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Burning magneto-hydrodynamics plasmas model: A port-based modelling approach

Benjamin Vincent *,** **Nicolas Hudon** *** **Laurent Lefèvre** ** **Denis Dochain** *

* ICTEAM, Université Catholique de Louvain, Bâtiment Euler, av. Lemaître 4-6, B-1348 Louvain-La-Neuve, Belgium. (e-mail:{benjamin.vincent,denis.dochain}@uclouvain.be)
** Université Grenoble Alpes, LCIS, F-26902, France. (e-mail:laurent.lefevre@lcis.grenoble-inp.fr)
*** Department of Chemical Engineering, Queen's University, Kingston, ON, K7L 3N6 Canada. (e-mail: nicolas.hudon@queensu.ca)

Abstract: In this paper, we propose a burning magneto-hydrodynamic (MHD) plasma model for the Tokamak reactor. Our proposal considers both electro-magnetic and material physical fields. While the electro-magnetic domain is ruled by Maxwell equations, the material physical domain is described by kinetic theory. The transport model is built at microscopic level and extended at the macroscopic one, by computation of the moments of the Boltzmann equation. A macroscopic fluid-like model is then derived for suitable control analysis of the physical model. The fusion reaction does not preserve mass, hence the reaction is included from the very beginning of the modelling. The thermonuclear reaction is embedded from the microscopic scale and couples mass and energy balances. An entropy balance is derived from the Gibbs–Duhem equation and the irreversible entropy production is discussed in the case of thermonuclear reactions.

Keywords: Non-equilibrium thermodynamics, Burning plasma, Transport equations, Kinetic theory, Fusion reaction, Tokamak reactor.

1. INTRODUCTION

Thermonuclear fusion is a safe and available candidate energy source to overcome the future energetic challenges. A prospective reactor to conduct the fusion reaction is the Tokamak design, where hydrogen isotopes (the combustibles) are magnetically confined and heated up to the required temperature. A Tokamak is a torus-shaped device that magnetically confines and heats plasma to reach fusion reaction conditions. At steady-state operation, stability and reactions conditions have to be maintained. For a detailed description of the Tokamak concept and the associated technological and operational challenges, the reader is referred to (Wesson and Campbell, 2004) and the references therein.

Advanced control strategies that rely on physics-based models are required to achieve high efficiency. A major control challenge concerns plasma profiles control (Pironti and Walker, 2005). The objectives are to follow density, temperature and current profiles to ensure magneto-hydrodynamics stability, high confinement modes and energy efficiency of the fusion reaction. Reliable physic-based models can address those control and diagnosis issues.

Plasma are multi-physics systems defined in different domains (electro-magnetic and material) with nonlinear couplings. The kinetic theory of gases describes the behavior of particles within the plasmas but has the disadvantage of being computationally intractable. On the other hand, fluid-like macroscopic models described by nonlinear partial differential equations are a good compromise for control purposes given the system complexity. Braginskii (1965) derived a fluid model from the kinetic theory that has been extended by Blum (1989) for numerical simulations and real-time applications such as tracking and control. Witrant et al. (2007) used a diffusion model for the control of current profile within the Tokamak. In later developments, a structured port-Hamiltonian single-fluid 3-D model has been proposed (Vu et al., 2016a) and passivity based control (Vu et al., 2016b) has successfully been applied for the current profile control by using with a diffusion-resistive model. Those models do not take into account the fusion reaction. For steady state operation, where the plasma sustains the fusion reaction, burning plasma models are required. Boyer and Schuster (2014, 2015) proposed an adaptive nonlinear control for density and energy. They successfully applied their 0D control strategy to a 1D radial study case.

In this paper, we propose a port-based fluid-like model for burning plasma in Tokamak reactors. The fusion reaction is developed from the microscopic level of description with the gas kinetic theory. Then we derive a multi-fluid model by defining macroscopic variables. As a result, we get continuity, momentum and energy balances equations for each species within the plasma. A one fluid-like model is derived (de Groot and Mazur, 1984) satisfying the port-based approach identified in (Duindam et al., 2009). To complete the model, we compute the internal energy

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and entropy balance equations. The system is closed with respect to Onsager's reciprocal relation for irreversible systems (Boozer, 1992; Onsager, 1931). We discuss the mass non-conservative and exothermic properties of the fusion reaction according to the irreversible production of entropy.

The paper is organized as follows. In Section 2, the electro-magnetic domain described by Maxwell equations is introduced. A first macroscopic multi-fluid model is derived from a kinetic point of view in Section 3. The model includes the fusion reaction from the very beginning. In Section 4, a one fluid-like model is derived from the previous section. In Section 5 a port-based approach is achieved by computing the internal energy and the entropy.

2. ELECTRO-MAGNETIC DOMAIN

In a Tokamak reactor three types of coils may generate the electro-magnetic fields. Their objectives are twofold: to confine the plasma within a toroidal shape, and to induce current to heat the plasma. The coils act as the primary circuit of a transformer while the plasma itself acts as a secondary circuit. The electro-magnetic domain is defined in a fixed frame and is modeled by Maxwell equations:

$$-\frac{\partial B}{\partial t} = \nabla \times E, \quad \text{and} \quad \nabla \times H = j + \frac{\partial D}{\partial t}, \tag{1}$$

where B, E, H and D are the magnetic flux, electric field, magnetic field and electric flux, respectively. The total current is denoted by j and include the inductive ohmic current and the non-inductive current generated by poloidal coils and antennae oriented towards the plasma (Wesson and Campbell, 2004), respectively. Maxwell equations (1) are subject to constraints:

$$\nabla . D = \rho_e, \quad \text{and} \quad \nabla . B = 0, \tag{2}$$

where ρ_e is the plasma total charge. The closure equations or constitutive relations link the electric flux with the electric field; and the magnetic flux with the magnetic field such that:

$$D = \epsilon E, \quad \text{and} \quad B = \mu H, \tag{3}$$

where ϵ and μ are the permittivity and the permeability tensors, respectively.

Particles inside the Tokamak vessel are subject to forces induced by the electro-magnetic domain. In the presence of electric field and magnetic flux, the force applied to a particle of charge q and velocity v, the Lorentz force, is given by:

$$F = q(E + v \times B). \tag{4}$$

This represents the main coupling in our model.

3. MICROSCOPIC TO MACROSCOPIC TRANSPORT FORMULATION

From a microscopic point of view, the fusion reaction is well understood and its description can be derived from the theory of gases (Chapman and Cowling, 1970). However, when considering the plasma control analysis and design in Tokamak reactors, the microscopic level of description is not well-suited to describe the transport phenomena.

3.1 Thermonuclear fusion reaction

The fusion reaction is the result of collisions between two light atoms with enough kinetic energy to overcome the Coulomb barrier. A heavier element is generated and a huge amount of energy is released. Fusion is a non-conservative mass reaction and the mass defect is responsible for the release of energy. This energy is converted into kinetic energy and distributed in inverse proportion to the reaction products' weight. The deuterium-tritium (D-T) fusion reaction is used for preliminary experiments in the existing Tokamak reactors (Wesson and Campbell, 2004). The products of this reaction are a helium atom and a neutron, and the released energy is $E_{\text{fus}} = \delta_m c^2 = 17.6\, MeV$, where the Einstein relation has been considered with c the speed of light in vacuum and δ_m the mass deficit.

We consider a reaction between two particles in a perfectly mixed gas. The total number of reaction in a elementary volume at time t is given by the double integral (Chapman and Cowling, 1970):

$$\mathcal{R} = \iint f_1(v_1) f_2(v_2) |v_1 - v_2| \sigma(v_1 - v_2)\, dv_1 dv_2, \tag{5}$$

where f_k and v_k for $k \in \{1, 2\}$ are the particle density distributions and velocities, respectively. The reactant cross-section is denoted by σ. At steady state, the plasma is described as a Maxwellian distribution. The reaction rate (5) becomes:

$$\mathcal{R} = n_1 n_2 \frac{(m_1 m_2)^{3/2}}{(2\pi\kappa_b T)^3} \int |v| \sigma(v) e^{\left(-\frac{m_1 m_2}{m_1+m_2}\frac{v^2}{2\kappa_b T}\right)}\, dv, \tag{6}$$

where $m_k, k \in \{1, 2\}$, T and κ_b are the particle masses, temperature, and Boltzmann constant, respectively. Reaction occurs if particles have enough kinetic energy, expressed as $\frac{v^2}{2}(m_1 m_2)/(m_1 + m_2)$ where v is the relative velocity. From this equation, the cross-section is identified such that the reaction rate takes the form (Thompson, 1957):

$$\mathcal{R} = n_1 n_2 \langle \sigma v \rangle_{DTr}. \tag{7}$$

In the literature, for the deuterium-tritium reaction, the cross-section $\langle \sigma v \rangle_{DTr}$ is usually approximated by scale laws (Bosch and Hale, 1992; Hively, 1977).

3.2 Boltzmann equation

In a plasma, each particle is defined by a position x and a velocity v at a time t such that the distribution function $f_k(x, v, t)$ describes all k species. The particle behavior is ruled by the Boltzmann equation:

$$\frac{\partial f_k}{\partial t} + v \nabla f_k + \frac{F_k}{m_k}\frac{\partial f_k}{\partial v} = \nu^k J_{\text{fus}}. \tag{8}$$

The plasma is magnetically confined. Hence particle motions are influenced by external forces. Here we neglect the gravity field and consider the Lorentz force $F_k = q_k(E + u_k \times B)$. The particle charges and average velocities are denoted by q_k and u_k, respectively. The right hand side of the Boltzmann equation (8) usually denotes the collision operator. Here we consider the fusion reaction as a Boltzmann operator (Dellacherie and Sentis, 2000) such that:

$$J_{\text{fus}} = \mathcal{R} = n_D\, n_{Tr}\, \langle \sigma v \rangle_{DTr}, \tag{9}$$

as defined in (7). Signed stoichiometric coefficients for the reaction are denoted by ν^k.

The description of plasma by macroscopic non-equilibrium variables is sought for control purposes. We therefore introduce macroscopic averaged values for each specie. The *particle density* $n_k(x,t)$ (m^{-3}), that describes the number of particles per unit volume is defined as follows:

$$n_k(x,t) = \int f_k(x,v,t)dv. \tag{10}$$

The average *fluid velocity* $u_k(x,t)$ $(m\,s^{-1})$ and the *pressure tensor* $\mathbf{P}_k(x,t)$ $(N\,m^{-2})$ are given by:

$$u_k(x,t) = \frac{1}{n_k(x,t)} \int v f_k(x,v,t)dv, \tag{11}$$

and

$$\mathbf{P}_k(x,t) = \frac{m_k}{3} \int (v-u_k)(v-u_k) f_k(x,v,t)dv. \tag{12}$$

The *temperature* $T_k(x,t)$ $(J,\, ev)$ is derived from the averaged particle velocity and is given by

$$T_k(x,t) = \frac{m_k}{3n_k} \int (v-u_k)^2 f_k(x,v,t)dv, \tag{13}$$

such that the scalar pressure $P_k(x,t)$ can be expressed as a function of temperature [1] :

$$P_k(x,t) = n_k(x,t)\, T_k(x,t). \tag{14}$$

The temperature T_k is the averaged velocity of the particle k. The *energy density* $\varepsilon_k(x,t)$ $(W\,m^{-3})$ is expressed as:

$$\varepsilon_k(x,t) = \frac{m_k}{2} \int v^2 f_k(x,v,t)dv = \frac{\varrho_k}{2}|u_k|^2 + \frac{3}{2} n_k T_k, \tag{15}$$

and is the sum of the particles kinetic energy and thermal energy. The *heat flux* $Q_k(x,t)$ $(W\,m^{-2})$ is given by:

$$Q_k(x,t) = \frac{m_k}{2} \int v(v-u_k)^2 f_k(x,v,t)dv. \tag{16}$$

The heat flux can be expressed qualitatively as the temperature times the fluid velocity.

Using the averaged quantities defined above, one can derived the balance equations by computing the moments of Boltzmann equation (8).

3.3 Balance equations

In this section we compute the zeroth, first and second moments of the Boltzmann equation (8) to derive the mass, momentum and, energy balance equations, respectively.

Continuity equation The continuity equation is obtained by pre-multiplying the Boltzmann equation by the particle mass m_k and then integrating it w.r.t. the velocities. With definitions (10) and (11), one obtains:

$$\frac{\partial \varrho_k}{\partial t} + u_k \nabla \varrho_k = -\varrho_k \nabla u_k + \nu^k m_k \int J_{\text{fus}}\, dv, \tag{17}$$

where $\varrho_k = m_k n_k$ is the particle mass density. On the left hand side of the continuity equation (17), one can identify the material derivative. The last term on the right hand side results from the mass non-conservative (mass defect) property of the fusion reaction.

Momentum balance The momentum balance equation is the first moment of Boltzmann equation: one multiply (8) by $m_k v$ and then integrate w.r.t. the velocity. With definitions (10)-(12), one obtains:

$$\begin{aligned} \varrho_k \left(\frac{\partial u_k}{\partial t} + u_k \nabla u_k \right) &= -\nabla \mathbf{P}_k + F_k\, n_k \\ &+ \nu^k\, m_k \int (v-u_k)\, J_{\text{fus}}\, dv. \end{aligned} \tag{18}$$

The contribution of the fusion reaction to the momentum balance appears as the last term in the right hand side of the equation.

Energy balance The energy balance equation is given by the second moment of the Boltzmann equation.

$$\begin{aligned} \frac{\partial \varepsilon_k}{\partial t} + u_k \nabla \varepsilon_k &= -\nabla(n_k\, T_k\, u_k) - \nabla Q_k + q_k n_k u_k E \\ &+ \nu^k\, m_k \int \frac{v^2}{2} J_{\text{fus}}\, dv. \end{aligned} \tag{19}$$

The fusion reaction term is given by the last term on the right hand side of the equation above.

Heat balance The total energy density is the sum of the kinetic and the thermal energy. It is more convenient (Braginskii, 1965) to remove the kinetic part such that we gets the heat balance equation:

$$\begin{aligned} \frac{3}{2} \left(\frac{\partial\, (n_k\, T_k)}{\partial t} + u_k \nabla (n_k T_k) \right) &= -\mathbf{P}_k \nabla u_k - \nabla Q_k \\ &+ \nu^k\, m_k \int \frac{1}{2} (v-u_k)^2\, J_{\text{fus}}\, dv. \end{aligned} \tag{20}$$

Entropy balance and discussion The ideal gas law can be applied to plasmas, hence the relation between the entropy $\mathcal{S}$ and the temperature T holds (Braginskii, 1965):

$$\mathcal{S} = \ln \left(\frac{T^{3/2}}{n} \right), \tag{21}$$

where n is the total particle density. One can derive the entropy directly from the multi-fluid model assuming that the total entropy $\mathcal{S}$ is the sum of the substantial entropy S_k. In the present contribution, we have made the choice to derive the total entropy and the irreversible entropy production using the one-fluid formulation described later.

4. FLUID-LIKE FORMULATION

The transport model derived in the above section is a set of $3 \times N$ balances equation, where N is the number of present species within the plasma (Braginskii, 1965). A one-fluid model is sufficient and more convenient for control purposes. In this section, we derive a single-fluid model for burning plasma.

4.1 Fluid-like variables

We first introduce hydrodynamics variables. The summations are carried over the N species that compose the plasma.

The *total mass density* ϱ $(kg\,m^{-3})$ is given by the sum of all densities:

$$\varrho = \sum_{k=1}^{N} m_k n_k, \tag{22}$$

[1] We consider here that the the Boltzmann constant κ_b is equal to one ($\mathbf{P}_k = n_k \kappa_b T_k$).

and the relative mass fractions

$$\omega_k = \frac{m_k n_k}{\varrho} \tag{23}$$

gives the particle proportions in the plasma. The *barycentric fluid velocity* v is equal to the *momentum per mass unit* $\mathbf{p}$ $(m\, s^{-1})$:

$$v = \mathbf{p} = \sum_{k=1}^{N} \frac{m_k n_k u_k}{\varrho}, \tag{24}$$

and describes the fluid velocity at the center of mass. The *diffusion flow* J_k $(m^{-2}\, s^{-1})$ of species k is defined with respect to the barycentric fluid velocity:

$$J_k = \rho_k (u_k - v). \tag{25}$$

The plasma is affected by a *current density* j $(A\, m^{-2})$:

$$j = \sum_{k=1}^{N} q_k n_k u_k, \tag{26}$$

where q_k represents the effective charge of species k. The *total pressure* $\mathbf{P}$ $(N\, m^{-2})$ is given by the sum of all partial pressures:

$$\mathbf{P} = \sum_{k=1}^{N} \mathbf{P}_k, \tag{27}$$

and the *total heat flux* Q $(W\, m^{-2})$ is given by

$$Q = \sum_{k=1}^{N} Q_k. \tag{28}$$

The *total energy* per mass unit $\mathcal{E}$ $(W\, kg^{-1})$ is given by:

$$\mathcal{E} = \frac{1}{\varrho} \sum_{k}^{N} \varepsilon_k. \tag{29}$$

The fusion reaction does not preserve the mass, therefore the following relation holds:

$$\sum_{k}^{N} \int \nu^k m_k J_{\text{fus}} dv = -\delta_m \mathcal{R}, \tag{30}$$

where δ_m is the mass deficit. Furthermore, for each reaction, energy is released. We have

$$\sum_{k}^{N} \int \nu^k \frac{v^2}{2} m_k J_{\text{fus}} dv = E_{\text{fus}} \mathcal{R}, \tag{31}$$

with E_{fus} the energy released by a single fusion reaction.

From these definitions, we now derive a one-fluid burning plasma transport balance system of equations.

4.2 Balance equations

We aim to develop a $N + 3$ system of balance equations, with N equations to characterize the species, plus one equation for the continuity equation, one for the momentum balance, and one the energy balance.

Continuity equation Starting from (17), we use the material derivative equation of the fluid defined as

$$\frac{D\,.}{Dt} = \frac{\partial\,.}{\partial t} + v\nabla. \tag{32}$$

and the properties defined in (30). We further use the relation

$$\sum_{k}^{N} J_k = 0 \tag{33}$$

which physically means that the sum of the diffusion flows is equal to zero. Summing (17) over all species k, one gets:

$$\frac{D\varrho}{Dt} = -\varrho \nabla v - \mathcal{R}\delta_m. \tag{34}$$

Alternatively, one can use the specific volume ($\overline{v} = 1/\varrho$) to express the continuity equation as:

$$\varrho \frac{D\,\overline{v}}{Dt} = \nabla v + \overline{v}\delta_m \mathcal{R}. \tag{35}$$

Species balance equation From (17), we use the material derivative (32), the diffusion flow (25) and the mass fraction (23) definitions, and derive the species balance equations:

$$\varrho \frac{D\,\omega_k}{Dt} = -\nabla J_k + \nu^k m_k \mathcal{R} + \omega_k \delta_m \mathcal{R}. \tag{36}$$

Momentum balance equation Assuming that the small collision time hypothesis holds (Braginskii, 1965), the material derivative $\frac{\partial \cdot}{\partial t} + u_k \nabla\cdot$ is approximated by the material derivative defined with respect to the barycentric velocity: $\frac{\partial \cdot}{\partial t} + v\nabla\cdot$. Hence, we get a fluid-like momentum balance by summing equation (18) over all species. Using definitions (26) to (28), we obtain:

$$\varrho \frac{Dv}{Dt} = -\nabla \mathbf{P} + j \times B. \tag{37}$$

Energy balance equation Again we assume the small collision time hypothesis and summing (15) over all species one gets:

$$\varrho \frac{D\,\mathcal{E}}{Dt} = -\sum_{k} \nabla(\mathbf{P}_k u_k) - \nabla Q + jE + E_{\text{fus}} \mathcal{R}. \tag{38}$$

5. PORT-BASED FORMULATION

Following the approach proposed in (Duindam et al., 2009, chap. 3), we aim to write the balance equations within a port-based formalism. For a quantity α of density ρ, the generic balance equation is of the form:

$$\frac{\partial\, \rho\alpha}{\partial t} = -\nabla f_\alpha + \sigma, \tag{39}$$

where f_α is the flux of α per surface area and σ is the input source term. In a moving material domain with respect to the velocity v, the balance equation is now expressed as:

$$\rho \frac{D\,\alpha}{Dt} = -\nabla(f - \rho v \alpha) + \sigma = -\nabla f_\alpha^R + \sigma, \tag{40}$$

where f_α^R is the relative flux and the time derivative $\frac{D}{Dt}$ is the material derivative defined in equation (32).

We now express the balance equations of a burning plasma using this formalism. Table 1 summarizes the balance equations for the port-based burning plasma model.

5.1 Closure equations

Momentum We assume the plasma to be a non-elastic fluid (Braginskii, 1965; de Groot and Mazur, 1984). Hence the relative momentum flux f_p^R is the sum of a scalar pressure P and a stress tensor τ responsible for the viscosity:

$$f_p^R = PI + \tau, \tag{41}$$

where I is the identity matrix. The scalar pressure is related to the density and the temperature with the ideal

balance	α	f^R	σ
Species	ω_k	J_k	$(\nu^k m_k + \omega_k \delta_m)\mathcal{R}$
Specific volume	$\overline{v}$	$-v$	$\overline{v}\,\delta_m\,\mathcal{R}$
Momentum	$\mathrm{p} = v$	$\mathbf{P}$	$j \times B$
Total energy	$\mathcal{E}$	$\sum_k \mathbf{P}_k u_k + Q$	$jE + E_{\text{fus}}\mathcal{R}$
Internal energy	$\overline{u}$	f_u^R	σ_u
Entropy	$\overline{s}$	f_s^R	σ_s

Table 1. Quantities for burning plasma balance equations

gas law. In the work of Braginskii (1965), a methodology is given to compute the stress tensor τ.

Internal energy The total energy is given by the sum of the kinetic and internal energies:

$$\mathcal{E} = \frac{v^2}{2} + \overline{u}(\overline{s}, \overline{v}, \omega_k). \tag{42}$$

The specific internal energy is given by $\overline{u}$ and is function of $\overline{s}$, $\overline{v}$ and ω_k the specific entropy, the specific volume and the mass fraction, respectively. From the total energy in (42), we want to derive the internal energy balance equation under the form given in (40). Therefore, we differentiate (42) by considering the material derivative:

$$\varrho \frac{D\,\mathcal{E}}{Dt} = v \varrho \frac{D\,v}{Dt} + \varrho \frac{D\,\overline{u}}{Dt}. \tag{43}$$

With the momentum balance (37), the energy balance (38), and some algebra one gets:

$$f_u^R = f_{\mathcal{E}}^R - v f_p^R = f_q. \tag{44}$$

The internal relative heat flux is set to be equal to the conduction heat flux per unit area f_q. The internal energy source term is given by:

$$\sigma_u = jE + E_{\text{fus}}\mathcal{R} - (P + \tau)\nabla v. \tag{45}$$

The internal energy increases with the Joule effect, the pressure and viscosity, and the energy released from the fusion reaction.

5.2 Irreversible entropy production

Here we follow the approach used in (Duindam et al., 2009, chap. 3), which was considered in the context of plasma transport without the fusion reaction in Vu et al. (2016a).

For macroscopic systems at thermodynamic equilibrium, the internal energy $\mathcal{U}$ is a function of the entropy $\mathcal{S}$, volume $\mathcal{V}$ and the mass of each components M_k. Upon internal energy differentiation, one gets Gibbs' equation for the free energy:

$$d\mathcal{U} = Td\mathcal{S} - Pd\mathcal{V} + \sum_k^N \mu_k dM_k, \tag{46}$$

where we have used the constitutive relations:

$$T = \left(\frac{\partial \mathcal{U}}{\partial \mathcal{S}}\right)_{\mathcal{V}, M_k}, \qquad P = -\left(\frac{\partial \mathcal{U}}{\partial \mathcal{V}}\right)_{\mathcal{S}, M_k}, \tag{47}$$

$$\mu_k = \left(\frac{\partial \mathcal{U}}{\partial M_k}\right)_{\mathcal{S}, \mathcal{V}, M_{j \neq k}} \tag{48}$$

for the temperature, the pressure and the chemical potential, respectively.

Our plasma model is a distributed parameter system. Hence we have to introduce local extensive variables: the specific entropy $\overline{s} = \mathcal{S}/M$, the specific volume $\overline{v} = \mathcal{V}/M$, and the mass fraction $\omega_k = M_k/M$. We assume local thermodynamic equilibrium (de Groot and Mazur, 1984) and derive a local version of Gibbs–Duhem free energy:

$$d\overline{u} = Td\overline{s} - Pd\overline{v} + \sum_k^N \mu_k d\omega_k. \tag{49}$$

This expression is equivalent to:

$$\varrho \frac{D\,\overline{s}}{Dt} = \frac{1}{T}\left(\varrho \frac{D\,\overline{u}}{Dt} + P\,\varrho \frac{D\,\overline{v}}{Dt} - \sum_{k=1}^N \mu_k\,\varrho \frac{D\,\omega_k}{Dt}\right), \tag{50}$$

where the entropy dynamics is put on the left hand side of the equation.

The idea is now to express the entropy as a function of other thermodynamic variables. Therefore substituting into (50), the balance equation for internal energy (44), the continuity equation (35), and the balance equations for the species (36), the entropy takes the form of (40) with the relative entropy flux:

$$f_s^R = \frac{1}{T}\left(f_q - \sum_{k=1}^N \mu_k\,J_k\right), \tag{51}$$

and the irreversible entropy production:

$$\begin{aligned} T\sigma_s = & \underbrace{-\frac{1}{T}\left(f_q - \sum_{k=1}^N \mu_k\,J_k\right)\nabla T}_{T\sigma_{\text{heat}}} - \underbrace{\tau\nabla v}_{T\sigma_{\text{viscous}}} \\ & \underbrace{-\sum_{k=1}^N J_k\,\nabla\mu_k}_{T\sigma_{\text{diff}}} + \underbrace{jE}_{T\sigma_{\text{Joule}}} \underbrace{-\sum_{k=1}^N \mu_k \nu^k m_k \mathcal{R}}_{T\sigma_{\text{reac}}} \\ & \underbrace{-\sum_{k=1}^N (\mu_k\omega_k + P\,\overline{v})\,\mathcal{R}\,\delta_m}_{T\sigma_{\text{def}}} + \underbrace{\delta_m c^2\,\mathcal{R}}_{T\sigma_{\text{rel}}}. \end{aligned} \tag{52}$$

According to the second law of thermodynamics, the entropy source term denotes the irreversible entropy production (52). This irreversibility arises from different physical phenomena. The first term ($T\sigma_{\text{heat}}$) is the heat conduction, the second one ($T\sigma_{\text{viscous}}$) is the heat transfer from the mechanical domain due to viscosity, the third one ($T\sigma_{\text{diff}}$) is due to diffusion, and the fourth one ($T\sigma_{\text{Joule}}$) is the entropy generation related to the Joule effect. The last terms are related to the fusion reaction, ($T\sigma_{\text{react}}$) is common for all reactions (Duindam et al., 2009), the last two generation terms encode the mass defect and the relativistic properties of the thermonuclear fusion reaction, respectively.

5.3 Transport coefficients

Onsager's theory (Onsager, 1931) implies the existence of a linear application, called a transport matrix, that maps the relation between the effort and flow variables. The irreversible entropy production given in (52) implies that the efforts and flows are given by:

$$e = (T\ v\ V_E\ \mu_k)^\top, \quad \text{and} \quad f = (f_q\ \tau\ j\ J_k)^\top, \tag{53}$$

where the variable V_E represents the plasma electric potential $E = \nabla V_E$. This variable is a reliable measure and thus commonly used for diagnosis and control purposes (Vu et al., 2016b; Witrant et al., 2007). There exists a symmetric positive semi-definite matrix Γ such that:

$$f = \Gamma \nabla e. \tag{54}$$

As a first approximation, a port-based model requires to find the diagonal coefficients in Γ:

$$\Gamma = diag\,(\chi \; \kappa \; \eta \; \gamma_k)\,, \tag{55}$$

where χ, κ, η and γ_k are the thermal diffusion, the viscosity, the resistivity, and the material diffusion coefficients, respectively. Those transport coefficients are functions of state variables. One set of transport coefficients can be identified from the plasma control literature:

- The thermal diffusion coefficient χ is function of the magnetic field and the temperature (Witrant et al., 2007);
- The viscosity coefficient can be approximated by a method developed in (Braginskii, 1965);
- The resistivity coefficient is function of the magnetic field and the temperature (Sauter et al., 1999); and,
- The material transport coefficients can be derived from Fick's law (de Groot and Mazur, 1984).

The cross terms in the Onsager's matrix may introduce coupling terms and have been investigated for the toroidal plasma by (Boozer, 1992) and (Garbet et al., 2012).

6. CONCLUSION

We have proposed a port-based burning plasma model for steady state operation in Tokamaks. This model is defined in both electro-magnetic and material domains. The originality of this contribution lies in the consideration of the fusion reaction developed from the microscopic level to the macroscopic level. Irreversible entropy production is discussed, and more specifically, the contribution of the fusion reaction to entropy generation. Further developments are concerned with the port-Hamiltonian formulation of this model by developing Stokes–Dirac structures (van der Schaft and Maschke, 2002). Then control applications will be investigated using thermodynamics properties of the system and approaches such as passivity-based control and power-shaping (Vu et al., 2016b).

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