} \ , \tag{6}$$

where α and β are the scale parameters of the Gumbell distributions of the idiosyncratic utilities, respectively at the second and first stage (Figure 2), such that $0 < \alpha/\beta < 1$ (Ben-Akiva and Lerman, 1985). $\mathbb{U}(R_i, \mathbf{h})$ is the attractiveness (or inclusive value) of the set of possible destinations, such as

$$\mathbb{U}(R_i, \mathbf{h}) = \alpha \ln \left(\sum_{j \in R_i} e^{U_j(\mathbf{h})/\alpha} \right) \ . \tag{7}$$

Consider now the relocation dynamics of a single worker. From an applied perspective, it is necessary to define in explicit time units at which rate the relocation decision takes place. The dynamic process is assumed to be memoryless: the worker considers only the current state of the system. For simplicity, it is also assumed that he does not develop any anticipating scenario on the basis of this current state. Concretely, the number of times he goes through the nested decision sequence is a Poisson process with (origin-specific) decision rate $v_i(\mathbf{h}) \in \mathbb{R}^+$. Thus, $v_i(\mathbf{h})^{-1}$ is the time interval at which only one decision is expected to occur.

From an empirical perspective, it means that if one wants to estimate the departure probability $p_o(i, \mathbf{h})$ from empirical migration data, he has to count the number of departures within a time interval $v_i(\mathbf{h})^{-1}$. Unfortunately, available migration data may have been reported within a time interval Δt that is not necessarily equal to $v_i(\mathbf{h})^{-1}$. Whilst Anderson and Papageorgiou (1994a) solved this problem by using a proportionality assumption, here the Poisson process enables a simpler solution. Indeed, one easily shows that the empirical departure probability $p_o^{\mathrm{E}}(i)$, measured by counting the number of departures from region i within a time interval Δt, is related to the theoretical probability $p_o(i, \mathbf{h})$, defined in a time interval $v_i(\mathbf{h})^{-1}$, by $p_o^{\mathrm{E}}(i) = [\Delta t / v_i(\mathbf{h})^{-1}] p_o(i, \mathbf{h})$.
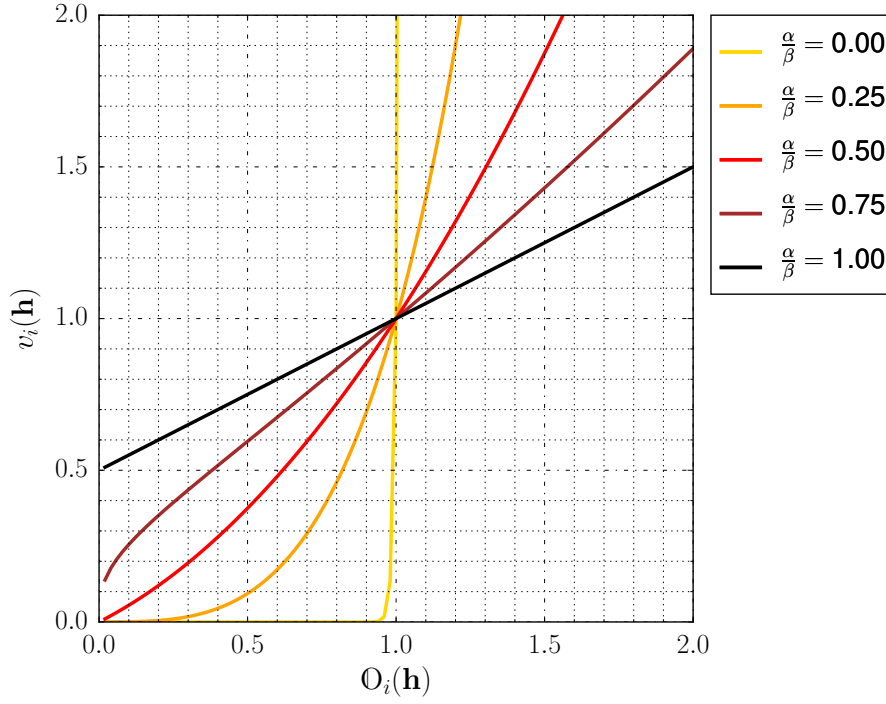
Figure 3: Relationship between the origin-specific individual decision rate and the odds of outmigration. Equation 8, which relates the origin-specific individual decision rate $v_i(\mathbf{h})$ to the odds of outmigration $\mathbb{O}_i(\mathbf{h})$, is plot for five different values of the ratio α/β.

Under those assumptions, the individual relocation dynamics is a continuous time Markov chain (Ross, 2009) which is simply a repetition of the relocation decision. Especially, its state set is $\{1, 2, 3\}$, the leaving rate of any location $i \in \{1, 2, 3\}$ is $w_i(\mathbf{h}) = v_i(\mathbf{h}) \, p_o(i, \mathbf{h})$ and the transition probabilities, that is, the probabilities to reach any destination $j \in R_i$ given that one is leaving region i, are given by $p(j|i, \mathbf{h})$. Finally, following Ross (2009), the infinitesimal transition rate from a location to another (or generator) is defined by $q(j|i, \mathbf{h}) = w_i(\mathbf{h}) \, p(j|i, \mathbf{h})$.

The functional form of the origin-specific decision rate v_i is difficult to express *a priori*. It seems reasonable that the worse a worker evaluates his current location compared to the others, the bigger his decision rate is. Thus, one may assume that v_i is an increasing function of the odds of outmigration $\mathbb{O}_i(\mathbf{h})$, which is the ratio of the probability to leave region i over the probability to stay. For example, take

$$v_i(\mathbf{h}) = \frac{v}{2} \left[1 + \mathbb{O}_i(\mathbf{h}) \right] \mathbb{O}_i(\mathbf{h})^{\beta/\alpha - 1} \; , \tag{8}$$

where the odds of outmigration is

$$\mathbb{O}_i(\mathbf{h}) = e^{\left[\mathbb{U}(R_i, \mathbf{h}) - U_i(\mathbf{h}) \right] / \beta} \; , \tag{9}$$

and where $v \in \mathbb{R}^+$ is the decision rate of an individual who evaluates his current location as good as the expected value of the potential destinations. The decision rate (8) increases with the odds of outmigration, and this increase is stronger for low values of the ratio α/β (Figure 3). From Ben-Akiva and Lerman (1985), if $\alpha = \beta$ then the nested structure disappears, which would mean in this case that there is no "home sweet home" effect. At the opposite, lower values of α/β strengthen this effect. Hence the rationale for (8) is that the stronger the "home sweet home" effect, the more sensitive the decision rate is with respect to the odds of outmigration.

Assuming (8), the transition rates write

$$q(j|i, \mathbf{h}) = \frac{v}{2} \, e^{\left[U_j(\mathbf{h}) - U_i(\mathbf{h}) \right] / \alpha} \; , \tag{10}$$

which is the exponential form of sociodynamics (Weidlich and Haag, 1983; Haag, 1989; Weidlich, 2006). Substituting (3) into (10) finally yields the individual transition rates from any region i to any other $j \in R_i$,

$$q(j \,|\, i, \mathbf{h}) = \frac{v}{2} \left[\frac{h_j + \phi\Big(h_i + h_k\Big)}{h_i + \phi\Big(h_j + h_k\Big)} \right]^{\frac{\mu}{\alpha(\sigma-1)}} \left[\frac{Ah_j^2 + B(h_i^2 + h_k^2) + Ch_j(h_i + h_k) + Dh_ih_k}{Ah_i^2 + B(h_j^2 + h_k^2) + Ch_i(h_j + h_k) + Dh_jh_k} \right]^{\frac{1}{\alpha}} , \quad (11)$$

where

$$A = 3\phi , \quad (12)$$

$$B = \theta(2\phi^2 - \phi - 1) + (\phi^2 + \phi + 1) , \quad (13)$$

$$C = \theta(\phi^2 + \phi - 2) + 2(\phi^2 + \phi + 1) , \quad (14)$$

$$D = \theta^2(2\phi^2 - 3\phi + \phi^{-1}) + 2\theta(\phi^2 - \phi^{-1}) + (2\phi^2 + 3\phi + \phi^{-1}) , \quad (15)$$

are constants.

3.2   Collective spatial dynamics

Consider now the entire population of high-skilled workers. The local share of total population is $s_i(t) = h_i(t)/H$, which is the ith component of the vector $\mathbf{s}(t) = \mathbf{h}(t)/H$. In this section, one is interested in the dynamics of the vector $\mathbf{s}(t)$. It is assumed that workers decisions' timings are independent. That is, each worker independently follows the individual relocation dynamics described before. As a result, the probability for two workers to move simultaneously is negligible. Thus, over an infinitesimal time step $\mathrm{d}t$, the infinitesimal change vector $\boldsymbol{\delta}(t) = \mathbf{s}(t + \mathrm{d}t) - \mathbf{s}(t)$ can only take 6 different values, each one describing the move of a single worker along one of the six possible origin-destination couples of regions. Those instances of the change vector can be ordered arbitrarily and grouped into a jump matrix $\mathbf{J}$. For example, take

$$\mathbf{J} = \frac{1}{H} \begin{pmatrix} -1 & -1 & 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & -1 & 0 & 1 \\ 0 & 1 & 0 & 1 & -1 & -1 \end{pmatrix} , \quad (16)$$

such that the kth change vector is the kth column of $\mathbf{J}$, which notes $\mathbf{J}_{.k}$. For convenience, let $O : \{\mathbf{J}_{.1}, ..., \mathbf{J}_{.6}\} \mapsto \{1, 2, 3\}$ and $I : \{\mathbf{J}_{.1}, ..., \mathbf{J}_{.6}\} \mapsto \{1, 2, 3\}$ be respectively the origin and destination functions (corresponding to outward and inward population fluxes) such that if $\mathbf{J}_{.k}$ is the change vector describing a worker move from region i to region j, then $O(\mathbf{J}_{.k}) = i$ and $I(\mathbf{J}_{.k}) = j$. With the matrix (16), this yields for example $O(\mathbf{J}_{.1}) = 1$ and $O(\mathbf{J}_{.1}) = 2$. Actually, under the assumption of independent departure times, the collective spatial dynamics of the H high-skilled workers is a continuous-time Markov chain as well[4]. Especially, its state set writes S and its transition rates from any state $\mathbf{s}$ to any other $\mathbf{s}'$ are given, for an interregional system with H high-skilled workers, by

$$Q(\mathbf{s}' \,|\, \mathbf{s}, H) = \begin{cases} Hs_{O(\mathbf{s}'-\mathbf{s})}\, q\Big(I(\mathbf{s}' - \mathbf{s}) \,|\, O(\mathbf{s}' - \mathbf{s}), H\mathbf{s}\Big), & \mathbf{s}' - \mathbf{s} \in \{\mathbf{J}_{.1}, ..., \mathbf{J}_{.6}\} \\ 0, & \mathbf{s}' - \mathbf{s} \notin \{\mathbf{J}_{.1}, ..., \mathbf{J}_{.6}\} \end{cases} . \quad (17)$$

One is interested in describing the transition probability function $P(\mathbf{s}, t \,|\, \mathbf{s_0}, H)$, denoting the probability that a three-regions system with H high-skilled workers in state $\mathbf{s_0}$ will be in state $\mathbf{s}$ a time t later. Its derivative with respect to time, written $\dot{P}(\mathbf{s}, t \,|\, \mathbf{s_0}, H)$, is described by the well-known discrete Kolmogorov-forward equation (or Master equation, see Gardiner, 1985, 2004), that is

$$\dot{P}(\mathbf{s}, t \,|\, \mathbf{s_0}, H) = \sum_{\mathbf{s}' \in S} \left\{ Q(\mathbf{s} \,|\, \mathbf{s}', H)P(\mathbf{s}', t \,|\, \mathbf{s_0}, H) - Q(\mathbf{s}' \,|\, \mathbf{s}, H)P(\mathbf{s}, t \,|\, \mathbf{s_0}, H) \right\} , \quad (18)$$

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[4]This can be demonstrated by lumping, see Tian and Kannan (2006)

whose solution has no general closed-form expression, so that numerical methods are required (Ross, 2009). This paper proposes instead to use an analytical approximation of the discrete Kolmogorov's forward equation as a continuous diffusion process (Fuchs, 2013), which can in turn be expressed as a set of Itô stochastic differential equations. The diffusion approximation is based on the idea that infinitesimal changes describing the evolution of the state vector $\mathbf{s}$ are somehow small compared to the total population H. Thus for a large population, the dynamic change of $\mathbf{s}$ can be approximated by a continuous process. Taking the Kramers-Moyal expansion of the discrete Kolmogorov-forward equation, and truncating the resulting expression to the second order term yields the Fokker-Planck (or continuous Kolmogorov forward) approximation[5] of the transition probability function for a system of population H,

$$\dot{P}(\mathbf{s}, t \,|\, \mathbf{s_0}, H) \simeq -\nabla \cdot \Big(\mathbf{E}(\boldsymbol{\delta} \,|\, \mathbf{s}, t) P(\mathbf{s}, t \,|\, \mathbf{s_0}, H) \Big) + \frac{1}{2H} \nabla^2 \Big(\mathbf{D^2}(\boldsymbol{\delta} \,|\, \mathbf{s}, t) P(\mathbf{s}, t \,|\, \mathbf{s_0}, H) \Big) \,, \qquad (19)$$

describing a diffusion process where ∇ is the nabla (or Del) operator, the central dot $\cdot$ stands for the scalar product, and where the drift coefficient $\mathbf{E}(\boldsymbol{\delta} \,|\, \mathbf{s}, t)$ and the diffusion coefficient $\mathbf{D^2}(\boldsymbol{\delta} \,|\, \mathbf{s}, t)$ are respectively the expected value and covariance matrix of the instantaneous change vector $\boldsymbol{\delta}$, given by

$$\mathbf{E}(\boldsymbol{\delta} \,|\, \mathbf{s}, t) = \mathbf{J} \times \mathbf{Q}(\mathbf{s}, H) \qquad (20)$$

$$\mathbf{D}^2(\boldsymbol{\delta} \,|\, \mathbf{s}, t) = H \Big[\mathbf{J} \times \mathbb{D}\Big(\mathbf{Q}(\mathbf{s}, H) \Big) \times \mathbf{J}^{\mathrm{tr}} \Big] \qquad (21)$$

where $\mathbf{J}^{\mathrm{tr}}$ is the transpose of $\mathbf{J}$, $\mathbf{Q}$ is the (6×1)-matrix whose k^{th} element is

$$\mathbf{Q}_k(\mathbf{s}, H) = Q(\mathbf{s} + \mathbf{J}_{.k} \,|\, \mathbf{s}, H) \,, \qquad (22)$$

and $\mathbb{D}\Big(\mathbf{Q}(\mathbf{s}, H) \Big)$ is the (6×6)-diagonal matrix whose main diagonal's components are the elements of $\mathbf{Q}(\mathbf{s}, H)$.

Equation 19 is an approximation since H is still in the right-hand side expression, at the denominator of the diffusion term. However, note that $Q(\mathbf{s}' \,|\, \mathbf{s}, H) = H\, Q(\mathbf{s}' \,|\, \mathbf{s})$ holds, where $Q(\mathbf{s}' \,|\, \mathbf{s})$ does not depends on H any more (see equations 11 and 17). Thus the H's simplify in (20) and (21), so that $\mathbf{E}$ and $\mathbf{D}^2$ do not depend on H neither, what has two important consequences. First, the H at the denominator of the diffusion term is the only one appearing in (19). As a result, the approximation becomes exact for the limit $H \to \infty$, which corresponds to the deterministic model. Second, Fuchs (2013) has shown that this property makes the method more robust since a van Kampen expansion (see van Kampen, 1992) would not yield different results than the Kramers-Moyal expansion.

Finally, the probability distribution exactly solving the right-hand side of (19) is identical to the distribution of solutions to

$$\begin{cases} \mathtt{d}\mathbf{s}(t) = \mathbf{E}(\boldsymbol{\delta} \,|\, \mathbf{s}, t)\mathtt{d}t + \dfrac{\mathbf{D}(\boldsymbol{\delta} \,|\, \mathbf{s}, t)}{\sqrt{H}} \mathtt{d}\mathbf{W}(t) \\[2ex] \mathbf{s}(0) = \mathbf{s_0} \end{cases} \qquad (23)$$

which is a system of Itô stochastic differential equations where $\mathbf{W}(t)$ is a Wiener process and $\mathbf{D}$ is the Cholesky decomposition of $\mathbf{D^2}$. Note that any other square root of $\mathbf{D^2}$ can be used (see Fuchs, 2013, p.40, referring to Stroock and Varadhan, 1997).

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[5]Note that the accuracy of the approximation does not necessarily improve when truncating after higher degrees since according to Pawula's theorem, the Kramers-Moyal expansion either terminates after the first or second order term, or it contains an infinity of terms (Pawula, 1967).

4 Equilibrium stability, selection and dynamic trajectories

This section starts by presenting the equilibrium stability, selection and dynamic trajectories of the deterministic system (the limit case $H \to \infty$) before discussing those concepts in the stochastic case. It follows from (11), (17) and (20) that differential equations of the system (23) have complex nonlinear forms, hence the results are obtained numerically. Thorough this section, the parameters are $\alpha = 10^{-2}$, $\mu = 0.5$, $\sigma = 5$, $\tau = 1.1504$ and $v = 1/365$. Those parameters where chosen to reproduce the intermediate case of Figure 1 (with $T = 1.9$), where even and uneven distributions of high-skilled workers are possible equilibria. Note that one parametrization is enough to support the results and so a comprehensive sensitivity analysis is out of the scope of this paper. Simulations are performed using the Euler method, on a daily basis, over 20 years.

Results are presented in a simplex whose summits are $\mathbf{S}_1 = (1, 0, 0)$, $\mathbf{S}_2 = (0, 1, 0)$ and $\mathbf{S}_3 = (0, 0, 1)$ (Figure 4). Just like in the intermediate case of Figure 1, in our deterministic case the regional system has four stable equilibria: one corresponding to the even dispersal of high-skilled workers ($\mathbf{s}_0^*$) and three corresponding to their agglomeration in each region ($\mathbf{s}_1^*$, $\mathbf{s}_2^*$ and $\mathbf{s}_3^*$, see Figure 4). Note that contrary to Krugman (1991) and Fujita (1999), the agglomeration is never complete and a small number of high-skilled workers remains in the other regions. This results from the dispersion strength of idiosyncratic preferences described by Tabuchi and Thisse (2002) and Murata (2003). There are also three saddle points which are crucial to understand dynamic trajectories of the system. Indeed, starting from e.g. $\mathbf{s} = (0.5, 0.1, 0.4)$, the dynamic trajectory will first be attracted to the saddle point before reaching its unstable manifold and ending up in equilibrium $\mathbf{s}_1^*$ (see the violet trajectory on Figure 4). Note that in the deterministic case, this is the only trajectory starting from $\mathbf{s} = (0.5, 0.1, 0.4)$, such that by extension there is a bijective correspondence between each point of the simplex and the associated long-term equilibrium. Thus equilibrium selection is not a issue.

In the stochastic case however, this bijective correspondence does not hold any more. In the simplest case, idiosyncratic preferences simply add noise around the deterministic trajectory such that the regional system finally oscillates around its deterministic equilibrium. Yet in other cases, they may also push the system towards another equilibrium. For example, a system in initial state $\mathbf{s} = (0.5, 0.1, 0.4)$ may end gravitating around state $\mathbf{s}_3^*$ instead of state $\mathbf{s}_1^*$ (Figure 4). Oscillations around equilibrium states require an adequate notion of equilibrium stability, and noise-induced transitions confuse the equilibrium selection pattern. Those two problems are now addressed. Note that due to the symmetry of the simplex, results will focus on its bottom left area (Figure 4).

Idiosyncratic preferences induce oscillating moves of the system state around the equilibrium states. Although their probability is negligible, large deviations from the expected trajectory may occur. In that case, the only meaningful definition of stability is that "*at any fixed time, the sample function should lie in the neighborhood of the origin with sufficiently high probability*" (Khasminskii, 2012, p.27). To assess the stabilities of equilibria $\mathbf{s}_0^*$ and $\mathbf{s}_1^*$ according to this definition, the following procedure is applied. Starting from an equilibrium, say $\mathbf{s}_0^*$, 1000 simulations are performed. For each simulation, the root mean square distance between $\mathbf{s}_0^*$ and the temporary state of the system is computed at the beginning of each year. This finally enables to plot, for each year, the estimated probability for the regional system to be closer than a given distance to the $\mathbf{s}_0^*$.
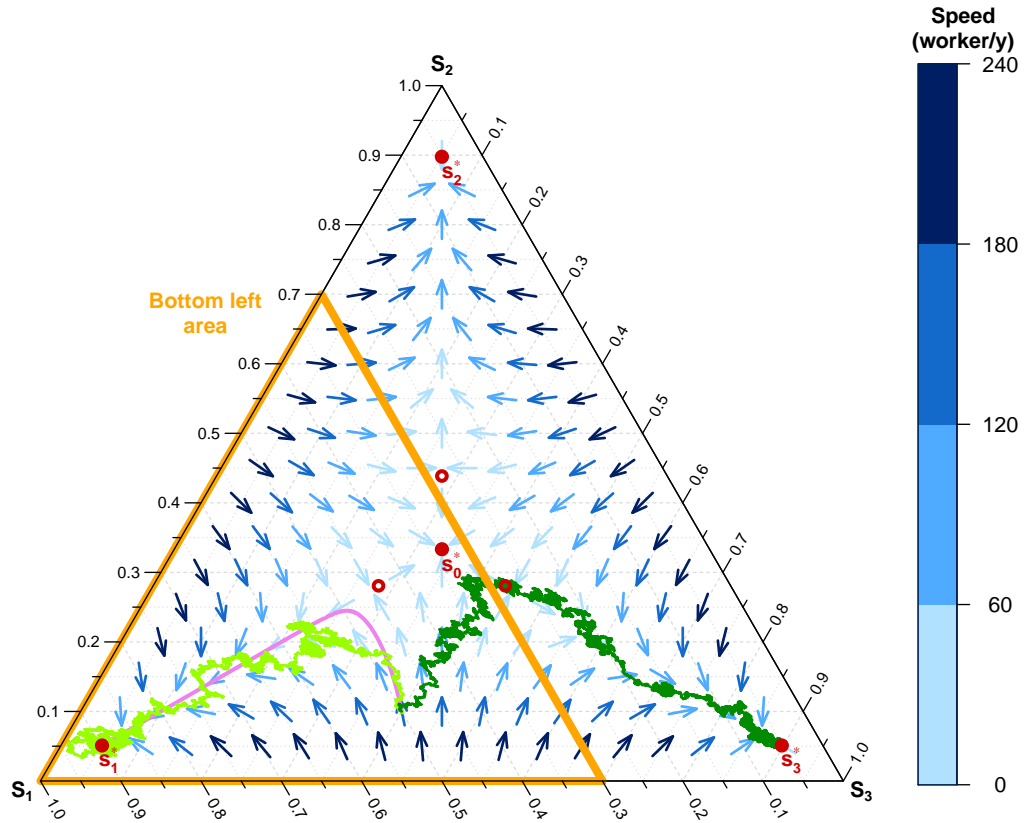
Figure 4: Noise-induces transitions in the footloose entrepreneur model with three regions. The black triangle is the simplex of the population distribution among the three regions, with summits $\mathbf{S}_1 = (1, 0, 0)$, $\mathbf{S}_2 = (0, 1, 0)$ and $\mathbf{S}_3 = (0, 0, 1)$. Blue arrows depict the expected values of the instantaneous change vector (20). Red dots are stable equilibria and red circles are saddle points. Violet line is the unique deterministic trajectory starting from $\mathbf{s} = (0.5, 0.1, 0.4)$, whilst green lines are two sample paths starting from $\mathbf{s} = (0.5, 0.1, 0.4)$. The orange triangle is the area that is explored for discussing equilibrium selection.

Figure 5: Probabilistic discussion of the stability of equilibrium s_0^*. Starting from equilibrium s_0^*, $1\,000$ simulations were used to estimate the probability (on ordinate) to be closer than a given root mean square distance (on abscissa) to s_0^*. Those probabilities were computed on a yearly basis during 20 years (see the color scale).

As expected, the estimated probability is an increasing function of the maximal root mean square distance to the equilibrium, may it be either s_0^* (Figure 5) or s_1^* (Figure 6). Starting from the dispersed equilibrium s_0^*, one sees that the probability for the interregional system not to be in a the neighbourhood of s_0^* becomes negligible for a radius larger than 0.4 units (Figure 5). This radius actually includes the other equilibria in the neighbourhood. For smaller radius, increasing the time interval decreases the probability that the system remains in the neighbourhood. The rationale is that a larger time interval increases the probability of large deviations from the expected trajectory to occur and to push the interregional system toward another equilibrium. The agglomerated equilibrium s_1^* is much more stable since starting from it, the probability for the interregional system to not remain in the neighbourhood of s_1^* become negligible for a radius larger than 0.04 units only (Figure 6). Moreover, the time interval has no effect on this relationship. The explanation is that the agglomerated equilibrium s_1^* is further from the saddle points than the dispersed equilibrium s_0^* (Figure 4). As a result, its basin of attraction is wider and deviations from the equilibrium are not amplified.

Complementary to equilibrium stability is the question of equilibrium selection. In the stochastic case, the question is: starting from an initial state that is not an equilibrium, what is the probability to lie, after a fixed time, in a given neighbourhood of the different equilibria? To study this question, the following procedure is applied. Exploiting the symmetry of the vector field, the bottom-left area of the simplex is evenly covered by 33 initial conditions. For each starting state, 1000 simulations are performed and for each simulation, the root mean square distances from s_0^* and s_1^* to the temporary state of the system are computed every five years. This finally enables to plot, for each starting initial state, the estimated probability for the regional system to be, at five years time intervals, in the vicinity of s_0^* of s_1^*.
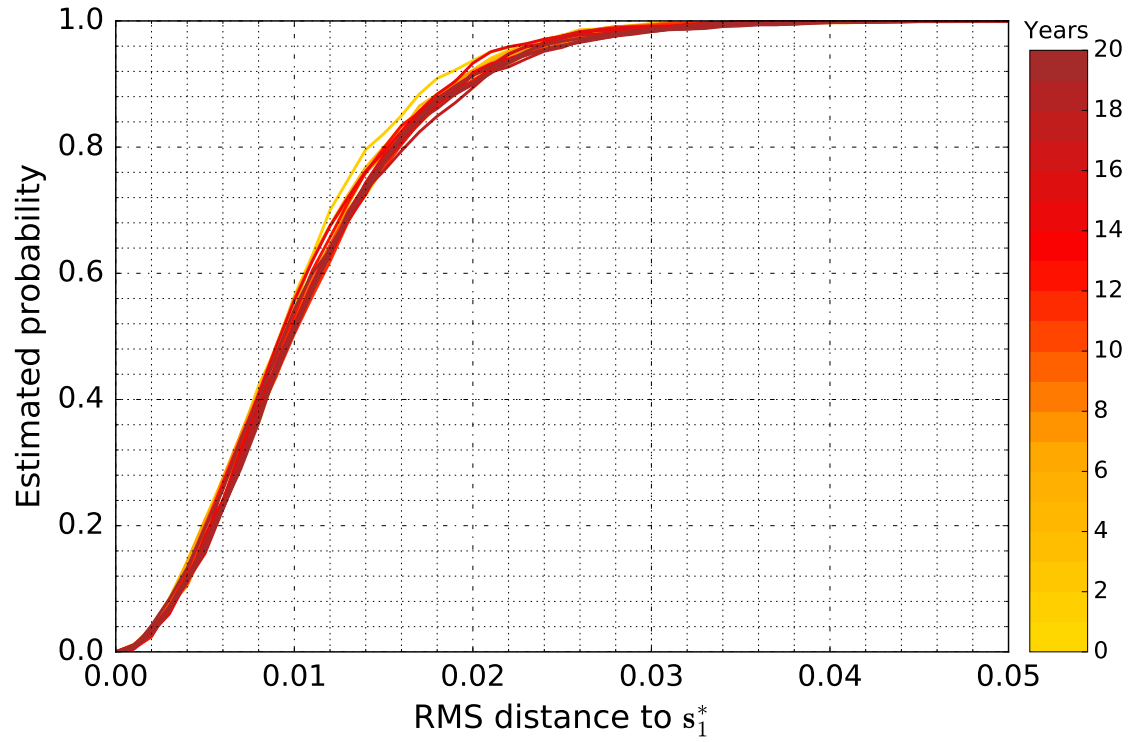
Figure 6: Probabilistic discussion of the stability of equilibrium s_1^*. Red dots are stable equilibria and red circles are saddle points. Starting from equilibrium s_1^*, $1\,000$ simulations were used to estimate the probability (on ordinate) to be closer than a given root mean square distance (on abscissa) to s_1^*. Those probabilities were computed on a yearly basis during 20 years (see the color scale).

Figure 7 depicts the probability for the interregional system to be, after five years, within 0.3 units from s_0^*. The probability function has been estimated by linear interpolation between the 33 initial states. Although the probability is globally decreasing with the distance to s_0^*, it decreases faster along the unstable manifold of the saddle point. Hence the saddle point does influence equilibrium selection and the effect of distance to s_0^* is not isotropic. Increasing the time interval reduces the probability to lie in the neighbourhood of s_0^* (Figure 8, 9 and 10). It also shows that isolines of the estimated probability function tend to be convex with respect to the saddle point (see for example 9). The agglomerated equilibrium s_1^* turns out to be more often selected since in most of the state space, the probability to end up in the neighbourhood of s_1^* is higher than 0.8 (Figure 11). Isolines of the estimated probability function are globally concentric around s_1^*, and they slowly move further away as the time interval growths (Figure 12, 13 and 14).
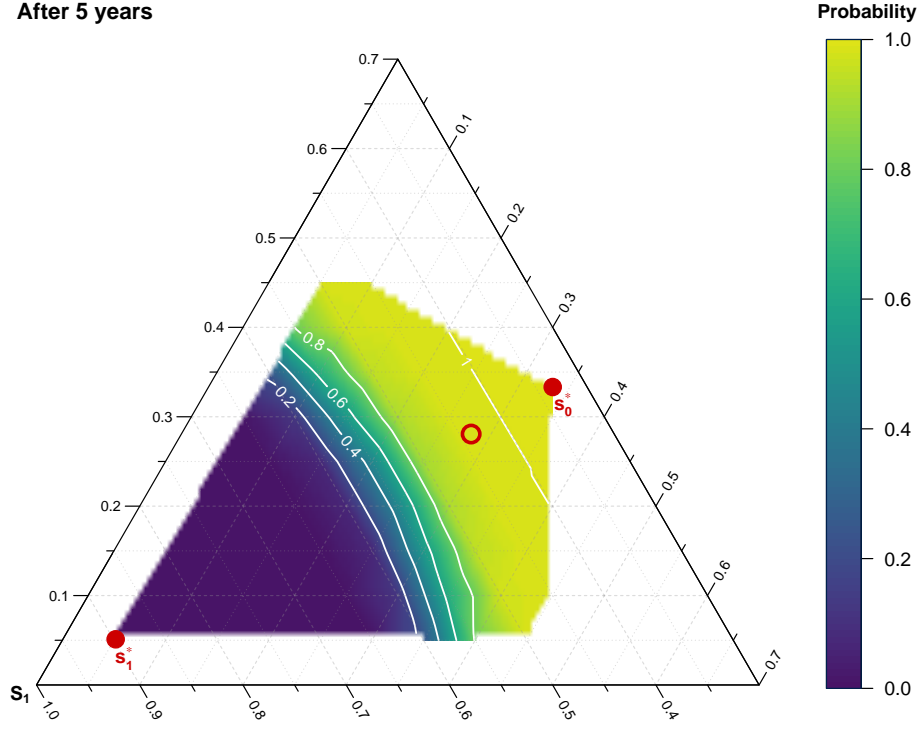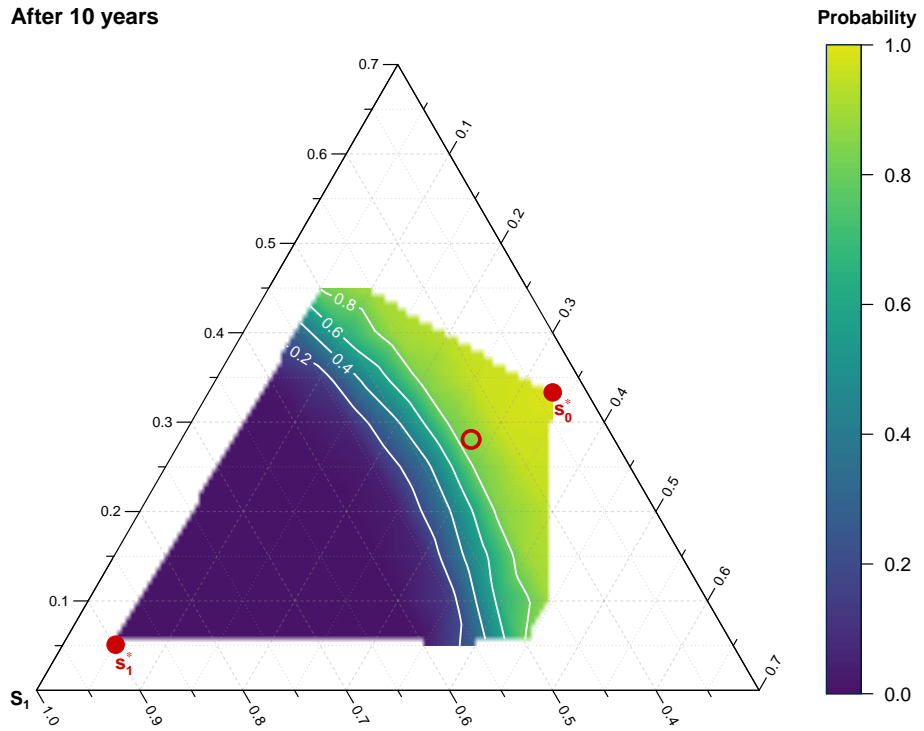
Figure 7: Probability of selecting equilibrium s_0^*. Taking any point within the triangle as a starting point, the color scale depicts the probability of the system to be closer than 0.3 units to s_0^* after 5 years. The area was sampled using 33 starting points with $1\,000$ simulations for each, and the probability functions was obtained by linear interpolation.
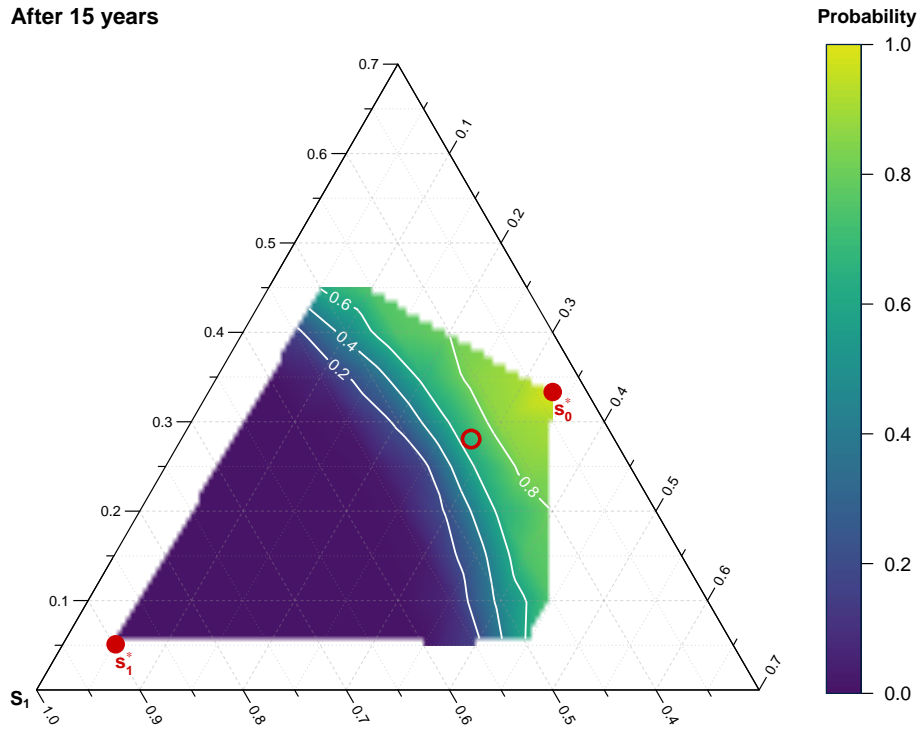

Figure 8: Probability of selecting equilibrium s_0^*. Taking any point within the triangle as a starting point, the color scale depicts the probability of the system to be closer than 0.3 units to s_0^* after 10 years. The area was sampled using 33 starting points with $1\,000$ simulations for each, and the probability functions was obtained by linear interpolation.
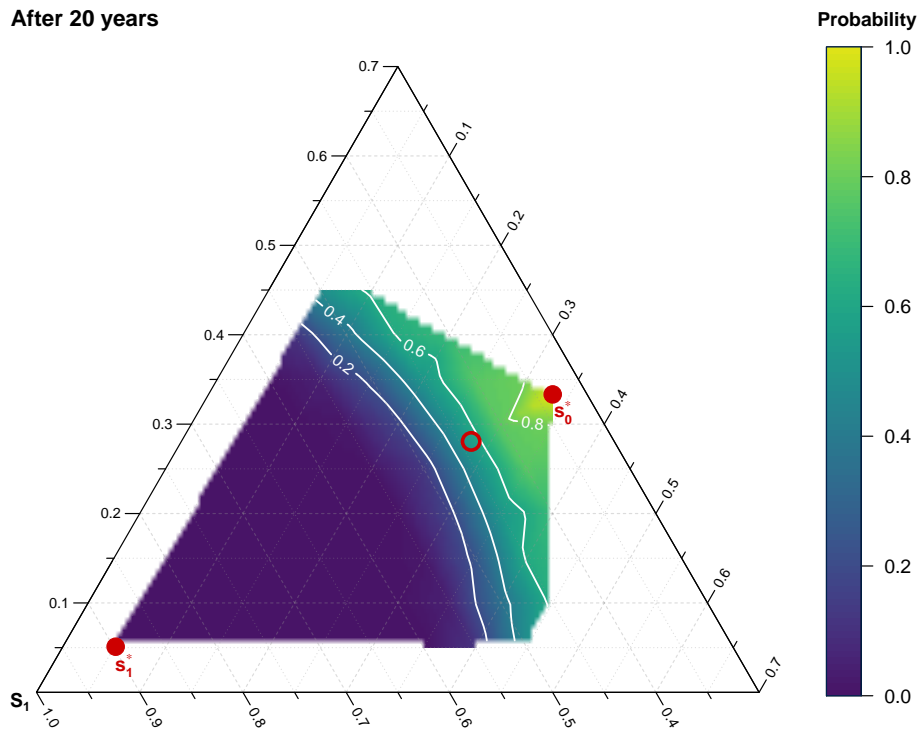
Figure 9: Probability of selecting equilibrium s_0^*. Taking any point within the triangle as a starting point, the color scale depicts the probability of the system to be closer than 0.3 units to s_0^* after 15 years. The area was sampled using 33 starting points with $1\,000$ simulations for each, and the probability functions was obtained by linear interpolation.

Figure 10: Probability of selecting equilibrium s_0^*. Taking any point within the triangle as a starting point, the color scale depicts the probability of the system to be closer than 0.3 units to s_0^* after 20 years. The area was sampled using 33 starting points with $1\,000$ simulations for each, and the probability functions was obtained by linear interpolation.
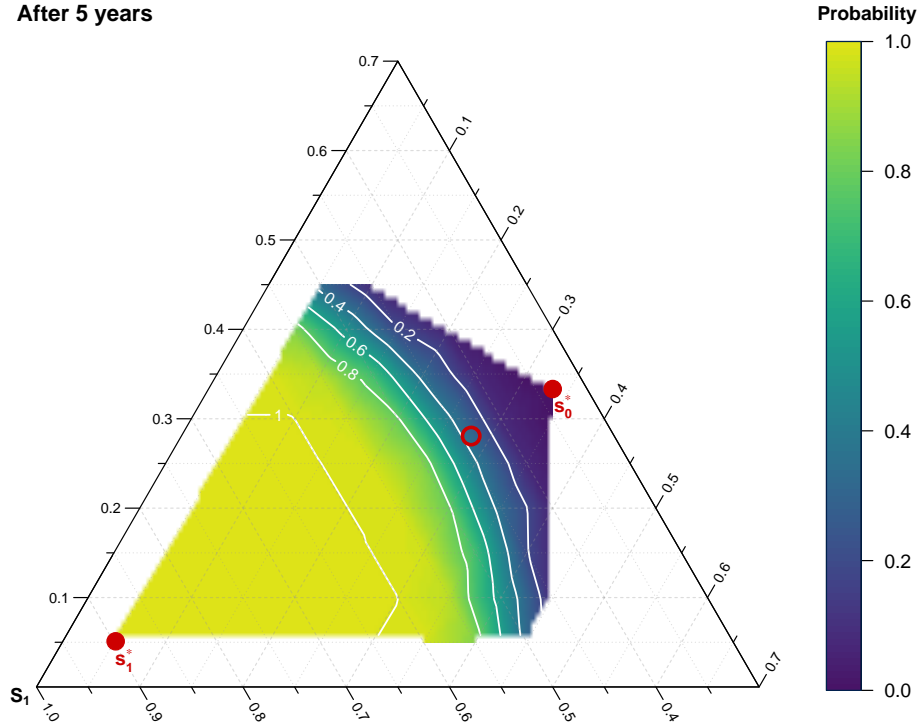
Figure 11: Probability of selecting equilibrium s_1^*. Taking any point within the triangle as a starting point, the color scale depicts the probability of the system to be closer than 0.3 units to s_1^* after 5 years. The area was sampled using 33 starting points with $1\,000$ simulations for each, and the probability functions was obtained by linear interpolation.
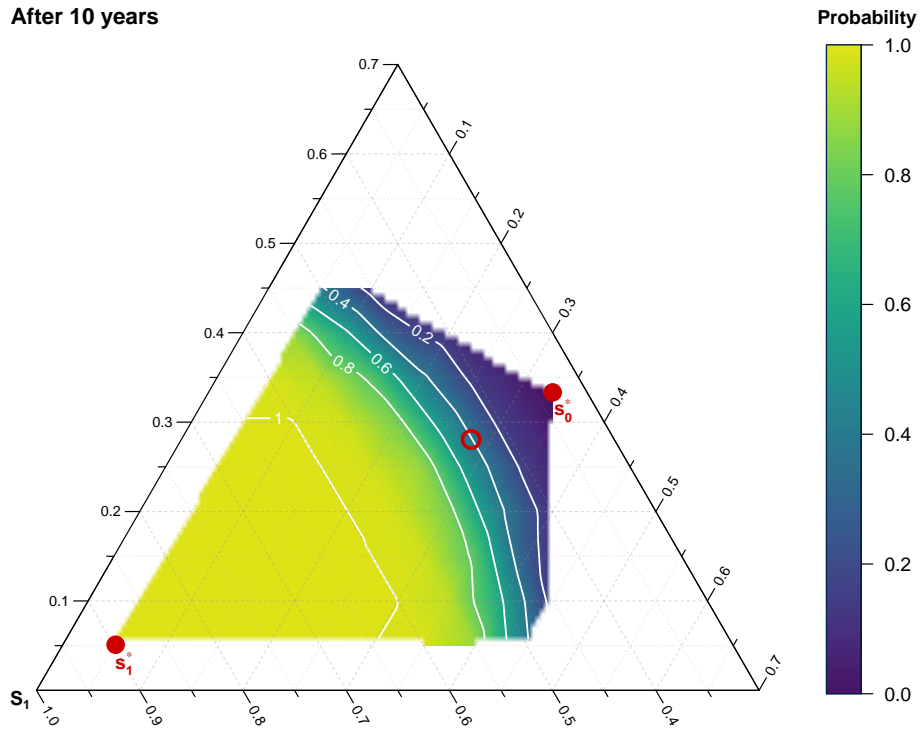

Figure 12: Probability of selecting equilibrium s_1^*. Taking any point within the triangle as a starting point, the color scale depicts the probability of the system to be closer than 0.3 units to s_1^* after 10 years. The area was sampled using 33 starting points with $1\,000$ simulations for each, and the probability functions was obtained by linear interpolation.
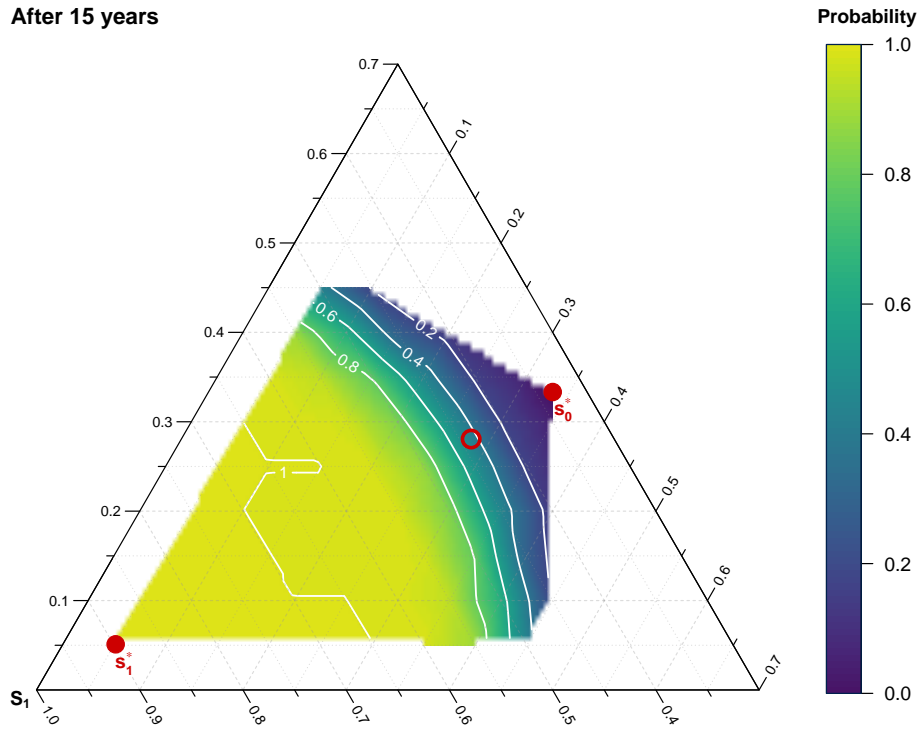
Figure 13: Probability of selecting equilibrium s_1^*. Taking any point within the triangle as a starting point, the color scale depicts the probability of the system to be closer than 0.3 units to s_1^* after 15 years. The area was sampled using 33 starting points with $1\,000$ simulations for each, and the probability functions was obtained by linear interpolation..
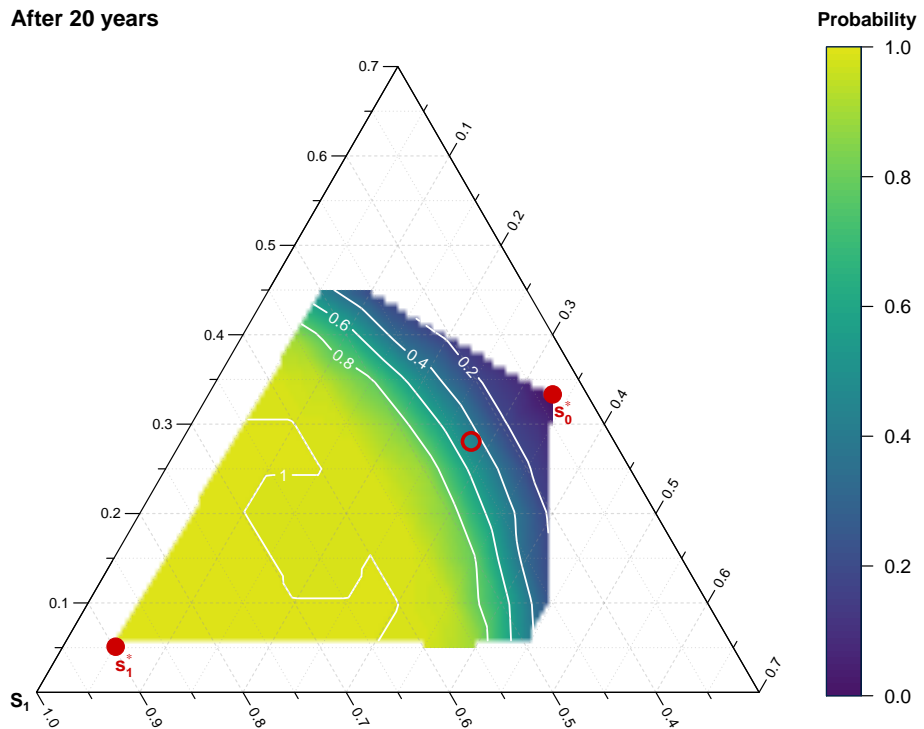

Figure 14: Probability of selecting equilibrium s_1^*. Taking any point within the triangle as a starting point, the color scale depicts the probability of the system to be closer than 0.3 units to s_1^* after 20 years. The area was sampled using 33 starting points with $1\,000$ simulations for each, and the probability functions was obtained by linear interpolation.
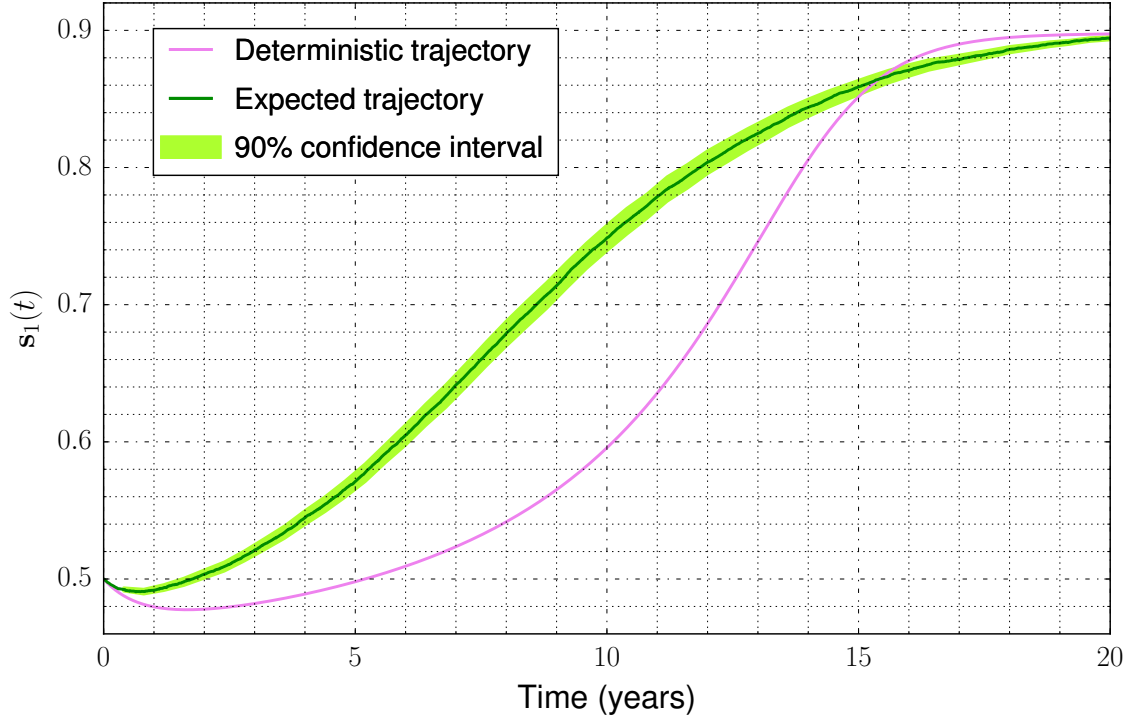
Figure 15: Expected and deterministic trajectories. Starting from the initial state $\mathbf{s} = (0.5, 0.1, 0.4)$, the violet line depicts the deterministic evolution of $\mathbf{s}_1(t)$ for the next 20 years. The green line is the expected trajectory of the sample paths ending up closer than 0.3 units to $\mathbf{s}_1^*$ after 20 years, and the lime buffer around it is the 90% confidence interval.

Thanks to the explicit time unit of the model, which can be adapted to empirical data (Section 3.1), dynamic trajectories can be analysed. It appears that along with equilibrium selection and stability, dynamic trajectories are also strongly influenced by noise induced transitions. To see this, consider again the initial state $\mathbf{s} = (0.5, 0.1, 0.4)$ (Figure 4), and take the viewpoint of region 1. According to the deterministic model, its share of high-skilled workers will slightly decrease before increasing such that after 20 years, region 1 contains 89.8% of the high-skilled workers (Figure 15). The stochastic model however conjectures that starting from this state, which is at 0.3 units from $\mathbf{s}_1^*$, the probability for the interregional system to be within, say, 0.1 units from $\mathbf{s}_1^*$ after 20 years is only about 0.7 (Figure 14). If one takes the expected trajectory of all the sample paths that have actually ended up in this neighbourhood, it appears that the decrease in $\mathbf{s}_1$ is smaller, and that growth starts sooner than in the deterministic case (Figure 15). The 90% confidence interval estimated by bootstrapping shows that this difference is significant. It results from the fact that the deterministic value aggregate the likeliness of different scenarios for region 1 (that it becomes the dominant region, that one of the two other regions become dominant or that the high-skilled workers spread evenly), whist the expected trajectory is only based on sample paths that actually ends up with the agglomeration of high-skilled workers in region 1. Thus the stochastic model provides a more accurate description of the different possible scenarios, and for this reason it constitutes a better tool for decision making.

5   Conclusion

This paper has addressed the question of equilibrium selection and stability in core-periphery models with heterogeneous preferences. It is made up with three major contributions. Firstly, starting from the footloose entrepreneur model with three equidistant regions, it has proposed an original individual relocation dynamics. This individual model infers the exponential form of sociodynamics from the nested logit structure of probabilistic migration models, and it can be calibrated using standard migration data. Secondly, the master equation describing the collective spatial dynamics of high-skilled workers has been deduced from the micro-model and, using a diffusion approximation, it has been reduced to a set of nonlinear stochastic differential equations, which is an original contribution to new economic geography. Thirdly, this set of equations has been simulated to uncover the existence of noise-induced transitions in the spatial dynamics of the population of high-skilled workers. Those transitions affect three essential properties of the footloose entrepreneur model.

First, it requires the notion of equilibrium stability to be discussed in stochastic terms. It has been shown that the dispersed equilibrium is less stable than the agglomerated ones in the sense that starting from this equilibrium, the probability for the interregional system to leave its neighbourhood becomes negligible for larger radius than for the agglomerated equilibria. Second, equilibrium selection does not follow a bijective relationship between initial conditions and equilibria any more. It has been shown that for most of the possible initial states, the probability for the interregional system to end up in the vicinity of the agglomerated equilibria is larger than the probability to end up in the vicinity of the dispersed equilibrium. Besides, increasing the time interval reduces the probability to lie in the neighbourhood of the dispersed equilibrium, but increases the probability to lie in the neighbourhood of the agglomerated one. However, the configuration of the state space, and especially the location of saddle points, has to be considered in evaluating this probability. Third, even if the interregional system ends up in the vicinity of the equilibrium that is selected by the deterministic model, the adjustment trajectory inferred from the stochastic model is significantly different from the deterministic path. This difference is lower, the larger the urban population is. This results from the ability of the stochastic model to differentiate scenarios. This constitutes, along with the use of explicit time units, a substantial improvement of the capacity of the footloose entrepreneur model (and other models of new economic geography) to support policy recommendations.

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