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Semisimple algebraic groups from a topological group perspective

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Looking back at these past four years doing my PhD at the Université Catholique de Louvain-la-Neuve, I feel fortunate.

In hindsight, I am convinced that choosing a suitable supervisor is mostly a random process, if only because it is impossible to predict the outcome of a collaboration lasting such a long time. Furthermore, in my case, I had barely seen Pierre-Emmanuel before beginning. In the end, our relation was very fruitful to me, and not only on the mathematical side. I only begin to grasp the tremendous influence he had on me, and I now reckon that this collaboration will nourish my mathematical life. It is fascinating to see the passion that animates Pierre-Emmanuel. His continuous enthusiasm has also proved to be of great help to surmount difficulties. I warmly thank him for all the mathematical background he transmitted, and for all the nice moments we had together, either at lunch times or in conferences around the world.

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Introduction

The present work revolves around the theme of algebraic groups, with a special emphasis on semisimple algebraic groups over local fields. It consists of three independent chapters. The first one aims for a structure theorem for totally disconnected locally compact groups having a linear open subgroup. The second chapter studies Chabauty limits of quasi-split simple algebraic groups acting on trees. Finally, the last chapter gives a condition for the group of semilinear automorphisms of a semisimple group G to decompose as a semidirect product of the group of algebraic automorphisms and the group of field automorphisms preserving G.

Algebraic groups over local fields

By an algebraic group over a scheme S, we mean a group object in the category of smooth affine S-scheme of finite type, and an algebraic group over a field k means an algebraic group over $\operatorname{Spec} k$. For a non-expert, this definition might appear as overly dry and sophisticated. To motivate it, we begin by discussing the historical development of algebraic group theory. It is also good to keep in mind the various levels of difficulty that arise when allowing various base schemes S, a fact which is well illustrated by the history of the development of the theory.

Algebraic groups were first only considered over $\mathbf{R}$ or $\mathbf{C}$, in which case they are instances of Lie groups. It took a long time before starting to vary the base scheme, so let us pause for a moment on this case and take the opportunity to discuss the early motivations for the theory of algebraic groups. A. Borel wrote a masterful account of the genesis of algebraic groups in [Bor01]. Relying on this reference, we quote the various influences.

The ancestors of algebraic groups are the so called Lie groups, a notion introduced in 1873 by S. Lie. His motivation was to develop a Galois theory for differential equations. Note in passing that abstract groups emerged from Galois theory. Lie groups started to play a leading role in mathematics after the fundamental contributions of W. Killing, E. Cartan and H. Weyl. Interestingly enough, H. Weyl came to study Lie groups because of his interest (amongst others) for relativity theory. In hindsight, his influence has been tremendous, since nowadays Lie groups play a central role in physics.

From a modern point of view, a Lie group is a group object in the category of smooth manifolds. This definition makes it clear that smooth manifolds are not far from smooth group schemes of finite type over the reals, the latter category being a subcategory of the former in which we restrict our attention to polynomial functions. Hence, it is not surprising that some works initiated by the theory of Lie groups dating from the end of the 19th century were early contributions to the theory of algebraic groups. Amongst those early contributions, we can cite the work of S. Lie and E. Study on projective representations of $\mathrm{SL}_n(\mathbf{C})$, the work of E. Picard on linear differential equations, and the work of E. Cartan on linear Lie groups that are algebraic. But, as A. Borel puts it in [Bor01, Chapter 5], those works did not appear to their authors as contributions to the theory of algebraic groups, since they had other goals in mind. From this period, the mathematician L. Maurer stands out as an exception. His main goal was to develop the theory of algebraic groups per se, but his work fell into oblivion.

Algebraic groups had then to wait for forty years before being revived by C. Chevalley and E. Kolchin. They had different motivations: the former was trying to extend the work of L. Maurer, while the latter was trying to develop a Galois theory of differential equations à la Picard. This leads us to the golden age of algebraic groups. In the fifties, C. Chevalley took the mathematical community by surprise by achieving a classification of simple algebraic groups over an algebraically closed field in arbitrary characteristic. To quote P. Cartier in [Che05, Postface, 25.2], "the announcement by C. Chevalley that the classification of simple algebraic groups over algebraically closed fields was independent of the characteristic was a crash of thunder". Soon afterwards, the needs for a better understanding of the classification of finite simple groups led mathematicians to push further Chevalley's work and to work over perfect fields, aiming for a classification of simple algebraic groups over finite fields. In retrospect, algebraic groups over finite fields indeed yields the core of the list of finite simple groups. The assumption of perfectness of the field was essential to make it possible to apply Galois-theoretic techniques when descending from the algebraically closed field case to the perfect field case.

Problems over local and global fields coming from number theory motivated the elimination of the perfectness assumption. According to [Con14, Appendix A], the initial breakthrough allowing to work over an arbitrary field came from a theorem of Grothendieck (namely Theorem A.1.1 in [Con14]), whose proof uses in an essential way scheme-theoretic ideas. At approximately the same time, Grothendieck and his school were developing the theory of schemes, and it was probably then natural to recast the theory of algebraic groups in the language of schemes, and to work over an arbitrary base. This notably led to the classification of split simple algebraic groups over an arbitrary scheme, massively generalising the classification achieved by Chevalley.

In our work, we mostly restrict our attention when the base scheme S is the spectrum of a non-discrete locally compact topological field. The classification of such fields is an early success of the development of the theory of locally compact groups impulsed by Hilbert's fifth problem. Note again in passing the inspiring role of Lie groups. That classification ensures that a non-discrete locally compact topological field is isomorphic to either $\mathbf{R}$, $\mathbf{C}$, a finite extension of the field of p-adic numbers $\mathbf{Q}_p$ or the field of formal Laurent series $\mathbf{F}_q(\!(T)\!)$ over the finite field $\mathbf{F}_q$, where q is a prime power. In this list, the disconnected ones, i.e. all of them apart from $\mathbf{R}$ and $\mathbf{C}$, are called (non-Archimedean) local fields.

To motivate in a few words the concept of a local field, let us mention their relevance in number theory, which is not surprising if one thinks of $\mathbf{Q}_p$ as the completion of $\mathbf{Q}$ with respect to a prime p. What makes

p-adic fields so useful in number theory is their historical use in the work of Hasse on the classification of quadratic forms over $\mathbf{Q}$, which was a first instance of his profound and amazing local-global principle.

Now that we have introduced the concept of an algebraic group and the concept of a local field, it remains to explain why considering algebraic groups over local fields is of interest. As we already pointed out, they play a central role in number theory. For example, the Langlands program is concerned with representation theory of algebraic groups over local fields. Let us also outline another remarkable link: according to [Buz], Tamagawa numbers of algebraic groups over local fields played an inspiring role in formulating the Birch and Swinnerton-Dyer conjecture, which is considered as one of the most important problems in mathematics.

The starting point of this thesis is to view the simple algebraic groups over local fields as the most prominent examples in the class $\mathscr{S}$ of nondiscrete compactly generated locally compact groups that are topologically simple. The study of the class $\mathscr{S}$ as a whole is a new trend that emerged in the past decade. However, its origins are to be traced within the historic interest for locally compact groups. In his famous so-called "fifth problem", D. Hilbert asked whether the differential structure on a Lie group can be reconstructed from the topological structure. In the fifties, A. Gleason and H. Yamabe answered this question in the affirmative, in a theorem nowadays deemed as "the solution to Hilbert's fifth problem". Their fundamental theorem (see for example [Tao14, Theorem 1.1.17]) provides structural information on general locally compact groups. In particular, it implies a first result on the class $\mathscr{S}$: if G is a connected, locally compact group which is topologically simple, then G is a Lie group. It is to be noted that Lie groups were again the main concern in those early developments.

But from the point of view of the classification of groups in $\mathscr{S}$, the result of A. Gleason and H. Yamabe is only a milestone, not the end of the story. Note that given any topological group G, we always have a short exact sequence of topological groups $1 \to G^0 \to G \to G/G^0 \to 1$ where G^0 is a connected closed normal subgroup of G, and G/G^0 is totally disconnected. In particular, by a result of D. van Dantzig (see

[CM11, Introduction] for a precise reference), G/G^0 has a basis of identity neighbourhood consisting of compact subgroups, so that the theorem of A. Gleason and H. Yamabe does not bring any new information. Hence, in order to attempt at a classification of groups in $\mathscr{S}$, the totally disconnected case and only this one remains open.

Before explaining how our results contribute to an exploration of the class $\mathscr{S}$, let us end this section by insisting on a technical difficulty. While finite extensions of $\mathbf{Q}_p$ are perfect fields since these are fields of characteristic 0, $\mathbf{F}_q(\!(T)\!)$ is not perfect. Indeed, using the valuation, we directly see that $T \in \mathbf{F}_q(\!(T)\!)$ has no n-th roots for $n > 0$. As we said earlier, considering algebraic groups over nonperfect fields makes it really necessary to use the schematic language. This is particularly well illustrated by the following example: for any prime p, the map $\mathrm{SL}_p \to \mathrm{PGL}_p$, considered as a map of algebraic group over $\mathbf{F}_p(\!(T)\!)$, is a purely inseparable isogeny, which makes this map wilder at the level of rational points (see Example 1.13 for an illustration of this).

Presentation of the results

We refer to [Cap16] for a very well-written and informative survey of results and problems pertaining to the class $\mathscr{S}$. Let us just list the most prominent examples of groups in the class $\mathscr{S}$: algebraic groups over local fields, Kac–Moody groups over finite fields, groups acting on trees and groups almost acting on trees (see [Cap16, Section 2] for more details). One of the main challenge that we now face is to find new examples of groups belonging to $\mathscr{S}$, hoping to obtain guiding principles to develop the general theory. This quest for new examples is the common theme of the first two chapters.

Chapter 1, which is joint work with P.-E. Caprace, is concerned with the structure of totally disconnected locally compact groups that are locally linear. A locally compact group is called **linear** if it admits a continuous faithful finite-dimensional linear representation over a local field. It is called **locally linear** if it has an open subgroup which is linear. The main motivation behind this work was a classification of groups G in $\mathscr{S}$ that are locally linear, and we indeed achieved our goal.

Theorem A (see Chapter 1, Corollary 1.4)**.** *Let G be a totally disconnected group in $\mathscr{S}$. If G is locally linear, there exists a local field k and an absolutely simple, simply connected, isotropic algebraic group H over k such that G is isomorphic to $H(k)/Z(H(k))$ as a Hausdorff topological group.*

By a result of D. van Dantzig (see [CM11, Introduction] for a precise reference), we can assume that the open subgroup U of G is compact. We are thus reduced to look for information about compact subgroups of $\mathrm{GL}_n(k)$. This is provided by the far reaching results of R. Pink (see [Pin98, Corollary 0.5]). But in order to use those results in the proof of Theorem A, one needs to quotient the linear compact open subgroup U of G by a normal soluble subgroup. An indication that the theory was ripe for the classification of locally linear totally disconnected locally compact groups in $\mathscr{S}$ is the result in [CRW14, Theorem 5.3] that U does not possess any non-trivial soluble closed normal subgroup.

In fact, our method allows us to give a general structure theorem for totally disconnected, locally compact groups having a linear open subgroup wothout assuming that it is a member of $\mathscr{S}$. We refer the reader to the introduction of Chapter 1 for a thorough overview of the results it contains. For now, we just highlight another corollary of our results, which is striking in the sense that it provides a uniform framework for three classical families of simple groups.

Theorem B. *Let G be a compactly generated, topologically simple, locally compact group. Then G is linear over a (possibly Archimedean) local field if and only if G belongs to one of the following class:*

1. *Finite simple groups.*
2. *Simple Lie group.*
3. *Simple algebraic groups over local fields.*

Chapter 1 has been published as a separate article, see [CS15]. Compared to the article, we have added two appendices. One is concerned with proving that the automorphism group of a simple algebraic group over a local field is simple-by-compact, while the other elucidates the connection between algebraic and analytic varieties.

The work in Chapter 2 originates from the same quest for more examples of groups in $\mathscr{S}$. As P.-E. Caprace and N. Radu show, one can try to build a topologically simple group by approximating it with a sequence of topologically simple groups.

Theorem ([CR16, Theorem 1.2])**.** *let T be a locally finite tree whose vertices are of degree ≥ 3. The set of topologically simple closed subgroups of $\mathrm{Aut}(T)$ acting 2-transitively on ∂T is Chabauty-closed.*

Recall that the Chabauty space $\mathbf{Sub}(G)$ of a locally compact group G is the set of all closed subgroup of G, endowed with the so called Chabauty topology. We refer the reader to [dlH08] for a survey of the Chabauty space. For example, it follows directly from the definition that $\mathbf{Sub}(\mathbf{R})$ is homeomorphic to a compact interval $[0, \infty]$. On the other hand, it is already a non trivial result to prove that $\mathbf{Sub}(\mathbf{R}^2)$ is homeomorphic to the 4-sphere, while a helpful description of $\mathbf{Sub}(\mathbf{R}^n)$ when $n \geq 3$ is not known.

For T_{p+1} the $(p+1)$-regular tree,

$$\{\mathrm{SL}_2(K)/Z(\mathrm{SL}_2(K)) \mid K \text{ a totally ramified extension of } \mathbf{Q}_p\}$$

is an infinite family in $\mathrm{Aut}(T_{p+1})$ of topologically simple closed subgroups acting 2-transitively on ∂T_{p+1}. Furthermore, since the Chabauty space is compact, this infinite family must accumulate. Hence the question of knowing whether an accumulation would be a new kind of group in $\mathscr{S}$ becomes a pressing issue. But N. Radu soon computed that the only accumulation point of this family is actually $\mathrm{SL}_2(\mathbf{F}_p(\!(T)\!))/Z(\mathrm{SL}_2(\mathbf{F}_p(\!(T)\!)))$. At first, this convergence of algebraic groups over fields of characteristic 0 to an algebraic groups over a field of positive characteristic was very surprising. In retrospect, our work shows that this convergence is very natural, since $\mathbf{F}_p[\![T]\!]$ is the (projective) limit of $\mathcal{O}_{K_n}/\mathfrak{m}_{K_n}^n$, where K_n is a totally ramified extension of $\mathbf{Q}_p$ of degree n, with ring of integers $\mathcal{O}_{K_n}$ and maximal ideal $\mathfrak{m}_{K_n}$.

Chapter 2 extends this result to cover the case of Chabauty limits of groups of the form $\mathrm{SL}_2(D)$ for D a division algebra over a local field, as well as quasi-split special unitary groups $\mathrm{SU}_3^{L/K}(K)$. In essence, we show that Chabauty limits of groups of this form are again groups of this form.

In order to be more precise, for T a tree, let us define a *topologically simple algebraic group acting on* T to be a locally compact group isomorphic to $H(K)/Z$, where K is a local field, H is an absolutely simple, simply connected, algebraic group over K of relative rank 1 whose Bruhat–Tits tree is isomorphic to T, and Z is the center of $H(K)$.

The first thing to observe is that, given a topologically simple algebraic group G acting on T, the action homomorphism $G \to \mathrm{Aut}(T)$ is not canonical, but depends on some choices. There is however a natural way to resolve this issue of canonicity, explained in [CR16]. Following that paper, we shall denote by $\mathcal{S}_T$ the space of topological isomorphism classes of topologically simple closed subgroups of $\mathrm{Aut}(T)$ acting 2-transitively on the set of ends. According to [CR16, Theorem 1.2], the space $\mathcal{S}_T$ endowed with the quotient topology induced from the Chabauty space $\mathbf{Sub}(\mathrm{Aut}(T))$ is compact Hausdorff.

Using this language, the main result of Chapter 2 goes as follows:

Theorem C (see Chapter 2, Theorem 2.1)**.** *Let T be a locally finite leafless tree, and let $\mathcal{S}_T^{\text{qs-alg}}$ be the set of isomorphism classes of topologically simple algebraic groups acting on T that are furthermore quasi-split. Then $\mathcal{S}_T^{\text{qs-alg}}$ is closed in $\mathcal{S}_T$.*

We are actually able to explicitly describe the space $\mathcal{S}_T^{\text{qs-alg}}$, as well as the convergences happening inside it. For an exhaustive description, we refer the reader to the introduction of Chapter 2. We just highlight here the case of the 3-regular tree, where more intricate convergences happen, indicating that this kind of questions can be delicate.

Theorem D (see Chapter 2, Theorem 2.3)**.** *Let T be the 3-regular tree.*

1. *The space $\mathcal{S}_T^{\text{qs-alg}}$ is homeomorphic to $\hat{\mathbf{N}}^2$, where $\hat{\mathbf{N}}$ denotes the one point compactification of $\mathbf{N}$. The first Cantor–Bendixson derivative consists of groups in positive characteristic, while the second Cantor–Bendixson derivatice consists of the group $\mathrm{SL}_2(\mathbf{F}_2(\!(T)\!))/Z$,*

where Z is the center of $\mathrm{SL}_2(\mathbf{F}_2(\!(T)\!))$ *(see Theorem 2.3 for a more precise formulation).*

2. *For each n, let $K = \mathbf{F}_2(\!(T)\!)$, let $L_n = K[X]/(X^2 - T^n X + T)$, and identify $\mathrm{SU}_3^{L_n/K}(K)/Z$, Z being the center of $\mathrm{SU}_3^{L_n/K}(K)$, with a closed subgroup G_n of $\mathrm{Aut}(T)$ (using Bruhat–Tits theory). Then $\{G_n\}_{n\in\mathbf{N}}$ converges in the Chabauty space of $\mathrm{Aut}(T_3)$ to (a closed subgroup of $\mathrm{Aut}(T)$ isomorphic to) $\mathrm{SL}_2(K)/Z(\mathrm{SL}_2(K))$. In particular, the Tits index need not be preserved under Chabauty limits.*

The main tool in proving those theorems is the so-called integral model of a reductive group over a local field. Those integral models are smooth group schemes over the ring of integers of the local field, and they play an essential role in the monumental work of F. Bruhat and J. Tits. Especially, we give a concrete description (via equations) of various models of rank 1 quasi-split, absolutely simple, simply connected groups over local fields. We believe that the explicit description of the integral model for SU_3 in the so-called ramified and residue characteristic 2 case is also of interest in its own. See Subsection 2.4.2 for more on this topic. This work also led us to consider more generally the description of combinatorial balls in Bruhat–Tits buildings. We come back to this in the next section.

The work in Chapter 2 has been submitted for publication as a separate article, and is available on the ArXiV (see [Stu16]).

The final chapter, on the contrary, is not motivated by questions pertaining to the class $\mathscr{S}$. In our work, we frequently encountered the group of abstract automorphisms $\mathrm{Aut}(G(k))$ of (the rational points of) a semisimple algebraic group G. As we recall in Chapter 3, the celebrated result of A. Borel and J. Tits provides conditions ensuring that it fits into a short exact sequence $1 \to (\mathrm{Aut}\, G)(k) \to \mathrm{Aut}(G(k)) \to \mathrm{Aut}_G(k) \to 1$, and the question of whether this sequence splits arises naturally. While it was known from a long time that this sequence splits when G is a split algebraic group (in which case $\mathrm{Aut}_G(k) = \mathrm{Aut}(k)$), the general case had not been addressed. Of course, before attacking this question, one needs in particular to be able to decide whether $\mathrm{Aut}_G(k)$ is trivial or not. We are not aware of any results in the literature towards that direction.

In fact, our results show that settling the question in general is rather tricky, since it depends on the arithmetic of field extensions associated to the algebraic groups. We obtain a necessary and sufficient criterion for this sequence to split when G is quasi-split. Our criterion is independent of the ground field.

Theorem E (Chapter 3, Theorem 3.2 and Proposition 3.29)**.** *Let G be a quasi-split, absolutely simple, simply connected algebraic group of type ${}^2A_n$, ${}^2D_n$ or ${}^2E_6$ over a field k, and let l be the separable quadratic extension k defining G.*

Let $\mathrm{Aut}_l(k) = \{\alpha \in \mathrm{Aut}(k) \mid$ there exists an automorphism of l whose restriction to k is $\alpha\}$, let $\mathrm{Aut}(l \geq k) = \{\alpha \in \mathrm{Aut}(l) \mid \alpha(k) = k\}$ and $\mathrm{Aut}(l/k) = \{\alpha \in \mathrm{Aut}(l) \mid \alpha$ acts trivially on $k\}$.

1. *The group $\mathrm{Aut}_G(k)$ is isomorphic to $\mathrm{Aut}_l(k)$.*

2. *The short exact sequence $1 \to (\mathrm{Aut}\, G)(k) \to \mathrm{Aut}(G(k)) \to \mathrm{Aut}_G(k) \to 1$ splits if and only if the short exact sequence $1 \to \mathrm{Aut}(l/k) \to \mathrm{Aut}(l \geq k) \to \mathrm{Aut}_l(k) \to 1$ splits.*

We also have a similar result for quasi-split groups of type ${}^3D_4$ or ${}^6D_4$. We refer the reader to Proposition 3.29 for precise statements. Our work shows that for quasi-split groups, the splitting question is actually controlled by the scheme of Dynkin diagrams. This useful notion introduced in [ABD$^+$64, Exposé 24, section 3] is not so widely known, and we discuss it in detail in Chapter 3. Schemes of Dynkin diagram allows us to treat the case of quasi-split groups in an uniform way, and the main result in Chapter 3 is formulated using this language (see Theorem 3.2).

Here is a concrete corollary of the main result in Chapter 3, providing many examples where the aforementioned short exact sequence does not split.

Corollary F (See Chapter 3, Corollary 3.32)**.** *Let K, L be finite Galois extensions of $\mathbf{Q}_p$, with L a quadratic extension of K. Let G be the quasi-split algebraic group $\mathrm{SU}_n^{L/K}$ over K. The short exact sequence $1 \to (\mathrm{Aut}\, G)(K) \to \mathrm{Aut}(G(K)) \to \mathrm{Aut}_G(K) \to 1$ splits if and only if $1 \to \mathrm{Gal}(L/K) \to \mathrm{Gal}(L/\mathbf{Q}_p) \to \mathrm{Gal}(K/\mathbf{Q}_p) \to 1$ splits.*

We also investigate the case of $\mathrm{SL}_n(D)$ over a local field K, where the results have a very different flavour. As mentioned above, before attacking splitting question, we need to control the group $\mathrm{Aut}_{\mathrm{SL}_n(D)}(K)$. Doing explicit computations, we obtain the following result.

Theorem G (see Chapter 3, Corollary 3.37)**.** $\mathrm{Aut}_{\mathrm{SL}_n(D)}(K) = \mathrm{Aut}(K)$

If $\alpha \in \mathrm{Aut}(K)$ is of finite order, the fact that α extends to an automorphism of D was already known (see Remark 3.38). This implies that $\alpha \in \mathrm{Aut}_{\mathrm{SL}_n(D)}(K)$. Hence, Theorem G was already known in characteristic 0, but we are not aware of such a result in the literature in positive characteristic.

Now, concerning the splitting question for $\mathrm{SL}_n(D)$, we did not completely settled it in the positive characteristic case (see Theorem 3.4 for a precise statement of what we know). Here, we just quote the result in characteristic 0.

Theorem H (see Chapter 3, Theorem 3.4)**.** *Let K be a local field of characteristic 0, and let D be a finite dimensional central division algebra of degree d over K. Let $G = \mathrm{SL}_n(D)$, and let K' be the fixed field of $\mathrm{Aut}(K)$ (so that K is a finite Galois extension of K'). Then $\mathrm{Aut}_G(K) = \mathrm{Aut}(K)$, and the short exact sequence $1 \to (\mathrm{Aut}\,G)(K) \to \mathrm{Aut}(G(K)) \to \mathrm{Aut}_G(K) \to 1$ splits if and only if $\gcd(nd, [K : K'])$ divides n.*

Further directions of research

Our work in Chapter 2 is part of a more global project aiming to describe and classify combinatorial balls in Bruhat–Tits buildings of arbitrary rank.

Given an irreducible Bruhat–Tits building of rank n and of order q, the combinatorial ball of radius r consists of a finite number (bounded by a function of n, q and r) of n-simplices glued together in a specific way. On the other hand, there are infinitely many pairwise non-isomorphic irreducible Bruhat–Tits building of rank n and of order q. Hence the question of when irreducible Bruhat–Tits buildings of a given rank and order have isomorphic balls of radius r arises naturally.

Another way to frame our investigation of balls in Bruhat–Tits buildings is to say that we study the metric space of all Bruhat-Tits buildings, endowed with the Gromov-Hausdorff topology. In [dT15], the authors show that the Gromov-Hausdorff space of buildings of type $\tilde{A}_n$ (with $n > 2$) has accumulations points, which are precisely those of positive characteristic. It is then natural to wonder what is the situation for other types of buildings. Our [Stu16] is closely related to that question, and indicates that buildings of type $C - BC_n$ accumulates on buildings of type A_n.

The question of classifying balls in buildings is already interesting when $r = 1$, i.e. for residues in buildings. In their classical paper [BT84a], F. Bruhat and J. Tits observed that reducing the integral model modulo the maximal ideal correspond geometrically to a localisation in the building. Hence, over local fields, the description of residues is straightforward. For example, given two split simple algebraic groups G_1 and G_2, defined respectively over the local fields k_1 and k_2, their residues are isomorphic if and only if they have isomorphic Dynkin diagram and the residue fields are isomorphic. However, describing residues in more general situations can be more involved. For example, one of the main focus in [MPW15] is to investigate residues of a euclidean building whose building at infinity is Moufang.

In view of the $r = 1$ case, it is natural to conjecture that for $r > 1$, reducing an integral model modulo the r-th power of the maximal ideal correspond geometrically to a localisation of radius r in the building. We implemented this idea in [Stu16], via an explicit description of the equations defining an integral model. The main goal of this project is to give a complete classification of isomorphism of balls in Bruhat–Tits buildings in the quasi-split case, using the explicit commutation relations in the quasi-split case (as given in [BT84a, Annexe]), together with the Artin–Weil theorem extending uniquely birational group laws on schemes (see [BLR90, § 6.6, Theorem 1]). It should then be possible to obtain the general case from the quasi-split case by unramified descent.

The application we have in mind for this work is a proof of the following conjecture, that we formulate in Chapter 2, where we use the same language than in Theorem C.

Conjecture. Let T be a locally finite leafless tree, and let $\mathcal{S}_T^{\mathrm{alg}}$ be the set of isomorphism classes of topologically simple algebraic groups acting on T. Then $\mathcal{S}_T^{\mathrm{alg}}$ is closed in $\mathcal{S}_T$.

It is also interesting to note that a complete characterisation of isomorphism classes of balls in Bruhat–Tits buildings involve questions about abstract isomorphisms of algebraic groups over local rings modulo power of their prime ideals. Indeed, let us for example try to show that if $\mathrm{SL}_n(k_1)$ and $\mathrm{SL}_n(k_2)$ have isomorphic balls of radius r, then $\mathcal{O}_{k_1}/\mathfrak{m}_{k_1}^r \cong \mathcal{O}_{k_2}/\mathfrak{m}_{k_2}^r$. First, by a standard argument of J. Tits, an isomorphism of balls implies that $\mathrm{PGL}_n(\mathcal{O}_{k_1}/\mathfrak{m}_{k_1}^r)$ and $\mathrm{PGL}_n(\mathcal{O}_{k_2}/\mathfrak{m}_{k_2}^r)$ are (abstractly) isomorphic. Then the desired result would follow from the algebraicity of abstract isomorphisms of PGL over $\mathcal{O}_{k_1}/\mathfrak{m}_{k_1}^r$. This particular case has actually already been proved in [Pet89], and our goal is to extend this result beyond the split case.

This project on balls in Bruhat–Tits might also give new points of view for the representation theory of reductive groups over local fields, since it is related with the Moy–Prasad filtration introduced in [MP94].

In the course of our research, we also got interested in the work of P.-E. Caprace, C. Reid and G. Willis (see [CRW13]) on the so-called structure lattice of a totally disconnected, locally compact (tdlc) group G, which consists of locally normal subgroups. A main tool in this study is the centraliser lattice $LC(G)$, which is a boolean algebra extracted from the lattice of locally normal subgroups. The authors then proceed to describe various properties of the action of G on the Stone space corresponding to the boolean algebra $LC(G)$, which enables them to prove structure theorems about G. Unfortunately, when $LC(G)$ is trivial, we get no information on G via this method.

We observed that it is actually possible to define a quotient $H(G)$ of the lattice of locally normal subgroups, such that $H(G)$ is a Heyting algebra whose associated boolean algebra is isomorphic to $LC(G)$. Furthermore, in situations where $LC(G)$ is trivial, it can still happen that $H(G)$ is non-trivial. This is the case for some Kac-Moody groups. We aim to characterise properties of the action of G on the Priestley space associated to $H(G)$. This approach could extend the results in [CRW13].

Chapter 1

Totally disconnected locally compact groups with a linear open subgroup

In this first chapter, we describe the global structure of totally disconnected locally compact groups having a linear open compact subgroup. Among the applications, we show that if a nondiscrete, compactly generated, topologically simple, totally disconnected locally compact group is locally linear, then it is a simple algebraic group over a local field. All the work in this chapter is joint work with P.-E. Caprace.

1.1 Introduction

A locally compact group is called **linear** if it admits a continuous faithful finite-dimensional linear representation over a local field. It is called **locally linear** if it has an open subgroup which is linear. The goal of this chapter is to study the class of totally disconnected locally compact groups (t.d.l.c.) that are locally linear. Roughly speaking, our main results ensure that such groups are built out of three kinds of elementary pieces: discrete groups, compact groups, and simple algebraic groups over local fields (note that abelian t.d.l.c. groups are compact-by-discrete). In order to be more precise, let us define a **topologically**

simple algebraic group over a local field to be a locally compact group isomorphic to $H(k)/Z$, where k is a local field, H is an absolutely simple, simply connected, isotropic algebraic group over k, and Z is the center of $H(k)$ (see Section 1.2.1 for more details on those groups). We also say that a locally compact group is **locally solvable** (respectively, **locally abelian of finite exponent**) if it has a solvable (respectively, abelian of finite exponent) open subgroup.

We can now state our main result.

Theorem 1.1. *Let G be a t.d.l.c. group having an open compact subgroup which is linear over a local field k. Then G has a series of closed normal subgroups:*

$$\{1\} \trianglelefteq R \trianglelefteq G_1 \trianglelefteq G_0 \trianglelefteq G$$

enjoying the following properties.

The group R is a closed characteristic subgroup and is locally solvable. The group G_0 is an open characteristic subgroup of finite index in G, and G_0/G_1 is locally abelian of finite exponent. Moreover, the quotient group $H_0 = G_0/R$, if nontrivial, has nontrivial closed normal subgroups, say $M_1, \dots, M_m$, satisfying the following properties.

(i) For some $l \leq m$ and all $i \leq l$, the group M_i is a topologically simple algebraic group over a local field k_i, of the same characteristic and residue characteristic as k. In particular, M_i is compactly generated and abstractly simple.

(ii) For all $j > l$, the group M_j is compact, hereditarily just-infinite (h.j.i.), and algebraic (in the sense of Definition 1.46) over a local field k_j, of the same characteristic and residue characteristic as k.

(iii) Every nontrivial closed normal subgroup N of H_0 contains M_i for some $i \leq l$, or contains an open subgroup of M_j for some $j > l$.

(iv) The quotient group $H_1 = G_1/R$ coincides with the direct product of subgroups $M_1 \dots M_m \cong M_1 \times \cdots \times M_m$, which is closed in H_0. In particular, H_1 is compactly generated.

Note the apparent analogy with the structure of general Lie groups, whose quotient by their solvable radical is semisimple. It should however be emphasized that the characteristic subgroup R afforded by Theorem 1.1 is locally solvable, but not solvable in general: indeed, it can contain discrete normal subgroups of G that are nonabelian free groups.

One nevertheless expects that the structure of the normal subgroup R is not too mysterious. In order to make that statement precise, we define the class of **elementary groups** as the smallest class of t.d.l.c. groups that contains all discrete groups, all profinite groups, and is closed under group extensions and directed unions of open subgroups. This class was first defined and investigated by Wesolek in [Wes15a] in the second countable case, and then extended to the general t.d.l.c. case in [Wes14]. We obtain the following consequence of Theorem 1.1.

Corollary 1.2. *Let G be a t.d.l.c. group having a compact open subgroup which is linear over a local field k. Then G has a series of closed characteristic subgroups*

$$\{1\} \trianglelefteq A \trianglelefteq G_1 \trianglelefteq G_0 \trianglelefteq G$$

enjoying the following properties.

The group A is elementary, G_0 is open of finite index in G, and the quotient G_0/G_1 is locally abelian of finite exponent. In particular, G_0/G_1 is elementary. Moreover, the quotient group $H_0 = G_0/A$, if nontrivial, satisfies the following.

(i) H_0 has finitely many minimal closed normal subgroups, that we call $M_1, \ldots, M_l$, and every nontrivial closed normal subgroup of H_0 contains some M_i.

(ii) Each M_i is a topologically simple algebraic group over a local field k_i, of the same characteristic and residue characteristic as k. In particular, M_i is compactly generated and abstractly simple.

(iii) The quotient group $H_1 = G_1/A$ coincides with the product of subgroups $M_1 \ldots M_l \cong M_1 \times \cdots \times M_l$, which is closed in H_0.

We will see in due course that the characteristic subgroup A afforded by Corollary 1.2 contains, as expected, the subgroup R afforded by Theorem 1.1. As a byproduct of Theorem 1.1, we also deduce that a locally linear t.d.l.c. group G is elementary if and only if the groups $M_1, \dots, M_m$ from the statement of Theorem 1.1 are all compact (equivalently the subquotient G_1/R is compact). This follows from the fact that topologically simple algebraic groups over local fields are not elementary (see Claim 4 in the proof of Corollary 1.2) combined with the fact that locally solvable groups are all elementary (see Proposition 1.29(ii)). This observation thus provides structural information on all locally linear elementary groups, and can be applied to the elementary group A appearing in Corollary 1.2, since A is itself locally linear.

Note that when k is of characteristic 0, the hypotheses of Theorem 1.1 imply that G is a p-adic Lie group, where p is the residue characteristic of k. The conclusions of Corollary 1.2 are then already known, due to Ph. Wesolek: indeed, they follow from [Wes15b, Corollary 1.5.]. Moreover, in that special case, the elementary quotient G_0/G_1 is even finite (the latter is however not true in positive characteristic, see Example 1.13). The main novelty of our results is that they hold in *all* characteristics. The key tool allowing for this uniformity is provided by the far-reaching results of Pink [Pin98] on compact subgroups of linear algebraic groups.

Another special case of particular interest is when G is assumed to be **topologically simple**, that is, its only closed normal subgroups are the trivial ones.

Corollary 1.3. *Let G be a t.d.l.c. group having a compact open subgroup which is linear over a local field k. If G is topologically simple, then one of the following holds.*

(a) G is discrete.

(b) G is nondiscrete, not compactly generated, and locally solvable.

(c) G is a topologically simple algebraic group over a local field k', of the same characteristic and residue characteristic as k. In particular, G is compactly generated and abstractly simple.

It should be emphasized that examples of topologically simple t.d.l.c. groups as in (b) of Corollary 1.3 do exist. Indeed, such examples can be produced using the construction described by Willis [Wil07, Section 3]. Those examples can be made locally isomorphic to the additive group of the field $\mathbf{F}_p(\!(t)\!)$ (hence they are indeed locally linear and locally abelian) and can be arranged to contain a copy of every finite group (hence they are not globally linear).

The following consequence of Corollary 1.3 is immediate.

Corollary 1.4. *Let G be a nondiscrete, compactly generated, topologically simple, t.d.l.c. group. If G is locally linear, then G is algebraic: indeed G is a topologically simple algebraic group over a local field.*

A systematic study of the class $\mathscr{S}$ of nondiscrete, compactly generated, topologically simple, t.d.l.c. groups has been initiated in [CRW14]. Corollary 1.4 implies that the locally linear members of $\mathscr{S}$ are precisely the algebraic ones, and are thus all known since the latter algebraic groups have been classified by Kneser and Bruhat–Tits. In particular, within the class $\mathscr{S}$, a group is locally linear if and only if it is globally linear. This matter of fact has the following consequence on irreducible complete Kac–Moody groups over finite fields, which form a subclass of $\mathscr{S}$ (see [Mar14] and references therein).

Corollary 1.5. *An irreducible complete Kac–Moody group over a finite field which is not globally linear, is also not locally linear: none of its compact open subgroups is linear.*

This applies to all Kac–Moody groups of irreducible nonspherical, nonaffine type which are either of rank at least 3 (by [CR09]) or of rank 2, and whose generalized Cartan matrix has -1 as an off-diagonal entry (by [CR13]). Corollary 1.5 confirms [CR14, Conjecture 1], except in the case of generalized Cartan matrices of size 2 with both off-diagonal entries different from -1.

Finally, we record the following application.

Corollary 1.6. *Let G be a compactly generated, topologically simple, locally compact group. Then G is linear over a (possibly Archimedean) local field if and only if G belongs to one of the following classes:*

- *Finite simple groups.*
- *Simple Lie groups.*
- *Simple algebraic groups over local fields.*

1.2 Algebraic t.d.l.c. groups

1.2.1 Linear algebraic groups

Let H be a **linear algebraic k-group**, where k is a field. By definition, this means that H is a smooth affine group scheme of finite type over k. Equivalently, H is (schematically) isomorphic to a smooth Zariski closed subgroup of $GL_{n,k}$. As all algebraic groups used in this thesis are linear, we omit this adjective in the sequel.

For $\varphi\colon H \to H_1$ a morphism of algebraic k-groups, we denote the evaluation of φ in a k-algebra A by $\varphi_A\colon H(A) \to H_1(A)$, or sometimes by $\varphi\colon H(A) \to H_1(A)$.

When k is a Hausdorff (i.e., not anti-discrete) topological field, the group $H(k)$ inherits a Hausdorff topology, which does not depend on the embedding into $GL_{n,k}$ (for more details, see [PR94]). We adopt the convention that any topological statement will refer to that topology, and not to the Zariski topology, unless we explicitly add the prefix Zariski (e.g., Zariski-connected or Zariski-dense). When k is a nondiscrete locally compact field, the group $H(k)$ is a locally compact second countable topological group.

A semisimple algebraic k-group H is called **k-simple** if it has no nontrivial Zariski-connected normal algebraic k-subgroup. It is called **absolutely simple** if for any field extension $k \to k'$, the algebraic k'-group $H \times_k k'$ is k'-simple. Equivalently, H is absolutely simple if its root system is irreducible.

The study of Zariski-connected semisimple algebraic groups reduces for the most part to that of absolutely simple ones. Namely, a Zariski-connected, semisimple, simply connected (respectively, adjoint) algebraic k-group is the direct product of k-simple, simply connected (respectively, adjoint) algebraic k-groups, and each factor is of the form $\mathcal{R}_{k'/k}H$ for some absolutely simple, simply connected (respectively, adjoint) algebraic k'-group H, where k' is a finite separable extension of k. Here, $\mathcal{R}_{k'/k}H$ denotes the Weil restriction of H. For a proof of those facts, see [Con14, Proposition 6.4.4, Remark 6.4.5 and Example 6.4.6].

Let H be a Zariski-connected algebraic k-group. As in [BT73, Section 6], we denote by $H(k)^+$ the normal subgroup of $H(k)$ generated by k-rational points of split unipotent k-subgroups of H.

In order to properly understand the definition of a topologically simple algebraic t.d.l.c. group, we need two results, that will also be invoked later on.

Proposition 1.7. *Let k be a local field, let H be a simply connected, k-simple algebraic k-group. Any continuous homomorphism $f\colon H(k) \to G$ to a locally compact group G is a closed map.*

Proof. See [BM96, Lemma 5.3]. Note that the assumption there that the target of the map should be second countable is superfluous. □

Theorem 1.8. *Let k be a local field and let H be a k-simple algebraic k-group. Then any proper open subgroup of $H(k)^+$ is compact. Moreover, $H(k)^+$ is compactly generated.*

Proof. See [Pra82, Theorem (T)] for the first assertion. To prove the compact generation, one can assume that H is isotropic and that $H(k) = H(k)^+$ (see Theorem 2.4). In this case, we can find a semismple element of norm greater than one in any nontrivial k-split torus, which thus generate an infinite discrete cyclic subgroup. So that this element together with a compact open subgroup will generate the whole group, in view of [Pra82, Theorem (T)]. □

Definition 1.9. Let G be a topologically simple t.d.l.c. group. We say that G is **algebraic** if there exists a local field k and a k-simple algebraic k-group H such that G is topologically isomorphic to $H(k)^+/Z(H(k)^+)$.

Note that by the main result of [Tit64], the group $H(k)^+/Z(H(k)^+)$ is abstractly simple.

The following theorem (which is a collection of results borrowed from [BT73, Section 6]) shows that in the above definition, one can assume that H is simply connected, absolutely simple and that $H(k) = H(k)^+$. This confirms that Definition 1.9 is consistent with the definition of a topologically simple algebraic group over a local field given in the introduction.

Theorem 1.10. *Let k be a local field and H be a k-simple algebraic k-group.*

(i) $H(k)^+$ is trivial if and only if H is anisotropic over k.

(ii) Assume that H is isotropic over k. Any central isogeny $f\colon H \to H_1$ induces an isomorphism $H(k)^+/Z(H(k)^+) \to H_1(k)^+/Z(H_1(k)^+)$.

(iii) If k is a finite separable extension of a subfield k', $(\mathcal{R}_{k/k'}H)(k')^+ \cong H(k)^+$.

(iv) If H is isotropic, simply connected, and absolutely simple, then we have $H(k) = H(k)^+$.

Proof. See [BT73, Section 6]. □

1.2.2 Hereditarily just-infinite groups

An essential point in the proof of Theorem 1.47 is that the linearity of the open compact subgroup, say U, implies that U has subquotients that are hereditarily just-infinite.

Definition 1.11. A profinite group G is called **just-infinite** if it is infinite and every nontrivial closed normal subgroup of G is of finite index. A profinite group G is called **hereditarily just-infinite** (h.j.i.) if every open subgroup of G is just-infinite.

Theorem 1.12. *Let U be an open compact subgroup of $H(k)$, where H is a k-simple, simply connected algebraic k-group, and k is a local field of residue characteristic p. Then $U/Z(U)$ is a nonvirtually abelian h.j.i. virtually pro-p group.*

Proof. First note that U is Zariski-dense in $H(k)$, hence U is infinite and $Z(U) = Z(H(k)) \cap U$. We want to show that if $U_1 \leq U/Z(U)$ is an open subgroup, then it is just-infinite. But the preimage of such a U_1 is an open compact subgroup of $H(k)$ as well, which has the same center than U. Hence, it suffices to prove that $U/Z(U)$ is just-infinite and virtually pro-p. Now, this follows directly from the main result of [Rie70a].

It remains to show that $U/Z(U)$ is not virtually abelian. But if it was, $H(k)$ would have an open, hence Zariski-dense, abelian subgroup, contradicting the fact that H is not abelian. □

We emphasize that when $\mathrm{char}(k) > 0$, the hypothesis that H is simply connected is essential in Theorem 1.12. Indeed, as one immediately deduces from [Rie70a], the above result does not hold if H is a k-simple group whose universal cover $\pi : \tilde{H} \to H$ is inseparable. Here is an explicit example.

Example 1.13. Consider the group $H = PSL_2$ over the local field $k = \mathbf{F}_2(\!(T)\!)$, and the open compact subgroup $U = PSL_2(\mathbf{F}_2[\![T]\!])$ in $PSL_2(k)$. We insist that we consider PSL_2 as the quotient scheme SL_2/μ_2 (over $\operatorname{Spec}\mathbf{Z}$), and that $PSL_2(\mathbf{F}_2[\![T]\!])$ denotes the group of $\mathbf{F}_2[\![T]\!]$ rational points of PSL_2, not to be confused with the quotient group $SL_2(\mathbf{F}_2[\![T]\!])/Z(SL_2(\mathbf{F}_2[\![T]\!]))$.

The universal cover is $\pi : SL_2 \to PSL_2$, which is purely inseparable over k. Now, $H(k)^+$ is a closed normal subgroup of H and is equal to $\pi_k(SL_2(k))$ (see [BT73, Section 6]). Let us show explicitly that $H(k)^+ \cap U$ is not open in U. It suffices to consider the sequence

$$h_i = \begin{pmatrix} 1 + T^{\frac{1+2i}{2}} & 0 \\ 0 & (1 + T^{\frac{1+2i}{2}})^{-1} \end{pmatrix}$$

whose elements are in $H(k) \setminus H(k)^+$ and which converge to the identity.

Let us check that $h_i \in H(k)$. If $k[X_{11}, X_{12}, X_{21}, X_{22}]/\det$ denotes the coordinate ring of SL_2 in its standard coordinates, then the coordinate ring of PSL_2 is the subring of $k[SL_2]$ generated by all products $X_{ij}X_{kl}$ where $i, j, k, l \in \{1, 2\}$. This shows that h_i is indeed in $H(k)$.

1.3 Locally normal subgroups

The interaction between the local and global structure of general t.d.l.c. groups has become more and more apparent in recent works. In this section, we review some tools from [CRW13] and [Wes15a] and use them to establish subsidiary facts that will be needed in the sequel.

1.3.1 Locally C-stable groups

Let G be a t.d.l.c. group.

Definition 1.14. (1) The **quasi-center** of G, denoted by $QZ(G)$, is the characteristic subgroup of G consisting of all elements whose centralizer is open. More generally, given $H \leq G$, we define the **quasi-centralizer** of H in G, denoted by $QC_G(H)$, to be the subgroup of G consisting of those elements that centralize an open subgroup of H.

(2) A subgroup $K \leq G$ is called **locally normal** if it is compact and normalized by an open subgroup of G.

(3) G is called **locally C-stable** if $QZ(G)$ is trivial and there is no nontrivial abelian locally normal subgroup.

Note that our definition of local C-stability is slightly less general than in [CRW13, Definition 3.17]. The following property of locally C-stable groups will be needed later.

Proposition 1.15. *Let G be a locally C-stable t.d.l.c. group. Then every locally normal subgroup of G has trivial quasi-center.*

Proof. See [CRW13, Theorem 3.18]. □

The locally C-stable assumption is indeed a weakening of the hypotheses that G is a compactly generated and topologically simple group, as asserted by the following.

Theorem 1.16. *Let G be a nondiscrete topologically simple t.d.l.c. group which is compactly generated. Then G is locally C-stable.*

Proof. See [CRW14, Theorem 5.3]. □

1.3.2 The structure lattice

Let G be a t.d.l.c. group.

Definition 1.17. Two subgroups H, K of G are **locally equivalent** if there exists a compact open subgroup U of G such that $H \cap U = K \cap U$. The set of all local equivalence classes having a locally normal representative is called the **structure lattice** of G, and is denoted by $\mathcal{LN}(G)$.

$\mathcal{LN}(G)$ is a lattice in a natural way: the meet operation is the intersection (of any representatives) and the join operation is the product (of well-chosen representatives). Obviously, $[\{e\}]$, the local equivalence class of the trivial subgroup, is the minimum of $\mathcal{LN}(G)$ and we denote it by 0. At the other extreme, the local equivalence class of compact open subgroups of G is the maximum of $\mathcal{LN}(G)$ and we denote it by ∞. We refer the reader to [CRW13, Section 2] for a more detailed discussion of $\mathcal{LN}(G)$.

An **atom** of $\mathcal{LN}(G)$ is a minimal nonzero element. The following lemma is a first elementary observation about the role of h.j.i. locally normal subgroups in the structure lattice.

Lemma 1.18. *Let G be a t.d.l.c. group and $\alpha \in \mathcal{LN}(G)$ have a locally normal representative V which is h.j.i. Then α is an atom of $\mathcal{LN}(G)$.*

Proof. First note that by definition, V is not discrete, hence $\alpha \neq 0$.

Let $\beta \in \mathcal{LN}(G)$, and assume that $\beta \leq \alpha$. We want to show that either $\beta = 0$ or $\beta = \alpha$. Let L be a locally normal representative of β (i.e., $\beta = [L]$), so that our assumption translates as $[L \cap V] = [L]$.

Consider an open compact subgroup W of $N_G(L) \cap N_G(V)$. Now, $L \cap W \cap V$ is normal closed in W, hence also in $W \cap V$. But since the latter is just-infinite, we obtain that $L \cap W \cap V$ is either trivial, or open in $W \cap V$, as wanted. □

When G is a product of h.j.i. groups, we can refine the previous lemma as follows.

Lemma 1.19. *Let $G \simeq V_1 \times \cdots \times V_m$ be a profinite group which splits as the direct product of finitely many h.j.i. closed subgroups, none of which* *is virtually abelian. Then the structure lattice $\mathcal{LN}(G)$ is isomorphic to the Boolean algebra of all subsets of $\{1,\dots,m\}$. Moreover, for a locally normal representative K of an element of $\mathcal{LN}(G)$, there exist $i_1,\dots,i_k \in \{1,\dots,m\}$ such that $U_{i_1}\dots U_{i_k} \le K \le V_{i_1}\dots V_{i_k}$, where U_{i_j} is an open subgroup of V_{i_j} for all j.*

Proof. Let $\alpha_i = [V_i]$ be the local equivalence class having representative V_i. Obviously, $\{\alpha_{i_1} \vee \dots \vee \alpha_{i_k} \mid 1 \le i_1 < \dots < i_k \le m,\ k = 1,\dots,m\}$ are different elements in $\mathcal{LN}(G)$. We want to show that all elements of $\mathcal{LN}(G)$ are of this form.

Let $\beta \in \mathcal{LN}(G)$, let K be a locally normal representative of β, and consider the projection $\pi_i(K)$ onto the i-th factor V_i. Since K is normalized by an open subgroup of G, it is normalized by an open subgroup of V_i. Furthermore, $\pi_i(K)$ is compact, hence closed. But V_i is h.j.i., so that $\pi_i(K)$ is either trivial or open in V_i. Reordering the V_i's if necessary, we can assume that $\pi_i(K)$ is open in V_i for $i \le k$, and $\pi_i(K)$ is trivial for $i > k$.

We obviously have that $[K] \le [V_1] \vee \dots \vee [V_k]$, and we now prove the reverse inequality. For this, consider $K \cap V_i$, for $i \le k$ (where V_i is seen as a subgroup of G via the natural injection). It is a locally normal subgroup of V_i, hence it is either trivial or open in V_i. But if $K \cap V_i$ were trivial, we would have $[K, N_{V_i}(K)] \subseteq K \cap V_i = \{e\}$, contradicting $\{e\} \neq [\pi_i(K), \pi_i(K) \cap N_{V_i}(K)] \subseteq [K, N_{V_i}(K)]$. We conclude that for $i \le k$, $[V_i \cap K] = [V_i]$, so that $[V_1] \vee \dots \vee [V_k] = [(N_{V_1}(K) \cap K) \dots (N_{V_k}(K) \cap K)] \le [K]$, as wanted.

For the last assertion, note that if K is a locally normal representative of an element in $\mathcal{LN}(G)$, then the second paragraph of this proof shows that $\pi_i(K)$ is open in V_i for $i \in \{i_1,\dots,i_k\}$ and is trivial otherwise, so that $K \le V_{i_1}\dots V_{i_k}$. But the third paragraph shows then that $N_{V_{i_j}}(K) \cap K$ is open in V_{i_j} for all j, and that $(N_{V_{i_1}}(K) \cap K) \dots (N_{V_{i_k}}(K) \cap K) \le K$, as wanted. □

We now consider profinite groups that are virtually a direct product of h.j.i. groups.

Lemma 1.20. *Let G be a profinite group without nontrivial finite normal subgroup, and having an open subgroup $G_0 \cong V_1 \times \dots \times V_m$ which splits as* *the direct product of finitely many h.j.i. closed subgroups, none of which is virtually abelian.*

Then G has a characteristic open subgroup $G_1 \simeq W_1 \times \cdots \times W_m$, contained in G_0, which splits as the direct product of finitely many h.j.i. closed subgroups, where W_i is an open subgroup of V_i.

Proof. We first claim that G is locally C-stable. Indeed, every h.j.i. group which is not virtually abelian has trivial quasi-center, by [BEW11, Proposition 5.1]. Therefore, the quasi-center of G must be finite, hence trivial by hypothesis. The fact that the only abelian locally normal subgroup of G is the trivial one follows from Lemma 1.19, since $\mathcal{LN}(G) = \mathcal{LN}(G_0)$. The claim stands proved.

Let now $\alpha_i = [V_i] \in \mathcal{LN}(G)$. Since the quasi-centralizer $QC_G(V_i)$ depends only on the local class α_i, we denote it by $QC_G(\alpha_i)$, following the convention adopted in [CRW13]. Next we set $L_i = QC_G(QC_G(\alpha_i))$. Since G is locally C-stable, we may invoke [CRW13, Lemma 3.15(ii)] which shows that $L_i = C_G(C_G(V_i))$. Thus, the subgroup L_i is closed in G. By Lemma 1.19, the automorphism group $\mathrm{Aut}(G)$ permutes the α_i and, hence, permutes the closed subgroups L_i.

We now claim that $[L_i] = \alpha_i$ and that L_i commutes with L_j for all $i \neq j$. Note that $V_i \subseteq L_i = C_G(C_G(V_i))$, hence we just have to show that V_i is open in L_i. Set $P_i := \prod_{j \neq i} V_j$. Then $G \geqslant L_i P_i \geqslant V_i P_i = G_0$. But G_0 is open in G, so that the index of V_i in L_i is finite, as wanted. For the second assertion of the claim, starting from $V_i \subseteq C_G(V_j)$ (which is true for all $i \neq j$), we get $L_i = C_G(C_G(V_i)) \subseteq C_G(C_G(C_G(V_j))) = C_G(V_j)$. So that finally, $C_G(L_i) \supseteq C_G(C_G(V_j)) = L_j$, as was to be shown.

Therefore, the subgroup $G_2 \leq G$ generated by the L_i's is characteristic, open, and isomorphic to the direct product $L_1 \times \cdots \times L_m$. Since each L_i is h.j.i., it has a basis of identity neighborhoods consisting of characteristic subgroups. Therefore, the same is true for G_2, and therefore G_2 has a characteristic subgroup G_1 contained in G_0, which has the desired form. □

Building upon Lemmas 1.18 and 1.19, we obtain the following more technical fact.

Lemma 1.21. *Let G be a profinite group having a closed normal subgroup $\Gamma \simeq V_1 \times \cdots \times V_m$ which splits as the direct product of finitely many h.j.i. closed subgroups, none of which is virtually abelian. Assume further that G contains no nontrivial abelian locally normal subgroup, and that G/Γ is abelian.*

Then for all $0 \neq \beta \in \mathcal{LN}(G)$, there exists $i \in \{1, \dots, m\}$ such that $[V_i] \leq \beta$. In particular, the set of all the atoms of $\mathcal{LN}(G)$ is precisely $\{[V_1], \dots, [V_m]\}$.

Proof. We first have to check that V_i has an open normalizer in G. But G acts on the atoms of $\mathcal{LN}(\Gamma)$, which is a finite set by Lemma 1.19. Also, in view of the last assertion of that lemma and the fact that Γ is normal, $[gV_ig^{-1}] = [V_i]$ if and only if $g \in N_G(V_i)$. Hence, $N_G(V_i)$ is of finite index in G. As it is also closed, because V_i is, we conclude that it is open, as wanted.

Now let β be a nontrivial element of $\mathcal{LN}(G)$ and set $\alpha_i = [V_i]$. Recall that by Lemma 1.18, the α_i's are atoms in $\mathcal{LN}(G)$. We want to show that $\beta \geq \alpha_i$ for some i, and we now separate the proof into two cases.

Case 1. $\beta \wedge (\alpha_1 \vee \cdots \vee \alpha_m) = 0$.

The assumption translates as $K \cap (V_1V_2 \dots V_m) = F$, a finite subgroup. Shrinking K again if necessary, we may and do assume that $K \cap (V_1V_2 \dots V_m) = \{e\}$.

Hence, we have a continuous injective map $K \to G/(V_1V_2 \dots V_m) \to G/\Gamma$. But since the latter is abelian, so is K. In view of the hypothesis, we conclude that $\beta = 0$, a contradiction.

Case 2. $\beta \wedge (\alpha_1 \vee \cdots \vee \alpha_m) \neq 0$.

Using Lemma 1.19, we conclude that for some i, we have $\alpha_i \leq \beta \wedge (\alpha_1 \vee \cdots \vee \alpha_m) \leq \beta$, as wanted. $\square$

1.3.3 Radical theories

In this section, we review the definition and basic properties of two characteristic subgroups of a general t.d.l.c. group. The first is the $[A]$-regular radical $R_{[A]}(G)$, defined in [CRW13], and the second is the elementary radical $R_{\mathcal{E}}(G)$, defined in [Wes15a] under the assumption that G is second countable.

Definition 1.22. Let $[A]$ be the smallest class of profinite groups, stable under isomorphism, such that the following conditions hold:

(a) $[A]$ contains all abelian profinite groups and all finite simple groups.

(b) If $U \in [A]$ and K is a closed normal subgroup of U, then $K \in [A]$ and $U/K \in [A]$.

(c) Given a profinite group U that is a (not necessarily direct) product of finitely many closed normal subgroups belonging to $[A]$, then $U \in [A]$.

Given a profinite group U, a subgroup K of U is called $[A]$**-regular** in U if for every closed normal subgroup L of U not containing K, the image of K in the quotient U/L contains a nontrivial locally normal subgroup of U/L belonging to the class $[A]$. Given a t.d.l.c. group G and a closed subgroup H, we say that H is $[A]$**-regular** in G if $H \cap U$ is $[A]$-regular in U for all open compact subgroups U of G.

Note that all groups in $[A]$ are virtually nilpotent by Fitting's theorem. The $[A]$**-regular radical** of a t.d.l.c. group G is defined to be the characteristic subgroup identified by the following result.

Theorem 1.23. *Let G be a t.d.l.c. group. Then G has a closed characteristic subgroup $R_{[A]}(G)$, which is characterized by either of the following properties:*

(i) $R_{[A]}(G)$ is the largest subgroup of G that is $[A]$-regular.

(ii) $R_{[A]}(G)$ is the smallest closed normal subgroup N such that G/N is locally C-stable.

Proof. See [CRW13, Theorem 6.11]. □

We move on to elementary groups following [Wes15a] and [Wes14]. We first restrict to t.d.l.c. groups that are second countable (t.d.l.c.s.c. for short).

Definition 1.24. The class $\mathcal{E}_{\mathrm{sc}}$ is defined as the smallest class of groups that are t.d.l.c.s.c. and such that

(a) $\mathcal{E}_{\mathrm{sc}}$ contains all metrizable profinite groups and all countable discrete groups.

(b) $\mathcal{E}_{\mathrm{sc}}$ is closed under taking group extensions.

(c) $\mathcal{E}_{\mathrm{sc}}$ is closed under countable directed unions of open subgroups.

A key feature, due to Wesolek, is the existence of a radical belonging to the class $\mathcal{E}_{\mathrm{sc}}$, asserted in the following.

Proposition 1.25. *Let G be a t.d.l.c.s.c. group. Then G has a largest closed normal subgroup $\mathrm{Rad}_{\mathcal{E}_{\mathrm{sc}}}(G)$ which belongs to the class $\mathcal{E}_{\mathrm{sc}}$.*

Proof. See [Wes15a, Proposition 7.4]. □

The relation between the two radicals introduced above is elucidated by the following.

Proposition 1.26. *Let G be a t.d.l.c.s.c. group. Then $R_{[A]}(G) \leq \mathrm{Rad}_{\mathcal{E}_{\mathrm{sc}}}(G)$. In particular, $G/\mathrm{Rad}_{\mathcal{E}_{\mathrm{sc}}}(G)$ is locally C-stable, and $R_{[A]}(G)$ belongs to $\mathcal{E}_{\mathrm{sc}}$.*

Proof. See [Wes15a, Corollary 9.12 and 9.13]. □

We now briefly explain how one can drop the second countability assumption, following the work in [Wes14]. This discussion was suggested to us by Wesolek.

Definition 1.27. The class of **elementary groups** is the smallest class $\mathcal{E}$ of t.d.l.c. groups such that

(a) $\mathcal{E}$ contains all profinite groups and all discrete groups.

(b) $\mathcal{E}$ is closed under taking group extensions.

(c) $\mathcal{E}$ is closed under directed unions of open subgroups.

It should be stressed that our choice of terminology is slightly different from Wesolek's: what he called the class of *elementary groups* and denoted by $\mathcal{E}$ in the references [Wes15a] and [Wes14] is denoted by $\mathcal{E}_{\mathrm{sc}}$ in the present chapter. Moreover, the class denoted here by $\mathcal{E}$ is denoted by $\mathcal{E}^*$ in [Wes14]. We believe that our choice is natural in the present context, and should not cause any confusion.

The inclusion $\mathcal{E}_{\mathrm{sc}} \subset \mathcal{E}$ is clear from the definitions. Conversely, we have the following.

Lemma 1.28. *Let $G \in \mathcal{E}$. If G is second countable, then $G \in \mathcal{E}_{\mathrm{sc}}$.*

Proof. This is a particular case of [Wes14, Proposition 4.3]. □

This lemma allows us to deduce that many properties of $\mathcal{E}_{\mathrm{sc}}$ generalize to $\mathcal{E}$, as follows (see also Theorem 1.31).

Proposition 1.29. *Let G be a t.d.l.c. group.*

(i) Let H be a t.d.l.c. group that maps continuously and injectively onto a dense normal subgroup of G. If $H \in \mathcal{E}$, then $G \in \mathcal{E}$.

(ii) If G is locally solvable, then $G \in \mathcal{E}$.

Proof. Write $G = \bigcup_{i \in I} O_i$ as a directed union of compactly generated open subgroups. By a result in [KK44], for each $i \in I$, there exists a compact normal subgroup $K_i \leq O_i$ such that O_i/K_i is a t.d.l.c.s.c. group.

(i) Let now H be a t.d.l.c. group that maps continuously and injectively onto a dense normal subgroup of G such that $H \in \mathcal{E}$. Then $(O_i \cap H)K_i/K_i$ is a dense normal subgroup of O_i/K_i. Moreover, by Lemma 1.28, it belongs to $\mathcal{E}_{\mathrm{sc}}$. In the second countable case, the desired result is known, namely $O_i/K_i \in \mathcal{E}_{\mathrm{sc}}$ by [Wes15a, Theorem 1.4]. Hence, for each i, the group O_i is compact-by-elementary, hence elementary. Therefore, G is itself elementary, as required.

(ii) For each i, the groups O_i and O_i/K_i are locally solvable. By [Wes15a, Theorem 8.1] we have $O_i/K_i \in \mathcal{E}_{\mathrm{sc}}$. Therefore, O_i is compact-by-elementary, and we conclude as in the proof of (i). □

The **elementary radical** is defined to be the characteristic subgroup identified by the following result, which is a straightforward adaptation of [Wes15a, Proposition 7.4].

Theorem 1.30. *Let G be a t.d.l.c. group. Then G has a largest closed normal subgroup $\mathrm{Rad}_{\mathcal{E}}(G)$ which is elementary.*

Proof. Let N_i be an ascending chain of closed normal subgroups of G that are elementary. Let $U < G$ be a compact open subgroup. Then N_iU is elementary for each i, hence so is the union $O = \bigcup_i N_iU$. It follows that $\overline{\bigcup_i N_i} \leq O$ is elementary, since $\mathcal{E}$ is closed under taking closed subgroups in view of [Wes14, Theorem 4.6(b)]. It follows from Zorn's lemma that the collection of elementary closed normal subgroups of G has maximal elements. In fact, there is a unique such, since the closure of the product of any two of them is itself elementary as a consequence of Proposition 1.29(i). □

Finally, we extend Theorem 1.26 to the general case.

Theorem 1.31. *Let G be a t.d.l.c. group. Then $R_{[A]}(G) \leq \mathrm{Rad}_{\mathcal{E}}(G)$. In particular, $G/\mathrm{Rad}_{\mathcal{E}}(G)$ is locally C-stable, and $R_{[A]}(G)$ is elementary.*

Proof. Write $G = \bigcup_{i \in I} O_i$ as a directed union of compactly generated open subgroups. By a result in [KK44], for each $i \in I$, there exists a compact normal subgroup $K_i \leq O_i$ such that O_i/K_i is a t.d.l.c.s.c. group.

In view of [CRW13, Proposition 6.15(ii)], for each $i \in I$, we have $O_i \cap R_{[A]}(G) = R_{[A]}(O_i)$. Let $\pi_i \colon O_i \to O_i/K_i$ be the projection. Since $[A]$-regularity is stable under quotients by closed normal subgroups, we have $R_{[A]}(O_i) \leq \pi_i^{-1}(R_{[A]}(O_i/K_i))$. Note that $R_{[A]}(O_i/K_i)$ is elementary by Theorem 1.26, hence so is $\pi_i^{-1}(R_{[A]}(O_i/K_i))$, and thus also $R_{[A]}(O_i)$ by [Wes14, Theorem 4.6(b)]. We conclude that $R_{[A]}(G)$ is a directed union of open elementary subgroups, and is thus elementary, as required. □

1.4 The compact subgroups of a linear algebraic group

As outlined in Section 1, Theorem 1.47 relies essentially on the results obtained by Pink in [Pin98]. The goal of this section is to review those results and to adapt them to our needs.

1.4.1 The group of abstract commensurators

Another important object used in the local-to-global transfer lying behind our main results is the group of abstract commensurators of a profinite group, first defined and investigated by Barnea–Ershov–Weigel in [BEW11].

Definition 1.32. Let U be a profinite group. The **group of abstract commensurators** of U, denoted $\mathrm{Comm}(U)$, is defined as follows. Consider the set E of isomorphisms $\alpha : U_1 \to U_2$, where the U_i's are open compact subgroups of U and α is a topological isomorphism. Define an equivalence relation $\sim$ on E by $\alpha \sim \beta$ if and only if they coincide on some open subgroup of U. We set $\mathrm{Comm}(U) = E/\sim$.

As explained in [BEW11], the group of abstract commensurators of an open compact subgroup of a simple algebraic group is described by Corollary 0.3 of Pink's paper [Pin98]. Let us record that result explicitly.

Theorem 1.33 ([Pin98], Corollary 0.3)**.** *Let G (respectively, G') be an absolutely simple, simply connected algebraic group over a local field k (respectively, k'). Let U (respectively, U') be an open compact subgroup of G (respectively, G'). Then for any topological isomorphism $\alpha : U \to U'$, there exists a unique isomorphism of algebraic groups $G \to G'$ over a unique isomorphism of topological fields $k \to k'$ such that the induced morphism $G(k) \to G(k')$ extends α.*

Given a topological group G, we denote by $\mathrm{Aut}(G)$ its group of bicontinuous automorphisms.

Corollary 1.34. *Let G be an absolutely simple, simply connected algebraic group over a local field k, and let U be an open compact subgroup of $G(k)$. Let Z denote the center of $G(k)$. Then* $\mathrm{Comm}(U)$ *is canonically isomorphic to* $\mathrm{Aut}(G(k)/Z)$.

Proof. For any open compact subgroup $V \leqslant G(k)$, we have $\mathrm{Comm}(U) \simeq \mathrm{Comm}(V)$. Hence, since Z is finite, we can assume that $U \cap Z = \{1\}$.

Now, we have a natural homomorphism $\varphi : \mathrm{Aut}(G(k)/Z) \to \mathrm{Comm}(U)$. Pink's result (see Theorem 1.33) directly implies that φ is surjective. For the injectivity, observe that $QZ(G(k)/Z) = \{1\}$. Indeed, any element of $G(k)$ whose centralizer is open belongs to the center, by Zariski-density of open subgroups. We finally conclude that φ is injective by [BEW11, Proposition 2.5]. $\square$

Remark 1.35. We have an exact sequence $1 \to \mathrm{Hom}(G(k), Z) \to \mathrm{Aut}(G(k)) \to \mathrm{Aut}(G(k)/Z))$. Once again by Pink's result (see [Pin98, Corollary 0.5]), $\mathrm{Aut}(G(k)) \to \mathrm{Aut}(G(k)/Z)$ is surjective. Note that if G is isotropic, $G(k)^+ = G(k)$ by Theorem 1.10 (4) and $G(k)/Z$ is abstractly simple by the main result of [Tit64], so that $\mathrm{Hom}(G(k), Z)$ is trivial and $\mathrm{Aut}(G(k)) \cong \mathrm{Aut}(G(k)/Z)$ if G is isotropic. On the other hand, by the classification of semisimple groups over local fields, G is anisotropic if and only if it is of the form $\mathrm{SL}_1(D)$ for some finite dimensional central division algebra D over k. In this case, it is not hard to find non-trivial homomorphisms from $G(k)$ to its center (a way to do that is to use the results in [Rie70b] describing the structure of $\mathrm{SL}_1(D)$).

A priori, the group $\mathrm{Comm}(U)$ is just an abstract group, but as discussed in [BEW11], there are several ways to endow it with a group topology. The identification provided by Corollary 1.34 suggests that, in our situation, the natural topology on $\mathrm{Comm}(U)$ should be the one which coincides with the Braconnier topology on $\mathrm{Aut}(G(k)/Z)$. Let us now address the details, following [BEW11, Section 7].

Definition 1.36. A profinite group U is called **countably characteristically based** if it has a countable basis of neighborhood of the identity consisting of characteristic subgroups. A profinite group is called **hereditarily countably characteristically based** (h.c.c.b.) if every open subgroup of U is countably characteristically based.

Example 1.37. Let G be a k-simple, simply connected algebraic k-group, where k is a local field, and let U be an open compact subgroup of $G(k)$. Then U is h.c.c.b. Indeed, U is a h.j.i. virtually pro-p group (see Theorem 1.12), hence is finitely generated. And as explained in [BEW11, Section 7.1], , every finitely generated profinite group is h.c.c.b.

Another way to see that U is h.c.c.b. is to exhibit by hand a countable characteristic neighborhood basis of any open subgroup by considering intersections of maximal open normal subgroups.

Definition 1.38. Let U be an h.c.c.b. profinite group. For any open subgroup $V \leq U$, let $\rho_V : \mathrm{Aut}(V) \to \mathrm{Comm}(U)$ be the natural homomorphism and endow $\mathrm{Aut}(V)$ with the compact open topology. The **Aut-topology** on $\mathrm{Comm}(U)$ is defined by the following sub-base of identity neighborhood :

$$\begin{aligned}\mathcal{B}_U = \{H \leq \mathrm{Comm}(U) \mid \\ \rho_V^{-1}(H) \text{ is open in } \mathrm{Aut}(V) \text{ for all open subgroups } V \text{ of } U\}.\end{aligned}$$

As explained in [BEW11, Proposition 7.3], this turns $\mathrm{Comm}(U)$ into a topological group.

Definition 1.39. Let G be a locally compact group. The **Braconnier** topology on $\mathrm{Aut}(G)$ is defined by the following sub-base of identity neighborhood :

$$\mathcal{U}(K, U) = \{\phi \in \mathrm{Aut}(G) \mid \forall\, x \in K,\ \phi(x) \in xU \text{ and } \phi^{-1}(x) \in xU\},$$

where $K \subseteq G$ is compact and $U \subseteq G$ is an identity neighborhood (see [CM11, Appendix I] for more comments on this topology).

Remark 1.40. The Braconnier topology is the natural one, in the sense that it turns Aut(G) into a topological group, while the compact open topology on Aut(G) does not in general. However it does in the special case where G is compact. Moreover, given any closed normal subgroup N of G, the adjoint map $\mathrm{Ad}\colon G \to \mathrm{Aut}(N)$ given by the conjugation action is continuous for the Braconnier topology (see [HR79, Theorem (26.7)]).

In order to prove that the Aut-topology on $\mathrm{Comm}(U)$ coincides with the Braconnier topology on $\mathrm{Aut}(G(k)/Z)$, we use the following result due to Barnea–Ershov–Weigel.

Proposition 1.41. *Let U be an h.c.c.b. profinite group. Assume that* $\mathrm{Comm}(U)$ *with the Aut-topology is Hausdorff. Suppose that* $\mathrm{Comm}(U)$ *is* *a topological group with respect to some topology $\mathcal{T}$ and that there exists an open subgroup V of U such that*

(i) The index $[\mathrm{Comm}(U) : \mathrm{Aut}(V)]$ *is countable.*

(ii) $\mathrm{Aut}(V)$ *is an open compact subgroup of* $(\mathrm{Comm}(U), \mathcal{T})$.

(iii) If N is an open subgroup of V and $(f_n)_{n=1}^{\infty}$ is a sequence in $\mathrm{Aut}(V)$ *such that $f_n \to 1$ with respect to $\mathcal{T}$, then $f_n(N) = N$ for sufficiently large n.*

Then $(\mathrm{Comm}(U), \mathcal{T})$ *is locally compact, second countable, and $\mathcal{T}$ coincides with the Aut-topology.*

Proof. The fact that the topologies coincide and that $\mathrm{Comm}(U)$ is a locally compact group is the exact content of [BEW11, Proposition 8.8]. It just remains to prove the second countability of $Comm(U)$. Since U is h.c.c.b., it follows that $\mathrm{Aut}(V)$ is a compact metrizable group with respect to the compact open topology. By definition of the Aut-topology, the natural embedding $\mathrm{Aut}(V) \to \mathrm{Comm}(U)$ is continuous, and is thus a homeomorphism onto its image. Therefore, $\mathrm{Comm}(U)$ is metrizable. Moreover, it is σ-compact since $[\mathrm{Comm}(U) : \mathrm{Aut}(V)]$ is countable. This confirms that $\mathrm{Comm}(U)$ is second countable. □

The following result is a straightforward adaptation of [BEW11, Example 8.1] to our situation.

Proposition 1.42. *Let H be an absolutely simple, simply connected algebraic group over a local field k, let U be an open compact subgroup of $H(k)$, and let Z denote the center of $H(k)$. Then the canonical isomorphism* $\mathrm{Comm}(U) \simeq \mathrm{Aut}(H(k)/Z)$ *of Corollary 1.34 is an isomorphism of topological groups, where* $\mathrm{Comm}(U)$ *has the* Aut*-topology and* $\mathrm{Aut}(H(k)/Z)$ *has the Braconnier topology. In particular,* $\mathrm{Aut}(H(k)/Z)$ *is a t.d.l.c.s.c. group.*

Proof. As noted in Example 1.37, U is h.c.c.b. We next claim that $\mathrm{Comm}(U)$ is Hausdorff. Indeed, $\mathrm{Comm}(U) = \mathrm{Comm}(V)$ for some open subgroup V having a trivial center, so that $QZ(V)$ is trivial (to prove this last assertion, one can argue as in [BEW11, Proposition 5.1]). Using

[BEW11, Proposition 2.5 and Theorem 8.6], we deduce the claim. For the rest of the proof, we can (and will) assume that $U \cap Z = \{1\}$.

The desired conclusion will follow from Proposition 1.41. In order to check the three conditions, we first remark that, since every automorphism of U extends to the whole of $H(k)$, we have

$$\mathrm{Aut}(U) = \{\varphi \in \mathrm{Aut}(H(k)/Z) \mid \varphi(U) = U\} = \mathcal{U}(U,U). \tag{1.1}$$

We now check the three conditions successively.

(i) The index of $\mathrm{Aut}(U)$ in $\mathrm{Comm}(U)$ is countable. Indeed, ϕ, $\psi \in \mathrm{Comm}(U)$ are in the same coset modulo $\mathrm{Aut}(U)$ if and only if $\phi(U) = \psi(U)$. Therefore, it suffices to check that U has a countable orbit under $\mathrm{Comm}(U) = \mathrm{Aut}(H(k)/Z)$. This is indeed the case, since $H(k)/Z$ is second countable, and thus has countably many compact open subgroups.

(ii) In view of (1.1), $\mathrm{Aut}(U)$ is open in $\mathrm{Aut}(H(k)/Z)$ by the definition of the Braconnier topology.

(iii) Let N be an open subgroup of U, and let $(f_n)_{n=1}^{\infty}$ be a sequence converging to 1 in $\mathrm{Aut}(H(k)/Z)$. Then, for n large enough, $f_n \in \mathcal{U}(N,N) = \{\varphi \in \mathrm{Aut}(H(k)/Z) \mid \varphi(N) = N\}$, as wanted. □

If U is an open compact subgroup of G, we have a canonical map $G \to \mathrm{Comm}(U)$; we end this section by verifying its continuity.

Lemma 1.43. *Let U be an h.c.c.b. profinite group and let G be a topological group containing U as a locally normal subgroup. Assume that G commensurates U. Then the canonical map $\varphi \colon G \to \mathrm{Comm}(U)$ is continuous, where $\mathrm{Comm}(U)$ has the Aut-topology.*

Proof. Let $W = N_G(U)$, which is open by assumption. It suffices to prove that the restriction of φ to W is continuous at the identity. Observe that φ factors through $\rho_U \colon \mathrm{Aut}(U) \to \mathrm{Comm}(U)$, which is continuous by definition of the Aut-topology. Moreover, the adjoint map $W \to \mathrm{Aut}(U)$ is continuous by Remark 1.40, so that the composed map $W \to \mathrm{Aut}(U) \to \mathrm{Comm}(U)$ is continuous as well. □

1.4.2 Decomposition into h.j.i. factors

We now come to the heart of our toolbox, which consists of Pink's results from [Pin98]. We start by repeating one of the main theorems from [Pin98].

Theorem 1.44. *Let k be a local field and let Γ be a compact subgroup of $GL_n(k)$. There exist closed normal subgroups $U_3 \leq U_2 \leq U_1$ of Γ such that*

(i) *U_1 is of finite index in Γ.*

(ii) *U_1/U_2 is abelian of finite exponent.*

(iii) *There exists a local field k' of the same characteristic and residue characteristic as k, a Zariski-connected, semisimple adjoint algebraic group H over k', with universal covering $\pi\colon \tilde{H} \to H$, and an open compact subgroup $\Delta \leq \tilde{H}(k')$ such that $U_2/U_3 \simeq \pi_{k'}(\Delta)$ as topological groups.*

(iv) *U_3 is solvable of derived length $\leq n$.*

Proof. See [Pin98, Corollary 0.5]. □

It will be crucial for our purposes to arrange that the subquotient U_2/U_3 is the direct product of h.j.i. groups. This is achieved by the following.

Theorem 1.45 (Extended version of [Pin98], Corollary 0.5)**.** *Let k be a local field and let Γ be a compact subgroup of $GL_n(k)$. There exist closed normal subgroups $U_3 \leq U_2 \leq U_1$ of Γ such that :*

(i) *U_1 is of finite index in Γ.*

(ii) *U_1/U_2 is abelian of finite exponent.*

(iii) *There exist local fields $k'_1, \dots, k'_m$ of the same characteristic and residue characteristic as k, Zariski-connected, absolutely simple adjoint algebraic k'_i-groups H_i, with universal covering $\pi_i\colon \tilde{H}_i \to H_i$, and open compact subgroups $\Delta_i \leq \tilde{H}_i(k'_i)$ such that*

$$U_2/U_3 \cong \pi_1(\Delta_1) \times \cdots \times \pi_m(\Delta_m)$$

as topological groups. In particular, in view of Theorem 1.12, the subquotient U_2/U_3 is a direct product of nonvirtually abelian h.j.i. groups.

(iv) U_3 is solvable of derived length $\leq n$.

Proof. Retain the notation of Theorem 1.44. As recalled in Section 1.2.1, we may and do decompose $\tilde{H}$ as the direct product of Weil restrictions $\prod_{i=1}^{m} \mathcal{R}_{k'_i/k'}\tilde{H}_i$, where each $\tilde{H}_i$ is an absolutely simple, simply connected algebraic group over a finite separable extension k'_i of k'. Let $G_i = \mathcal{R}_{k'_i/k'}\tilde{H}_i(k')$.

Now, the compact group Δ appearing in (iii) of Theorem 1.44 is an open compact subgroup of $G_1 \times \cdots \times G_m$. Therefore, there exists an open compact subgroup Δ_i for G_i such that $\Lambda = \Delta_1 \times \cdots \times \Delta_m$ is contained in Δ. Thus, $\pi(\Lambda) \simeq (\pi_1)_{k'_1}(\Delta_1) \times \cdots \times (\pi_m)_{k'_m}(\Delta_m)$ is an open subgroup of U_2/U_3. However, it is not clear a priori that it is normalized by Γ/U_3. In order to ensure that, it suffices to apply Lemma 1.20 to the group U_2/U_3 (note that $U_2/U_3 \simeq \pi_{k'}(\Delta)$ has indeed no nontrivial finite normal subgroups, in view of Theorem 1.12). This shows that, upon replacing each Δ_i by a suitable open subgroup, the image $\pi(\Lambda)$ is indeed an open subgroup of U_2/U_3 of the desired form, which is moreover normalized by Γ/U_3.

Let U'_2 denote the preimage of $\pi(\Lambda)$. Now the quotient U_1/U'_2 is finite-by-{abelian of finite exponent}. We may thus replace U_1 by a smaller open normal subgroup U'_1 of Γ containing U'_2 to ensure that U'_1/U'_2 is indeed abelian of finite exponent. Now the normal chain $U_3 \leq U'_2 \leq U'_1$ of Γ satisfies all the requested properties. □

To capture the properties of the compact factors appearing in (iii) of Theorem 1.45, we introduce the following terminology.

Definition 1.46. A compact h.j.i. group Γ is called **algebraic** if there is a local field k and a Zariski-connected, absolutely simple, adjoint algebraic k-group H with universal cover $\pi\colon \tilde{H} \to H$, and a compact open subgroup Δ of $\tilde{H}(k)$ such that Γ is isomorphic to $\pi(\Delta)$.

1.5 The global structure of locally linear groups

1.5.1 Proof of the main theorem

The following result is a reformulation of Theorem 1.1 from Section 1, using the terminology introduced in Section 1.3.3.

Theorem 1.47. *Let G be a t.d.l.c. group having an open compact subgroup which is linear over a local field k. Then G has a series of closed normal subgroups:*

$$\{1\} \trianglelefteq R \trianglelefteq G_1 \trianglelefteq G_0 \trianglelefteq G$$

enjoying the following properties.

The group $R = R_{[A]}(G)$ is the $[A]$-regular radical of G and is locally solvable. The group G_0 is an open characteristic subgroup of finite index in G, and G_0/G_1 is locally abelian of finite exponent. Moreover, the quotient group $H_0 = G_0/R$, if nontrivial, has nontrivial closed normal subgroups, say $M_1, \dots, M_m$, satisfying the following properties.

(i) For some $l \leq m$ and all $i \leq l$, the group M_i is a topologically simple algebraic group over a local field k_i, of the same characteristic and residue characteristic as k. In particular, M_i is compactly generated and abstractly simple.

(ii) For all $j > l$, the group M_j is compact, h.j.i., and algebraic (in the sense of Definition 1.46) over a local field k_j, of the same characteristic and residue characteristic as k.

(iii) Every nontrivial closed normal subgroup N of H_0 contains M_i for some $i \leq l$, or contains an open subgroup of M_j for some $j > l$.

(iv) The quotient group $H_1 = G_1/R$ coincides with the product of subgroups $M_1 \dots M_m \cong M_1 \times \cdots \times M_m$, which is closed in H_0. In particular, H_1 is compactly generated.

Proof. Let $U \leq G$ be a compact open subgroup which is linear over k. Let also $H = G/R$ and V denote the image of U in H. Theorem 1.45 applied to the group U yields closed normal subgroup $U_3 \leq U_2 \leq U_1$ satisfying the properties (i)-(iv) from that statement.

Claim 1. *U_3 is contained in R as an open subgroup. In particular, R is locally solvable.*

The image of U_3 in H is a solvable locally normal subgroup. It must therefore be trivial, since H is locally C-stable by Theorem 1.23. Thus, $U_3 \leq R$.

Assume now for a contradiction that U_3 is not open in R. Then $U \cap R$ contains U_3 as a closed normal subgroup of infinite index. Since $U \cap R$ is $[A]$-regular in U by Theorem 1.23, it follows that the image of $U \cap R$ in U/U_3 contains a nontrivial locally normal subgroup belonging to $[A]$. However, by Lemma 1.21 and Theorem 1.45, every nontrivial locally normal subgroup of U/U_3 contains a locally normal subgroup which is h.j.i. and algebraic. Those subgroups do not belong to $[A]$. This is a contradiction, and the claim stands proved.

Claim 2. *There exist closed normal subgroups $V_2 \leq V_1$ of V such that*

(i) V_1 is of finite index in V.

(ii) V_1/V_2 is abelian of finite exponent.

(iii) There exist local fields $k'_1, \dots, k'_m$ of the same characteristic and residue characteristic than k, Zariski-connected, absolutely simple adjoint algebraic k'_i-group H_i, with universal covering $\pi_i \colon \tilde{H}_i \to H_i$, and open compact subgroups $\Delta_i \leq \tilde{H}_i(k'_i)$ such that $V_2 \simeq \pi_1(\Delta_1) \times \cdots \times \pi_m(\Delta_m)$ as topological groups. In particular, the group V_2 is a direct product of nonvirtually abelian h.j.i. groups.

We denote by V_i the image of U_i in V. Then V_i is a closed normal subgroup of V (because U_i is compact), and V_3 is trivial by Claim 1.

Note that V_1 and V_2 satisfy conditions (i) and (ii) from the claim, in view of the corresponding properties of U_1 and U_2. It remains to check that V_2 satisfies (iii). Since $V_2 \simeq U_2R/R \simeq U_2/U_2 \cap R$, it suffices to show that $U_2 \cap R = U_3$ in view of Theorem 1.45(iii). By Claim 1, the group U_3 is contained as an open subgroup of $U_2 \cap R$, so that the image of $U_2 \cap R$ in U_2/U_3 is a finite normal subgroup, and is thus trivial by Theorem 1.45(iii) and Lemma 1.19. This shows that $U_2 \cap R = U_3$, so that $V_2 \simeq U_2/U_3$. The claim stands proved.

Claim 3. *The set of atoms of $\mathcal{LN}(H)$ coincides with the finite set $\{[\pi_1(\Delta_1)], \dots, [\pi_m(\Delta_m)]\}$, and every nonzero element of $\mathcal{LN}(H)$ contains an atom. In particular, H has an open characteristic subgroup H_0 containing V_2, which commensurates $\pi_i(\Delta_i)$ for all $i = 1, \dots, m$.*

Since V_1 is open in H, we have $\mathcal{LN}(H) = \mathcal{LN}(V_1)$. In view of Claim 2, the hypotheses of Lemma 1.21 are satisfied by V_1. This proves the desired assertions on $\mathcal{LN}(H)$.

Now the H-action on $\mathcal{LN}(H)$ permutes the atoms, and we define H_0 to be the kernel of that permutation action. Then H_0 is indeed open, characteristic and of finite index in H, and commensurates $\pi_i(\Delta_i)$ for all i. Since V_2 normalizes $\pi_i(\Delta_i)$ for all i, we have $V_2 \leq H_0$, as claimed.

Claim 4. *For each $i \in \{1, \dots, m\}$, let $\varphi_i \colon H_0 \to \mathrm{Comm}(\pi_i(\Delta_i))$ be the homomorphism induced by Claim 3. Then the product homomorphism*

$$\varphi = \varphi_1 \times \cdots \times \varphi_m \colon H_0 \to \mathrm{Comm}(\pi_1(\Delta_1)) \times \cdots \times \mathrm{Comm}(\pi_m(\Delta_m))$$

is continuous and injective, where each factor is endowed with the Aut-topology.

In view of Lemma 1.43, the map φ is a product of continuous homomorphisms, and is thus continuous. Let us now check its injectivity.

In view of Definition 1.14, we have $\operatorname{Ker}\varphi \leq QC_{H_0}(V_2)$, and it suffices to check that V_2 has trivial quasi-centralizer in H_0.

Recalling that H, and thus also H_0, is locally C-stable, we deduce from [CRW13, Lemma 3.11 and Theorem 3.18] that $QC_{H_0}(V_2 C_{H_0}(V_2)) = 1$. Since V_1 is open in H, $1 = QC_{H_0}(V_2 C_{H_0}(V_2)) = QC_{H_0}((V_2 C_{H_0}(V_2)) \cap V_1) = QC_{H_0}(V_2 C_{V_1}(V_2))$. Therefore, it is enough to show that the centralizer $C_{V_1}(V_2)$ is trivial. Now $C_{V_1}(V_2) \cap V_2 \leq QZ(V_2)$, which is trivial by Proposition 1.15. Thus, $C_{V_1}(V_2)$ embeds into V_1/V_2, and is thus abelian by Claim 2. But $C_{V_1}(V_2)$ is also locally normal in H (see, e.g., [CRW13, Lemma 2.1]), and must therefore be trivial because H is locally C-stable.

Claim 5. *Let $i \in \{1, \dots, m\}$, and let $Z_i = Z(\tilde{H}_i(k'_i))$. Then there is an isomorphism of topological groups*

$$\mathrm{Comm}(\pi_i(\Delta_i)) \simeq \mathrm{Aut}(\tilde{H}_i(k'_i)/Z_i),$$

where $\mathrm{Comm}(\pi_i(\Delta_i))$ *has the Aut-topology and* $\mathrm{Aut}(\tilde{H}_i(k_i')/Z_i)$ *has the Braconnier topology.*

By Corollary 1.34 and Proposition 1.42, we have an isomorphism of topological groups $\mathrm{Comm}(\Delta_i) \simeq \mathrm{Aut}(\tilde{H}_i(k_i')/Z_i)$. Since $\mathrm{Ker}(\pi_i)$ is finite, there is an open subgroup $\Delta_i' \leq \Delta_i$ which intersects $\mathrm{Ker}(\pi_i)$ trivially. Thus, π_i induces an isomorphism of profinite groups between Δ_i' and its image, so that

$$\mathrm{Comm}(\pi_i(\Delta_i)) = \mathrm{Comm}(\pi_i(\Delta_i')) \simeq \mathrm{Comm}(\Delta_i') = \mathrm{Comm}(\Delta_i).$$

The claim follows.

Claim 6. *Let* $i \in \{1, \dots, m\}$ *and set*

$$M_i = \varphi_i^{-1}(\mathrm{Inn}(\tilde{H}_i(k_i'))) \cap \bigcap_{j \neq i} \mathrm{Ker}\,(\,\varphi_j\,),$$

where $\mathrm{Inn}(\tilde{H}_i(k_i'))$ *is viewed as a subgroup of* $\mathrm{Comm}(\pi_i(\Delta_i))$ *by means of Claim 5. Then* M_i *is a closed normal subgroup of* H_0*, and exactly one of the following assertions holds:*

(a) M_i *is a compact, h.j.i. group which is algebraic over* k_i'*.*

(b) $M_i \simeq \tilde{H}_i(k_i')/Z_i$*, and* $\tilde{H}_i$ *is isotropic over* k_i'*. In particular,* M_i *is a topologically simple algebraic group over* k_i'*.*

We first check that the quotient group $\tilde{H}_i(k_i')/Z_i$ is isomorphic to $\mathrm{Inn}(\tilde{H}_i(k_i'))$ endowed with the Braconnier topology. Indeed, by Proposition 1.42, the group $\mathrm{Aut}(\tilde{H}_i(k_i')/Z_i)$ is locally compact, and by Remark 1.40 the canonical embedding $\tilde{H}_i(k_i')/Z_i \to \mathrm{Aut}(\tilde{H}_i(k_i')/Z_i)$ is continuous. From Proposition 1.7, we deduce that the latter embedding is a homeomorphism onto its image, namely $\mathrm{Inn}(\tilde{H}_i(k_i'))$, and that the latter is closed in $\mathrm{Aut}(\tilde{H}_i(k_i')/Z_i)$. This also implies that M_i is a closed normal subgroup of H_0.

We next observe that the restriction of φ_i to M_i is a homeomorphism onto its image. Indeed $(\varphi_i)|_{M_i}$ is injective by Claim 4. Moreover, by Claim 2, we have $\pi_i(\Delta_i) \leq M_i$, and $\varphi_i(\pi_i(\Delta_i))$ is open in $\mathrm{Inn}(\tilde{H}_i(k_i')) \simeq$

$\tilde{H}_i(k_i')/Z_i$. Thus, $(\varphi_i)|_{M_i}$ is a continuous isomorphism onto an open, hence closed, subgroup of $\mathrm{Inn}(\tilde{H}_i(k_i'))$.

Since $\mathrm{Inn}(\tilde{H}_i(k_i'))$ is second countable (see Proposition 1.42) and $(\varphi_i)|_{M_i}$ is injective, we deduce that the compact group $\pi_i(\Delta_i)$ is of countable index in M_i. It follows that M_i is σ-compact. By the Open Map Theorem (see [HR79, Theorem (5.29)]), we deduce that the map $(\varphi_i)|_{M_i}$ is open, as requested.

Now if M_i is compact, the desired claim follows by construction (see Theorem 1.12 and Definition 1.46). If M_i is noncompact, then $\varphi_i(M_i)$ is a noncompact open subgroup of $\mathrm{Inn}(\tilde{H}_i(k_i'))$ so that $\tilde{H}_i(k_i')$ is noncompact. Hence, $\tilde{H}_i$ is isotropic by [Pra82, Theorem (BTR)]. By Theorem 1.8, the only noncompact open subgroup of $\mathrm{Inn}(\tilde{H}_i(k_i')) \simeq \tilde{H}_i(k_i')/Z_i = \tilde{H}_i(k_i')^+/Z(\tilde{H}_i(k_i')^+)$ is the whole group (see Theorem 1.10 for the last equality). The claim follows.

Claim 7. *We have $[M_i, M_j] = 1$ for $i \neq j$. Moreover, the product $H_1 = M_1 \dots M_m \cong M_1 \times \cdots \times M_m$ is closed in H_0, and the quotient $G_0/G_1 = H_0/H_1$ is locally abelian of finite exponent.*

The injectivity of φ, established in Claim 4, ensures that the M_i's commute pairwise, and that the canonical map from $M_1 \times \cdots \times M_m$ onto the subgroup $S = M_1 \dots M_m$ is a continuous isomorphism. To see that H_1 is closed, consider the canonical projection $H_0 \to H_0/\overline{M_2 \dots M_m}$. If M_1 is compact, then it has closed image. If M_1 is not compact, then Claim 6 and Proposition 1.7 ensure that M_1 has closed image as well. Hence, the product $M_1\overline{M_2 \dots M_m}$ is closed in H_0, and a straightforward induction shows that H_1 is closed as well.

Finally, since $V_2 \leq H_1$, it follows from Claim 2 that H_0/H_1 is locally abelian of finite exponent.

Claim 8. *Every nontrivial closed normal subgroup N of H_0 contains some noncompact M_i, or an open subgroup of some compact M_j.*

The group $V \cap N$ is a locally normal subgroup of H_0, and therefore there is some i such that $[\pi_i(\Delta_i)] \leq [V \cap N]$ by Claim 3. If M_i is compact, this yields the desired assertion. Otherwise we see that $N \cap M_i$ is an open normal subgroup of M_i, so that $M_i \leq N$ by Claim 6. The claim stands proved.

To conclude the proof, we denote by G_0 (respectively, G_1) the preimage of H_0 (respectively, H_1) in G, and re-order the set $\{M_1, \dots, M_m\}$ so that the noncompact elements come first. We see that all the requested assertions have been established in the claims above: Assertion (iv) and the fact that G_0/G_1 is locally abelian of finite exponent in Claim 7, Assertions (i) and (ii) in Claim 6 and Assertion (iii) in Claim 8. □

Remark 1.48. We remark that the closedness of the product of the M_i in H_0 asserted in Theorem 1.47(iv) is not an automatic property. Clearly, the subgroup generated by two distinct minimal normal subgroups is abstractly isomorphic to their direct product, but it need not be closed. This phenomenon naturally yields to the concept of **quasi-products**. Further discussions and concrete examples may be found in [CM11, Appendix II].

Remark 1.49. The subgroup R is also characterised as the maximal closed normal locally solvable subgroup. Indeed, if N is a closed normal locally solvable subgroup, its image $\tilde{N}$ in G/R is a locally solvable group, so has a non-trivial quasi-center. But G/R is locally C-stable by Theorem 1.23, hence $\tilde{N}$ is trivial by 1.15.

Remark 1.50. Defining the **discrete residual** of a topological group to be the intersection of all its open normal subgroups, we prove in Section 1.A that, with the notation of Theorem 1.1, $M_1 \times \cdots \times M_l$ is the discrete residual of G_0/R.

1.5.2 Corollaries

Proof of Corollary 1.2. We apply Theorem 1.47, which yields subgroups M_i and G_0 of G. Let A be the elementary radical of G_0, see Theorem 1.30.

Claim 1. *We have $R \leq A$. Moreover, A/R coincides with the elementary radical of $H_0 = G_0/R$.*

By Theorem 1.31, we have $R \leq A$ and R is elementary. Therefore, A/R contains the elementary radical of H_0. The claim follows, since the quotient group A/R is elementary by [Wes14, Theorem 4.6(c)].

Claim 2. *Set $W = \bigcap_{i=1}^{l} C_{H_0}(M_i) \leq H_0$, where l is as in Theorem 1.47. Then W is compact-by-{locally abelian of finite exponent}. In particular, it is elementary.*

We have $M_j \leq W$ for all $j > l$ by Theorem 1.47(i). Thus, $\tilde{W} = M_{l+1} \ldots M_m$ is a compact normal subgroup of W. Moreover, $W \cap (M_1 \ldots M_l) = 1$, since the latter product has trivial center in view of Theorem 1.47(ii) and (iv). It follows that $W/\tilde{W}$ embeds into H_0/H_1, which is locally abelian of finite exponent. This implies that W is elementary by Proposition 1.29(ii).

Claim 3. *Every nontrivial closed normal subgroup N of H_0 which is not contained in W contains some M_i with $i \in \{1, \ldots, l\}$.*

Assume that N does not contain any noncompact M_i. Then $1 = N \cap M_i \geq [N, M_i]$ since M_i is topologically simple. Thus, $N \leq W$ as desired.

Claim 4. *W coincides with the elementary radical of H_0.*

That W is contained in the elementary radical follows from Claim 2. If that inclusion were proper, then the elementary radical of H_0 would contain some noncompact M_i by Claim 3. This is impossible because every closed subgroup of an elementary group is elementary (see [Wes14, Theorem 4.6(b)]), while nondiscrete compactly generated topologically simple groups are not elementary (see [Wes15a, Proposition 6.2]).

To conclude the proof, we remark that $H' = H_0/W$ is isomorphic to G_0/A in view of Claims 1 and 4. Thus, it suffices to show all the desired assertions for the quotient H_0/W. For each $i \in \{1, \ldots, l\}$, we define a group M_i' as the image of M_i in the quotient $H' = H_0/W$. That image is injective because M_i is simple, and closed by Proposition 1.7. Thus, each M_i' is a topologically simple algebraic group over k_i'. The assertions that the M_i' are precisely the minimal normal subgroups of H_0/W, and that every nontrivial closed normal subgroup contains one of them, follow from Claim 3. That $H_1' = M_1' \dots M_l'$ is closed follows from the same argument as in the proof of Claim 7 in Theorem 1.47. Finally, that $(H_0/W)/H_1'$ is locally abelian of finite exponent follows from the last assertion of Theorem 1.47. Therefore, that quotient is elementary by Proposition 1.29(ii). $\square$

Proof of Corollary 1.3. Assume that G is nondiscrete. Since it is topologically simple, its $[A]$-regular radical R is either trivial or the whole of G.

Assume that $R = G$. Then G is locally solvable by Theorem 1.47. Moreover, [Wil07, Theorem 2.2] implies that G is not compactly generated.

Assume now that $R = 1$. By the definition of the $[A]$-regular radical, G is not locally abelian. Hence, the product $M_1 \times \cdots \times M_m$ from Theorem 1.47 is nontrivial. Since G is topologically simple, we have $m = 1$ and $G = M_1$. Since G is not compact (because a topologically simple profinite group is finite, hence discrete), we obtain the desired conclusion. $\square$

Proof of Corollary 1.4. Immediate from Corollary 1.3. $\square$

Proof of Corollary 1.6. Each class of groups listed in the statement consists of linear groups. Assume conversely that G is a compactly generated, topologically simple, locally compact group that is linear. If G is connected, then it is a simple Lie group, as a consequence of the solution to Hilbert's fifth problem. Otherwise G is totally disconnected. If it is nondiscrete, then it is algebraic by Corollary 1.4. If it is discrete, then it is residually finite by a theorem of Mal'cev (see, e.g., [LS03, Window 7, Proposition 8]), hence a finite simple group. $\square$

1.5.3 Some examples

In this section, we describe a family of examples of t.d.l.c. groups satisfying the hypotheses of Theorem 1.1, and illustrating some peculiar properties that the quotient $H_0 = G_0/R$ can have in general. For the construction, we use the Nottingham group.

Definition 1.51. The **Nottingham group**, denoted by $J(p^i)$, is the group of normalized continuous automorphisms of the ring $\mathbf{F}_{p^i}[\![T]\!]$. Otherwise stated, an element $g \in J(p^i)$ is defined by its action on T and is of the form $g(T) = T + \sum_{i=2}^{\infty} a_i T^i, \quad a_i \in \mathbf{F}_{p^i}$.

We will use the universality of the Nottingham group, asserted in the following.

Theorem 1.52 (Main result in [Cam97])**.** *Every countably based pro-p group can be embedded, as a closed subgroup, in the Nottingham group.*

The following construction shows that the group H_0 from Theorem 1.1 need not be second countable, and that it need not have any maximal compact normal subgroup.

Example 1.53. Consider the algebraic group SL_n over the local field $\mathbf{F}_p(\!(T)\!)$. Then, the quotient $U = SL_n(\mathbf{F}_p[\![T]\!])/Z(SL_n(\mathbf{F}_p[\![T]\!]))$ is a compact linear group which is h.j.i. by Theorem 1.12. Let L be a t.d.l.c. group admitting a continuous embedding into the Nottingham group $J(p)$. Then the semidirect product $G = U \rtimes L$ is a t.d.l.c. group.

We claim that G is locally C-stable. Let us first check that $QZ(G)$ is trivial. First observe that $QZ(U)$ is trivial by [BEW11, Proposition 5.1]. Hence, if $ul \in QZ(U \rtimes L)$, then $l \in L$ must be nontrivial. Since a nontrivial element of $J(p)$ acts by an outer automorphism on any open subgroup V of U, we deduce that $QZ(G)$ is trivial.

We now show that G has no nontrivial locally normal abelian subgroup. Arguing by contradiction, let K be such a subgroup. Since U is h.j.i. but not virtually abelian, the intersection $N_U(K) \cap K$ must be trivial. Thus, K commutes with $N_U(K)$. This is impossible, because L acts on $N_U(K)$ by outer automorphisms. This confirms the claim.

The claim implies that $R = 1$, and that $G = G_0 = H_0$ in the notation of Theorem 1.1. We now specialize this family of examples in two ways.

Taking $L = J(p)$, endowed with the discrete topology, we see that G is a metrizable, locally linear, t.d.l.c. group which is not second countable.

Now consider $L = \bigoplus_{n \in \mathbf{N}} \mathbf{Z}/p^n\mathbf{Z}$, with the discrete topology. It embeds in the pro-p group $\prod_{n\in\mathbf{N}} \mathbf{Z}/p^n\mathbf{Z}$, which itself embeds in $J(p)$ by Theorem 1.52. In this situation, we see that G is a locally linear, t.d.l.c.s.c. group, but has no maximal compact normal subgroup.

1.A Structure of the group of abstract automorphisms of an algebraic group over a local field

Defining the **discrete residual** of a topological group to be the intersection of all its open normal subgroups, we aim to prove that, with the notation of Theorem 1.1, $M_1 \times \cdots \times M_l$ is the discrete residual of G_0/R. We take this opportunity to elucidate related fundamental questions on the structure of $\mathrm{Aut}(G(k))$.

We begin by recalling the structure of $\mathrm{Aut}(k)$ for k a local field.

Lemma 1.54. *Let k be a local field. Then* $\mathrm{Aut}(k)$ *is either finite or a (topologically) finitely generated, locally pro-p group.*

Proof. In [Sch33], the author proves that if a field is complete with respect to two inequivalent norms, then it is algebraically closed. Since local fields are not algebraically closed, this implies that every automorphism of a local field is continuous.

Now, recalling that $\mathbf{Q}$ is dense in $\mathbf{Q}_p$, we see that every automorphism of $\mathbf{Q}_p$ is trivial. But a local field k of characteristic 0 is a finite extension of $\mathbf{Q}_p$ for some prime number p, and hence $\mathrm{Aut}(k) = \mathrm{Aut}(k/\mathbf{Q}_p)$ is finite.

On the other hand, local fields of positive characteristic are isomorphic to $\mathbf{F}_{p^i}(\!(T)\!)$. But $\mathrm{Aut}(\mathbf{F}_{p^i}(\!(T)\!))$ contains the Nottingham group $J(p^i)$ as an open finite index subgroup (see Definition 1.51 for the definition of the Nottingham group). As explained in [Cam97, Preliminaries], $J(p^i)$ is a pro-p group. It i also known that $J(p^i)$ is just-infinite: see [Cam00, Proposition 2] when $p \neq 2$ and [Heg01, Theorem 7] for $p = 2$. The (topological) finite generation follows from the general fact that a just-infinite pro-p group is (topologically) finitely generated. Indeed, it suffices to note that the Frattini subgroup of such a group is closed and normal, hence of finite index, so that this general fact follows from [DdSMS91, Proposition 1.9]. □

We go on in our investigation of the automorphism group of algebraic groups over local fields.

Definition 1.55. Let G be an absolutely simple, simply connected algebraic group over a local field k. By Pink's result (see Theorem 1.33), the group $\mathrm{Aut}(G(k))$ of abstract automorphisms of $G(k)$ fits inside the exact sequence $1 \to \mathrm{Aut}\, G(k) \to \mathrm{Aut}(G(k)) \to \mathrm{Aut}(k)$. Let $\mathrm{Aut}_G(k)$ be the image of $\mathrm{Aut}(G(k)) \to \mathrm{Aut}(k)$ (see Section 3.1 for more discussion on this).

Proposition 1.56. *Let G be an absolutely simple, simply connected algebraic group over a local field k. If G is isotropic, the homomorphism* $\mathrm{Aut}(G(k)) \to \mathrm{Aut}_G(k)$ *is continuous.*

Proof. Given an (abstract) automorphism $\alpha \in \mathrm{Aut}(G(k))$, we denote the underlying field automorphism by $\varphi_\alpha \in \mathrm{Aut}(k)$. Let S be a maximal k-split torus in G. We claim that $\alpha(S(k))$ is (the group of rational points of) a k-split torus of G of the same rank than S. Indeed, by Pink's result (see [Pin98, Corollary 0.5]), there exists a unique automorphism f of G over a unique automorphism φ_α of k such that the induced automorphism of $G(k)$ is α (in Chapter 3, we call f a semilinear automorphism).

For φ an automorphism of k, let φ_*^{-1} denotes the base change of a scheme along φ^{-1} (following the notation we introduce in Chapter 3). We can reformulate Pink's result as saying that there exists a k-algebraic isomorphism of algebraic groups $\tilde{f}\colon G \to \varphi_* G$ such that the induced automorphism of $G(k)$ is α (our discussion before Lemma 3.8 explain this in more details). But $\tilde{f}(S)$ is a split torus of the same rank than S because $\tilde{f}$ is injective and a closed immersion (see [DG70, Chap 2, §5, n°5 Proposition 5.1]). Hence, $\alpha(S(k)) \simeq (\varphi_*^{-1}\tilde{f}(S))(k)$, as wanted.

Let $\{\alpha_n\}_{n\in\mathbf{N}}$ be a sequence in $\mathrm{Aut}(G(k))$ converging to the identity. Let A_S be the apartment of the Bruhat–Tits building of G corresponding to S, let C be a chamber of A and let U be the pointwise fixator of C. We claim that there exists $N \in \mathbf{N}$ such that for $n \geq N$, there exists $g_n \in U$ such that $g_n S(k) g_n^{-1} = \alpha_n(S(k))$. Indeed, since U is a compact open subgroup of $G(k)$ and $\{\alpha_n\}_{n\in\mathbf{N}}$ converges to the identity, $\alpha_n(U) = U$ for n big enough. Hence, the apartment $A_{\alpha_n(S)}$ corresponding to $\alpha_n(S)$ contains C for n big enough. Now, since $G(k)$ acts strongly transitively on its building, there exists $g_n \in G(k)$ fixing pointwise C such that $g_n.A_S = A_{\alpha_n(S)}$, or equivalently $g_nS(k)g_n^{-1} = \alpha_n(S(k))$, as wanted.

Hence, up to passing to a subsequence, we can and do assume that $\alpha_n(S(k)) = g_nS(k)g_n^{-1}$, where g_n belongs to the compact (open) subgroup U. Furthermore, passing again to a subsequence, we can assume that $\{g_n\}_{n\in\mathbf{N}}$ converges to g. Let $\mathrm{Int}(g)$ be the interior automorphism of $G(k)$ induced by g. Defining $\tilde{\beta}_n = \mathrm{Int}(g_n)\alpha_n$, we have that $\tilde{\beta}_n(S(k)) = S(k)$ and $\{\tilde{\beta}_n\}_{n\in\mathbf{N}}$ converges to $\mathrm{Int}(g)$. Now the sequence $\beta_n = \mathrm{Int}\, g \circ \tilde{\beta}_n$ is such that $\beta_n(S(k)) = S(k)$, $\varphi_{\beta_n} = \varphi_{\alpha_n}$ and $\{\beta_n\}_{n\in\mathbf{N}}$ converges to the identity. Finally, since $\mathrm{Aut}(k)$ is compact, up to passing to a subsequence, we can and do assume that φ_{β_n} converges to φ.

Note that β_n restricts to a semilinear automorphism of S. Since the k-split torus S is defined over $\mathbf{Z}$, the group of semilinear automorphims $\mathrm{Aut}(S \to \mathrm{Spec}\, k)$ of S decomposes as the semidirect product $\mathrm{Aut}(S_{\mathbf{Z}}) \rtimes \mathrm{Aut}(k)$ (see Lemma 3.6). Note that by [DG70, II, §1, n°2.11]), automorphisms of S as a $\mathbf{Z}$-group can naturally be seen as elements of $\mathrm{Aut}(\mathbf{Z}^r, \mathbf{Z}^r)$ (where r is the rank of S). Write the decomposition of β_n in $\mathrm{Aut}(S_{\mathbf{Z}}) \rtimes \mathrm{Aut}(k)$ as (f_n, φ_{β_n}). Since the section $\mathrm{Aut}(k) \to \mathrm{Aut}(S \to \mathrm{Spec}\, k) \leq \mathrm{Aut}(S(k))$ is continuous, $\{f_n\}_{n\in\mathbf{N}}$ converges to φ^{-1}. This readily implies that $\{f_n\}_{n\in\mathbf{N}}$ eventually preserves a chosen decomposition of S as $\mathbb{G}_m \times \cdots \times \mathbb{G}_m$. But this forces f_n to eventually be the identity, so that φ^{-1} is trivial, as wanted. $\square$

We do not know how to prove Proposition 1.56 in the anisotropic case, i.e. for groups of the form $\mathrm{SL}_1(D)$. But on the other hand, when G is anisotropic, we have the following result on $\mathrm{Aut}(G(k))$.

Proposition 1.57. *For D a finite dimensional central division algebra over a local field k, the group of abstract automorphism $\mathrm{Aut}(\mathrm{SL}_1(D))$ is compact.*

Proof. By Lemma 1.12, $\mathrm{SL}_1(D)$ is a non-abelian hereditarily just-infinite virtually pro-p group.

We claim that such a group G is (topologically) finitely generated. Indeed, up to passing to an open finite index subgroup, we can assume that G is pro-p. Then, its Frattini subgroup $\Phi(G)$ is a closed normal subgroup which is non-trivial (because the Frattini subgroup contains the commutator subgroup and G is not virtually abelian). Hence, because G is just infinite, $G/\Phi(G)$ is finite, and any set of representatives topologically generates G by [DdSMS91, Proposition 1.9], as was to be shown.

Hence, according to [BEW11, Section 7.1], $\mathrm{Aut}(\mathrm{SL}_1(D))$ is a profinite group, as wanted. □

We continue by proving that the group of inner automorphisms is cocompact in the group of abstract automorphisms.

Proposition 1.58. *Let G be an absolutely simple, simply connected algebraic group over a local field k.* $\mathrm{Aut}(G(k))/\,\mathrm{Inn}(G(k))$ *is compact.*

Proof. If G is anisotropic, $\mathrm{Aut}(G(k))$ is compact by Proposition 1.57 and $\mathrm{Inn}(G(k))$ is a closed subgroup of it so that the result holds. We now assume that G is isotropic and prove this cocompactness phenomenon by looking at the action of $G(k)$ on its Bruhat–Tits building X.

Let $G(k) \to \mathrm{Aut}(X)$ be the map induced by the action of $G(k)$ on its building. We let $\mathrm{Aut}(X)$ be a topological group for the pointwise convergence. Note that $G(k) \to \mathrm{Aut}(X)$ is proper, so that $G(k)/Z$ is identified with a closed subgroup of $\mathrm{Aut}(X)$. We claim that $\mathrm{Aut}(G(k)) = N_{\mathrm{Aut}(X)}(G(k)/Z)$. Indeed, let α be an abstract automorphism of G. By Pink's result (see [Pin98, Corollary 0.5]), there exists $\varphi_\alpha \in \mathrm{Aut}(k)$ and an algebraic isomorphism $\tilde{f}_\alpha \colon G \to (\varphi_\alpha)_*(G)$ inducing the automorphism α. But $\tilde{f}$ preserves a maximal k-split torus and its associated root groups. Furthermore, φ_α preserves the valuation on root groups, since any automorphism of k is continuous. Hence, α preserves any valued root group datum, so that it induces an automorphism a_α of the building. Furthermore, α normalizes $G(k)/Z$ in $\mathrm{Aut}(X)$. Conversely, consider the homomorphism $N_{\mathrm{Aut}(X)}(G(k)/Z) \to \mathrm{Aut}(G(k))$ given by conjugation. The kernel is in the centraliser of $G(k)/Z$, and hence is trivial, because $G(k)$ acts strongly transitively on X.

Let $A = N_{\mathrm{Aut}(X)}(G(k)/Z)$ and let C be a chamber of X. Since $G(k)$ acts strongly transitively on X, $N_{\mathrm{Aut}(X)}(G(k)/Z) = G(k)/Z.\,\mathrm{Stab}_A(C)$. But $\mathrm{Stab}_A(C)$ is compact, because X is locally finite. Hence, the quotient $\mathrm{Aut}(G(k))/\,\mathrm{Inn}(G(k))$ is compact, which concludes the proof. □

Corollary 1.59. *Let G be an absolutely simple, simply connected algebraic group over a local field k. Then $\mathrm{Aut}_G(k)$ is compact.*

Proof. If G is anisotropic, this follows from the fact that $\mathrm{Aut}_G(k) = \mathrm{Aut}(k)$ (see Chapter 3, Theorem 3.3) and Lemma 1.54. In the isotropic case, since $\mathrm{Aut}(G(k))/\,\mathrm{Inn}(G(k))$ is compact by Proposition 1.58, the result follows directly from Proposition 1.56. □

Remark 1.60. For k a local field of characteristic $p \geq 5$ and for G an isotropic, simply connected, absolutely simple algebraic group over k, the index of $\mathrm{Aut}_G(k)$ in $\mathrm{Aut}(k)$ is finite. Indeed, $\varphi \in \mathrm{Aut}(k)$ belongs to $\mathrm{Aut}_G(k)$ if and only if φ_*G is k-isomorphic to G. But φ_*G and G have the same Tits index. Hence, according to the classification of algebraic groups over local fields (see especially [Tit79, 4.5]), there only are a finite number of isotropic, simply connected, absolutely simple algebraic group over k (up to k-isomorphism) having the same Tits index than G.

When the characteristic is 2 or 3, we do not know if $\mathrm{Aut}_G(k)$ is always of finite index. A computation seems to indicate that for an absolutely simple, simply connected, quasi-split group G of type ${}^2A_n$ over $k = \mathbf{F}_2(\!(T)\!)$, this is the case.

We end this section with a slight improvement on Theorem 1.47 G_0/R, which describes the discrete residual of G_0/R.

Proposition 1.61. *With the notations of Theorem 1.47, the product $M_1.\dots.M_l$ is the discrete residual of G_0/R.*

Proof. Recall that we have an injective map $\varphi_1 \times \cdots \times \varphi_m \colon G_0/R \to \mathrm{Aut}(\tilde{H}_1(k_1')/Z_1) \times \cdots \times \mathrm{Aut}(\tilde{H}_m(k_m')/Z_m)$ (see Claim 4). For $i \leq l$ (respectively $i > l$), $M_i \cong \tilde{H}_i(k_i')/Z_i$ (respectively M_i is an open normal subgroup of $\tilde{H}_i(k_i')/Z_i$) and φ_i restricted to M_i maps an element to the inner automorphism corresponding to this element.

We claim that $\operatorname{Aut}(\tilde{H}_i(k_i')/Z_i)/\operatorname{Inn}(\tilde{H}_i(k_i')/Z_i)$ is compact for all $i \in \{1, \dots m\}$. Indeed, as we observed in Remark 1.35, $\operatorname{Aut}(\tilde{H}_i(k_i'))$ surjects onto $\operatorname{Aut}(\tilde{H}_i(k_i')/Z_i)$. But by Proposition 1.58, $\operatorname{Aut}(\tilde{H}_i(k_i'))/\operatorname{Inn}(\tilde{H}_i(k_i'))$ is compact for all $i \in \{1, \dots m\}$.

This already proves that $M_1. \dots .M_l$ is cocompact in G_0/R, so that it is indeed in the discrete residual of G_0/R. On the other hand, $M_1. \dots .M_l$ is a direct product of topologically simple groups by Theorem 1.47, and hence $M_1. \dots .M_l$ is the discrete residual of G_0/R. □

1.B Analytificaton of a finite type k-scheme

We aim to prove that the Lie algebra of an algebraic group G over a local field k is isomorphic to the Lie algebra of $G(k)$ considered as an analytic manifold. This fact is at the heart of the proof of Theorem 1.12. Indeed, it is the crucial ingredient in the proof of the main theorem in [Rie70a]. However, when using this fact, C. Riehm does not provide explanations about how to compare the analytic and the algebraic Lie algebra, and despite a literature search, we could not find a comprehensive treatment of the matter.

Let us first discuss what we found in the literature. In [PR94, Section 3.1], the authors explain in great details how to "analytify" a k-variety. But when it comes to the comparison of the analytic tangent space versus the algebraic tangent space, they only make the elliptic comment "the proof follows easily from a comparison of the appropriate definitions". However, in view of the definitions involved (we recall them in Definition 1.67), this fact does not seem to be a formal consequence of the definitions. On the contrary, our proof relies on explicit computations in a chosen chart. Another issue that makes their treatment of the subject incomplete for us is that they insist on the fact that they work only in characteristic 0.

Another reference that touches the matter is [Mar91, Proposition 2.6.11]. But again, when comparing analytic and algebraic tangent space, the author use a reference to ([Bou72, Chapter 3, §3, n°11]) where the result is not clearly stated. Hence, we decided to include a discussion here. However, nothing is new in our treatment of the subject, and we essentially follow the presentation given in [PR94].

Let us begin by discussing the "topologification" of the rational points of a finite type k-scheme, when k is a topological field. The following result is a mix of a question on the website MathOverflow (see [BCn]) and [PR94, Section 3.1].

Proposition 1.62. *Let k be a topological field. There exists a unique (up to natural isomorphism) functor* Top_k *from the category of finite type k-schemes to the category of topological spaces such that*

1. *Composing* Top_k *with the forgetful functor from the category of topological spaces to the category of sets yields a functor naturally isomorphic to the functor of taking k-points.*
2. *For $\mathbf{A}_1$ the affine line over k,* $\mathrm{Top}_k(\mathbf{A}_1)$ *is homeomorphic to k.*
3. *(Compatibility with fiber products). For $X \to S$ and $Y \to S$ two morphisms in the category of finite type k-schemes,* $\mathrm{Top}_k(X \times_S Y)$ *is (naturally in all the variables) homeomorphic to the fiber product* $\mathrm{Top}_k(X) \times_{\mathrm{Top}_k(S)} \mathrm{Top}_k(Y)$.
4. *(Compatibility with closed immersions). If $Z \to X$ is a closed immersion in the category of finite type k-scheme, then* $\mathrm{Top}_k(Z) \to \mathrm{Top}_k(X)$ *is a closed embedding in the category of topological space (i.e. an injective continuous closed map).*
5. *(Compatibility with open immersions). If $U \to X$ is an open immersion in the category of finite type k-scheme, then* $\mathrm{Top}_k(U) \to \mathrm{Top}_k(X)$ *is a closed embedding in the category of topological space (i.e. an injective continuous open map).*

Proof. For X a finite type k-scheme, we define a collection of subsets of $X(k)$: for U an open subscheme of X, for $f_1, \ldots, f_n$ regular functions on U and for V a neighbourhood of 0 in k, consider $X(U, f_1, \ldots, f_n, V) = \{x \in X(k) \mid x \in U(k) \text{ and } f_i(x) \in V \text{ for } i = 1, \ldots n\}$.

We define $\mathrm{Top}_k(X)$ to be the set $X(k)$ together with the topology having for base the sets $X(U, f_1, \ldots, f_n, V)$. Note that for $f\colon X \to Y$ a morphism in the category of finite type k-schemes, $\mathrm{Top}_k(f) =$

$f(k)\colon X(k) = \mathrm{Top}_k(X) \to \mathrm{Top}_k(Y) = Y(k)$ is continuous, because it is locally given by polynomial, so that it follows from the fact that the field is topological. Furthermore, Top_k satisfies all the conditions of the proposition.

Uniqueness is a direct consequence of the conditions we imposed on the functor. Indeed, at the level of set, the functor has to be naturally isomorphic to the functor of k-points. But a finite type k-scheme is covered by finitely many open affine subschemes of finite type. Finally, an affine scheme of finite type is a closed subscheme of a finite dimensional affine space, i.e. a finite direct product of affine lines. Hence, the topology is entirely determined by the remaining conditions. □

We can go further and also give the structure of an analytic manifold to $X(k)$ when X is smooth and k is a complete field. As pointed out in a the MathOverflow question [BCn], the natural tool to use is to cover X by etale maps over affine space, and then use the inverse function theorem for analytic manifolds. We first recall the implicit function theorem in the analytic case.

Definition 1.63. Let k be a local field and let $f\colon k^n \to k\colon x \to \sum a_\alpha x^\alpha$ be an analytic map around 0 (we use the multi-index notation as in [Ser92, Chapter II]). Let δ_i be the multi-index $(0,\ldots,0,1,0,\ldots,0)$, where the 1 appears in the i-th position. We define the partial derivative of f with respect to x_i to be $\frac{\partial f}{\partial x_i}\colon k^n \to k\colon x \to \sum_{\alpha\geq\delta_i} a_\alpha \binom{\alpha}{\delta_i} x^{\alpha-\delta_i}$

Remark 1.64. In [Ser92, Chapter II], $\frac{\partial f}{\partial x_i}$ is denoted Δ^{δ_i}.

Theorem 1.65 (Analytic Implicit Function Theorem)**.** *Let k be a local field and let $g\colon k^c \times k^n \to k^c$ be an analytic map around 0 such that $g(0) = 0$. Assume that $\det(\frac{\partial g_i}{\partial x_j}(0))_{i,j\in\{1,\ldots,c\}}$. Then there exist $V \subset k^n$, $W \subset k^c$, both neighbourhoods of 0, and an analytic map $\Phi\colon k^n \to k^c$ such that for all $v \in V$ and $w \in W$, $g(v,w) = 0$ if and only if $\Phi(v) = w$.*

Proof. The argument deducing the implicit function theorem from the inverse function theorem is classical. See for example the proof of Theorem C.40 in [Lee13] for how this argument works in the smooth case. Note that this argument carries over verbatim in the analytic case, up to assuming the analytic inverse function theorem. For the proof of the analytic inverse function theorem, we refer the reader to [Ser92, Chapter II]. $\square$

Proposition 1.66. *Let k be a complete field with respect to a non-trivial absolute value. We construct an "analytification functor" from the category of smooth, finite type k-schemes to the category of analytic varieties whose composition with the forgetful functor from analytic varieties to topological space is the functor* Top_k *from Proposition 1.62. For $f\colon X \to Y$ in the category of smooth, finite type k-schemes, we denote by $f_{\mathrm{an}}\colon X_{\mathrm{an}} \to Y_{\mathrm{an}}$ its image under this analytification functor.*

Proof. Let X be a smooth finite type k-scheme. Let $\{U_i\}_{1,\dots,l}$ be a cover of X by affine open subschemes such that each U_i is standard etale over $\mathbf{A}_k^{n_i}$, the affine space over k of dimension n_i. Such a cover exists because X is smooth (see [Sta17, Tag 054L]). This implies that $U_i(k)$ is the set of points in $\mathbf{A}_k^{c_i+n_i}(k)$ where $f_{i,1}, \dots, f_{i,c_i}$ vanish, for some polynomials $f_{i,1}, \dots, f_{i,c_i}$ such that $\det(\frac{\partial f_{i,j}}{\partial x_m})_{j,m\in\{1,\dots,c_i\}}$ is a polynomial that does not vanish on $U_i(k)$ (see [Sta17, Tag 00U9]). By the Analytic Implicit Function Theorem (Theorem 1.65), $U_i(k)$ is locally the graph of an analytic map. By gluing all those maps, we deduce that there exists an open $V_i \subset k^{n_i}$ and an analytic map $\Phi_i\colon V_i \to k^{c_i}$ such that $U_i(k) = \{(v, \Phi_i(v)) \mid v \in V_i\}$.

For each $i = 1, \dots, l$, let π_i be the restriction to $U_i(k)$ of the projection $k^{c_i+n_i} \to k^{n_i}$. It follows from its definition that Φ_i is a continuous inverse to π_i, so that $(U_i(k), \pi_i, n_i)$ is a chart on $X(k)$. We check the compatibility of those charts. For $i \neq j \in \{1, \dots, l\}$, let $\tilde{U}_i = U_i \cap U_j \subset U_i$, so that $\tilde{U}_i(k)$ is seen as a subset of $U_i(k) \subset k^{c_i+n_i}$, and let $\tilde{U}_j = U_i \cap U_j \subset U_j$, so that $\tilde{U}_j(k)$ is seen as a subset of $U_j(k) \subset k^{c_j+n_j}$. It is now readily seen that $(U_i(k), \pi_i, n_i)$ and $(U_j(k), \pi_j, n_j)$ are compatible, since the isomorphism $\tilde{U}_i(k) \cong \tilde{U}_j(k)$ is algebraic, and hence given by a rational map. Note that rational maps are analytic because inverse of non-vanishing polynomials are analytic (see [Ser92, Chapter III, §7, Lemma]). Furthermore, this argument shows that the analytic structure we defined on X does not depend on the choice of the cover by affine open subschemes that are etale over $\mathbf{A}_k^n$. We denote the corresponding analytic variety by X_{an}.

Finally, an algebraic map $f\colon X \to Y$ is locally given by polynomials, so that $f_{\mathrm{an}} = f(k)\colon X_{\mathrm{an}} \to Y_{\mathrm{an}}$ is analytic. This concludes the construction of the analytic functor. $\square$

We now compare the analytic and the algebraic tangent space. We begin by recalling the corresponding definitions.

Definition 1.67.

1. Let X be a k-scheme of finite type and let $x \in X(k)$. Let $\mathcal{O}_{X,x}$ be the stalk at x of the structure sheaf, and let $\mathfrak{m}_x$ be its maximal ideal (we also denote it $\mathfrak{m}_x^{\mathrm{alg}}$ when we need to distinguish if from its analytic counterpart). The algebraic tangent space at x is defined to be the dual of the k-vector space $\mathfrak{m}_x/\mathfrak{m}_x^2$ (see [Vak17, Definition 12.1.1])

2. Let X be an analytic variety over a local field k and let $x \in X$. Let $\underline{H}_{X,x}$ be the k-algebra of germs of analytic function at x, and let $\mathfrak{m}_x$ be its maximal ideal (we also denote it $\mathfrak{m}_x^{\mathrm{an}}$ when we need to distinguish if from its algebraic counterpart). The analytic tangent space at x is defined to be the dual of the k-vector space $\mathfrak{m}_x/\mathfrak{m}_x^2$ (see [Ser92, Chapter III, §8]).

Theorem 1.68. *Let X be a smooth k-scheme of finite type and let $x \in X(k)$. Since every regular function is analytic, we have a homomorphism of k-algebra $\theta_x\colon \mathcal{O}_{X,x} \to \underline{H}_{X_{\mathrm{an}},x}$, such that $\theta_x(\mathfrak{m}_x^{\mathrm{alg}}) \subset \mathfrak{m}_x^{\mathrm{an}}$. The induced map $\overline{\theta}_x\colon \mathfrak{m}_x^{\mathrm{alg}}/(\mathfrak{m}_x^{\mathrm{alg}})^2 \to \mathfrak{m}_x^{\mathrm{an}}/(\mathfrak{m}_x^{\mathrm{an}})^2$ is an isomorphism of k-vector space.*

Proof. Let U be an affine (Zariski)-open neighbourhood of x which is etale over $\mathbf{A}_k^n$. In particular, as we saw in Proposition 1.66, U is isomorphic (as a k-scheme) to a closed subscheme of $\mathbf{A}_k^{n+r}$, say $U \cong \operatorname{Spec} k[X_1, \dots, X_n, X_{n+1}, \dots, X_{n+r}]/(f_1, \dots, f_r)$ and $\det(\frac{\partial f_i}{\partial X_{n+j}})_{i,j=1,\dots,r}$ does not vanish on U. To simplify notations, we can and do assume that $x = 0 \in \mathbf{A}_k^{n+r}$. We then readily check that $\{X_1, \dots X_n\}$ is a basis of the k-vector space $\mathfrak{m}_x^{\mathrm{alg}}/(\mathfrak{m}_x^{\mathrm{alg}})^2$.

Now, recall that the projection on the first n components $\mathbf{A}_k^{n+r}(k) \to \mathbf{A}_k^n(k)$ restricted to $U(k)$ is a chart for X_{an}, by definition of the analytification functor. Hence, using this chart, we get an isomorphism of $\mathfrak{m}_x^{\mathrm{an}}/(\mathfrak{m}_x^{\mathrm{an}})^2$ with the cotangent space at 0 of $\mathbf{A}_k^n(k)$, that we denote $\mathfrak{m}_0^{\mathrm{an}}/(\mathfrak{m}_0^{\mathrm{an}})^2$. By [Ser92, Chapter III, §7, Lemma], we deduce that $\mathfrak{m}_x^{\mathrm{an}}/(\mathfrak{m}_x^{\mathrm{an}})^2$ also has a basis given by $\{X_1, \dots X_n\}$. Furthermore, $\theta(X_i) = X_i$ for all $i \in \{1, \dots, n+r\}$, which concludes the proof. □

Recall that the Lie algebra of a Lie group is just the tangent space at the identity, endowed with a bracket operation coming from the differential of the commutator map.

Corollary 1.69. *Let G be an algebraic group over a local field k, and let $\mathfrak{g}$ be its Lie algebra. Let G_{an} be the corresponding analytic group as defined in Proposition 1.66, and let $\mathfrak{g}_{\mathrm{an}}$ be the Lie algebra of G_{an}. The dual $\overline{\theta}_e^*$ of the map $\overline{\theta}_e$ defined in Theorem 1.68 gives an isomorphism of Lie algebras $\mathfrak{g}_{\mathrm{an}} \cong \mathfrak{g}$.*

Proof. Let $c\colon G \times G \to G\colon (x, y) \mapsto xyx^{-1}y^{-1}$ be the commutator map. By definition, the bracket operation $[\ ,\ ]$ on $\mathfrak{g}$ or $\mathfrak{g}_{\mathrm{an}}$ is the total derivative of c at the identity. So by definition, for X, Y in $\mathfrak{g} = (\mathfrak{m}_e^{\mathrm{alg}}/(\mathfrak{m}_e^{\mathrm{alg}})^2)^*$ (respectively $\mathfrak{g}_{\mathrm{an}} = (\mathfrak{m}_e^{\mathrm{an}}/(\mathfrak{m}_e^{\mathrm{an}})^2)^*$) and $v \in \mathfrak{m}_e^{\mathrm{alg}}/(\mathfrak{m}_e^{\mathrm{alg}})^2$ (respectively $v \in \mathfrak{m}_e^{\mathrm{an}}/(\mathfrak{m}_e^{\mathrm{an}})^2$), we have $[X, Y](v) = (X \oplus Y)(v \circ c)$ (there are a few abuse of notations in this formula, but the intended meaning should be clear). We can now readily check that $\overline{\theta}_e^*$ is an isomorphism of Lie algebras:

$$\begin{aligned}
&\text{For } X, Y \in \mathfrak{g}_{\mathrm{an}},\ v \in \mathfrak{m}_e^{\mathrm{alg}}/(\mathfrak{m}_e^{\mathrm{alg}})^2, \\
&(\overline{\theta}_e^*[X, Y])(v) &&= [X, Y](\overline{\theta}_e(v)) \\
& &&= (X \oplus Y)(\overline{\theta}_e(v) \circ c) \\
& &&= (X \oplus Y)(\overline{\theta}_e(v \circ c)) \\
& &&= (\overline{\theta}_e^*(X) \oplus \overline{\theta}_e^*(Y))(v \circ c) \\
& &&= [\overline{\theta}_e^*(X), \overline{\theta}_e^*(Y)](v)
\end{aligned}$$

□

Chapter 2

Chabauty limits of algebraic groups acting on trees

Given a locally finite leafless tree T, various algebraic groups over local fields might appear as closed subgroups of $\mathrm{Aut}(T)$. We show that the set of closed cocompact subgroups of $\mathrm{Aut}(T)$ that are isomorphic to a quasi-split simple algebraic group is a closed subset of the Chabauty space of $\mathrm{Aut}(T)$. This is done via a study of the integral Bruhat–Tits model of SL_2 and $\mathrm{SU}_3^{L/K}$, that we carry on over arbitrary local fields, without any restriction on the (residue) characteristic. In particular, we show that in residue characteristic 2, the Tits index of simple algebraic subgroups of $\mathrm{Aut}(T)$ is not always preserved under Chabauty limits.

2.1 Introduction

Ta vague monte avec la rumeur d'un prodige
C'est ici ta limite. Arrête-toi, te dis-je.
(Victor Hugo, *L'année terrible*, 1872)

According to well-known rigidity results of J. Tits (see [Tit74, Theorem 5.8], together with [Tit86, Théorème 2] or [Wei09, Theorem 27.6]), a Bruhat–Tits building of rank ≥ 2 determines uniquely the simple algebraic group and the underlying ground field to which it is associated. In particular, two simply connected absolutely simple algebraic groups over local fields of relative rank ≥ 2 have isomorphic Bruhat–Tits buildings if and only if they are isomorphic as locally compact groups. This contrasts drastically with the rank 1 case, where infinitely many pairwise non-isomorphic simple algebraic groups of relative rank 1 can have the same Bruhat–Tits tree. Therefore, given a locally finite leafless tree T, the set $\mathbf{Sub}(\mathrm{Aut}(T))$ of closed subgroups of the locally compact group $\mathrm{Aut}(T)$ may contain infinitely many pairwise non-isomorphic algebraic groups. For example, the Bruhat–Tits tree of the split group $\mathrm{SL}_2(K)$ is completely determined by the order of the residue field of K, while the isomorphism type of $\mathrm{SL}_2(K)$ depends on the isomorphism type of the local field K. Since $\mathbf{Sub}(\mathrm{Aut}(T))$ carries a natural compact Hausdorff topology, namely the Chabauty topology, we are naturally led to the following question: *what are the Chabauty limits of algebraic groups in* $\mathbf{Sub}(\mathrm{Aut}(T))$*?* The goal of this chapter is to initiate the study of that problem. In particular, we provide a complete solution in the case of quasi-split groups.

In order to be more precise, for T a tree, let us define a *topologically simple algebraic group acting on* T to be a locally compact group isomorphic to $H(K)/Z$, where K is a local field, H is an absolutely simple, simply connected, algebraic group over K of relative rank 1 whose Bruhat–Tits tree is isomorphic to T, and Z is the center of $H(K)$.

The first thing to observe is that, given a topologically simple algebraic group G acting on T, the action homomorphism $G \to \mathrm{Aut}(T)$ is not canonical, but depends on some choices. There is however a natural way to resolve this issue of canonicity, explained in [CR16]. Following that paper, we shall denote by $\mathcal{S}_T$ the space of (topological) isomorphism classes of topologically simple closed subgroups of $\mathrm{Aut}(T)$ acting 2-transitively on the set of ends. According to [CR16, Theorem 1.2], the space $\mathcal{S}_T$ endowed with the quotient topology induced from the Chabauty space $\mathbf{Sub}(\mathrm{Aut}(T))$ is compact Hausdorff.

We can therefore reformulate the question mentioned above as follows. Let $\mathcal{S}_T^{\mathrm{alg}}$ be the set of isomorphism classes of topologically simple algebraic groups acting on T. *What are the accumulation points in* $\mathcal{S}_T$ *of the elements of* $\mathcal{S}_T^{\mathrm{alg}}$ *?* It seems reasonable to conjecture that $\mathcal{S}_T^{\mathrm{alg}}$ is closed in $\mathcal{S}_T$. Our main theorem is a partial result in this direction.

Theorem 2.1. *Let T be a locally finite leafless tree, and let $\mathcal{S}_T^{\text{qs-alg}}$ be the set of isomorphism classes of topologically simple algebraic groups acting on T that are furthermore quasi-split. Then $\mathcal{S}_T^{\text{qs-alg}}$ is closed in $\mathcal{S}_T$.*

As recalled in Section 2.2.1, the classification of the simple algebraic groups over local fields implies that absolutely simple, simply connected, quasi-split algebraic groups over K of relative rank 1 are of the form $\mathrm{SL}_2(K)$ or $\mathrm{SU}_3^{L/K}(K)$ (see Lemma 2.9). So that in effect, the main goal of the chapter is only to dispose of those two "types" of groups.

Since the Bruhat–Tits tree of $\mathrm{SL}_2(K)$ or $\mathrm{SU}_3^{L/K}(K)$ for L a ramified extension of K (respectively $\mathrm{SU}_3^{L/K}(K)$ for L an unramified extension of K) is isomorphic to the (p^n+1)-regular tree (respectively the semiregular tree of bidegree $(p^{3n}+1; p^n+1)$), where p^n is the order of the residue field of K, the space $\mathcal{S}_T^{\text{qs-alg}}$ is empty unless T is one of those trees.

It should also be noted that for some trees T, every algebraic group having T as Bruhat–Tits tree is actually quasi-split. According to the classification tables in [Tit79, 4.2 and 4.3], this is the case if and only if T is the regular tree of degree $p+1$ or the semiregular tree of bidegree $(p^{3n}+1; p^n+1)$. Combining this observation with Theorem 2.1, we get the following corollary.

Corollary 2.2. *Let p be a prime number, and let T be the $(p+1)$-regular tree, or the $(p^{3n}+1; p^n+1)$-semiregular tree. Then the set $\mathcal{S}_T^{\text{alg}}$ coincides with $\mathcal{S}_T^{\text{qs-alg}}$, so that it is closed in $\mathcal{S}_T$.*

In fact, we give an explicit description of the topological space $\mathcal{S}_T^{\text{qs-alg}}$. To achieve it, we proceed in two steps. We first describe the space $\mathcal{L}$ of quadratic pairs of local fields (as defined in Definition 2.62), and the purpose of Section 2.5.1 is to give an explicit description of $\mathcal{L}$, which appears in Proposition 2.71. The process is a bit lengthy, but only uses elementary facts about local fields. In a second step, we show in the proof of Theorem 2.3 that the map

$$\mathcal{L} \to \mathcal{S}_T \colon (K, L) \mapsto \hat{G}_{(K,L)}$$

is a homeomorphism onto its image (see Definition 2.74 and Proposition 2.77 for the definition of this map). Note that we make an abuse of notation: we represent a point in $\mathcal{S}_T$, which is an isomorphism class, by a representative of that class. This abuse should not cause any confusion, and will simplify notations throughout the rest of the chapter.

To ease the statement of the explicit form of the main theorem, let us introduce some terminology. Recall that a countable totally disconnected topological space X is classified by two invariants (see [MS20, Théorème 1]). More precisely, let $\hat{\mathbf{N}}$ be the one point compactification of $\mathbf{N}$ (or in other words, a topological space homeomorphic to $\{1, \frac{1}{2}, \frac{1}{3}, \dots, 0\} \subset \mathbf{R}$). If $X^{(k)}$ is the last non-empty Cantor-Bendixson derivative of X, and if $X^{(k)}$ has n connected components, then X is homeomorphic to $\hat{\mathbf{N}}^k \times \{1, \dots, n\}$.

Theorem 2.3.

1. *Let p be an odd prime number, and let T be the (p^n+1)-regular tree. Then there exists a homeomorphism $f\colon \hat{\mathbf{N}} \times \{1,2\} \to \mathcal{S}_T^{\text{qs-alg}}$ such that*

$$\begin{aligned} f(\hat{\mathbf{N}} \times \{1\}) &= \{\mathrm{SL}_2(K)/Z \mid \overline{K} \cong \mathbf{F}_{p^n}\} \\ f((\infty,1)) &= \mathrm{SL}_2(\mathbf{F}_{p^n}(\!(X)\!))/Z \\ f(\hat{\mathbf{N}} \times \{2\}) &= \{\mathrm{SU}_3^{L/K}(K)/Z \mid \overline{K} \cong \mathbf{F}_{p^n} \\ &\qquad \textit{and } L \textit{ is (separable) quadratic ramified}\} \\ f((\infty,2)) &= \mathrm{SU}_3^{L_0/\mathbf{F}_{p^n}(\!(X)\!)}(\mathbf{F}_{p^n}(\!(X)\!))/Z \end{aligned}$$

   where ∞ denotes the accumulation point of $\hat{\mathbf{N}}$, $\overline{K}$ denotes the residue field of K, and L_0 is any (separable) quadratic ramified extension of $\mathbf{F}_{p^n}(\!(X)\!)$.

2. *Let T be the (2^n+1)-regular tree. Then $\mathcal{S}_T^{\text{qs-alg}}$ is homeomorphic to $\hat{\mathbf{N}}^2$. More precisely,*

$$\begin{aligned} \mathcal{S}_T^{\text{qs-alg}} = \{\mathrm{SL}_2(K)/Z \mid \overline{K} \cong \mathbf{F}_{2^n}\} \cup \{\mathrm{SU}_3^{L/K}(K)/Z \mid \overline{K} \cong \mathbf{F}_{2^n} \\ \textit{and } L \textit{ is separable quadratic ramified}\} \end{aligned}$$

   The first Cantor-Bendixson derivative of $\mathcal{S}_T^{\text{qs-alg}}$ is

$$\{\mathrm{SU}_3^{L/\mathbf{F}_{2^n}(\!(X)\!)}(\mathbf{F}_{2^n}(\!(X)\!))/Z \mid L \textit{ is separable quadratic ramified}\}$$

$$\cup \{\mathrm{SL}_2(\mathbf{F}_{2^n}(\!(X)\!))/Z\}$$

while its second Cantor-Bendixson derivative contains the single element $\mathrm{SL}_2(\mathbf{F}_{2^n}(\!(X)\!))/Z$*.*

3. *Let p be any prime number, and let T be the $(p^{3n}+1; p^n+1)$-semiregular tree. Then $\mathcal{S}_T^{\text{qs-alg}}$ is homeomorphic to $\hat{\mathbf{N}}$. More precisely,*

$$\mathcal{S}_T^{\text{qs-alg}} = \{\, \mathrm{SU}_3^{L/K}(K)/Z \mid \overline{K} \cong \mathbf{F}_{p^n} \\ \text{and } L \text{ is (separable) quadratic unramified}\}$$

Furthermore, the accumulation point of $\mathcal{S}_T^{\text{qs-alg}}$ is the quotient group $\mathrm{SU}_3^{L/\mathbf{F}_{p^n}(\!(X)\!)}(\mathbf{F}_{p^n}(\!(X)\!))/Z$, where L is the (separable) quadratic unramified extension of $\mathbf{F}_{p^n}(\!(X)\!)$.

As one can see from Theorem 2.3, we face a more complex situation in residue characteristic 2. Indeed, that statement implies that the split group $SL_2(\mathbf{F}_{2^n}(\!(X)\!))/Z$ is a limit of unitary groups, thereby illustrating the fact that the Tits index need not be preserved under Chabauty limits in residue characteristic 2. In other words, the map associating to an isomorphism class in $\mathcal{S}_T^{\text{alg}}$ its Tits index is not continuous.

Since the map $\mathcal{L} \to \mathcal{S}_T$ is a homeomorphism onto its image, the complexity of the residue characteristic 2 case should already be visible at the level of the space $\mathcal{L}$ of quadratic pairs of local fields. And indeed, Proposition 2.71 reflects this fact. The specific features of Chabauty limits in residue characteristic 2 highlight the complexity of the aforementioned conjecture, which will be addressed in full generality in a forthcoming paper, but with different methods.

The strategy to prove our results is the same for all algebraic groups under consideration (i.e. SL_2 or SU_3). Let us outline it in the SL_2 case (our notational conventions for local fields are spelled out at the beginning of Section 2.2.1).

1. In Definition 2.21, we recall the definition of the Bruhat–Tits tree:

$$\mathcal{I} = \mathrm{SL}_2(K) \times \mathbf{R}/\sim$$

2. In Definition 2.25, we define a pointed version (around 0) of the Bruhat–Tits tree:

$$\mathcal{I}_0 = \mathrm{SL}_2(\mathcal{O}_K) \times \mathbf{R}/\sim_0$$

and in Lemma 2.27, we show that the homomorphism $\mathrm{SL}_2(\mathcal{O}_K) \to \mathrm{SL}_2(K)$ induces an $(\mathrm{SL}_2(\mathcal{O}_K) \to \mathrm{SL}_2(K))$-equivariant bijection $\mathcal{I}_0 \to \mathcal{I}$.

3. In Definition 2.56, we define the ball around 0 of radius r:

$$B_0(r) = \{[(g,x)]_0 \in \mathcal{I}_0 \mid x \in [-\omega(\pi_K^r), \omega(\pi_K^r)] \subset \mathbf{R}, g \in \mathrm{SL}_2(\mathcal{O}_K)\}$$

4. In Definition 2.36, we define a local version (around 0 and of radius r) of the Bruhat–Tits tree:

$$\mathcal{I}^{0,r} = \mathrm{SL}_2(\mathcal{O}_K/\mathfrak{m}_K^r) \times [-\omega(\pi_K^r), \omega(\pi_K^r)]/\sim_{0,r}$$

and we show in Theorem 2.59 that the homomorphism $\mathrm{SL}_2(\mathcal{O}_K) \to \mathrm{SL}_2(\mathcal{O}_K/\mathfrak{m}_K^r)$ induces an $(\mathrm{SL}_2(\mathcal{O}_K) \to \mathrm{SL}_2(\mathcal{O}_K/\mathfrak{m}_K^r))$-equivariant bijection $B_0(r) \to \mathcal{I}^{0,r}$.

5. Following an idea dating back to M. Krasner (see [Del84] for references, this idea is also used in e.g. [Kaz86]), we define a metric d on the space $\mathcal{K}$ of (isomorphism classes of) local fields by declaring that for $r \in \mathbf{N}$ and $K_1, K_2 \in \mathcal{K}$, $d(K_1; K_2) \leq \frac{1}{2^r}$ if and only if $\mathcal{O}_{K_1}/\mathfrak{m}_{K_1}^r \cong \mathcal{O}_{K_2}/\mathfrak{m}_{K_2}^r$ (see Lemma 2.66). We observe in Proposition 2.71 that the space $\mathcal{K}_{p^n}$ of (isomorphism classes of) local fields having residue field $\mathbf{F}_{p^n}$ is homeomorphic to $\hat{\mathbf{N}}$.

6. Points 1 to 4 imply that if K_1 and K_2 are close to each other in $\mathcal{K}_{p^n}$, then $\mathrm{SL}_2(\mathcal{O}_{K_1})$ and $\mathrm{SL}_2(\mathcal{O}_{K_2})$ are close to each other in the Chabauty space of $\mathrm{Aut}(T_{p^n+1})$ (where T_{p^n+1} is the (p^n+1)-regular tree). Indeed, up to isomorphism, they act in the same way on a large ball centred at 0. This is the key step in the proof of Theorem 2.78.

7. We are then able to conclude effortlessly, using a rigidity argument,

that the map $\mathcal{K}_{p^n} \to \mathcal{S}^{\mathrm{alg}}_{T_{p^n+1}} : K \mapsto \mathrm{SL}_2(K)/Z$ is a homeomorphism onto its image.

A key tool to implement our strategy is the existence of good functors from $\mathcal{O}_K$-algebras (such as $\mathcal{O}_K/\mathfrak{m}_K^r$) to groups (like $\mathrm{SL}_2(\mathcal{O}_K/\mathfrak{m}_K^r)$). The integral model provided by Bruhat–Tits theory plays the role of this good functor. In the SL_2 case, this is just the algebraic group SL_2 considered over $\mathcal{O}_K$. But a description of the integral model is not always so straightforward, and an important feature of this article is an explicit computation of Bruhat–Tits models for $\mathrm{SU}_3^{L/K}$, especially in the more delicate case when the residue characteristic is 2 and L is ramified.

The complexity of the integral model of $\mathrm{SU}_3^{L/K}$ when the residue characteristic is 2 and L is ramified also explains why we get a different behaviour for regular trees of degree $2^n + 1$ in Theorem 2.3. As often in the theory of algebraic groups, the characteristic 2 case is more involved to work out (and in our situation, it is again because of the presence of orthogonal groups in characteristic 2 lurking in the background, see Remark 2.41), but as was strongly advocated by J. Tits, this case is also of great interest. Our results seem to be another illustration of this philosophy.

It also appears that studying convergence of groups isomorphic to $\mathrm{SL}_2(D)/Z$ (where D is a finite dimensional central division algebra over a local field K) can be done in parallel to the $\mathrm{SL}_2(K)$ case. Hence we decided to treat this case as well in this chapter. We stress that this is only an opportunistic choice, and that the other cases should be settled by first considering similar questions in arbitrary rank for quasi-split groups, and then by applying a descent method.

Nevertheless, thanks to this treatment, we get the following results as well.

Theorem 2.4. *Let T be a locally finite leafless tree, and let $\mathcal{S}_T^{\mathrm{SL}_2(D)}$ be the set of isomorphism classes of topologically simple algebraic groups acting on T that are furthermore isomorphic to $\mathrm{SL}_2(D)/Z$ for some central division algebra D. Then $\mathcal{S}_T^{\mathrm{SL}_2(D)}$ is closed in $\mathcal{S}_T$.*

Hence, for the reasons explained before Corollary 2.2 and according to the tables in [Tit79, 4.2 and 4.3], we obtain the following strengthening of Corollary 2.2.

Corollary 2.5. *Let p be a prime number, and let T be the (p^n+1)-regular tree where n is not divisible by 3, or the $(p^{3n}+1; p^n+1)$-semiregular tree. Then the set $\mathcal{S}_T^{\mathrm{alg}}$ coincides with $\mathcal{S}_T^{\text{qs-alg}} \cup \mathcal{S}_T^{\mathrm{SL}_2(D)}$, so that it is closed in $\mathcal{S}_T$.*

Again, just as for the quasi-split case, we are actually able to describe explicitly the topological space $\mathcal{S}_T^{\mathrm{SL}_2(D)}$ and all the convergences in this space.

Theorem 2.6. *Let T be the (p^n+1)-regular tree.*

1. *The topological space $\mathcal{S}_T^{\mathrm{SL}_2(D)}$ is homeomorphic to the Cartesian product $\hat{\mathbf{N}} \times \{1, \ldots, \lceil \frac{n+1}{2} \rceil\}$. The first Cantor-Bendixson derivative of $\mathcal{S}_T^{\mathrm{SL}_2(D)}$ is*

$$\{\mathrm{SL}_2(D)/Z \mid \overline{D} \cong \mathbf{F}_{p^n} \text{ and } D \text{ is of characteristic } p\}$$

2. *For $i \in \mathbf{N}$, let D_i (respectively D) be a finite dimensional central division algebra over K_i (respectively K) having residue field of cardinality p^n. Let d_i (respectively d) be the degree of D_i (respectively D), so that $|\overline{K}_i|^{d_i} = p^n = |\overline{K}|^d$, where $\overline{K}_i$ (respectively $\overline{K}$) denotes the residue field of K_i (respectively K). Let r_i (respectively r) be the Hasse invariant of D_i (respectively D), as in Definition 2.90. If $(\mathrm{SL}_2(D_i))_{i \in \mathbf{N}}$ converges to $\mathrm{SL}_2(D)$ in the Chabauty space $\mathbf{Sub}(\mathrm{Aut}(T))$, then for all i large enough, $r_i = \pm r$ and $d_i = d$, so that $|\overline{K}_i| = |\overline{K}|$ as well.*

We conclude this introduction by mentioning the recent work of M. de la Salle and R. Tessera [dT15], who used independently closely related ideas in their study of the space of Bruhat–Tits buildings of type $\tilde{A}_n$ (with $n > 2$) endowed with the Gromov–Hausdorff topology.

2.2 Definitions of the algebraic groups SL_2 and SU_3

For the rest of the chapter, K will denote a local field (all our local fields are assumed to be non-archimedean), and D will denote a finite dimensional central simple division algebra over K. Let us spell out our notational conventions for the objects associated with K (respectively D): the ring of integers is denoted $\mathcal{O}_K$ (respectively $\mathcal{O}_D$), its maximal ideal by $\mathfrak{m}_K$ (respectively $\mathfrak{m}_D$), a uniformiser by π_K (respectively π_D) and $\overline{K}$ (respectively $\overline{D}$) denotes the residue field. The valuation of K (respectively D), and also its unique extension to any finite extension of K, is denoted by ω. We use the notation $\mathbf{Q}_{p^n}$ for the unique (up to isomorphism) unramified extension of $\mathbf{Q}_p$ of degree n.

Also, in order to avoid the repetition of long lists of adjectives, in this section, by an algebraic group, we mean an absolutely simple, simply connected algebraic group over a local field.

2.2.1 Quasi-split groups of relative rank 1

As mentioned in the introduction, the Bruhat–Tits building of an algebraic group G is a tree if and only if G is of relative rank 1. Instead of giving the general definition of quasi-split algebraic groups, and then specialising to those that are of relative rank 1, we take a practical approach and give an explicit description of those groups, the result being that they are all of the form SL_2 or SU_3. We begin by recalling the definition of SU_3.

Definition 2.7. Let L be a separable quadratic extension of K, and let σ be the nontrivial element of $\mathrm{Aut}(L/K)$, whose action by conjugation on L is denoted $x \mapsto \bar{x}$. Consider the transposition along the anti-diagonal ${}^S(.)\colon \mathrm{SL}_3(L) \to \mathrm{SL}_3(L)\colon g \mapsto {}^S g$. More explicitly, $({}^S g)_{-j,-i} = g_{ij}$, for $i, j \in \{-1, 0, 1\}$. Then we define

$$\mathrm{SU}_3^{L/K}(K) = \{g \in \mathrm{SL}_3(L) \mid {}^S\bar{g}^{-1} = g\}$$

We denote $\mathrm{SU}_3^{L/K}$ (or simply SU_3 when the pair of field (K, L) is arbitrary or understood from the context) the corresponding algebraic group over K. Note that the equations $\det(g) - 1$ and ${}^S\bar{g}g - \mathrm{Id}$ (together with the embedding $L \hookrightarrow M_2(K)$) realise $\mathrm{SU}_3^{L/K}$ as a closed subspace of the affine space $\mathbf{A}_K^n$ of dimension $n = 4 \times 3^2$. Using this, it is readily seen that SU_3 is an algebraic group over K.

Remark 2.8. The group SU_3 defined above is the special unitary group with respect to the following hermitian form of L^3:

$$((x_{-1}, x_0, x_1), (y_{-1}, y_0, y_1)) \mapsto \overline{x}_{-1}y_1 + \overline{x}_0 y_0 + \overline{x}_1 y_{-1}$$

The advantage of taking this peculiar hermitian form is that the associated involution preserves the group of upper triangular matrices. As Lemma 2.9 shows, up to isomorphism, there is only one "type" of non-split, quasi-split algebraic group of relative rank 1 over local fields. Hence, choosing the above hermitian form is in fact not restrictive.

We can now describe quasi-split algebraic groups of relative rank 1 (recall that by the convention of this section, all our algebraic groups are absolutely simple, simply connected, algebraic groups over a local field).

Lemma 2.9. *Let K be a local field and let G be a quasi-split algebraic group of relative rank 1 over K. Then G is one of the following group:*

1. SL_2 *over K.*
2. $\mathrm{SU}_3^{L/K}$*, where L is as in Definition 2.7.*

Proof. If G is quasi-split, then by definition, its anisotropic kernel is trivial. Hence, by [Tit66, 2.7.1, Theorem 2], G is entirely determined (up to K-isomorphism) by its Dynkin diagram together with the $*$-action on it (or in other words, G is determined by its index). Also note that the number of orbit under this $*$-action is the relative rank, so that according to [Tit66, Table II], the only possibilities for the index are

 or 

The first index is the index of $\mathrm{SU}_3^{L/K}$, where L is any separable quadratic extension of K, while the second index is the index of SL_2. $\square$

2.2.2 The algebraic group $\mathrm{SL}_2(D)$

As outlined in the introduction, treating the case of the group $\mathrm{SL}_2(D)$ (where D is a finite dimensional central division algebra) is very close to treating the case of $\mathrm{SL}_2(K)$, so that we decided to include this case as well. Let us recall the definition of the group $\mathrm{SL}_2(D)$.

Definition 2.10. Let D be a finite dimensional central division algebra over K. We define the group $\mathrm{SL}_2(D) = \{u \in \mathrm{End}_D(D^2) \mid \mathrm{Nrd}(u) = 1\}$, where $\mathrm{Nrd}(u)$ stands for the reduced norm of u (we recall the definition of the reduced norm in Definition 2.92), and D^2 is considered as a right D-vector space.

Let us stress again that the case of main interest is the case of quasi-split groups, i.e. the case $D = K$. We advice the reader to consider only this case in a first reading.

When $D = K$, the group $\mathrm{SL}_2(K)$ is the group of rational points of a closed subspace SL_2 of the affine space $\mathbf{A}^4_K$ defined by the polynomial equation $\det -1$. It is then straightforward to check that SL_2 is indeed an algebraic group over K.

For arbitrary D, it is well-known that $\mathrm{SL}_2(D)$ can be seen as the group of rational point of an algebraic group over K. We recall in Appendix 2.B the standard facts about division algebras, and we also discuss in Appendix 2.C the representation of $\mathrm{SL}_2(D)$ as an algebraic group over K.

2.3 The Bruhat–Tits tree of $\mathrm{SL}_2(D)$ and SU_3

The aim of this section is to give a streamlined definition of the Bruhat–Tits tree associated with $\mathrm{SL}_2(D)$ and SU_3, together with the action on it. As outlined in the introduction, our definition of the Bruhat–Tits tree follows [BT72, §7].

In order to be as efficient as possible, we only describe concretely the objects needed, and give unmotivated definitions. Our description is easily obtained from the explicit description given in [BT72, §10], and we give in Appendix 2.A more details about the connection with [BT72].

Recall from the introduction (or from general Bruhat–Tits theory) that the Bruhat–Tits tree $\mathcal{I}$ should be isomorphic to $G(K) \times \mathbf{R} / \sim$. For $x \in \mathbf{R}$, we define a group $P_x \leq G(K)$ which will eventually turn out to be the stabiliser of $[(\mathrm{Id}, x)] \in \mathcal{I}$ (see Remark 2.22).

Definition 2.11. Let D be a finite dimensional central division algebra over K and let g be a $n \times n$ matrix with coefficients in D. Given a $n \times n$ matrix m with coefficient in $\mathbf{R}$, we say that g has a valuation greater than m if $\omega(g_{ij}) \geq m_{ij}$ (for all $i, j \in \{1, \ldots, n\}$), and we denote it by $\omega(g) \geq m$.

Definition 2.12. In the $\mathrm{SL}_2(D)$ case, for $x \in \mathbf{R}$, we define

$$P_x = \{g \in \mathrm{SL}_2(D) \mid \omega(g) \geq \left(\begin{smallmatrix} 0 & -x \\ x & 0 \end{smallmatrix}\right)\}$$

The definition of P_x in the SU_3 case is less straightforward when the residue characteristic is 2 and the extension L is ramified. Following [BT84a, 4.3.3], we define a parameter to handle the complication.

Lemma 2.13. *Let L be a separable quadratic extension of K. There exists $t \in L$ and $\alpha, \beta \in K$ such that:*

1. *$L = K[t]$ and $t^2 - \alpha t + \beta = 0$.*
2. *$\omega(\beta) = 0$ when L is unramified, and β is a uniformiser of K when L is ramified.*
3. *$\alpha = 0$, or $0 = \omega(\beta) = \omega(\alpha) < \omega(2)$, or $0 < \omega(\beta) \leq \omega(\alpha) \leq \omega(2)$.*

Proof. See [BT84a, Lemme 4.3.3, (ii)]. The fact that α can be chosen so that $\omega(\alpha) = 0$ in the unramified case is a direct consequence of the theory of unramified extensions of local fields (see for example [FV02, Chapter II, Section 3.2, Proposition]). With this in mind, the equivalence with [BT84a, Lemme 4.3.3, (ii)] is clear. □

Remark 2.14. To make Lemma 2.13 possibly clearer, let us state what is the valuation of α on a case-by-case analysis:

1. If L is unramified, $\begin{cases} \alpha = 0 \text{ if the residue characteristic is not 2} \\ \omega(\alpha) = 0 \text{ if the residue characteristic is 2} \end{cases}$

2. If L is ramified,
$$\begin{cases} \alpha = 0 \text{ if the residue characteristic is not } 2 \\ \alpha = 0 \text{ or } 0 < \omega(\alpha) \leq \omega(2) \text{ if the residue characteristic is } 2 \end{cases}$$

The only difference between Remark 2.14 and Lemma 2.13 is that the latter allows the possibility that $\alpha = 0$ in the unramified residue characteristic 2 case. But this clearly cannot happen.

Definition 2.15. Let L be a separable quadratic extension of K, and let t, α, β be chosen as in Lemma 2.13. Let $l = t\alpha^{-1} \in L$ if $\alpha \neq 0$, and $l = \frac{1}{2} \in L$ if $\alpha = 0$, where α is as in Lemma 2.13 (note that $\alpha = 0$ implies $2 \neq 0$ in K, since L is assumed to be a separable extension). We then define $\gamma = -\frac{1}{2}\omega(l) \in \mathbf{R}$.

Remark 2.16. Note that $\gamma \geq 0$. Furthermore, in view of Remark 2.14, $\gamma > 0$ if and only if the residue characteristic is 2 and L is a ramified extension.

In fact, the parameter γ associated with a quadratic separable extension L/K only depends on the normalisation of the valuation on K.

Proposition 2.17. *Let L/K be a separable quadratic extensions of local fields. Then the parameter γ introduced in Definition 2.15 does not depend on the choices of t, α and β. We call γ the parameter associated with the extension L of K*

Proof. This is a direct corollary of the work of Bruhat–Tits. Indeed, according to [BT84a, Proposition 4.3.3, (ii)], the element l appearing in Definition 2.15 has a maximal valuation amongst elements of L of trace 1. □

Definition 2.18. In the $\mathrm{SU}_3^{L/K}$ case, let γ be the parameter associated with the extension L of K as in Definition 2.15. For $x \in \mathbf{R}$, we define

$$P_x = \{g \in \mathrm{SU}_3^{L/K}(K) \mid \omega(g) \geq \begin{pmatrix} 0 & -\frac{x}{2}-\gamma & -x \\ \frac{x}{2}+\gamma & 0 & -\frac{x}{2}+\gamma \\ x & \frac{x}{2}-\gamma & 0 \end{pmatrix}\}$$

The final ingredient in the definition of the Bruhat–Tits tree is the definition of a subgroup N, together with its affine action on $\mathbf{R}$.

Definition 2.19. 1. In the $\mathrm{SL}_2(D)$ case, consider the following subsets

(a) $T = \{\left(\begin{smallmatrix} x & 0 \\ 0 & x^{-1} \end{smallmatrix}\right) \mid x \in D^\times\} < \mathrm{SL}_2(D)$

(b) $M = \{\left(\begin{smallmatrix} 0 & -x \\ x^{-1} & 0 \end{smallmatrix}\right) \mid x \in D^\times\} \subset \mathrm{SL}_2(D)$

2. In the $\mathrm{SU}_3^{L/K}$ case, consider the following subsets

(a) $T = \{\left(\begin{smallmatrix} x & 0 & 0 \\ 0 & x^{-1}\bar{x} & 0 \\ 0 & 0 & \bar{x}^{-1} \end{smallmatrix}\right) \mid x \in L^\times\} < \mathrm{SU}_3^{L/K}(K)$

(b) $M = \{\left(\begin{smallmatrix} 0 & 0 & x \\ 0 & -x^{-1}\bar{x} & 0 \\ \bar{x}^{-1} & 0 & 0 \end{smallmatrix}\right) \mid x \in L^\times\} \subset \mathrm{SU}_3^{L/K}(K)$

In both cases, let $N = T \sqcup M$.

Definition 2.20. In both cases, we define a map $\nu\colon N \to \mathrm{Aff}(\mathbf{R})$ as follows. In the $\mathrm{SL}_2(D)$ case (respectively the SU_3 case), for $m = \left(\begin{smallmatrix} 0 & -x \\ x^{-1} & 0 \end{smallmatrix}\right) \in M$ (respectively $m = \left(\begin{smallmatrix} 0 & 0 & x \\ 0 & -x^{-1}\bar{x} & 0 \\ \bar{x}^{-1} & 0 & 0 \end{smallmatrix}\right) \in M$), $\nu(m)$ is the reflection through $-\omega(x)$, while for $t = \left(\begin{smallmatrix} x & 0 \\ 0 & x^{-1} \end{smallmatrix}\right) \in T$ (respectively for $t = \left(\begin{smallmatrix} x & 0 & 0 \\ 0 & x^{-1}\bar{x} & 0 \\ 0 & 0 & \bar{x}^{-1} \end{smallmatrix}\right) \in T$), $\nu(t)$ is the translation by $-2\omega(x)$.

We finally arrive at the definition of the Bruhat–Tits tree.

Definition 2.21 ([BT72, 7.4.1 and 7.4.2])**.** Let G be either $\mathrm{SL}_2(D)$ or $\mathrm{SU}_3(K)$. Define an equivalence relation on $G \times \mathbf{R}$ as follows: $(g, x) \sim (h, y)$ if and only if there exists $n \in N$ such that $y = \nu(n)(x)$ and $g^{-1}hn \in P_x$. The Bruhat–Tits tree of G is $\mathcal{I} = G \times \mathbf{R} / \sim$, and $[(g, x)]$ stands for the equivalence class of (g, x) in $\mathcal{I}$. The group G acts on $\mathcal{I}$ by multiplication on the first component.

Remark 2.22. We discuss in Appendix 2.A why our groups P_x coincide with the groups $\hat{P}_x$ appearing in the definition of the Bruhat–Tits building in [BT72, 7.4.1 and 7.4.2]. Since the definition of N together with its action ν on $\mathbf{R}$ also coincide with [BT72] (see also Appendix 2.A for more details), the space $\mathcal{I}$ of Definition 2.21 is really the Bruhat–Tits building of G as defined in [BT72]. In particular, for $g \in G$, the map $f_g\colon \mathbf{R} \to \mathcal{I}\colon x \mapsto g.[(\mathrm{Id}, x)]$ is injective (by the discussion in [BT72], below Definition 7.4.2), an apartment of $\mathcal{I}$ is a subset of the form $f_g(\mathbf{R})$ for some $g \in G$, and we can endow $\mathcal{I}$ with a metric which gives the usual metric on $\mathbf{R}$ when restricted to any apartment. Furthermore, in view of [BT72, Proposition 7.4.4], P_x is in fact the stabiliser of $[(\mathrm{Id}, x)] \in \mathcal{I}$.

Remark 2.23. The metric space $\mathcal{I}$ is indeed a tree, whose regularity depends on G. If $G = \mathrm{SL}_2(D)$ (respectively $\mathrm{SU}_3^{L/K}(K)$ where L is ramified), then $\mathcal{I}$ is the regular tree of degree $|\overline{D}| + 1$ (respectively $|\overline{K}| + 1$), while if $G = \mathrm{SU}_3^{L/K}(K)$ with L unramified, then $\mathcal{I}$ is the semiregular tree of bidegree $(|\overline{K}|^3 + 1; |\overline{K}| + 1)$. Indeed, this follows from the fact that our definition of $\mathcal{I}$ agrees with the one given in [BT72, 7.4.1 and 7.4.2], and from the tables in [Tit79, 4.2 and 4.3].

Remark 2.24. Note that in Definition 2.21, it is equivalent to say that $(g, x) \sim (h, y)$ if and only if for all $\tilde{n} \in N$ such that $\nu(\tilde{n})(x) = y$, we have $g^{-1}h\tilde{n} \in P_x$. Indeed, if there exists $n \in N$ such that $\nu(n)(x) = y$ and $g^{-1}hn \in P_x$, let $\tilde{n}$ be any element of N such that $\nu(\tilde{n})(x) = y$. Then $g^{-1}h\tilde{n} = g^{-1}hnn^{-1}\tilde{n}$. But $n^{-1}\tilde{n}$ stabilises $[(\mathrm{Id}, x)]$, and hence belongs to P_x by Remark 2.22. Thus, $g^{-1}hnn^{-1}\tilde{n}$ belongs to P_x as well, as wanted.

We pass to another equivalent definition of the Bruhat–Tits tree, which can be thought of as a pointed version of $\mathcal{I}$ around $[(\mathrm{Id}, 0)]$.

Definition 2.25. In the $\mathrm{SL}_2(D)$ case or the $\mathrm{SU}_3(K)$ case, we define an equivalence relation on $P_0 \times \mathbf{R}$ as follows: $(g, x) \sim_0 (h, y)$ if and only if there exists $n \in N \cap P_0$ such that $y = \nu(n)(x)$ and $g^{-1}hn \in P_x \cap P_0$. The Bruhat–Tits tree of G centred at 0 is $\mathcal{I}_0 = P_0 \times \mathbf{R} / \sim_0$, and $[(g, x)]_0$ stands for the equivalence class of (g, x) in $\mathcal{I}_0$. The group P_0 acts on $\mathcal{I}_0$ by multiplication on the first component.

To prove that $\mathcal{I}_0$ is naturally in equivariant bijection with $\mathcal{I}$, we need the following observation.

Lemma 2.26. *Let $g, h \in P_0$, and let $x, y \in \mathbf{R}$. If $(g, x) \sim (h, y)$, there exists $n \in N \cap P_0$ such that $\nu(n)(x) = y$*

Proof. Recall that P_0 is the stabiliser of $[(\mathrm{Id}, 0)] \in \mathcal{I}$ in G (see Remark 2.22). Since G acts by isometries on $\mathcal{I}$, and since $g, h \in P_0$, we have

$$|x| = d_{\mathcal{I}}([(\mathrm{Id}, x)], [(\mathrm{Id}, 0)]) = d_{\mathcal{I}}([(g, x)], [(\mathrm{Id}, 0)])$$

$$|y| = d_{\mathcal{I}}([(\mathrm{Id}, y)], [(\mathrm{Id}, 0)]) = d_{\mathcal{I}}([(h, y)], [(\mathrm{Id}, 0)])$$

where $d_{\mathcal{I}}$ denotes the distance in the metric space $\mathcal{I}$ (see Remark 2.22). But if $(g, x) \sim (h, y)$, we have in particular $d_{\mathcal{I}}([(g, x)], [(\mathrm{Id}, 0)]) = d_{\mathcal{I}}([(h, y)], [(\mathrm{Id}, 0)])$, and hence $|x| = |y|$. Thus, the existence of $n \in N \cap P_0$ such that $\nu(n)(x) = y$ follows from Definition 2.20, □

Lemma 2.27. *Let G be either $\mathrm{SL}_2(D)$ or $\mathrm{SU}_3(K)$. The map $\mathcal{I}_0 \to \mathcal{I}\colon [(g, x)]_0 \mapsto [(g, x)]$ is a $(P_0 \hookrightarrow G)$-equivariant bijection.*

Proof.

- Injectivity: assume $(g, x) \sim (h, y)$, where g, h are in P_0. By Lemma 2.26, there exists $n \in N \cap P_0$ such that $y = \nu(n)(x)$ and since $(g, x) \sim (h, y)$, $g^{-1}hn \in P_x$ by Remark 2.24. But $g^{-1}hn$ also belongs to P_0, so that $(g, x) \sim_0 (h, y)$, as wanted.

- Surjectivity: let $[(g, x)] \in \mathcal{I}$. Since G acts strongly transitively on $\mathcal{I}$ ([BT72, Corollaire 7.4.9]), there exists $h \in P_0$ such that $h.[(g, x)] = [(\mathrm{Id}, y)]$, for some $y \in \mathbf{R}$. Hence, $[(g, x)]$ is the image of $[(h^{-1}, y)]_0 \in \mathcal{I}_0$.

- Equivariance: the image of $h.[(g, x)]_0$ is $[(hg, x)] = h.[(g, x)]$. □

2.4 Local description of the Bruhat–Tits tree

We now aim to give a local description of balls of the Bruhat–Tits tree, together with the group action on it. Recall that the ball of radius 1 around $[(\mathrm{Id}, 0)] \in \mathcal{I}$ (together with the action of P_0 on it), is in some sense encoded in P_0 considered over the residue field, i.e. over $\mathcal{O}_K/\mathfrak{m}_K$ (see [BT84a, Théorème 4.6.33] for a precise meaning). It is then natural to think that more generally, the ball of radius r around $[(\mathrm{Id}, 0)] \in \mathcal{I}$ (together with the action of P_0 on it) is encoded in P_0 considered over the ring $\mathcal{O}_K/\mathfrak{m}_K^r$. We prove in this section that this is indeed true.

2.4.1 Local models for the Bruhat–Tits tree

We just mimic the definition of the Bruhat–Tits tree, except that the coefficients of all groups under consideration are now taken in the ring

$\mathcal{O}_D/\mathfrak{m}_D^r$ (or $\mathcal{O}_L/\mathfrak{m}_L^r$ in the SU_3 case). All groups defined in this section are adorned by the superscript $0, r$ to reflect the fact that they are local version around 0 of radius r.

Definition 2.28. In the $\mathrm{SL}_2(D)$ case, let $r \in \mathbf{N} \cup \{\infty\}$. We only need to describe balls of radius rd, where d is the degree of D over its center K. Let $x \in [-\omega(\pi_D^{rd}), \omega(\pi_D^{rd})]$. Note that the valuation ω induces a well-defined map on $\mathcal{O}_D/\mathfrak{m}_D^{rd}$, that we still denote ω. By convention, $\mathfrak{m}_D^\infty = (0)$. Mimicking Definition 2.12, we define $P_x^{0,rd} = \{g \in \mathrm{SL}_2(\mathcal{O}_D/\mathfrak{m}_D^{rd}) \mid \omega(g) \geq \left(\begin{smallmatrix} 0 & -x \\ x & 0 \end{smallmatrix}\right)\}$ (see Definition 2.95 for the definition of $\mathrm{SL}_2(\mathcal{O}_D/\mathfrak{m}_D^{rd})$. When $D = K$, we obtain the group $\mathrm{SL}_2(\mathcal{O}_K/\mathfrak{m}_K^r)$ in its usual meaning, i.e. the group of 2×2 matrices with coefficient in $\mathcal{O}_K/\mathfrak{m}_K^r$ having determinant 1).

We also need the local version of the subgroup N.

Definition 2.29. In the $\mathrm{SL}_2(D)$ case, we define

1. $H^{0,rd} = \{\left(\begin{smallmatrix} x & 0 \\ 0 & x^{-1} \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathcal{O}_D/\mathfrak{m}_D^r) \mid \omega(x) = 0\}$
2. $M^{0,rd} = \{\left(\begin{smallmatrix} 0 & -x \\ x^{-1} & 0 \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathcal{O}_D/\mathfrak{m}_D^r) \mid \omega(x) = 0\}$

And then, we set $N^{0,rd} = H^{0,rd} \sqcup M^{0,rd}$

In the SU_3 case, some complications arise due to the fact that the group P_0 of Definition 2.18 is not naturally described as living in $\mathrm{SL}_3(\mathcal{O}_L)$ when the parameter γ of Definition 2.15 is strictly positive, i.e. when the residue characteristic is 2 and the extension L is ramified. This is related to the fact that if one considers the algebraic group $\underline{G} = \underline{\mathrm{SU}}_3^{L/K}$ over $\mathcal{O}_K$ as in Definition 2.39 (which is possible since the equations $\det(g) - 1$ and ${}^S\bar{g}g - \mathrm{Id}$ only involve coefficients belonging to $\mathcal{O}_K$), then it is not smooth (as an $\mathcal{O}_K$-scheme) if and only if the residue characteristic is 2 and the extension L is ramified. Indeed, in this case, $\dim_{\overline{K}} \mathrm{Lie}(\underline{G}_{\overline{K}}) = \dim_K \mathrm{Lie}(\underline{G}_K) + 3$, while smoothness of $\underline{\mathrm{SU}}_3^{L/K}$ when $\gamma = 0$ is proved in Theorem 2.40.

By contrast, the correct definition of the local version of the Bruhat–Tits tree in the SU_3 case when $\gamma = 0$ is the "natural" one.

Definition 2.30. In the SU_3 case when $\gamma = 0$, let $r \in \mathbf{N} \cup \{\infty\}$. Let $x \in [-\omega(\pi_L^r), \omega(\pi_L^r)]$. Note that the valuation ω induces a well-defined map on $\mathcal{O}_L/\mathfrak{m}_L^r$, that we still denote ω. Also, the Galois action on L induce an action on $\mathcal{O}_L/\mathfrak{m}_L^r$, that we also denote by $x \mapsto \bar{x}$. By convention, $\mathfrak{m}_L^\infty = (0)$. Mimicking Definition 2.18, we define $P_x^{0,r} = \{g \in \mathrm{SL}_3(\mathcal{O}_L/\mathfrak{m}_L^r) \mid {}^S\bar{g}g = \mathrm{Id}, \omega(g) \geq \begin{pmatrix} 0 & -\frac{x}{2} & -x \\ \frac{x}{2} & 0 & -\frac{x}{2} \\ x & \frac{x}{2} & 0 \end{pmatrix}\}$.

Again, we need the local version of the subgroup N.

Definition 2.31.

1. $H^{0,r} = \{\begin{pmatrix} x & 0 & 0 \\ 0 & x^{-1}\bar{x} & 0 \\ 0 & 0 & \bar{x}^{-1} \end{pmatrix} \in \mathrm{SL}_3(\mathcal{O}_L/\mathfrak{m}_L^r) \mid \omega(x) = 0\}$
2. $M^{0,r} = \{\begin{pmatrix} 0 & 0 & x \\ 0 & -x^{-1}\bar{x} & 0 \\ \bar{x}^{-1} & 0 & 0 \end{pmatrix} \in \mathrm{SL}_3(\mathcal{O}_L/\mathfrak{m}_L^r) \mid \omega(x) = 0\}$

And then, we set $N^{0,r} = H^{0,r} \sqcup M^{0,r}$.

When $\gamma > 0$ (i.e. when the residue characteristic is 2 and L is ramified), we only need to give the local description for small radii. We introduce a new parameter which controls the meaning of small in this case.

Definition 2.32. Set $i_0 = \min\{r \in \mathbf{N} \mid \omega(\pi_L^r) \geq \gamma\}$. Equivalently, let α be as in Lemma 2.13. If $\alpha = 0$ (respectively if $\alpha \neq 0$), i_0 is such that $\omega(\pi_K^{i_0}) = \omega(2)$ (respectively $\omega(\pi_K^{i_0}) = \omega(\alpha)$).

Definition 2.33. In the SU_3 case when $\gamma > 0$, let $r \in \mathbf{N}$ be such that $r \leq 2i_0$. Let $x \in [-\omega(\pi_L^r), \omega(\pi_L^r)]$. Note that the valuation ω induces a well-defined map on $\mathcal{O}_L/\mathfrak{m}_L^r$, that we still denote ω. we define $P_x^{0,r} = \{g \in \mathrm{SL}_2(\mathcal{O}_L/\mathfrak{m}_L^r) \mid \omega(g) \geq \begin{pmatrix} 0 & -x \\ x & 0 \end{pmatrix}\}$

We also need the local version of the subgroup N.

Definition 2.34. In the SU_3 case when $\gamma > 0$ and for $r \leq 2i_0$, we define

1. $H^{0,r} = \{\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} \in \mathrm{SL}_2(\mathcal{O}_L/\mathfrak{m}_L^r) \mid \omega(x) = 0\}$
2. $M^{0,r} = \{\begin{pmatrix} 0 & -x \\ x^{-1} & 0 \end{pmatrix} \in \mathrm{SL}_2(\mathcal{O}_L/\mathfrak{m}_L^r) \mid \omega(x) = 0\}$

And then, we set $N^{0,r} = H^{0,r} \sqcup M^{0,r}$

We can also easily define an action of $N^{0,r}$ by affine isometries on $\mathbf{R}$.

Definition 2.35. In all cases ($\mathrm{SL}_2(D)$ and SU_3 for $\gamma \geq 0$), we let $H^{0,r}$ acts trivially on $\mathbf{R}$, and we let all elements of $M^{0,r}$ act as a reflection through $0 \in \mathbf{R}$. This gives an affine action of $N^{0,r}$ on $\mathbf{R}$, and we denote again the resulting map $N^{0,r} \to \mathrm{Aff}(\mathbf{R})$ by ν.

We are now able to give a definition of the ball of radius r around $[(\mathrm{Id}, 0)] \in \mathcal{I}$ which only depends on the ring $\mathcal{O}/\mathfrak{m}^r$, and not on the whole division algebra D or the field L.

Definition 2.36. Let $r \in \mathbf{N} \cup \{\infty\}$. In the $\mathrm{SL}_2(D)$ case (respectively the SU_3 case), let $\pi = \pi_D$ and $d = \sqrt{[D:K]}$ (respectively $\pi = \pi_L$ and $d = 1$). Also assume that $r \leq 2i_0$ in the SU_3 case when $\gamma > 0$. We define an rd-local equivalence on $P_0^{0,rd} \times [-\omega(\pi^{rd}), \omega(\pi^{rd})]$ as follows. For $g, h \in P_0^{0,rd}$ and $x, y \in [-\omega(\pi^{rd}), \omega(\pi^{rd})]$

$$(g,x) \sim_{0,rd} (h,y) \Leftrightarrow \text{ there exists } n \in N^{0,rd} \text{ such that } \nu(n)(x) = y \text{ and } g^{-1}hn \in P_x^{0,rd}$$

The resulting space $\mathcal{I}^{0,rd} = P_0^{0,rd} \times [-\omega(\pi^{rd}), \omega(\pi^{rd})]/\sim_{0,rd}$ is called the local Bruhat–Tits tree of radius rd around 0, and $[(g,x)]^{0,rd}$ stands for the equivalence class of (g,x) in $\mathcal{I}^{0,rd}$. The group $P_0^{0,rd}$ acts on $\mathcal{I}^{0,rd}$ by multiplication on the first component.

Remark 2.37. Note that as for Definition 2.21, it is equivalent to say that $(g,x) \sim_{0,rd} (h,y)$ if and only if for all $\tilde{n} \in N^{0,rd}$ such that $\nu(\tilde{n})(x) = y$, we have $g^{-1}h\tilde{n} \in P_x^{0,rd}$. Indeed, if there exists $n \in N^{0,rd}$ such that $\nu(n)(x) = y$ and $g^{-1}hn \in P_x^{0,rd}$, let $\tilde{n}$ be any element of $N^{0,rd}$ such that $\nu(\tilde{n})(x) = y$. We have $g^{-1}h\tilde{n} = g^{-1}hnn^{-1}\tilde{n}$, and a case-by-case analysis shows that $n^{-1}\tilde{n} \in P_x^{0,rd}$. Hence $g^{-1}hnn^{-1}\tilde{n}$ belongs to $P_x^{0,rd}$ as well, as wanted.

2.4.2 Integral models

We have just defined the space $\mathcal{I}^{0,rd}$, where d is the degree of D in the $\mathrm{SL}_2(D)$ case, and is equal to one otherwise. In order to show that it encodes the ball of radius rd together with the action of P_0 on it (as will be done in Theorem 2.59), we need to prove that there exists a surjective homomorphism $P_0 \to P_0^{0,rd}$. In the $\mathrm{SL}_2(D)$ case (respectively the SU_3 case when $\gamma = 0$), the homomorphism $P_0 \to P_0^{0,rd}$ is just the one induced by the projection $\mathcal{O}_D \to \mathcal{O}_D/\mathfrak{m}_D^{rd}$ (respectively $\mathcal{O}_L \to \mathcal{O}_L/\mathfrak{m}_L^r$). But in the SU_3 case when $\gamma > 0$, even the existence of such a homomorphism is not obvious at first sight.

We solve the question by defining (for each case separately) a smooth $\mathcal{O}_K$-scheme $\underline{G}$, such that $\underline{G}(\mathcal{O}_K) \cong P_0$ and $\underline{G}(\mathcal{O}_K/\mathfrak{m}_K^r) \cong P_0^{0,\epsilon rd}$ (where $\epsilon = 2$ in the SU_3 case when L is ramified, and is equal to 1 otherwise). Then the desired surjectivity follows by an application of Hensel's lemma for smooth schemes (that we recall in Theorem 2.52).

The smooth $\mathcal{O}_K$-scheme $\underline{G}$ is in fact the Bruhat–Tits integral model $\hat{\mathfrak{G}}_\varphi$ associated with a standard valuation φ (see [BT84a, 4.6.26]). A potential interest of this section is that we also give an explicit description of this integral model in the more complicated case of SU_3 when $\gamma > 0$. But let us begin with the $\mathrm{SL}_2(D)$ case and the SU_3 case when $\gamma = 0$.

Definition 2.38. Let $\underline{\mathrm{SL}_2}$ be the group SL_2 considered over $\mathcal{O}_K$. Concretely, this is the $\mathcal{O}_K$-scheme associated with the $\mathcal{O}_K$-algebra $\mathcal{O}_K[\underline{\mathrm{SL}_2}] = \mathcal{O}_K[X_{11}, X_{12}, X_{21}, X_{22}]/(X_{11}X_{22} - X_{12}X_{21} - 1)$. In the case of a central division algebra of degree $d > 1$ over K, the definition of an integral model $\underline{\mathrm{SL}_{2,D}}$ over $\mathcal{O}_K$ is a bit less straightforward to define. We give it in the appendix (see Definition 2.96).

Definition 2.39. When the parameter γ associated with L/K is 0, let $\underline{\mathrm{SU}_3}^{L/K}$ be the group SU_3 considered over $\mathcal{O}_K$. We often omit the superscript L/K. Concretely, $\underline{\mathrm{SU}_3}$ is the $\mathcal{O}_K$-scheme associated with the $\mathcal{O}_K$-algebra $\mathcal{O}_K[\underline{\mathrm{SU}_3}] = \mathcal{O}_K[X_{ij}^{kl}]/I$ ($i, j \in \{1,2,3\}$, $k, l \in \{1,2\}$), where I is the ideal generated by the following equations

$$\text{For all } i,j \in \{1,2,3\}, \quad \begin{cases} X_{ij}^{12} = -\beta X_{ij}^{21} \\ X_{ij}^{22} = X_{ij}^{11} + \alpha X_{ij}^{21} \end{cases}$$

$$\sum_{\sigma \in \mathrm{Sym}(3)} [(-1)^{sgn(\sigma)} \prod_{i=1}^{3} X_{i\sigma(i)}] - 1$$

$$\begin{pmatrix} \overline{X}_{33} & \overline{X}_{23} & \overline{X}_{13} \\ \overline{X}_{32} & \overline{X}_{22} & \overline{X}_{12} \\ \overline{X}_{31} & \overline{X}_{21} & \overline{X}_{11} \end{pmatrix} \begin{pmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Here α and β are as in Lemma 2.13, so that the first equations encode the ring embedding $\mathcal{O}_L \hookrightarrow M_2(\mathcal{O}_K)$. Also, for a 2×2 matrix $M = \left(\begin{smallmatrix} M^{11} & M^{12} \\ M^{21} & M^{22} \end{smallmatrix}\right)$, we denote $\overline{M} = \left(\begin{smallmatrix} M^{22} & -M^{12} \\ -M^{21} & M^{11} \end{smallmatrix}\right)$ (this operation reflects the conjugation on $\mathcal{O}_L$). Finally note that a 1 (respectively a 0) in the above equations denotes the 2×2 identity matrix (respectively the 2×2 zero matrix), i.e. it corresponds to the $1 \in L$ (respectively $0 \in L$).

Theorem 2.40. *$\underline{\mathrm{SL}}_{2,D}$ and $\underline{\mathrm{SU}}_3^{L/K}$ (when $\gamma = 0$) are smooth $\mathcal{O}_K$-scheme.*

Proof. Smoothness of $\underline{\mathrm{SL}}_2$ over $\mathcal{O}_K$ (and in fact of $\underline{\mathrm{SL}}_n$ over any ring) is easily checked using the infinitesimal lifting criterion (see [Sta17, Tag 02H6]). The case of $\underline{\mathrm{SL}}_{2,D}$ is discussed in the appendix (see Theorem 2.97).

We now prove the smoothness of $\underline{\mathrm{SU}}_3$. It suffices to prove that it is flat and that the fibres are smooth. The generic fibre is $\mathrm{SU}_3^{L/K}$, and is a form of SL_3, hence is smooth over K. The closed fibre is the $\overline{K}$-functor $(\underline{\mathrm{SU}}_3)_{\overline{K}}$ which associates to any $\overline{K}$-algebra R the group

$$(\underline{\mathrm{SU}}_3)_{\overline{K}}(R) = \{g \in \mathrm{SL}_3(R \otimes_{\overline{K}} \mathcal{O}_L/\mathfrak{m}_L^{\epsilon}) \mid {}^S\bar{g}g = \mathrm{Id}\}$$

where $\epsilon = 1$ if L is unramified, and $\epsilon = 2$ if L is ramified. When L is unramified, this algebraic group becomes isomorphic to SL_3 after base change to $\overline{L}$, and hence is smooth and connected. We now treat the ramified case. Let SO_3 be the special orthogonal group associated with the quadratic form $(x_{-1}, x_0, x_1) \mapsto x_{-1}x_1 + x_0^2$, considered over $\overline{K}$. More explicitly, for a $\overline{K}$-algebra R,

$$(\mathrm{SO}_3)_{\overline{K}}(R) = \{g \in \mathrm{SL}_3(R) \mid {}^S gg = \mathrm{Id}\}$$

Since by assumption $\gamma \neq 0$, the characteristic of $\overline{K}$ is not 2, and it is then well known that SO_3 is isomorphic to PGL_2 over $\overline{K}$, hence is smooth and connected of dimension 3. There exists a homomorphism of algebraic groups $f\colon (\underline{\mathrm{SU}}_3)_{\overline{K}} \to (\mathrm{SO}_3)_{\overline{K}}$ induced by the homomorphism of $\overline{K}$-algebra $\mathcal{O}_L/\mathfrak{m}_L^2 \to \overline{K}$. The kernel of this map can be computed by hand, and we obtain that for any $\overline{K}$-algebra R,

$$\ker f(R) = \{g \in \mathrm{SL}_3(R \otimes_{\overline{K}} \mathcal{O}_L/\mathfrak{m}_L^2) \mid \\ g = \begin{pmatrix} \begin{pmatrix} 1 & 0 \\ g_{11}^{21} & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ g_{12}^{21} & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ g_{13}^{21} & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ g_{21}^{21} & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ -2g_{11}^{21} & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ g_{12}^{21} & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ g_{31}^{21} & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ g_{21}^{21} & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ g_{11}^{21} & 1 \end{pmatrix} \end{pmatrix} \}$$

This description makes it clear that $\ker f$ is of dimension 5 and connected. Hence, using [DG70, II, §5, Proposition 5.1] (note that it does not use smoothness), we conclude that $\dim(\underline{\mathrm{SU}}_3)_{\overline{K}} = 8$. But we can also easily compute that the Lie algebra of $(\underline{\mathrm{SU}}_3)_{\overline{K}}$ is

$$(\mathfrak{su}_3)_{\overline{K}} = \{g \in M_3(\mathcal{O}_L/\mathfrak{m}_L^2) \mid {}^S\bar{g} + g = 0, \mathrm{trace}(g) = 0\}$$

This is readily seen to be of dimension 8 (recall that we are in the case $\gamma = 0$ and L ramified, so that the residue characteristic is not 2), and hence, we conclude that $(\underline{\mathrm{SU}}_3)_{\overline{K}}$ is smooth, as wanted. Also note that the homomorphism $f\colon (\underline{\mathrm{SU}}_3)_{\overline{K}} \to (\mathrm{SO}_3)_{\overline{K}}$ is surjective onto a connected algebraic group, with connected kernel, hence $(\underline{\mathrm{SU}}_3)_{\overline{K}}$ is also connected.

It remains to prove flatness. Since $\mathcal{O}_K$ is a pruferian ring, flatness is equivalent to being without torsion (see [BT84a, 2.2.2]). In other words, to prove flatness, it suffices to prove that $(\underline{\mathrm{SU}}_3)_K$ is dense in $\underline{\mathrm{SU}}_3$. Since we proved that $(\underline{\mathrm{SU}}_3)_{\overline{K}}$ is connected, one can argue as in the conclusion of the proof of Lemma 2.49, when we show that $\mathfrak{Y}_K$ is dense in $\mathfrak{Y}$. $\square$

Remark 2.41. In passing, note that the group $(\underline{\mathrm{SU}}_3^{L/K})_{\overline{K}}$ is not a reductive group over $\overline{K}$ when L is ramified (as predicted by [BT84a, 4.6.31]). In fact, we just showed in the above proof that its reductive quotient is naturally described as the orthogonal group in 3 variables. Again, this might be seen as a reason for the complication of the ramified, residue characteristic 2, since philosophically, it involves orthogonal group in characteristic 2.

Remark 2.42. There is also a more direct way to prove the smoothness of $\underline{\mathrm{SU}}_3$ in the unramified case, since in this case $(\underline{\mathrm{SU}}_3)_{\mathcal{O}_L}$ is isomorphic to SL_3 over $\mathcal{O}_L$. But this does not work in the ramified case. Indeed, if $(\underline{\mathrm{SU}}_3)_{\mathcal{O}_L}$ were isomorphic to SL_3 over $\mathcal{O}_L$ in the ramified case, then its closed fibre $(\underline{\mathrm{SU}}_3)_{\overline{K}}$ would be isomorphic to SL_3 over $\overline{K} \cong \mathcal{O}_L/\mathfrak{m}_L$, which is not true, as we have just seen in the above proof.

We now give the explicit equation of the integral model in the SU_3 case when $\gamma > 0$.

Definition 2.43. Let $K[\mathrm{SU}_3^{L/K}]$ be the standard representation of the coordinate ring of $\mathrm{SU}_3^{L/K}$. More explicitly, $K[\mathrm{SU}_3^{L/K}] = K[X_{ij}^{kl}]/I$ ($i, j \in \{1,2,3\}$, $k, l \in \{1,2\}$), where I is the ideal generated by the equations displayed in Definition 2.39. We also use the ring $\mathcal{O}_K[\mathbf{A}^{36}] = \mathcal{O}_K[X_{ij}^{kl}]$ ($i, j \in \{1,2,3\}$, $k, l \in \{1,2\}$).

Notation 2.44. We use the following notations: $\lambda_k = \begin{pmatrix} \pi_K^{k+1} & 0 \\ 0 & \pi_K^k \end{pmatrix}$, $\upsilon_k = \begin{pmatrix} \pi_K^k & 0 \\ 0 & \pi_K^{k+1} \end{pmatrix}$ and $\tau_k = \begin{pmatrix} \pi_K^k & 0 \\ 0 & \pi_K^k \end{pmatrix}$.

Recall the definition of i_0 in Definition 2.32, and let $n_0 = \lfloor \frac{i_0}{2} \rfloor$. The integral model depends on the parity of i_0. If i_0 is odd, we define the $\mathcal{O}_K$-algebra map

$$\varphi_{i_0} \colon \mathcal{O}_K[\mathbf{A}^{36}] \to K[\mathrm{SU}_3^{L/K}] \colon \begin{pmatrix} Y_{11} & Y_{12} & Y_{13} \\ Y_{21} & Y_{22} & Y_{23} \\ Y_{31} & Y_{32} & Y_{33} \end{pmatrix} \mapsto \begin{pmatrix} X_{11} & X_{12}\lambda_{n_0} & X_{13} \\ \lambda_{n_0}^{-1}X_{21} & \lambda_{n_0}^{-1}X_{22}\lambda_{n_0} & \lambda_{n_0}^{-1}X_{23} \\ X_{31} & X_{32}\lambda_{n_0} & X_{33} \end{pmatrix}$$

while if i_0 is even, we define the $\mathcal{O}_K$-algebra map

$$\varphi_{i_0} \colon \mathcal{O}_K[\mathbf{A}^{36}] \to K[\mathrm{SU}_3^{L/K}] \colon \begin{pmatrix} Z_{11} & Z_{12} & Z_{13} \\ Z_{21} & Z_{22} & Z_{23} \\ Z_{31} & Z_{32} & Z_{33} \end{pmatrix} \mapsto \begin{pmatrix} X_{11} & X_{12}\tau_{n_0} & X_{13} \\ \tau_{n_0}^{-1}X_{21} & X_{22} & \tau_{n_0}^{-1}X_{23} \\ X_{31} & X_{32}\tau_{n_0} & X_{33} \end{pmatrix}$$

Remark 2.45. The above notation for the map φ_{i_0} means that (for example in the i_0 odd case) $\varphi_{i_0}(Y_{11}) = X_{11}$, $\varphi_{i_0}(Y_{21}) = \lambda_{n_0}^{-1}X_{21}$, and so on.

Definition 2.46. Let $\underline{\mathrm{SU}}_3^{L/K}$ be the closed subscheme of $\mathbf{A}^{36}$ (over $\mathcal{O}_K$) defined by the ideal $\ker \varphi_{i_0}$. We often omit the superscript L/K when it is not necessary to insist on the pair of field (K, L) under consideration.

Remark 2.47. Note that φ_{i_0} is just the equation for a base change. Also note that by definition, $\underline{\mathrm{SU}}_3$ is the schematic adherence of SU_3 in $\mathbf{A}^{36}$ (see [BT84a, 1.2.6] for the definition of the schematic adherence). Actually, $\underline{\mathrm{SU}}_3$ is the integral model $\hat{\mathfrak{G}}_\varphi$ associated in the sense of [BT84a, 4.6.26] to the standard valuation of SU_3. The concrete description given here was found following the concrete description given in [BT87], see especially section 3.9 and the Theorem in section 5 in loc. cit. But we provide a concrete proof that $\underline{\mathrm{SU}}_3$ is a smooth $\mathcal{O}_K$-group scheme, without referring to [BT87].

To not lengthen too much the chapter, we now make all arguments when i_0 is odd, the case i_0 even being similar, if not simpler. A first important observation is that $P_0 \cong \underline{\mathrm{SU}}_3(\mathcal{O}_K)$.

Lemma 2.48. *The map φ_{i_0} gives an isomorphism $\mathrm{SU}_3 \to (\underline{\mathrm{SU}}_3)_K$, and the inverse image of $(\underline{\mathrm{SU}}_3)_K(\mathcal{O}_K) \subset \{g \in \mathbf{A}^{36} \mid \omega(g_{ij}^{kl}) \geq 0\}$ is just $\{g \in \mathrm{SU}_3(K) \mid \omega(g) \geq \begin{pmatrix} 0 & -\gamma & 0 \\ \gamma & 0 & \gamma \\ 0 & -\gamma & 0 \end{pmatrix}\}$. In view of Definition 2.18, we indeed have that $P_0 \cong \underline{\mathrm{SU}}_3(\mathcal{O}_K)$.*

Proof. By definition, for $g \in \mathrm{SU}_3(K)$, $\varphi_{i_0}(g)$ is equal to the product $\begin{pmatrix} \mathrm{Id} & 0 & 0 \\ 0 & \lambda_{n_0}^{-1} & 0 \\ 0 & 0 & \mathrm{Id} \end{pmatrix}\begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix}\begin{pmatrix} \mathrm{Id} & 0 & 0 \\ 0 & \lambda_{n_0} & 0 \\ 0 & 0 & \mathrm{Id} \end{pmatrix}$. For example, let us examine what we get for g_{12}. The 2×2 matrix $\begin{pmatrix} g_{12}^{11} & -\beta g_{12}^{12} \\ g_{12}^{21} & g_{12}^{11}+\alpha g_{12}^{21} \end{pmatrix}$ is thus sent to $\begin{pmatrix} \pi_K^{n_0+1} g_{12}^{11} & -\pi_K^{n_0}\beta g_{12}^{12} \\ \pi_K^{n_0+1} g_{12}^{21} & \pi_K^{n_0}(g_{12}^{11}+\alpha g_{12}^{21}) \end{pmatrix}$. All coefficients of this latter matrix are integral if and only if $g_{12}^{11} \in (\pi_K^{-n_0})$ and $g_{12}^{21} \in (\pi_K^{-n_0-1})$. We have $(\pi_K^{-n_0}) = (\pi_L^{-2n_0}) = (\pi_L^{-(i_0-1)})$ and $(\pi_K^{-n_0-1}) = (\pi_L^{-2n_0-2}) = (\pi_L^{-(i_0+1)})$ (recall that we are just treating the case i_0 odd). But by Definition 2.32, i_0 is the smallest integer such that $\omega(\pi_L^{i_0}) \geq \gamma$. Hence, all coefficients of $\begin{pmatrix} \pi_K^{n_0+1} g_{12}^{11} & -\pi_K^{n_0}\beta g_{12}^{12} \\ \pi_K^{n_0+1} g_{12}^{21} & \pi_K^{n_0}(g_{12}^{11}+\alpha g_{12}^{21}) \end{pmatrix}$ are integral if and only if $\omega(g_{12}) \geq -\gamma$. The other cases are similar. □

Lemma 2.49. *The ideal defining $\underline{\mathrm{SU}}_3$ in $\mathbf{A}^{36}$ is generated by the following equations*

1. *If i_0 is odd*

$$\begin{pmatrix} Y_{11} & Y_{12} & Y_{13} \\ Y_{21} & Y_{22} & Y_{23} \\ Y_{31} & Y_{32} & Y_{33} \end{pmatrix}\begin{pmatrix} \tau_{i_0} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \tau_{i_0} \end{pmatrix}\begin{pmatrix} \overline{Y}_{33} & \overline{Y}_{23} & \overline{Y}_{13} \\ \overline{Y}_{32} & \overline{Y}_{22} & \overline{Y}_{12} \\ \overline{Y}_{31} & \overline{Y}_{21} & \overline{Y}_{11} \end{pmatrix} = \begin{pmatrix} \tau_{i_0} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \tau_{i_0} \end{pmatrix} \tag{2.1}$$

$$\begin{pmatrix} \overline{Y}_{33} & \overline{Y}_{23} & \overline{Y}_{13} \\ \overline{Y}_{32} & \overline{Y}_{22} & \overline{Y}_{12} \\ \overline{Y}_{31} & \overline{Y}_{21} & \overline{Y}_{11} \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \tau_{i_0} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} Y_{11} & Y_{12} & Y_{13} \\ Y_{21} & Y_{22} & Y_{23} \\ Y_{31} & Y_{32} & Y_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \tau_{i_0} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.2}$$

$$\frac{2}{\pi_K^{i_0}}(Y_{31}^{11}Y_{11}^{11}+\beta Y_{31}^{21}Y_{11}^{21})+\frac{\alpha}{\pi_K^{i_0}}(Y_{31}^{21}Y_{11}^{11}+Y_{11}^{21}Y_{31}^{11})=-(\overline{Y}_{21}Y_{21})^{11} \tag{2.3}$$

$$\frac{2}{\pi_K^{i_0}}(Y_{13}^{11}Y_{33}^{11}+\beta Y_{13}^{21}Y_{33}^{21})+\frac{\alpha}{\pi_K^{i_0}}(Y_{13}^{21}Y_{33}^{11}+Y_{33}^{21}Y_{13}^{11})=-(\overline{Y}_{23}Y_{23})^{11} \tag{2.4}$$

$$\textit{for } (i,j)\in\{(1,1);(1,3);(3,1);(3,3)\},\begin{cases}Y_{ij}^{12}=-\beta Y_{ij}^{21}\\ Y_{ij}^{22}=Y_{ij}^{11}+\alpha Y_{ij}^{21}\end{cases} \tag{2.5}$$

$$\begin{cases}Y_{21}^{12}=-\frac{\beta}{\pi_K}Y_{21}^{21}\\ Y_{21}^{22}=\pi_K Y_{21}^{11}+\alpha Y_{21}^{21}\end{cases}\qquad\begin{cases}Y_{23}^{12}=-\frac{\beta}{\pi_K}Y_{23}^{21}\\ Y_{23}^{22}=\pi_K Y_{23}^{11}+\alpha Y_{23}^{21}\end{cases} \tag{2.6}$$

$$\begin{cases}Y_{12}^{12}=-\frac{\beta}{\pi_K}Y_{12}^{21}\\ \pi_K Y_{12}^{22}=Y_{12}^{11}+\alpha Y_{12}^{21}\end{cases}\qquad\begin{cases}Y_{32}^{12}=-\frac{\beta}{\pi_K}Y_{32}^{21}\\ \pi_K Y_{32}^{22}=Y_{32}^{11}+\alpha Y_{32}^{21}\end{cases} \tag{2.7}$$

$$\begin{cases}\pi_K Y_{22}^{12}=-\frac{\beta}{\pi_K}Y_{22}^{21}\\ Y_{22}^{22}=Y_{22}^{11}+\frac{\alpha}{\pi_K}Y_{22}^{21}\end{cases} \tag{2.8}$$

$$\begin{aligned}&\lambda_0Y_{22}\lambda_0^{-1}Y_{11}Y_{22}+Y_{12}Y_{23}Y_{31}+Y_{13}Y_{32}Y_{21}\\ &-\lambda_0Y_{22}\lambda_0^{-1}Y_{31}Y_{13}-Y_{11}Y_{32}Y_{23}-Y_{33}Y_{12}Y_{21}=1\end{aligned} \tag{2.9}$$

2. *If i_0 is even*

$$\begin{pmatrix}Z_{11}&Z_{12}&Z_{13}\\Z_{21}&Z_{22}&Z_{23}\\Z_{31}&Z_{32}&Z_{33}\end{pmatrix}\begin{pmatrix}\tau_{i_0}&0&0\\0&1&0\\0&0&\tau_{i_0}\end{pmatrix}\begin{pmatrix}\overline{Z}_{33}&\overline{Z}_{23}&\overline{Z}_{13}\\\overline{Z}_{32}&\overline{Z}_{22}&\overline{Z}_{12}\\\overline{Z}_{31}&\overline{Z}_{21}&\overline{Z}_{11}\end{pmatrix}=\begin{pmatrix}\tau_{i_0}&0&0\\0&1&0\\0&0&\tau_{i_0}\end{pmatrix} \tag{2.10}$$

$$\begin{pmatrix}\overline{Z}_{33}&\overline{Z}_{23}&\overline{Z}_{13}\\\overline{Z}_{32}&\overline{Z}_{22}&\overline{Z}_{12}\\\overline{Z}_{31}&\overline{Z}_{21}&\overline{Z}_{11}\end{pmatrix}\begin{pmatrix}1&0&0\\0&\tau_{i_0}&0\\0&0&1\end{pmatrix}\begin{pmatrix}Z_{11}&Z_{12}&Z_{13}\\Z_{21}&Z_{22}&Z_{23}\\Z_{31}&Z_{32}&Z_{33}\end{pmatrix}=\begin{pmatrix}1&0&0\\0&\tau_{i_0}&0\\0&0&1\end{pmatrix} \tag{2.11}$$

$$\frac{2}{\pi_K^{i_0}}(Z_{31}^{11}Z_{11}^{11}+\beta Z_{31}^{21}Z_{11}^{21})+\frac{\alpha}{\pi_K^{i_0}}(Z_{31}^{21}Z_{11}^{11}+Z_{11}^{21}Z_{31}^{11})=-(\overline{Z}_{21}Z_{21})^{11} \tag{2.12}$$

$$\frac{2}{\pi_K^{i_0}}(Z_{13}^{11}Z_{33}^{11}+\beta Z_{13}^{21}Z_{33}^{21})+\frac{\alpha}{\pi_K^{i_0}}(Z_{13}^{21}Z_{33}^{11}+Z_{33}^{21}Z_{13}^{11})=-(\overline{Z}_{23}Z_{23})^{11} \tag{2.13}$$

$$\textit{for all } i,j\in\{1,2,3\},\begin{cases}Z_{ij}^{12}=-\beta Z_{ij}^{21}\\ Z_{ij}^{22}=Z_{ij}^{11}+\alpha Z_{ij}^{21}\end{cases} \tag{2.14}$$

$$\sum_{\sigma \in \mathrm{Sym}(3)} [(-1)^{sgn(\sigma)} \prod_{i=1}^{3} Z_{i\sigma(i)}] = 1 \tag{2.15}$$

Proof. Recall that we only write down the case i_0 odd. Let I be the ideal in $\mathcal{O}_K[\mathbf{A}^{36}]$ generated by those equations. We want to show that $I = \ker \varphi_{i_0}$ (see Definition 2.46).

Claim 1. $I \leq \ker \varphi_{i_0}$

Proof of the claim: This can easily be checked equation by equation. For example, $\varphi_{i_0}(\lambda_{n_0} Y_{21}) = X_{21}$, hence $\varphi_{i_0}(\overline{Y}_{21} \upsilon_{n_0}) = \overline{X}_{21}$. Hence, $\varphi_{i_0}^{-1}$ of the equalities

$$\begin{pmatrix} \overline{X}_{33} & \overline{X}_{23} & \overline{X}_{13} \\ \overline{X}_{32} & \overline{X}_{22} & \overline{X}_{12} \\ \overline{X}_{31} & \overline{X}_{21} & \overline{X}_{11} \end{pmatrix} \begin{pmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

in $K[\mathrm{SU}_3]$ gives the equalities

$$\begin{pmatrix} \overline{Y}_{33} & \overline{Y}_{23}\upsilon_{n_0} & \overline{Y}_{13} \\ \upsilon_{n_0}^{-1}\overline{Y}_{32} & \upsilon_{n_0}^{-1}\overline{Y}_{22}\upsilon_{n_0} & \upsilon_{n_0}^{-1}\overline{Y}_{12} \\ \overline{Y}_{31} & \overline{Y}_{21}\upsilon_{n_0} & \overline{Y}_{11} \end{pmatrix} \begin{pmatrix} Y_{11} & Y_{12}\lambda_{n_0}^{-1} & Y_{13} \\ \lambda_{n_0} Y_{21} & \lambda_{n_0} Y_{22} \lambda_{n_0}^{-1} & \lambda_{n_0} Y_{23} \\ Y_{31} & Y_{32}\lambda_{n_0}^{-1} & Y_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Now, multiplying with $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda_0^{-1}\tau_{n_0+1} & 0 \\ 0 & 0 & 1 \end{pmatrix}$ on the left and $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \tau_{n_0+1}\upsilon_0^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}$ on the right, we get $\begin{pmatrix} \overline{Y}_{33} & \overline{Y}_{23}\upsilon_{n_0} & \overline{Y}_{13} \\ \overline{Y}_{32} & \overline{Y}_{22}\upsilon_{n_0} & \overline{Y}_{12} \\ \overline{Y}_{31} & \overline{Y}_{21}\upsilon_{n_0} & \overline{Y}_{11} \end{pmatrix} \begin{pmatrix} Y_{11} & Y_{12} & Y_{13} \\ \lambda_{n_0} Y_{21} & \lambda_{n_0} Y_{22} & \lambda_{n_0} Y_{23} \\ Y_{31} & Y_{32} & Y_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \tau_{i_0} & 0 \\ 0 & 0 & 1 \end{pmatrix}$, which is Equation 2.2.

As another example, from Equation 2.2, we get in particular

$$\overline{Y}_{33} Y_{13} + \tau_{i_0} \overline{Y}_{23} Y_{23} + \overline{Y}_{13} Y_{33} = 0 \tag{2.2,L1C3}$$

The 11 component of Equation 2.2,L1C3 reads

$$Y_{33}^{22} Y_{13}^{11} - Y_{33}^{12} Y_{13}^{21} + Y_{13}^{22} Y_{33}^{11} - Y_{13}^{12} Y_{33}^{21} = -\tau_{i_0} (\overline{Y}_{23} Y_{23})^{11}$$

Now, using $Y_{33}^{22} = Y_{33}^{11} + \alpha Y_{33}^{21}$ and $Y_{33}^{12} = -\beta Y_{33}^{21}$ (and similarly for Y_{13}), we get

$$2(Y_{33}^{11} Y_{13}^{11} + \beta Y_{33}^{21} Y_{13}^{21}) + \alpha (Y_{33}^{21} Y_{13}^{11} + Y_{13}^{21} Y_{33}^{11}) = -\tau_{i_0} (\overline{Y}_{23} Y_{23})^{11}$$

But note that all coefficients in this equation have valuation greater than or equal to $\omega(\pi_K^{i_0})$ (because $\omega(\pi_K^{i_0}) = \begin{cases} \omega(2) \text{ if } \alpha = 0 \\ \omega(\alpha) \text{ if } \alpha \neq 0 \end{cases}$ by Definition 2.32, and if $\alpha \neq 0$, $\omega(\alpha) \leq \omega(2)$ by Lemma 2.13). Hence we can divide both sides by $\pi_K^{i_0}$ and still have an equation with coefficients in $\mathcal{O}_K$. Checking the other equations is a similar task. ■

Let $\mathfrak{Y}$ be the closed $\mathcal{O}_K$-subscheme of $\mathbf{A}^{36}$ defined by the the ideal I. By Claim 1, $\underline{\mathrm{SU}}_3$ is a closed subscheme of $\mathfrak{Y}$, and we want to prove that they are equal. The crux of the proof relies on investigating the closed fibre of the $\mathcal{O}_K$-scheme $\mathfrak{Y}$, or in other words, the scheme $\mathfrak{Y}_{\overline{K}}$ over $\overline{K}$. As it will be needed later, we elucidate what $\mathfrak{Y}_{\mathcal{O}_K/\mathfrak{m}_K^{i_0}}$ looks like, and then deduce what we want about $\mathfrak{Y}_{\overline{K}}$.

Claim 2. *$\mathfrak{Y}_{\mathcal{O}_K/\mathfrak{m}_K^{i_0}}$ is the following $\mathcal{O}_K/\mathfrak{m}_K^{i_0}$-algebraic group: for any $\mathcal{O}_K/\mathfrak{m}_K^{i_0}$-algebra R, $\mathfrak{Y}_{\mathcal{O}_K/\mathfrak{m}_K^{i_0}}(R) =$*

$$\left\{ \begin{array}{l} \left(\begin{array}{ccc} \begin{pmatrix} w_{11}^{11} & -\beta w_{11}^{21} \\ w_{11}^{21} & w_{11}^{11} \end{pmatrix} & \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right) & \begin{pmatrix} w_{13}^{11} & -\beta w_{13}^{21} \\ w_{13}^{21} & w_{13}^{11} \end{pmatrix} \\ \begin{pmatrix} w_{21}^{11} & -\frac{\beta}{\pi_K} w_{21}^{21} \\ w_{21}^{21} & \pi_K w_{21}^{11} \end{pmatrix} & \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) & \begin{pmatrix} w_{23}^{11} & -\frac{\beta}{\pi_K} w_{23}^{21} \\ w_{23}^{21} & \pi_K w_{23}^{11} \end{pmatrix} \\ \begin{pmatrix} w_{31}^{11} & -\beta w_{31}^{21} \\ w_{31}^{21} & w_{31}^{11} \end{pmatrix} & \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right) & \begin{pmatrix} w_{33}^{11} & -\beta w_{33}^{21} \\ w_{33}^{21} & w_{33}^{11} \end{pmatrix} \end{array} \right) \Bigg| \; w_{ij} \in R, w_{11}w_{33} - w_{13}w_{31} = 1 \\ \frac{2}{\pi_K^{i_0}}(w_{31}^{11}w_{11}^{11} + \beta w_{31}^{21}w_{11}^{21}) + \frac{\alpha}{\pi_K^{i_0}}(w_{31}^{21}w_{11}^{11} + w_{11}^{21}w_{31}^{11}) = -(\overline{w}_{21}w_{21})^{11} \\ \frac{2}{\pi_K^{i_0}}(w_{13}^{11}w_{33}^{11} + \beta w_{13}^{21}w_{33}^{21}) + \frac{\alpha}{\pi_K^{i_0}}(w_{13}^{21}w_{33}^{11} + w_{33}^{21}w_{13}^{11}) = -(\overline{w}_{23}w_{23})^{11} \end{array} \right\} \tag{$*$}$$

where the group structure is the one coming from the representation of elements as forming a 3×3 matrix.

Proof of the claim: We have to analyse our equations modulo $\pi_K^{i_0}$, or in other words, work in the ring $(\mathcal{O}_K/\mathfrak{m}_K^{i_0})[Y_{ij}^{kl}]/I$. In particular, in view of Definition 2.32 and Lemma 2.13, we are now working in characteristic 2.

From Equation 2.1, we get $Y_{22}\overline{Y}_{22} = 1$, so that in particular, Y_{22} and $\overline{Y}_{22}$ are invertible matrices. Still from Equation 2.1, we also have $Y_{22}\overline{Y}_{32} = 0 = Y_{22}\overline{Y}_{12}$. Hence, $\overline{Y}_{32} = 0 = \overline{Y}_{12}$, so that also $Y_{32} = 0 = Y_{12}$. This implies that Equation 2.9 simplifies to

$$\lambda_0 Y_{22} \lambda_0^{-1}(Y_{11}Y_{33} - Y_{31}Y_{13}) = 1$$

On the other hand, Equation 2.2 gives $\overline{Y}_{33}Y_{11} + \overline{Y}_{13}Y_{31} = 1$. But $\overline{Y}_{33} = Y_{33}$ and $\overline{Y}_{13} = Y_{13}$ (which follows from Equation 2.5 and the fact that the characteristic is 2). Hence, we conclude that $\lambda_0 Y_{22}\lambda_0^{-1} = 1$, and hence that $Y_{22} = 1$ (using Equation 2.8). Combining what we know so far with Equations 2.5 and 2.6, we get the claim. ∎

Let $\mathcal{R}\,\mathrm{SL}_2$ be the Weil restriction from $\mathcal{O}_L/\mathfrak{m}_L^{2i_0}$ to $\mathcal{O}_K/\mathfrak{m}_K^{i_0}$ of the algebraic group SL_2. In more concrete terms, for any $\mathcal{O}_K/\mathfrak{m}_K^{i_0}$-algebra R,

$$\mathcal{R}\,\mathrm{SL}_2(R) = \{\begin{pmatrix} \begin{pmatrix} w_{11}^{11} & -\beta w_{11}^{21} \\ w_{11}^{21} & w_{11}^{11} \end{pmatrix} & \begin{pmatrix} w_{13}^{11} & -\beta w_{13}^{21} \\ w_{13}^{21} & w_{13}^{11} \end{pmatrix} \\ \begin{pmatrix} w_{31}^{11} & -\beta w_{31}^{21} \\ w_{31}^{21} & w_{31}^{11} \end{pmatrix} & \begin{pmatrix} w_{33}^{11} & -\beta w_{33}^{21} \\ w_{33}^{21} & w_{33}^{11} \end{pmatrix} \end{pmatrix} \mid w_{ij} \in R\}$$

Claim 3. *For any $\mathcal{O}_K/\mathfrak{m}_K^{i_0}$-algebra R, there exists a (functorial in R) group homomorphism*

$$f_R \colon \mathfrak{Y}_{\mathcal{O}_K/\mathfrak{m}_K^{i_0}}(R) \to \mathcal{R}\,\mathrm{SL}_2(R) \colon$$
$$\begin{pmatrix} \begin{pmatrix} w_{11}^{11} & -\beta w_{11}^{21} \\ w_{11}^{21} & w_{11}^{11} \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} w_{13}^{11} & -\beta w_{13}^{21} \\ w_{13}^{21} & w_{13}^{11} \end{pmatrix} \\ \begin{pmatrix} w_{21}^{11} & -\frac{\beta}{\pi_K} w_{21}^{21} \\ w_{21}^{21} & \pi_K w_{21}^{11} \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} w_{23}^{11} & -\frac{\beta}{\pi_K} w_{23}^{21} \\ w_{23}^{21} & \pi_K w_{23}^{11} \end{pmatrix} \\ \begin{pmatrix} w_{31}^{11} & -\beta w_{31}^{21} \\ w_{31}^{21} & w_{31}^{11} \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} w_{33}^{11} & -\beta w_{33}^{21} \\ w_{33}^{21} & w_{33}^{11} \end{pmatrix} \end{pmatrix} \mapsto \begin{pmatrix} \begin{pmatrix} w_{11}^{11} & -\beta w_{11}^{21} \\ w_{11}^{21} & w_{11}^{11} \end{pmatrix} & \begin{pmatrix} w_{13}^{11} & -\beta w_{13}^{21} \\ w_{13}^{21} & w_{13}^{11} \end{pmatrix} \\ \begin{pmatrix} w_{31}^{11} & -\beta w_{31}^{21} \\ w_{31}^{21} & w_{31}^{11} \end{pmatrix} & \begin{pmatrix} w_{33}^{11} & -\beta w_{33}^{21} \\ w_{33}^{21} & w_{33}^{11} \end{pmatrix} \end{pmatrix}$$

Furthermore, $f_{\mathcal{O}_K/\mathfrak{m}_K^{i_0}}$ is surjective.

Proof of the claim: The map f_R is readily seen to be a group homomorphism. Let us check that $f_{\mathcal{O}_K/\mathfrak{m}_K^{i_0}}$ is surjective. On the left hand side, apart from a determinant-like equation, we have also equations like $\frac{2}{\pi_K^{i_0}}(w_{31}^{11}w_{11}^{11} + \beta w_{31}^{21}w_{11}^{21}) + \frac{\alpha}{\pi_K^{i_0}}(w_{31}^{21}w_{11}^{11} + w_{11}^{21}w_{31}^{11}) = -(\overline{w}_{21}w_{21})^{11}$. But note that $(\overline{w}_{21}w_{21})^{11} = \pi_K(w_{21}^{11})^2 + (w_{21}^{21})^2$. Since squaring is a surjective map on $\mathcal{O}_K/\mathfrak{m}_K$, we see that for any $x \in \mathcal{O}_K/\mathfrak{m}_K^{i_0}$, there exists $w_{21}^{11}, w_{21}^{21} \in \mathcal{O}_K/\mathfrak{m}_K^{i_0}$ such that $x = \pi_K(w_{21}^{11})^2 + (w_{21}^{21})^2$. The surjectivity follows. ∎

Claim 4. *The $\overline{K}$-group $\mathfrak{Y}_{\overline{K}}$ is smooth of dimension* 8, *and is connected.*

Proof of the claim: Smoothness and the computation of the dimension over $\overline{K}$ follows directly from Claim 2. Indeed, one can see directly that the dimension is in between 8 and 12. But the tangent space at the identity is obviously of codimension 4 in $\mathbf{A}^{12}_{\overline{K}}$, as wanted.

We now prove that $\mathfrak{Y}_{\overline{K}}$ is connected. We have a morphism

$$f\colon \mathfrak{Y}_{\overline{K}} \to (\mathrm{SL}_2)_{\overline{K}} \colon \begin{pmatrix} \begin{pmatrix} w_{11}^{11} & 0 \\ w_{11}^{21} & w_{11}^{11} \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} w_{13}^{11} & 0 \\ w_{13}^{21} & w_{13}^{11} \end{pmatrix} \\ \begin{pmatrix} w_{21}^{11} & w_{21}^{21} \\ w_{21}^{21} & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} w_{23}^{11} & w_{23}^{21} \\ w_{23}^{21} & 0 \end{pmatrix} \\ \begin{pmatrix} w_{31}^{11} & 0 \\ w_{31}^{21} & w_{31}^{11} \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} w_{33}^{11} & 0 \\ w_{33}^{21} & w_{33}^{11} \end{pmatrix} \end{pmatrix} \mapsto \begin{pmatrix} w_{11}^{11} & w_{13}^{11} \\ w_{31}^{11} & w_{33}^{11} \end{pmatrix}$$

which is surjective. The kernel is

$$\left\{ \begin{matrix} \begin{pmatrix} \begin{pmatrix} 1 & 0 \\ w_{11}^{21} & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ w_{13}^{21} & 0 \end{pmatrix} \\ \begin{pmatrix} w_{21}^{11} & w_{21}^{21} \\ w_{21}^{21} & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} w_{23}^{11} & w_{23}^{21} \\ w_{23}^{21} & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ w_{31}^{21} & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ w_{33}^{21} & 1 \end{pmatrix} \end{pmatrix} & \Bigg| & w_{11}^{21} + w_{33}^{21} = 0, \\ (w_{21}^{21})^2 = \frac{\alpha}{\pi_K^{i_0}} w_{31}^{21}, (w_{23}^{21})^2 = \frac{\alpha}{\pi_K^{i_0}} w_{13}^{21} & & \end{matrix} \right\}$$

which is clearly a product of connected schemes, hence is connected. So $f\colon \mathfrak{Y}_{\overline{K}} \to (\mathrm{SL}_2)_{\overline{K}}$ is a surjective morphism, whose kernel and image are connected. Hence, $\mathfrak{Y}_{\overline{K}}$ is connected. ■

Claim 5. *We have* $(\underline{\mathrm{SU}}_3)_K = \mathfrak{Y}_K$*, and* $(\underline{\mathrm{SU}}_3)_{\overline{K}} = \mathfrak{Y}_{\overline{K}}$*.*

Proof of the claim: Over K, we have a composition of closed embeddings $\mathrm{SU}_3 \hookrightarrow (\underline{\mathrm{SU}}_3)_K \hookrightarrow \mathfrak{Y}_K$. But $\mathrm{SU}_3 \hookrightarrow \mathfrak{Y}_K$ is clearly an isomorphism, hence the claim. We now prove the equality of the closed fibre, i.e. $(\underline{\mathrm{SU}}_3)_{\overline{K}} = \mathfrak{Y}_{\overline{K}}$. Our argument is based on [Ele]. By Claim 4, $\mathfrak{Y}_{\overline{K}}$ is a smooth irreducible affine $\overline{K}$-scheme of dimension 8. But $(\underline{\mathrm{SU}}_3)_{\overline{K}}$ is a closed subscheme of $\mathfrak{Y}_{\overline{K}}$ of the same dimension. Hence $(\underline{\mathrm{SU}}_3)_{\overline{K}} = \mathfrak{Y}_{\overline{K}}$. Indeed, the $\overline{K}$-group scheme $\mathfrak{Y}_{\overline{K}}$ is connected (respectively smooth) by Claim 4, thus irreducible (respectively reduced). Hence $\overline{K}[\mathfrak{Y}_{\overline{K}}]$ is a domain. But the kernel of $\overline{K}[\mathfrak{Y}_{\overline{K}}] \twoheadrightarrow \overline{K}[(\underline{\mathrm{SU}}_3)_{\overline{K}}]$ is contained in the nilradical of $\overline{K}[\mathfrak{Y}_{\overline{K}}]$ (by Krull's principal ideal theorem), which shows that $\overline{K}[\mathfrak{Y}_{\overline{K}}] \twoheadrightarrow \overline{K}[(\underline{\mathrm{SU}}_3)_{\overline{K}}]$ is injective as well (because being a domain, $\overline{K}[\mathfrak{Y}_{\overline{K}}]$ has in particular a trivial nilradical). ■

We can now conclude the proof of Lemma 2.49. We have some closed embeddings $\underline{\mathrm{SU}}_3 \hookrightarrow \mathfrak{Y} \hookrightarrow \mathbf{A}^{36}$, and by Claim 5, $\underline{\mathrm{SU}}_3 \hookrightarrow \mathfrak{Y}$ is an equality on fibres. Hence, since $(\underline{\mathrm{SU}}_3)_K$ is dense in $\underline{\mathrm{SU}}_3 = (\underline{\mathrm{SU}}_3)_K \sqcup (\underline{\mathrm{SU}}_3)_{\overline{K}}$ (because $\underline{\mathrm{SU}}_3$ is a schematic adherence, see [BT84a, 1.2.6]), we conclude that $\mathfrak{Y}_K$ is dense in $\mathfrak{Y} = \mathfrak{Y}_K \sqcup \mathfrak{Y}_{\overline{K}}$ as well. In the terminology of [BT84a, 1.2.3], this precisely means that $\mathfrak{Y}$ is without torsion. But there is a $1-1$ correspondence between closed K-schemes of $\mathbf{A}^{36}_K$ and closed $\mathcal{O}_K$-schemes of $\mathbf{A}^{36}_{\mathcal{O}_K}$ without torsion ([BT84a, 1.2.6]). Since $(\underline{\mathrm{SU}}_3)_K = \mathfrak{Y}_K$, this concludes the proof. □

For $G = \mathrm{SL}_2$ or SU_3, we have just defined an integral model $\underline{G}$. We now check that in each case, $\underline{G}(\mathcal{O}_K) \cong P_0$.

Lemma 2.50. *1.* $\underline{\mathrm{SL}}_{2,D}(\mathcal{O}_K) \cong P_0$

2. When $\gamma = 0$*,* $\underline{\mathrm{SU}}_3(\mathcal{O}_K) \cong P_0$

3. When $\gamma > 0$*,* $\underline{\mathrm{SU}}_3(\mathcal{O}_K) \cong P_0$

Proof. 1. When $D = K$, $\underline{\mathrm{SL}}_2(\mathcal{O}_K) = \mathrm{Mor}_{\mathcal{O}_K}(\mathcal{O}_K[\underline{\mathrm{SL}}_2], \mathcal{O}_K)$ by definition, which is clearly isomorphic to $\mathrm{SL}_2(\mathcal{O}_K)$. The case $\underline{\mathrm{SL}}_{2,D}$ when $[D : K] > 1$ is done in the appendix (see Lemma 2.98).

2. Using the fact that $\mathcal{O}_L \cong \mathcal{O}_K \oplus t.\mathcal{O}_K$ (where $t \in \mathcal{O}_L$ is as in Lemma 2.13), one can check that $\mathrm{Mor}_{\mathcal{O}_K}(\mathcal{O}_K[\underline{\mathrm{SU}}_3], \mathcal{O}_K) \cong \{g \in \mathrm{SL}_3(\mathcal{O}_L) \mid {}^S\bar{g}g = \mathrm{Id}\}$, as wanted.

3. This has already been proved in Lemma 2.48. □

We now spell out what the group $\underline{G}(\mathcal{O}_K/\mathfrak{m}_K^r)$ is, together with the homomorphism $p_r \colon P_0 \to P_0^{0,r}$.

Lemma 2.51. *1.* $\underline{\mathrm{SL}}_2(\mathcal{O}_K/\mathfrak{m}_K^r) \cong P_0^{0,r}$*. Following the identifications*

$$\begin{array}{ccc} \underline{\mathrm{SL}}_2(\mathcal{O}_K) & \cong \mathrm{SL}_2(\mathcal{O}_K) & = P_0 \\ \downarrow & \downarrow & \\ \underline{\mathrm{SL}}_2(\mathcal{O}_K/\mathfrak{m}_K^r) & \cong \mathrm{SL}_2(\mathcal{O}_K/\mathfrak{m}_K^r) & = P_0^{0,r} \end{array}$$

the homomorphism $p_r \colon P_0 \to P_0^{0,r}$ *is the one induced by the projection of the coefficients* $\mathcal{O}_K \to \mathcal{O}_K/\mathfrak{m}_K^r$*.*

2. *More generally, for D a central division algebra of degree d over K, $\underline{\mathrm{SL}}_{2,D}(\mathcal{O}_K/\mathfrak{m}_K^r) \cong P_0^{0,rd}$. Following the identifications*

$$\begin{array}{ccc} \underline{\mathrm{SL}}_{2,D}(\mathcal{O}_K) & \cong \mathrm{SL}_2(\mathcal{O}_D) = P_0 \\ \downarrow & \downarrow \\ \underline{\mathrm{SL}}_{2,D}(\mathcal{O}_K/\mathfrak{m}_K^r) & \cong \mathrm{SL}_2(\mathcal{O}_D/\mathfrak{m}_D^{rd}) = P_0^{0,rd} \end{array}$$

the homomorphism $p_{rd}\colon P_0 \to P_0^{0,rd}$ is the one induced by the projection of the coefficients $\mathcal{O}_D \to \mathcal{O}_D/\mathfrak{m}_D^{rd}$.

3. *When $\gamma = 0$, let $\epsilon = \begin{cases} 1 \text{ if } L \text{ is unramified} \\ 2 \text{ if } L \text{ is ramified} \end{cases}$. $\underline{\mathrm{SU}}_3(\mathcal{O}_K/\mathfrak{m}_K^r) \cong P_0^{0,\epsilon r}$. Following the identifications*

$$\begin{array}{ccc} \underline{\mathrm{SU}}_3(\mathcal{O}_K) & \cong P_0 & \leq \mathrm{SL}_3(\mathcal{O}_L) \\ \downarrow & \downarrow & \\ \underline{\mathrm{SU}}_3(\mathcal{O}_K/\mathfrak{m}_K^r) & \cong P_0^{0,\epsilon r} & \leq \mathrm{SL}_3(\mathcal{O}_L/\mathfrak{m}_L^{\epsilon r}) \end{array}$$

the homomorphism $p_{\epsilon r}\colon P_0 \to P_0^{0,\epsilon r}$ is the one induced by the projection of the coefficients $\mathcal{O}_L \to \mathcal{O}_L/\mathfrak{m}_L^{\epsilon r}$.

4. *When $\gamma > 0$, there exists a surjective homomorphism of groups $\underline{\mathrm{SU}}_3(\mathcal{O}_K/\mathfrak{m}_K^{i_0}) \twoheadrightarrow P_0^{0,2i_0}$. For $r \leq 2i_0$, we thus have the following diagram*

$$\begin{array}{ccc} \underline{\mathrm{SU}}_3(\mathcal{O}_K) & \cong P_0 & \leq \mathrm{SL}_3(L) \\ \downarrow & \downarrow f_1 & \\ \underline{\mathrm{SU}}_3(\mathcal{O}_K/\mathfrak{m}_K^{i_0}) & \twoheadrightarrow P_0^{0,2i_0} & = \mathrm{SL}_2(\mathcal{O}_L/\mathfrak{m}_L^{2i_0}) \\ & \downarrow f_2 & \\ & P_0^{0,r} & = \mathrm{SL}_2(\mathcal{O}_L/\mathfrak{m}_L^r) \end{array}$$

where f_2 is induced by the ring homomorphism $\mathcal{O}_L/\mathfrak{m}_L^{2i_0} \to \mathcal{O}_L/\mathfrak{m}_L^r$. The resulting homomorphism $p_r\colon P_0 \to P_0^{0,r}$ is given by the following formula:

$$f_2 \circ f_1\colon P_0 \leq \mathrm{SL}_3(L) \to \mathrm{SL}_2(\mathcal{O}_L/\mathfrak{m}_L^r)$$

$$\begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix} \mapsto \begin{pmatrix} p(g_{11}) & p(g_{13}) \\ p(g_{31}) & p(g_{33}) \end{pmatrix}$$

(*where* $p\colon \mathcal{O}_L \to \mathcal{O}_L/\mathfrak{m}_L^r$ *denotes the projection modulo* $\mathfrak{m}_L^r$).

Proof. 1. By definition, $\underline{\mathrm{SL}}_2(\mathcal{O}_K/\mathfrak{m}_K^r) = \mathrm{Mor}_{\mathcal{O}_K}(\mathcal{O}_K[\underline{\mathrm{SL}}_2], \mathcal{O}_K/\mathfrak{m}_K^r)$, which is clearly isomorphic to $\mathrm{SL}_2(\mathcal{O}_K/\mathfrak{m}_K^r)$, as wanted.

2. This is treated in the appendix (see Lemma 2.98).

3. Using the fact that if L is unramified (respectively L is ramified), $\mathcal{O}_L/\mathfrak{m}_L^r \cong \mathcal{O}_K/\mathfrak{m}_K^r \oplus t.\mathcal{O}_K/\mathfrak{m}_K^r$ (respectively $\mathcal{O}_L/\mathfrak{m}_L^{2r} \cong \mathcal{O}_K/\mathfrak{m}_K^r \oplus t.\mathcal{O}_K/\mathfrak{m}_K^r$), one can check that

$$\mathrm{Mor}_{\mathcal{O}_K}(\mathcal{O}_K[\underline{\mathrm{SU}}_3], \mathcal{O}_K/\mathfrak{m}_K^r) \cong \{g \in \mathrm{SL}_3(\mathcal{O}_L/\mathfrak{m}_L^r) \mid {}^S\bar{g}g = \mathrm{Id}\}$$
(respectively
$$\mathrm{Mor}_{\mathcal{O}_K}(\mathcal{O}_K[\underline{\mathrm{SU}}_3], \mathcal{O}_K/\mathfrak{m}_K^r) \cong \{g \in \mathrm{SL}_3(\mathcal{O}_L/\mathfrak{m}_L^{2r}) \mid {}^S\bar{g}g = \mathrm{Id}\})$$

as wanted.

4. Recall the definition of the Weil restriction $\mathcal{R}\,\mathrm{SL}_2$ of SL_2 from $\mathcal{O}_L/\mathfrak{m}_L^{2i_0}$ to $\mathcal{O}_K/\mathfrak{m}_K^{i_0}$ that we discussed before Claim 3 in the proof of Lemma 2.49. Note that $\mathcal{R}\,\mathrm{SL}_2(\mathcal{O}_K/\mathfrak{m}_K^{i_0}) \cong \mathrm{SL}_2(\mathcal{O}_L/\mathfrak{m}_L^{2i_0})$. Now, the existence of a surjective homomorphism $\underline{\mathrm{SU}}_3(\mathcal{O}_K/\mathfrak{m}_K^{i_0}) \to P_0^{0,r}$ was proved in Claim 3. □

Our work on integral models, and especially the fact that they are smooth schemes over $\mathcal{O}_K$, allows us to deduce the surjectivity of $P_0 \to P_0^{0,r}$. For this, we use a well-known generalised version of Hensel's lemma for smooth schemes, that we now recall.

Theorem 2.52 (Hensel's lemma for smooth scheme)**.** *Let X be a smooth $\mathcal{O}_K$-scheme and $r_1 \geq r_2 \in \mathbf{N} \cup \{\infty\}$. Then $X(\mathcal{O}_K/\mathfrak{m}_K^{r_1}) \to X(\mathcal{O}_K/\mathfrak{m}_K^{r_2})$ is surjective* (*where by convention,* $\mathfrak{m}_K^\infty = (0)$).

Proof. It suffices to prove that for all $r \in \mathbf{N}$, $X(\mathcal{O}_K) \to X(\mathcal{O}_K/\mathfrak{m}_K^r)$ is surjective. For $r = 1$, this is [Gro67, Théorème 18.5.17]. In the general case, note that as remarked below [Gro67, Définition 18.5.5], (S, S_0) is a Henselian couple if and only if $(S_{red}, (S_0)_{red})$ is so. We deduce that $(\operatorname{Spec}\mathcal{O}_K, \operatorname{Spec}\mathcal{O}_K/\mathfrak{m}_K^r)$ is a Henselian couple. Thus the proof of Théorème 18.5.17 applies verbatim to our situation, upon making one change: replace the reference to 18.5.11(b) to a reference to 18.5.4(b) (taking $S = \operatorname{Spec}\mathcal{O}_K$ and $S_0 = \operatorname{Spec}\mathcal{O}_K/\mathfrak{m}_K^r$ in the notation of 18.5.4). □

Corollary 2.53. *1. In the* $\mathrm{SL}_2(D)$ *case, let* d *be the degree of* D *over* K*. The map* $p_r\colon P_0 \to P_0^{0,rd}$ *is surjective, for all* $r \in \mathbf{N}$*.*

2. In the SU_3 *case when* $\gamma = 0$*, the map* $p_{\epsilon r}\colon P_0 \to P_0^{0,\epsilon r}$ *is surjective, for all* $r \in \mathbf{N}$ *(where* $\epsilon = 1$ *if* L *is unramified, and* $\epsilon = 2$ *if* L *is ramified).*

3. In the SU_3 *case when* $\gamma > 0$*, the map* $p_r\colon P_0 \to P_0^{0,r}$ *is surjective, for all* $r \in \mathbf{N}$ *such that* $r \leq 2i_0$*.*

Proof. In each case, this is a direct consequence of the commutative square involving $P_0 \to P_0^{0,r}$ given in Lemma 2.51, together with the fact that the integral model is smooth, so that Theorem 2.52 applies to the left hand side of the diagram.

In the SU_3 case when $\gamma > 0$, we furthermore have to argue that the map f_2 appearing in Lemma 2.51 is surjective, but this is just another instance of Hensel's Lemma (Theorem 2.52) in the SL_2 case. □

Remark 2.54. In the SU_3 case when $\gamma = 0$ and L is ramified, we did not prove that the map $p_r\colon P_0 \to P_0^{0,r}$ is surjective when r is odd. We did not take the time to investigate further whether such a surjectivity holds, since we do not need it.

Along with the surjectivity of the restriction map $p_r\colon P_0 \to P_0^{0,r}$, one of the key result in our local description of the ball of radius r is that p_r is also somehow injective enough. This result can be seen as a natural generalisation of [BT84a, Corollaire 4.6.8].

Lemma 2.55. *Let* $r \in \mathbf{N}$*.*

1. In the $\mathrm{SL}_2(D)$ *case, let* $x \in [-\omega(\pi_K^{rd}), \omega(\pi_K^{rd})]$*. Then* $p_{rd}^{-1}(P_x^{0,rd}) \subset P_x$ *(where* d *is the degree of* D *over* K*).*

2. *In the* SU_3 *case when* $\gamma = 0$, *let* $x \in [-\omega(\pi_L^{\epsilon r}), \omega(\pi_L^{\epsilon r})]$. *Then* $p_{\epsilon r}^{-1}(P_x^{0,\epsilon r}) \subset P_x$ (*where* $\epsilon = 1$ *if* L *is unramified, and* $\epsilon = 2$ *if* L *is ramified*).

3. *In the* SU_3 *case when* $\gamma > 0$, *assume that* $r \leq 2i_0$, *and let* $x \in [-\omega(\pi_L^r), \omega(\pi_L^r)]$. *Then* $p_r^{-1}(P_x^{0,r}) \subset P_x$.

Proof. In the $\mathrm{SL}_2(D)$ case (respectively the SU_3 case when $\gamma = 0$), belonging to $p_{rd}^{-1}(P_x^{0,rd})$ (taking $d = 1$ in the SU_3 case) implies that the valuation of the off diagonal entries are big enough. Hence, the result follows directly from Definition 2.12 and Definition 2.18.

In the SU_3 case when $\gamma > 0$, let $g \in p_r^{-1}(P_x^{0,r})$. We want to show that $g \in P_x$. We can assume that $x \in [0, \omega(\pi_L^r)]$, the argument when x is negative being similar.

By assumption, we know that $\omega(g_{31}) \geq x$, and we want to show that this implies $\omega(g_{21}) \geq \frac{x}{2} + \gamma$ and $\omega(g_{32}) \geq \frac{x}{2} - \gamma$. Since $g \in \mathrm{SU}_3(K)$, $\varphi_{i_0}(g) \in \underline{SU}_3(\mathcal{O}_K)$. In particular, the coefficients of g satisfy

$$\frac{2}{\pi_K^{i_0}}(g_{31}^{11}g_{11}^{11} + \beta g_{31}^{21}g_{11}^{21}) + \frac{\alpha}{\pi_K^{i_0}}(g_{31}^{21}g_{11}^{11} + g_{11}^{21}g_{31}^{11}) = -(\overline{g}_{21}\tau_{i_0}^{-1}g_{21})^{11}$$

Note that $(\overline{g}_{21}g_{21})^{11} = (g_{21}^{11} + \alpha g_{21}^{21})g_{21}^{11} + \beta(g_{21}^{21})^2$, an that $\tau_{i_0}^{-1}$ is just multiplication by $\pi_K^{-i_0}$. Also recall that if $\alpha = 0$, $\omega(2) = \omega(\pi_K^{i_0}) = 2\gamma$, while if $\alpha \neq 0$, $\omega(\alpha) = \omega(\pi_K^{i_0}) = 2\gamma + \omega(\pi_L)$ (see Definition 2.15 and Definition 2.32). Furthermore, if $\alpha \neq 0$, $\omega(\alpha) \leq \omega(2)$ by Lemma 2.13. Hence, we get

$$\omega((g_{21}^{11} + \alpha g_{21}^{21})g_{21}^{11} + \beta(g_{21}^{21})^2) \geq 2\gamma + x$$

But $\omega((g_{21}^{11} + \alpha g_{21}^{21})g_{21}^{11} + \beta(g_{21}^{21})^2) = \min\{\omega((g_{21}^{11})^2); \omega(\beta(g_{21}^{21})^2)\} = 2\omega(g_{21})$, so that $\omega(g_{21}) \geq \gamma + \frac{x}{2}$, as wanted.

Finally, using again that $g \in \mathrm{SU}_3(K)$, we also find $g_{21}\overline{g}_{33} + g_{22}\overline{g}_{32} + g_{23}\overline{g}_{31} = 0$. By Claim 2 of Lemma 2.49, if i_0 is odd (respectively even) $\lambda_{n_0}^{-1}g_{22}\lambda_{n_0}$ (respectively g_{22}) is equal to $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ modulo $\mathfrak{m}_K^{i_0}$. Thus g_{22} is in particular of valuation 0. Hence, we get that $\overline{g}_{32}$ is of the same valuation than $g_{21}\overline{g}_{33} + g_{23}\overline{g}_{31}$. Since $g \in P_0$, $\omega(g_{33}) \geq 0 \leq \omega(g_{23})$, and we know that $\omega(g_{31}) \geq x \geq \frac{x}{2} \leq \omega(g_{21})$. This concludes the proof. $\square$

We arrive finally at our main result, which is that the ball of radius rd (respectively r), together with the action of $\mathrm{SL}_2(\mathcal{O}_D)$ (respectively $\mathrm{SU}_3(\mathcal{O}_K)$), is encoded in $P_0^{0,rd}$ (respectively $P_0^{0,r}$). Let us first state our definition of the ball of radius r.

Definition 2.56. Let G be $\mathrm{SL}_2(D)$ (respectively $\mathrm{SU}_3^{L/K}(K)$). Let $\pi = \pi_D$ and $d = \sqrt{[D:K]}$ (respectively $\pi = \pi_L$ and $d = 1$). The ball of radius rd around $0 \in \mathbf{R}$ is

$$B_0(rd) = \{[(g,x)]_0 \in \mathcal{I}_0 \mid x \in [-\omega(\pi^{rd}), \omega(\pi^{rd})] \subset \mathbf{R}, g \in P_0\}$$

Remark 2.57. Recall that the map $B_0(rd) \to \mathcal{I}\colon [(g,x)]_0 \mapsto [(g,x)]$ is an equivariant embedding by Lemma 2.27.

The following result explains why we call $B_0(rd)$ the ball of radius rd.

Lemma 2.58. *In the* $\mathrm{SL}_2(D)$ *case (respectively the* $\mathrm{SU}_3(K)$ *case), let* $\pi = \pi_D$ *and* $d = \sqrt{[D:K]}$ *(respectively* $\pi = \pi_L$ *and* $d = 1$*). Let us identify* $B_0(rd)$ *with its image in* $\mathcal{I}$ *under* $\mathcal{I}_0 \to \mathcal{I}$*. Renormalise the distance on* $\mathbf{R}$ *so that* $d_{\mathbf{R}}(0, \omega(\pi)) = 1$*, and put the metric* $d_{\mathcal{I}}$ *on* $\mathcal{I}$ *arising from the distance* $d_{\mathbf{R}}$ *(see Remark 2.22). Then* $B_0(rd) = \{p \in \mathcal{I} \mid d_{\mathcal{I}}([\mathrm{Id}, 0], p) \le rd\}$*.*

Proof. Looking at the embedding $\mathbf{R} \hookrightarrow \mathcal{I}\colon x \mapsto [(\mathrm{Id}, x)]$, it is easy to identify which $x \in \mathbf{R}$ are vertices of the tree $\mathcal{I}$. Indeed, $x \in \mathbf{R}$ is a vertex of $\mathcal{I}$ if and only if P_x strictly contains $P_{x+\varepsilon}$ (where ε is a real number such that $|\varepsilon| < \omega(\pi)$). From our description of P_x, one readily check that $x \in \mathbf{R}$ is a vertex of $\mathcal{I}$ if and only if $x \in \mathbf{Z}.\omega(\pi) = \omega(\pi^{\mathbf{Z}})$. Now, if $[(g,x)] \in B_0(rd)$, then $d_{\mathbf{R}}(0,x) \le rd$ by our normalisation of the distance on $\mathbf{R}$, while if $d_{\mathcal{I}}([(\mathrm{Id},0)], [(g,x)]) \le rd$, then $(g,x) \sim (\mathrm{Id}, y)$ with $d_{\mathbf{R}}(0,y) \le rd$, so that $[(g,x)] \in B_0(rd)$, as wanted. □

Theorem 2.59. *Let* $r \in \mathbf{N}$*. Depending on cases, we assume the following:*

1. *In the* SU_3 *case when* $\gamma = 0$ *and* L *is ramified, we assume that* r *is even.*

2. *In the* SU_3 *case when* $\gamma > 0$, *we assume that* $r \leq 2i_0$.

Also, let $d = \sqrt{[D:K]}$ *in the* $\mathrm{SL}_2(D)$ *case* (*respectively* $d = 1$ *in the* SU_3 *case*). *The map* $B_0(rd) \to \mathcal{I}^{0,rd} \colon [(g,x)]_0 \mapsto [(p_{rd}(g), x)]^{0,rd}$ *is a* ($p_{rd} \colon P_0 \to P_0^{0,rd}$)*-equivariant bijection.*

Proof. It is readily seen that the given map is well-defined.

- Injectivity: let $[(g,x)]_0, [(h,y)]_0 \in B_0(rd)$ be such that they have the same image in $\mathcal{I}^{0,rd}$. By Remark 2.37, it means that for all $\tilde{n} \in N^{0,rd}$ such that $\nu(\tilde{n})(x) = y$, $p_{rd}(g)^{-1} p_{rd}(h)\tilde{n} \in P_x^{0,rd}$. So, we can assume that $\tilde{n}$ is either equal to Id, or is of the form $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ (respectively $\left(\begin{smallmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{smallmatrix}\right)$) in the $\mathrm{SL}_2(D)$ case (respectively the SU_3 case). Hence, there exists $n \in N$ such that $p_{rd}(n) = \tilde{n}$. But $\nu(n)(x) = y$, and $g^{-1}hn \in p_{rd}^{-1}(P_x^{0,rd}) \subset P_x$ by Lemma 2.55. Hence, $[(g,x)]_0 = [(h,y)]_0$, as wanted.

- Surjectivity: follows directly from the surjectivity of $p_{rd} \colon P_0 \to P_0^{0,rd}$ (Corollary 2.53).

- Equivariance:

$$h.[(g,x)]_0 = [(hg,x)]_0 \mapsto [(p_{rd}(hg), x)]^{0,rd} = p_{rd}(h).[(p_{rd}(g), x)]^{0,rd}$$

□

2.5 Convergences on the arithmetic side

2.5.1 The topological space of quadratic pairs of local fields

Definition 2.60. Consider the set of pairs of local fields (K, L) where either

1. $K = L$ (equipped with the trivial conjugation action).

2. L is a separable quadratic extension of K.

We say that a pair (K, L) is trivial (respectively ramified, respectively unramified) if $L = K$ (respectively L is quadratic ramified, respectively

L is quadratic unramified). We also use those adjectives for L, when the pair under consideration is implicit. Furthermore, we freely amalgamate the notions of local fields and trivial pair of local fields.

Remark 2.61. Strictly speaking, a trivial extension of a local field is both ramified and unramified, but we nevertheless adopt the above vocabulary to be able to easily differentiate the three kinds of pairs.

Definition 2.62. We say that two pairs (K_1, L_1) and (K_2, L_2) are isomorphic if there exists a conjugation equivariant isomorphism between the two pairs. Let $\mathcal{L}$ be the set of pairs of local fields as in Definition 2.60, up to isomorphism. For each prime p, let us also define $\mathcal{L}_{p^n} = \{(K, L) \in \mathcal{L} \mid |\overline{K}| = p^n\}$.

Following an idea dating back to Krasner (see [Del84] for references, this idea is also used in e.g. [Kaz86]), we define a metric on the space $\mathcal{L}$.

Definition 2.63. Let (K_1, L_1) and (K_2, L_2) be in $\mathcal{L}$. The conjugation induces an automorphism of $\mathcal{O}_{L_i}/\mathfrak{m}_{L_i}^r$, for any $r \in \mathbf{N}$. We say that (K_1, L_1) is r-close to (K_2, L_2) if and only if there exists a conjugation equivariant isomorphism $\mathcal{O}_{L_1}/\mathfrak{m}_{L_1}^r \to \mathcal{O}_{L_2}/\mathfrak{m}_{L_2}^r$ inducing an isomorphism $\mathcal{O}_{K_1}/(\mathfrak{m}_{L_1}^r \cap \mathcal{O}_{K_1}) \to \mathcal{O}_{K_2}/(\mathfrak{m}_{L_2}^r \cap \mathcal{O}_{K_2})$.

Remark 2.64. If L is unramified or if the residue characteristic is not 2, a conjugation equivariant isomorphism $\mathcal{O}_{L_1}/\mathfrak{m}_{L_1}^r \to \mathcal{O}_{L_2}/\mathfrak{m}_{L_2}^r$ always induces an isomorphism $\mathcal{O}_{K_1}/(\mathfrak{m}_{L_1}^r \cap \mathcal{O}_{K_1}) \to \mathcal{O}_{K_2}/(\mathfrak{m}_{L_2}^r \cap \mathcal{O}_{K_2})$, since in those cases, $\mathcal{O}_{K_1}/(\mathfrak{m}_{L_1}^r \cap \mathcal{O}_{K_1})$ is the invariant subring of $\mathcal{O}_{L_1}/\mathfrak{m}_{L_1}^r$. We do not know if it still holds in the ramified and residue characteristic 2 case. Also note that $(\mathfrak{m}_{L_1}^r \cap \mathcal{O}_{K_1}) = \begin{cases} \mathfrak{m}_{K_1}^{\lceil \frac{r}{2} \rceil} & \text{if } L_1 \text{ is ramified} \\ \mathfrak{m}_{K_1}^r & \text{if } L_1 \text{ is unramified} \end{cases}$.

Remark 2.65. Note that being r-close is an equivalence relation, and that if $r \geq l$ and (K_1, L_1) is r-close to (K_2, L_2), then (K_1, L_1) is l-close to (K_2, L_2).

We now observe that this notion of closeness induces a metric which is non-archimedean on $\mathcal{L}$. Let $d \colon \mathcal{L} \times \mathcal{L} \to \mathbf{R}_{\geq 0}$ be defined by

$$d((K_1, L_1), (K_2, L_2)) = \inf\{\frac{1}{2^r} \mid (K_1, L_1) \text{ is } r\text{-close to } (K_2, L_2)\}$$

Lemma 2.66. *$d(.,.)$ is a non-archimedean metric on $\mathcal{L}$.*

Proof. If $d((K_1, L_1), (K_2, L_2)) = 0$, then $\mathcal{O}_{L_1}$ and $\mathcal{O}_{L_2}$ are equivariantly isomorphic. But then, the pairs of field of fraction are isomorphic in $\mathcal{L}$, as wanted. The fact that this distance is non-archimedean is a consequence of Remark 2.65. □

Remark 2.67. With this definition, if $d((K_1, L_1), (K_2, L_2)) \leq \frac{1}{2}$, then L_1 is unramified if and only if L_2 is unramified. In other words, unramified pairs are always at distance 1 from other kind of pairs. This is because L is unramified if and only if the conjugation action is non-trivial on the residue field.

A crucial fact about the space $\mathcal{L}_{p^n}$ (for a fixed prime power p^n, as in Definition 2.62) is that it is a compact space. As was outlined in the introduction, this is one of the key observation to prove that $\mathcal{S}_T^{\text{qs-alg}}$ is closed in $\mathcal{S}_T$. In fact, it is even possible to give an explicit description of the metric space $\mathcal{L}_{p^n}$. It takes some time to establish this explicit description, but it only uses basic facts from the theory of local fields. The corner stone in this description is Theorem 2.68 which is certainly well known to experts (this is for example used implicitly in [Kaz86]). While working on this chapter, we learnt that it had also been obtained and used independently in [dT15, Lemma 1.3]. Given its importance, we decide nevertheless to include our own proof.

Theorem 2.68. *Let K be a totally ramified extension of degree k of $\mathbf{Q}_{p^n}$. The distance between K and $\mathbf{F}_{p^n}(\!(X)\!)$ is $\frac{1}{2^k}$. More explicitly, let $\{a_x\}_{x \in \mathbf{F}_{p^n}} \subset \mathbf{Q}_{p^n}$ be a set of representative of $\overline{K}$. Then the bijection*

$$\varphi_{\pi_K} \colon \mathcal{O}_K \to \mathbf{F}_{p^n}[\![X]\!]$$
$$\sum_{i=0}^{\infty} a_{x_i} \pi_K^i \mapsto \sum_{i=0}^{\infty} x_i X^i$$

(which depends on a choice of uniformiser of K) induces an isomorphism of ring

$$\overline{\varphi}_{\pi_K} \colon \mathcal{O}_K / \mathfrak{m}_K^k \to \mathbf{F}_{p^n}[\![X]\!]/(X^k)$$

Proof. Let $\{a_x\}_{x \in \mathbf{F}_{p^n}}$ be a set of representative of $\overline{K}$. Since $\mathbf{Q}_{p^n} \leq K$, we can choose the a_x's so that they all lie in $\mathbf{Q}_{p^n}$. Now, we have $(a_x + a_y) - a_{x+y} \in (p)$ and $(a_x a_y) - a_{xy} \in (p)$. But also, since K is totally ramified, $(p) = \mathfrak{m}_K^k$. Hence, this implies that the map φ_K (which is always a bijection, by the general theory of local fields) is a homomorphism modulo $\mathfrak{m}_K^k$ and (X^k).

To conclude that K and $\mathbf{F}_{p^n}(\!(X)\!)$ are at distance $\frac{1}{2^k}$, it suffices to observe that $\mathcal{O}_K/\mathfrak{m}_K^{k+1}$ is not isomorphic to $\mathbf{F}_{p^n}[\![X]\!]/(X^{k+1})$. But this is clear, since $p \notin \mathfrak{m}_K^{k+1}$, hence $\sum_{i=1}^{p} 1 \neq 0$ in $\mathcal{O}_K/\mathfrak{m}_K^{k+1}$. □

We need a series of variations on Theorem 2.68, that we now state as corollaries.

Corollary 2.69. *1. Let K be a totally ramified extension of degree k of $\mathbf{Q}_{p^n}$, and let L be the unramified quadratic extension of K. The distance between (K, L) and $(\mathbf{F}_{p^n}(\!(X)\!), \mathbf{F}_{p^{2n}}(\!(X)\!))$ is $\frac{1}{2^k}$.*

2. Let K be a totally ramified extension of degree k of $\mathbf{Q}_{p^n}$, where p is an odd prime, and let L be a ramified quadratic extension of K. The distance between (K, L) and $(\mathbf{F}_{p^n}(\!(X)\!), \mathbf{F}_{p^n}(\!(\sqrt{X})\!))$ is $\frac{1}{2^{2k}}$.

3. Let $\mathbf{F}_{2^n}(\!(X)\!)[T]/T^2 - \alpha T + X$ be a separable quadratic ramified extension of $\mathbf{F}_{2^n}(\!(X)\!)$, with $\alpha \in (X)$. Let K be a totally ramified extension of degree k of $\mathbf{Q}_{2^n}$, and let $\varphi_{\pi_K} \colon \mathcal{O}_K \to \mathbf{F}_{2^n}[\![X]\!]$ be the bijection defined in Theorem 2.68. Finally, let $a = \varphi_{\pi_K}^{-1}(\alpha) \in \mathcal{O}_K$. Then $(K, K[T]/T^2 - aT + \pi_K)$ is $2k$-close to the ramified pair $(\mathbf{F}_{2^n}(\!(X)\!), \mathbf{F}_{2^n}(\!(X)\!)[T]/T^2 - \alpha T + X)$.

Proof. 1. As in the proof of Theorem 2.68, let $\{a_x\}_{x \in \mathbf{F}_{p^n}}$ be a set of representative of $\overline{K}$ such that $a_x \in \mathbf{Q}_{p^n}$, for all $x \in \mathbf{F}_{p^n}$.

Since unramified extensions correspond to extensions of the residue field, there exists $\alpha, \beta \in \mathbf{F}_{p^n}$ such that

$$L \cong K[T]/T^2 - a_\alpha T + a_\beta$$
$$\mathbf{F}_{p^{2n}}(\!(X)\!) \cong \mathbf{F}_{p^n}(\!(X)\!)[T]/T^2 - \alpha T + \beta$$

Observing furthermore that

$$\mathcal{O}_L/\mathfrak{m}_L^k \cong \mathcal{O}_K/\mathfrak{m}_K^k \oplus t.\mathcal{O}_K/\mathfrak{m}_K^k$$
$$\mathbf{F}_{p^{2n}}[\![X]\!]/(X^k) \cong \mathbf{F}_{p^n}[\![X]\!]/(X^k) \oplus t.\mathbf{F}_{p^n}[\![X]\!]/(X^k)$$

it is clear (in view of Theorem 2.68) that (K, L) is k-close to $(\mathbf{F}_{p^n}(\!(X)\!), \mathbf{F}_{p^{2n}}(\!(X)\!))$.

To conclude that the distance is $\frac{1}{2^k}$ it suffices to note that if (K, L) and $(\mathbf{F}_{p^n}(\!(X)\!), \mathbf{F}_{p^{2n}}(\!(X)\!))$ were r-close for $r > k$, then K and $\mathbf{F}_{p^n}(\!(X)\!)$ would be r-close as well, contradicting Theorem 2.68.

2. First note that by Lemma 2.13, there exists a uniformiser $\pi_K \in K$ such that $L \cong K[T]/T^2 + \pi_K$ (since we avoid by assumption the residue characteristic 2). Also note that for any uniformiser $\beta \in \mathbf{F}_{p^n}(\!(X)\!)$, the pair $(\mathbf{F}_{p^n}(\!(X)\!), \mathbf{F}_{p^n}(\!(X)\!)[T]/T^2 + \beta)$ is isomorphic to the pair $(\mathbf{F}_{p^n}(\!(X)\!), \mathbf{F}_{p^n}(\!(\sqrt{X})\!))$ (so that despite appearances, there is only one ramified pair on $\mathbf{F}_{p^n}(\!(X)\!)$).

   Since

$$\mathcal{O}_L/\mathfrak{m}_L^{2k} \cong \mathcal{O}_K/\mathfrak{m}_K^k \oplus t.\mathcal{O}_K/\mathfrak{m}_K^k$$
$$\mathbf{F}_{p^n}[\![\sqrt{X}]\!]/(\sqrt{X}^{2k}) \cong \mathbf{F}_{p^n}[\![X]\!]/(X^k) \oplus \sqrt{X}.\mathbf{F}_{p^n}[\![X]\!]/(X^k)$$

   it is clear (in view of Theorem 2.68) that (K, L) is $2k$-close to $(\mathbf{F}_{p^n}(\!(X)\!), \mathbf{F}_{p^n}(\!(\sqrt{X})\!))$.

   To conclude that the distance is $\frac{1}{2^k}$ it suffices to note that if (K, L) and $(\mathbf{F}_{p^n}(\!(X)\!), \mathbf{F}_{p^n}(\!(\sqrt{X})\!))$ were r-close for $r > 2k$, then K and $\mathbf{F}_{p^n}(\!(X)\!)$ would be $\lceil \frac{r}{2} \rceil$-close as well, contradicting Theorem 2.68.

3. The ingredients are similar than for the previous assertions: by Theorem 2.68, $\mathcal{O}_K/\mathfrak{m}_K^k \cong \mathbf{F}_{2^n}[\![X]\!]/(X^k)$. Observing that for a ramified quadratic extension $L = K[t]$ of K, we have $\mathcal{O}_L/\mathfrak{m}_L^{2r} \cong \mathcal{O}_K/\mathfrak{m}_K^r \oplus t.\mathcal{O}_K/\mathfrak{m}_K^r$, we directly obtain the conclusion. We could also easily conclude that the distance is $\frac{1}{2^{2k}}$, but we do not need this information. □

We also need two further results in the residue characteristic 2 case.

Lemma 2.70. *1. $(\mathbf{F}_{2^n}((X)), \mathbf{F}_{2^n}((X))[T]/T^2 - X^iT + X)$ is at distance $\frac{1}{2^{2i}}$ from $\mathbf{F}_{2^n}((X))$.*

2. Any separable quadratic ramified extension of $\mathbf{F}_{2^n}((X))$ is of the form $\mathbf{F}_{2^n}((X))[T]/T^2 - \alpha T + X$, for some non zero $\alpha \in (X)$. Also, given $i \in \mathbf{N}$, there are only finitely many extensions (up to isomorphism) of the form $\mathbf{F}_{2^n}((X))[T]/T^2 - \alpha T + X$ where $\alpha \in (X^i) \setminus (X^{i+1})$.

Proof. 1. To simplify notations, let $L = \mathbf{F}_{2^n}((X))[T]/T^2 - X^iT + X$. Observe that the conjugation action is trivial on $\mathcal{O}_L/\mathfrak{m}_L^{2i}$, so that $\mathcal{O}_L/\mathfrak{m}_L^{2i} \cong \mathbf{F}_{2^n}[\![X]\!]/(X^i) \oplus \sqrt{X}.\mathbf{F}_{2^n}[\![X]\!]/(X^i)$, with trivial conjugation action. Hence, $(\mathbf{F}_{2^n}((X)), \mathbf{F}_{2^n}((X))[T]/T^2 - X^iT + X)$ is $2i$-close from $\mathbf{F}_{2^n}((X))$. Now, the conjugation action is non-trivial on $\mathcal{O}_L/\mathfrak{m}_L^{2i+1}$, so that the distance is $\frac{1}{2^{2i}}$.

2. By Lemma 2.13, any quadratic ramified extension is of the form $\mathbf{F}_{2^n}((X))[T]/T^2 - \alpha T + \beta$, where $\beta \in (X) \setminus (X^2)$ and $\alpha \in (X)$. Now, because $\mathbf{F}_{2^n}((X))$ has many isomorphisms, such an extension is always (equivariantly) isomorphic to an extension of the desired form. For the last statement, mimicking the proof of [Lan94, Chapter II, §5, Proposition 14], the finiteness follows directly from the compactness of $(X^i) \setminus (X^{i+1})$. □

As in the introduction, let $\hat{\mathbf{N}}$ denote the one point compactification of $\mathbf{N}$.

Proposition 2.71. *Let p be an odd prime number. Then $\mathcal{L}_{p^n}$ is homeomorphic to $\hat{\mathbf{N}} \times \{1,2,3\}$. On the other hand, $\mathcal{L}_{2^n}$ is homeomorphic to $\hat{\mathbf{N}}^2$. Furthermore, in $\mathcal{L}_{2^n}$, the set of unramified pairs form a clopen subset homeomorphic to $\hat{\mathbf{N}}$.*

Proof.

Claim 1. *Let K be a local field. If $|\overline{K}| = p^n$, then K is a totally ramified extension of $\mathbf{Q}_{p^n}$, or it is isomorphic to $\mathbf{F}_{p^n}((X))$.*

Proof of the claim: By the classification of local fields, K is either a finite extension of $\mathbf{Q}_p$, or isomorphic to $\mathbf{F}_{p^n}((X))$ for some prime power

p^n. Since $\overline{\mathbf{F}_{p^n}(\!(X)\!)} = \mathbf{F}_{p^n}$, the latter case is clear. For the first case, $\overline{K} = \mathbf{F}_{p^n}$ if and only if the maximal unramified subextension of K is $\mathbf{Q}_{p^n}$. ■

Claim 2. *Let (K_k, L_k) and (K_l, L_l) be trivial pairs (respectively unramified, respectively ramified and of residue characteristic not 2). Assume that K_k and K_l are totally ramified extension of $\mathbf{Q}_{p^n}$ such that $[K_k : \mathbf{Q}_{p^n}] = k < [K_l : \mathbf{Q}_{p^n}] = l$. Then the distance between (K_k, L_k) and (K_l, L_l) is $\frac{1}{2^k}$.*

Proof of the claim: We observed in Lemma 2.66 that $\mathcal{L}$ is a metric space which is non-archimedean, and hence every triangle is isosceles. Thus, the distance between (K_k, L_k) and (K_l, L_l) is either $\frac{1}{2^k}$ or $\frac{1}{2^l}$ (taking in each case as a comparison point the corresponding pair in positive characteristic, and using Theorem 2.68, or Corollary 2.69). But in the latter case, since being l-close is an equivalence relation, we would conclude that (K_k, L_k) is l-close to $\mathbf{F}_{p^n}(\!(X)\!)$, which would contradict Theorem 2.68 or Corollary 2.69. ■

Claim 3. *There are only finitely many totally ramified extension of degree $\leq k$ of a local field of characteristic 0.*

Proof of the claim: This is just a well-known corollary of the so called Krasner's Lemma. A proof of Claim 3 can be found in [Lan94, Chapter II, §5, Proposition 14]. ■

Claim 4. *Let $(K, L) \in \mathcal{L}_{p^n}$. If K is of characteristic 0, the pair (K, L) is isolated in $\mathcal{L}_{p^n}$.*

Proof of the claim: Since unramified pairs are at distance 1 from other kind of pairs, it follows from Claim 2 and Claim 3 that unramified pairs of characteristic 0 are isolated in $\mathcal{L}_{p^n}$.

When p is an odd prime, ramified pairs are at distance $\frac{1}{2}$ from trivial pairs, and there are only 2 different quadratic ramified extension of a given local field (since p is odd) hence the result follows again from Claim 2 and Claim 3 in this case.

Finally, when $p = 2$, let (K, L) be a trivial or ramified pair of characteristic 0 belonging to $\mathcal{L}_{2^n}$. By definition, if (K, L) is r-close to $(\tilde{K}, \tilde{L})$, then K is $\lceil \frac{r}{2} \rceil$-close to $\tilde{K}$. Hence, by Claim 2 for trivial pairs, (K, L) is isolated from pairs $(\tilde{K}, \tilde{L})$ where $\tilde{K}$ is either of characteristic 2, or $[\tilde{K} : \mathbf{Q}_{2^n}] \neq [K : \mathbf{Q}_{2^n}]$. But there are only finitely many pairs (K, L) with $[K : \mathbf{Q}_{2^n}] = k$ by Claim 3. Hence, the conclusion. ■

Claim 5. *$\mathcal{L}_{p^n}$ is a countable space.*

Proof of the claim: By Claim 3, there are only countably many pairs of characteristic 0. For pairs of positive characteristic, if p is odd, there is only one pair of each type (recall that we consider pairs up to isomorphism). If $p = 2$, there is one trivial pair and one unramified pair, and there are countably many ramified pairs of characteristic 2 by Lemma 2.70. ■

We are now able to deduce the homeomorphism type of $\mathcal{L}_{p^n}$: for p any prime, the unramified pairs are isolated from other kind of pairs in $\mathcal{L}_{p^n}$. Furthermore, unramified pairs of characteristic 0 are isolated by Claim 4, and the unramified pair of positive characteristic is an accumulation point by Corollary 2.69. Hence, by [MS20, Théorème 1], unramified pairs account for one disjoint copy of $\hat{\mathbf{N}}$.

When p is odd, trivial pairs (respectively ramified pairs) are isolated from ramified pairs (respectively trivial pairs), the characteristic 0 ones are isolated by Claim 4, and the unique pair of positive characteristic is an accumulation points by Theorem 2.68 and Corollary 2.69, so that we obtain two more disjoint copies of $\hat{\mathbf{N}}$.

Finally, when $p = 2$, since pairs of characteristic 0 are isolated by Claim 4, the first Cantor Bendixson derivative $\mathcal{L}_{2^n}^{(1)}$ contains only pairs of positive characteristic, and $\mathcal{L}_{2^n}^{(1)}$ contains all of them by Corollary 2.69. Also, by Lemma 2.70, ramified pairs are isolated in $\mathcal{L}_{2^n}^{(1)}$, and the trivial pair $\mathbf{F}_{2^n}(\!(X)\!)$ is an accumulation point in $\mathcal{L}_{2^n}^{(1)}$. So that again by [MS20, Théorème 1], we get the result. □

2.5.2 The topological space of division algebras

In Section 2.5.1, we studied convergence in the space $\mathcal{L}$ of pairs of local fields. This subsequently allows us to conclude convergence in a corresponding Chabauty space (see Theorem 2.78), in the case of quasi-split (absolutely simple, simply connected) algebraic groups of rank 1 (i.e. in the $\mathrm{SL}_2(K)$ case and the SU_3 case). It turns out that groups of the form $\mathrm{SL}_2(D)$ with $[D : K] > 1$ do not converge to quasi-split groups in the Chabauty space, and hence we can treat arithmetical convergence for division algebras separately from arithmetical convergences of pairs of local fields.

Definition 2.72. Let $\mathcal{D}$ be the set of finite dimensional division algebras D over a local field K, up to isomorphism. Let also $\mathcal{D}_{p^n} = \{D \in \mathcal{D} \mid |\overline{D}| = p^n\}$. As in Section 2.5.1, we say that D_1 is r-close to D_2 if and only if there exists an isomorphism $\mathcal{O}_{D_1}/\mathfrak{m}^r_{D_1} \to \mathcal{O}_{D_2}/\mathfrak{m}^r_{D_2}$.

Again, this notion of closeness induces a non-archimedean metric on $\mathcal{D}$, by defining

$$d \colon \mathcal{D} \times \mathcal{D} \to \mathbf{R}_{\geq 0} \colon d(D_1, D_2) = \inf\{\frac{1}{2^r} \mid D_1 \text{ is } r\text{-close to } D_2\}$$

It is then quite straightforward to work out the homeomorphism type of $\mathcal{D}_{p^n}$.

Proposition 2.73. *Let p be a prime number. Then $\mathcal{D}_{p^n}$ is homeomorphic to $\hat{\mathbf{N}} \times \{1, \dots, n\}$.*

Proof. Let $D \in \mathcal{D}_{p^n}$. By Definition 2.89, D is isomorphic to the cyclic algebra $(E/K, \sigma^r, \pi_K)$, where $[E : K] = d$ divides n, $r \in (\mathbf{Z}/d\mathbf{Z})^\times$ and $|\overline{K}| = p^{\frac{n}{d}}$. Furthermore, it is easily seen that if $D_1 = (E_1/K_1, \sigma^{r_1}, \pi_{K_1})$ is 2-close to $D_2 = (E_2/K_2, \sigma^{r_2}, \pi_{K_2})$, then $[E_1 : K_1] = [E_2 : K_2]$ and $r_1 = r_2$.

Using Theorem 2.68 and the explicit description of central division algebra given in Appendix 2.B, we see that a point in $\mathcal{D}_{p^n}$ is isolated if and only if the corresponding division algebra is of characteristic 0 (see also [dT15, Theorem 1.2]). Finally, $\mathcal{D}_{p^n}$ is a countable space, and it is readily seen that the number of positive characteristic division algebras in $\mathcal{D}_{p^n}$ is equal to $\sum_{d|n} |(\mathbf{Z}/d\mathbf{Z})^\times| = n$. Hence, the result follows from [MS20, Théorème 1]. $\square$

2.6 Continuity from local fields to subgroups of $\mathrm{Aut}(T)$

Definition 2.74. Let $(K, L) \in \mathcal{L}$.

1. If (K, L) is trivial, we associate to it the group $\mathrm{SL}_2(K)$.
2. if (K, L) is ramified or unramified, we associate to it the group $\mathrm{SU}_3^{L/K}(K)$.

The associated group is denoted $G_{(K,L)}$. Similarly, we associate to $D \in \mathcal{D}$ the group $G_D = \mathrm{SL}_2(D)$ (note that if $D = K$, the two definitions coincide).

Proposition 2.75. *Let (K_1, L_1) and (K_2, L_2) be two elements in $\mathcal{L}$ that are r-close, with $r > 1$. Let G_i be the algebraic group associated with (K_i, L_i). Then $(P_0^{0,r})_{G_1} \cong (P_0^{0,r})_{G_2}$, and $\mathcal{I}_{G_1}^{0,r}$ is equivariantly in bijection with $\mathcal{I}_{G_2}^{0,r}$, except when (K_1, L_1) is a ramified pair and (K_2, L_2) is trivial. In this latter case, $(P_0^{0,r-1})_{G_1} \cong (P_0^{0,r-1})_{G_2}$, and $\mathcal{I}_{G_1}^{0,r-1}$ is equivariantly in bijection with $\mathcal{I}_{G_2}^{0,r-1}$*

Proof. We prove it on a case by case analysis.

1. When the pair are both trivial, the isomorphism $\mathcal{O}_{K_1}/\mathfrak{m}_{K_1}^r \cong \mathcal{O}_{K_2}/\mathfrak{m}_{K_2}^r$ induces an isomorphism $\varphi\colon (P_0^{0,r})_{G_1} = \mathrm{SL}_2(\mathcal{O}_{K_1}/\mathfrak{m}_{K_1}^r) \cong \mathrm{SL}_2(\mathcal{O}_{K_2}/\mathfrak{m}_{K_2}^r) = (P_0^{0,r})_{G_2}$. Define a linear map $f\colon \mathbf{R} \to \mathbf{R}\colon x \mapsto x\frac{\omega(\pi_{K_2})}{\omega(\pi_{K_1})}$. It is clear that for all $x \in [-\omega(\pi_{K_1}^r); \omega(\pi_{K_1}^r)]$, φ restricts to an isomorphism $(P_x^{0,r})_{G_1} \cong (P_{f(x)}^{0,r})_{G_2}$. Furthermore,
$$\varphi(T^{0,r})_{G_1} = (T^{0,r})_{G_2}$$
$$\varphi(M^{0,r})_{G_1} = (M^{0,r})_{G_2}$$
and for all $n \in N^{0,r}$, $f(n.x) = \varphi(n).f(x)$. Thus, the map $\mathcal{I}_{G_1}^{0,r} \to \mathcal{I}_{G_2}^{0,r}\colon [(g, x)]^{0,r} \mapsto [(\varphi(g); f(x))]^{0,r}$ is a φ-equivariant bijection.

2. When the pair are both ramified of both unramified, the argument is the same than for the previous case: the conjugation equivariant isomorphism $\mathcal{O}_{L_1}/\mathfrak{m}_{L_1}^r \cong \mathcal{O}_{L_2}/\mathfrak{m}_{L_2}^r$ induces an isomorphism φ

$$\begin{array}{ccc} \mathrm{SL}_3(\mathcal{O}_{L_1}/\mathfrak{m}_{L_1}^r) & \cong & \mathrm{SL}_3(\mathcal{O}_{L_2}/\mathfrak{m}_{L_2}^r) \\ \vee & & \vee \\ (P_0^{0,r})_{G_1} & \xrightarrow[\varphi]{} & (P_0^{0,r})_{G_2} \end{array}$$

Define a linear map $f\colon \mathbf{R} \to \mathbf{R}\colon x \mapsto x\frac{\omega(\pi_{L_2})}{\omega(\pi_{L_1})}$. It is clear that for all $x \in [-\omega(\pi_{L_1}^r); \omega(\pi_{L_1}^r)]$, φ restricts to an isomorphism $(P_x^{0,r})_{G_1} \cong (P_{f(x)}^{0,r})_{G_2}$. Furthermore,

$$\varphi(T^{0,r})_{G_1} = (T^{0,r})_{G_2}$$
$$\varphi(M^{0,r})_{G_1} = (M^{0,r})_{G_2}$$

and for all $n \in N^{0,r}$, $f(n.x) = \varphi(n).f(x)$. Hence, the map $\mathcal{I}_{G_1}^{0,r} \to \mathcal{I}_{G_2}^{0,r}\colon [(g,x)]^{0,r} \mapsto [(\varphi(g); f(x))]^{0,r}$ is a φ-equivariant bijection.

3. Recall that unramified pairs are isolated from pairs of other types, and that ramified pairs in residue characteristic not 2 are at distance $\frac{1}{2}$ from trivial pairs. Since we assume that $r > 1$, there just remains to examine the case when a trivial pair is r-close to a ramified pair in residue characteristic 2.

   Without loss of generality, (K_1, L_1) is the ramified pair. Let $t \in L_1$ be such that $t^2 = \alpha t - \beta$, where t, α and β are as in Lemma 2.13. Since (K_1, L_1) is r-close to (K_2, L_2) and (K_2, L_2) is a trivial pair, in particular the conjugation is trivial modulo $\mathfrak{m}_L^r$. Hence, if $\alpha \neq 0$ (respectively if $\alpha = 0$), $\omega(2) \geq \omega(\alpha) = \omega(\pi_K^{i_0}) \geq \omega(\pi_L^r)$ (respectively $\omega(2) = \omega(\pi_K^{i_0}) \geq \omega(\pi_L^{r-1})$), so that we have $r - 1 \leq 2i_0$, as needed to apply Definition 2.33 to $r-1$.

   That being said, we can proceed as for the other cases: the isomorphism $\mathcal{O}_{L_1}/\mathfrak{m}_{L_1}^{r-1} \cong \mathcal{O}_{L_2}/\mathfrak{m}_{L_2}^{r-1}$ induces in turn an isomorphism $\varphi\colon (P_0^{0,r-1})_{G_1} = \mathrm{SL}_2(\mathcal{O}_{L_1}/\mathfrak{m}_{L_1}^{r-1}) \cong \mathrm{SL}_2(\mathcal{O}_{L_2}/\mathfrak{m}_{L_2}^{r-1}) = (P_0^{0,r-1})_{G_2}$. Define a linear map $f\colon \mathbf{R} \to \mathbf{R}\colon x \mapsto x\frac{\omega(\pi_{L_2})}{\omega(\pi_{L_1})}$. It is clear that for all $x \in [-\omega(\pi_{L_1}^{r-1}); \omega(\pi_{L_1}^{r-1})]$, φ restricts to an isomorphism

$(P_x^{0,r-1})_{G_1} \cong (P_{f(x)}^{0,r-1})_{G_2}$. Furthermore,

$$\varphi(T^{0,r-1})_{G_1} = (T^{0,r-1})_{G_2}$$
$$\varphi(M^{0,r-1})_{G_1} = (M^{0,r-1})_{G_2}$$

and for all $n \in N^{0,r-1}$, $f(n.x) = \varphi(n).f(x)$. Hence, the map $\mathcal{I}_{G_1}^{0,r-1} \to \mathcal{I}_{G_2}^{0,r-1}$: $[(g,x)]^{0,r} \mapsto [(\varphi(g); f(x))]^{0,r}$ is a φ-equivariant bijection. $\square$

Proposition 2.76. *Let D_1 and D_2 be two elements in $\mathcal{D}$ that are rd_1-close, with $r \geq 1$ and with $d_1 = \sqrt{[D_1 : K_1]}$. We have $\sqrt{[D_1 : K_1]} = \sqrt{[D_2 : K_2]} = d$. Let G_i be the algebraic group associated with D_i. Then $(P_0^{0,rd})_{G_1} \cong (P_0^{0,rd})_{G_2}$, and $\mathcal{I}_{G_1}^{0,rd}$ is equivariantly in bijection with $\mathcal{I}_{G_2}^{0,rd}$.*

Proof. The proof is the same than the proof of Proposition 2.75. The isomorphism $\mathcal{O}_{D_1}/\mathfrak{m}_{D_1}^{rd} \cong \mathcal{O}_{D_2}/\mathfrak{m}_{D_2}^{rd}$ induces in turn an isomorphism $\varphi\colon (P_0^{0,rd})_{G_1} = \mathrm{SL}_2(\mathcal{O}_{D_1}/\mathfrak{m}_{D_1}^{rd}) \cong \mathrm{SL}_2(\mathcal{O}_{D_2}/\mathfrak{m}_{D_2}^{rd}) = (P_0^{0,rd})_{G_2}$. Define a linear map $f\colon \mathbf{R} \to \mathbf{R}\colon x \mapsto x\frac{\omega(\pi_{D_2})}{\omega(\pi_{D_1})}$. It is clear that for all $x \in [-\omega(\pi_{D_1}^{rd}); \omega(\pi_{D_1}^{rd})]$, φ restricts to an isomorphism $(P_x^{0,rd})_{G_1} \cong (P_{f(x)}^{0,rd})_{G_2}$. Furthermore,

$$\varphi(T^{0,rd})_{G_1} = (T^{0,rd})_{G_2}$$
$$\varphi(M^{0,rd})_{G_1} = (M^{0,rd})_{G_2}$$

and for all $n \in N^{0,rd}$, $f(n.x) = \varphi(n).f(x)$. Hence, the map $\mathcal{I}_{G_1}^{0,rd} \to \mathcal{I}_{G_2}^{0,rd}$: $[(g,x)]^{0,rd} \mapsto [(\varphi(g); f(x))]^{0,rd}$ is a φ-equivariant bijection. $\square$

We can finally go back to our original problem, which is to study convergence of algebraic groups in the Chabauty space of $\mathrm{Aut}(T)$. We first discuss the homomorphism $G \to \mathrm{Aut}(T)$ (for G equal to $\mathrm{SL}_2(D)$ or $\mathrm{SU}_3^{L/K}(K)$).

Proposition 2.77. *Let G be either $\mathrm{SL}_2(D)$ or $\mathrm{SU}_3(K)$, and let T_G be its associated Bruhat–Tits tree (Definition 2.21). The induced homomorphism $\hat{}\,: G \to \mathrm{Aut}(T_G)$ is continuous with closed image, and the kernel is equal to the center of G.*

Proof. In each case, the group P_x is really the stabiliser of $[(\mathrm{Id}, x)] \in \mathcal{I}$ (see Remark 2.22). Since a basic identity neighbourhood in $\mathrm{Aut}(T)$ is given by intersecting finitely many vertices stabilisers, the continuity follows. The fact that the image is closed follows from the general argument in [BM96, Lemma 5.3]. Finally, the kernel can also be seen directly from the explicit description of P_x. Indeed, if g is in the intersection $\bigcap_{x\in\mathbf{R}} P_x$, then g is diagonal. But also, the conjugation action of g on root groups needs to be trivial, so that g is in the center of G. Conversely, the center of G clearly acts trivially on $\mathcal{I}$, which concludes the proof. $\square$

The convergence is then a more or less direct consequence of Theorem 2.59.

Theorem 2.78. *Let $((K_i, L_i))_{i\in\mathbf{N}}$ be a sequence in $\mathcal{L}$ which converges to (K, L). Let $T = T_{(K,L)}$ (respectively $T_i = T_{(K_i,L_i)}$) be the Bruhat–Tits tree of $G = G_{(K,L)}$ (respectively $G_i = G_{(K_i,L_i)}$). For N big enough and for all $i \geq N$, there exist isomorphisms $T_i \cong T$ such that the induced embeddings $\hat{G}_i \hookrightarrow \mathrm{Aut}(T)$ make $(\hat{G}_i)_{i\geq N}$ converge to $\hat{G}$ in the Chabauty topology of $\mathrm{Aut}(T)$.*

Remark 2.79. The convergence depends on a choice of specific isomorphisms $T_i \cong T$, or in other words it depends on choosing how $\hat{G}_i$ sits in $\mathrm{Aut}(T)$. This dependence is not problematic since for two isomorphic closed subgroups H, H' of $\mathrm{Aut}(T)$ both acting 2-transitively on ∂T, there exists g in the fixator of e_0 such that $gHg^{-1} = H'$, where e_0 is any edge of T (see [Rad15, Proposition A.1], and recall also that H acts transitively on the edges of T). Hence, for other choices of embeddings, the sequence converges to a conjugate of $\hat{G}$ in $\mathrm{Aut}(T)$. Recall also that we introduced the space $\mathcal{S}_T$ in the introduction precisely to avoid this dependence.

The main step of the proof is to establish that the sequence of stabilisers $((\hat{P}_0)_{G_i})_{i\geq N}$ converges to the stabiliser $(\hat{P}_0)_G$ in $\mathrm{Aut}(T)$. From there, we can conclude that $(\hat{G}_i)_{i\geq N}$ converges to $\hat{G}$ from general theory.

Proof. As we recall in the introduction, the Bruhat–Tits tree T is regular of degree p^n+1 (respectively semiregular of degree $(p^{3n}+1; p^n+1)$) if the pair (K, L) is trivial or ramified (respectively unramified) and belongs to $\mathcal{L}_{p^n}$. This shows that there exists N such that for all $i \geq N$, $T_i \cong T$.

Passing to a subsequence, we can assume that (K_i, L_i) is $(2i+1)$-close to (K, L). We now define an explicit isomorphism $f_i \colon T_i = \mathcal{I}_{G_i} \to \mathcal{I}_G = T$ as follows: let $\mathcal{I}_{G_i}^{0,2i} \cong \mathcal{I}_G^{0,2i}$ be the isomorphism given by Proposition 2.75. By Theorem 2.59, this gives an isomorphism on balls of radius $2i$: $\mathcal{I}_{G_i} \supset B_0(2i) \cong B_0(2i) \subset \mathcal{I}_G$ (recall that by Lemma 2.58, $B_0(2i)$ is really the ball of radius $2i$ on the tree $\mathcal{I}_G$). As $\mathcal{I}_{G_i}$ is a semiregular tree of the same bidegree than $\mathcal{I}_G$, we can extend this isomorphism of balls to an isomorphism $f_i \colon \mathcal{I}_{G_i} \to \mathcal{I}_G$ (this extension is of course not unique, but we choose one such). By means of f_i, we get an embedding $\hat{G}_i \to \mathrm{Aut}(T)$.

We claim that $((\hat{P}_0)_{G_i})_{i \in \mathbf{N}}$ converges to $(\hat{P}_0)_G$. Proving this claim amounts to prove two things (according to [CR16, Lemma 2.1]).

1. Let $(\hat{h}_i)$ be a sequence such that $\hat{h}_i \in (\hat{P}_0)_{G_i}$, and assume that $\hat{h}_i$ converges to $\hat{h}$ in $\mathrm{Aut}(T)$. We have to show that $\hat{h} \in (\hat{P}_0)_G$. For all i, let $h_i \in (P_0)_{G_i}$ be an inverse image of $\hat{h}_i$ under $\hat{}\, : G_i \to \mathrm{Aut}(T)$. Let $\bar{h}_i = p_{2i}(h_i) \in (P_0^{0,2i})_{G_i}$. Let $\varphi_{2i} \colon (P_0^{0,2i})_{G_i} \cong (P_0^{0,2i})_G$ be the isomorphism given in Proposition 2.75. By Corollary 2.53, there exists $\tilde{h}_i \in (P_0)_G$ which is an inverse image of $\varphi_{2i}(\bar{h}_i)$ under $p_{2i} \colon (P_0)_G \to (P_0^{0,2i})_G$. Now, because all the identifications were equivariant, the action of $\tilde{h}_i$ on the ball of radius $2i$ around 0 is the same than the action of $\hat{h}_i$ on this ball. Hence, $(\hat{\tilde{h}}_i)$ converges to $\hat{h}$ as well. But $(\hat{P}_0)_G$ is a closed subgroup of $\mathrm{Aut}(T)$ (by Proposition 2.77), hence $\hat{h} \in (\hat{P}_0)_G$, as wanted.

2. Conversely, given an element $\hat{h} \in (\hat{P}_0)_G$, we have to find a sequence $(\hat{h}_i)$ of elements in $(\hat{P}_0)_{G_i}$ such that $\hat{h}_i$ converges to $\hat{h} \in \mathrm{Aut}(T)$. It suffices to follow the path of identifications in reverse : let h be an inverse image of $\hat{h}$ under $\hat{}\, : G \to \mathrm{Aut}(T)$. Let $\bar{h}_i = p_{2i}(h) \in (P_0^{0,2i})_G$, and let $\varphi_{2i} \colon (P_0^{0,2i})_G \cong (P_0^{0,2i})_{G_i}$ be the isomorphism given in Proposition 2.75. For all i, let h_i be an inverse image of $\varphi_{2i}(\bar{h}_i)$ under $p_{2i} : (P_0)_{G_i} \to (P_0^{0,2i})_{G_i}$, which exists by Corollary 2.53. Now, because all the identifications were equivariant, the action of h_i on the ball of radius i around 0 is the same than

the action of h on this ball. Hence, $(\hat{h}_i)$ converges to $\hat{h}$, as wanted.

Finally, from the convergence of $((\hat{P}_0)_{G_i})_{i\geq N}$ to $(\hat{P}_0)_G$, we can formally deduce the convergence of $(\hat{G})_{i\geq N}$ to $\hat{G}$. Indeed by [CR16, Theorem 1.2], $(\hat{G}_i)_{i\geq N}$ subconverges to a topologically simple group H. But since $((\hat{P}_0)_{G_i})_{i\geq N}$ converges to $(\hat{P}_0)_G$, H has an open compact subgroup isomorphic to $(\hat{P}_0)_G$. Hence, by [CS15, Corollary 1.3], H is algebraic. And hence, by [Pin98, Corollary 0.3], $H \cong G$. Since by the same argument, any subsequence of $(\hat{G}_i)_{i\geq N}$ subconverges to $\hat{G}$, we conclude that $(\hat{G}_i)_{i\geq N}$ converges to $\hat{G}$. □

Similarly, we can prove the corresponding statement for sequences in $\mathcal{D}$.

Theorem 2.80. *Let $(D_i)_{i\in\mathbf{N}}$ be a sequence in $\mathcal{D}$ which converges to D. Let $T = T_D$ (respectively $T_i = T_{D_i}$) be the Bruhat–Tits tree of $G = G_D$ (respectively $G_i = G_{D_i}$). For N big enough and for all $i \geq N$, there exist isomorphisms $T_i \cong T$ such that the induced embeddings $\hat{G}_i \hookrightarrow \mathrm{Aut}(T)$ make $(\hat{G}_i)_{i\geq N}$ converge to $\hat{G}$ in the Chabauty topology of* $\mathrm{Aut}(T)$.

Proof. The Bruhat–Tits tree T_{D_i} is the regular tree of degree $p^n + 1$ if and only if D_i belongs to $\mathcal{D}_{p^n}$. Hence there exists N such that for all $i \geq N$, $T_i \cong T$.

Passing to a subsequence, we can assume that D_i is (di)-close to D, where D is of degree d over its center. Hence, for $i \geq 1$, D_i is also of degree d over its center. We now define an explicit isomorphism $f_i\colon T_i = \mathcal{I}_{G_i} \to \mathcal{I}_G = T$ as follows: let $\mathcal{I}_{G_i}^{0,di} \cong \mathcal{I}_G^{0,di}$ be the isomorphism given by Proposition 2.76. By Theorem 2.59, this gives an isomorphism on balls of radius di: $\mathcal{I}_{G_i} \supset B_0(di) \cong B_0(di) \subset \mathcal{I}_G$ (recall that by Lemma 2.58, $B_0(di)$ is really the ball of radius di on the tree $\mathcal{I}_G$). As $\mathcal{I}_{G_i}$ is a regular tree of the same degree than $\mathcal{I}_G$, we can extend this isomorphism of balls to an isomorphism $f_i\colon \mathcal{I}_{G_i} \to \mathcal{I}_G$ (this extension is of course not unique, but we choose one such). By means of f_i, we get an embedding $\hat{G}_i \hookrightarrow \mathrm{Aut}(T)$.

Now, the end of the proof is word for word the same than the corresponding end of the proof of Theorem 2.78, upon replacing all 2's with

d's, and upon replacing the reference to Proposition 2.75 with a reference to Proposition 2.76. $\square$

Proof of Theorem 2.3. Let T be a semiregular tree and $\mathcal{L}_T = \{(K, L) \in \mathcal{L} \mid \text{the Bruhat–Tits tree of } G_{(K,L)} \text{ is isomorphic to } T\}$. By Remark 2.23 and Proposition 2.71, $\mathcal{L}_T$ is a compact space. Now, by Theorem 2.78, the map $\mathcal{L}_T \to \mathcal{S}_T\colon (K, L) \mapsto \hat{G}_{(K,L)}$ is continuous. We claim that it is injective as well. Indeed, if $\hat{G}_{(K_1,L_1)}$ is abstractly isomorphic to $\hat{G}_{(K_2,L_2)}$, then by [BT73, Corollaire 8.13], the corresponding adjoint algebraic groups $\mathrm{Ad}\,\mathbf{G}_1$ and $\mathrm{Ad}\,\mathbf{G}_2$ are algebraically isomorphic over an isomorphism of fields $K_1 \cong K_2$. Since $\mathrm{Ad}\,\mathbf{G}_1$ (respectively $\mathrm{Ad}\,\mathbf{G}_2$) is quasi-split, there exists a smallest extension splitting it ([BT84a, 4.1.2]), namely L_1 (respectively L_2). Hence, $(K_1, L_1) \cong (K_2, L_2)$, as wanted.

To summarise, $\mathcal{L}_T \to \mathcal{S}_T\colon (K, L) \mapsto \hat{G}_{(K,L)}$ is an injective continuous map whose source is a compact space, hence it is a homeomorphism onto its image. Now, the explicit description given in Theorem 2.3 follows from Remark 2.23 and Proposition 2.71. $\square$

Proof of Theorem 2.6. Let T be a regular tree and let $\mathcal{D}_T = \{D \in \mathcal{D} \mid \text{the Bruhat–Tits tree of } G_D \text{ is isomorphic to } T\}$. By Remark 2.23 and Proposition 2.73, $\mathcal{D}_T$ is a compact space. Now, by Theorem 2.80, the map $\mathcal{D}_T \to \mathcal{S}_T\colon D \mapsto \hat{G}_D$ is continuous. Let D_1 and D_2 be central division algebras over K_1 and K_2 respectively, with respective degree d_1, d_2 and Hasse invariant r_1, r_2 (as defined in Definition 2.90). We claim that $\hat{G}_{D_1} = \hat{G}_{D_2}$ if and only if $K_1 \simeq K_2$, $d_1 = d_2$ and $r_1 = \pm r_2$. Indeed, if $\hat{G}_{D_1}$ is abstractly isomorphic to $\hat{G}_{D_2}$, then by [BT73, Corollaire 8.13], the corresponding adjoint algebraic groups $\mathrm{Ad}\,\mathbf{G}_1$ and $\mathrm{Ad}\,\mathbf{G}_2$ are algebraically isomorphic over an isomorphism of fields $K_1 \cong K_2$. Now, according to [KMRT98, Remark 26.11], this is only possible if $D_1 \cong D_2$ or $D_1 \cong D_2^{opp}$, which is equivalent to the given condition.

To summarise, let $\mathcal{D}_T/\sim_{opp}$ be the space $\mathcal{D}_T$ modulo the equivalence relation $D_1 \sim_{opp} D_2$ if and only if $D_1 \cong D_2$ or $D_1 \cong D_2^{opp}$. We proved that $\mathcal{D}_T/\sim_{opp} \to \mathcal{S}_T\colon D \mapsto \hat{G}_D$ is an injective continuous map whose source is a compact space, hence it is a homeomorphism onto its image. Now, the explicit description given in Theorem 2.6 follows from Remark 2.23 and Proposition 2.73.

To be able to conclude that for T the (p^n+1)-regular tree, $\mathcal{S}_T^{\mathrm{SL}_2(D)}$ is homeomorphic to $\hat{\mathbf{N}} \times \{1, \ldots, \lceil \frac{n+1}{2} \rceil\}$, one has to count the number of division algebras in $\mathcal{D}_T/\sim_{opp}$ of characteristic p. But there is only one such division algebra in $\mathcal{D}_T/\sim_{opp}$ of degree 1 over its center, one such division algebra in $\mathcal{D}_T/\sim_{opp}$ of degree 2 over its center if 2 divides n, and for all $3 \leq d$ dividing n, there are $\frac{\varphi(d)}{2}$ such division algebras in $\mathcal{D}_T/\sim_{opp}$ of degree d over their center (where φ denotes Euler's totient function). Hence, if n is even (respectively odd), we have $2+\sum_{d|n, d\geq 3} \frac{\varphi(d)}{2}$ (respectively $1+\sum_{d|n,d\geq 3} \frac{\varphi(d)}{2}$) division algebras of characteristic p in $\mathcal{D}_T/\sim_{opp}$. Using that $\sum_{d|n} \varphi(d) = n$, we readily get the conclusion. $\square$

2.A Comparison with the original Bruhat–Tits definitions

The purpose of this appendix is to show that our definition of the Bruhat–Tits tree agrees with the one in [BT72, 7.4.1 and 7.4.2]. Since the relative rank of $\mathrm{SL}_2(D)$ and SU_3 is 1, it is already clear that the apartment A is indeed isomorphic to $\mathbf{R}$. The main task is to show that our group P_x is the same as the group $\hat{P}_x$ used to define the equivalence relation in [BT72, 7.4.1].

In the $\mathrm{SL}_2(D)$ case, the explicit description of $\hat{P}_x$ is given in [BT72, Corollaire 10.2.9], that we take as a definition.

Definition 2.81 ([BT72, Corollaire 10.2.9])**.** Let $\{a_1, a_2\}$ be the canonical basis of $\mathbf{R}^2$, and let $a_{ij} = a_j - a_i$ $(i, j \in \{1, 2\})$. We can see $\mathbf{R}$ as a vector space V, dual of the vector space $V^* = \mathbf{R}\,.a_{12}$. Now, for $x \in \mathbf{R}$, $\hat{P}_x = \{g \in \mathrm{SL}_2(K) \mid \omega(g_{ij}) \geq a_{ji}(x), \text{ for all } 1 \leq i, j \leq 2\}$.

Note that we can omit the factor $(r+1)^{-1}\delta$ appearing in loc. cit. since by definition, $\delta = \omega(\det(g)) = \omega(1) = 0$.

This description obviously depends on the identification of $\mathbf{R}$ as the dual of V^*. Now, if we furthermore impose the condition $a_{12} = \mathrm{Id}\colon \mathbf{R} \to \mathbf{R}$, then $\hat{P}_x$ is indeed equal to the group P_x of Definition 2.12. To end the comparison between [BT72, Définition 7.4.2] and our Definition 2.21, one has to show that the maps $\nu\colon N \to \mathrm{Aff}(\mathbf{R})$ are the same. This is easily obtained by comparing [BT72, Proposition 10.2.5 (ii)] with our Definition 2.20.

In the SU_3 case, as in Definition 2.7, we index the rows and the columns of a 3-by-3 matrix by $\{-1, 0, 1\}$. Let a_1 be a non-trivial element of $\mathbf{R}^*$, and set $a_{-1} = -a_1$ and $a_0 = 0$. We now take some time to spell out the definition of $\overline{\omega}_{ij}$ as defined in [BT72, 10.1.27].

Definition 2.82. Recall the definition of the element $l \in L$ we introduced in Definition 2.15. Namely, $l = \begin{cases} t\alpha^{-1} \text{ if } \alpha \neq 0 \\ \frac{1}{2} \text{ if } \alpha = 0 \end{cases}$, where t and α are as in Lemma 2.13.

Lemma 2.83. *Let $L^1 = \{x \in L \mid x + \bar{x} = 1\}$ and $L^1_{\max} = \{x \in L^1 \mid \omega(x) = \sup\{\omega(x) \mid x \in L^1\}\}$. The element $l \in L$ in Definition 2.82 belongs to $L^1_{\max}$*

Proof. See [BT84a, 4.3.3 (ii)]. □

Definition 2.84 ([BT72, 10.1.20])**.** Let q be the pseudo-quadratic form associated with the hermitian form used to defined SU_3 (see Remark 2.8). Explicitly, for $x \in L^3$, $q(x) = lf(x, x) + L^0$, where $L^0 = \{x \in L \mid x + \bar{x} = 0\}$ (see [BT72, 10.1.1 (7), (8)]). For $x \in L$, we define $\omega_q(x) = \frac{1}{2}\sup\{\omega(k) \mid k \in q((0, x, 0))\} = \frac{1}{2}\sup\{\omega(k) \mid k \in l\bar{x}x + L^0\}$.

We can actually compute explicitly the value of ω_q.

Lemma 2.85.

1. $\omega_q(x) = \omega(x) + \omega_q(1)$
2. $\omega_q(1) = \frac{1}{2}\omega(l)$

Hence, $\omega_q(x) = \omega(x) + \frac{1}{2}\omega(l)$

Proof. The first property follows from the definition, and the second one is Lemma 2.83. □

Definition 2.86 ([BT72, 10.1.27])**.** Let $\{e_{-1}, e_0, e_1\}$ be the canonical basis of L^3. For $g \in End(L^3)$, let (g_{ij}) be the matrix of g in the basis $\{e_{-1}, e_0, e_1\}$. For $i, j \in \{-1, 0, 1\}$, we define $\bar{\omega}_{ij}(g) = \tilde{\omega}_i(g_{ij}) - \tilde{\omega}_j(1)$, where $\tilde{\omega}_{\pm 1} = \omega$, while $\tilde{\omega}_0 = \omega_q$.

Remark 2.87. The definition given in [BT72, 10.1.27] is readily checked to agree with ours. Indeed, we can take advantage of the fact that X_0 is one dimensional. Let us identify $\mathrm{Hom}(X_j, X_i)$ with L, through the basis $\{e_{-1}, e_0, e_1\}$, and define ω_i as in [BT72, 10.1.27]. Then, for $x \in L$ and $\alpha \in \mathrm{Hom}(X_j, X_i) \cong L$, we have $\omega_i(\alpha(xe_j)) - \omega_j(xe_j) = \omega_i((\alpha x)e_i) - \omega_j(xe_j) = \tilde{\omega}_i(\alpha x) - \tilde{\omega}_j(x) = \tilde{\omega}_i(\alpha) - \tilde{\omega}_j(1)$.

Definition 2.88 ([BT72, Corollaire 10.1.33])**.** With these notations, $\hat{P}_x = \{g \in \mathrm{SU}_3(K) \mid \overline{\omega}_{ij}(g) \geq a_i(x) - a_j(x), i, j \in \{-1, 0, 1\}\}$.

Note that we can omit the factor $\frac{1}{2}\omega c(g)$ appearing in loc. cit. since by definition, $c(g)$ is the similitude ratio (see [BT72, Definition 10.1.4]) and is equal to 1 for $g \in \mathrm{SU}_3$.

Again, this description depends on the choice of a non-trivial element in $\mathbf{R}^*$. Now, if we choose $a_1 \colon \mathbf{R} \to \mathbf{R} \colon x \to \frac{x}{2}$, then for $x \in \mathbf{R}$, the group $\hat{P}_x$ of Definition 2.88 is indeed equal to the group P_x of Definition 2.18. To end the comparison between [BT72, Definition 7.4.2] and our Definition 2.21, one has to show that the maps $\nu \colon N \to \mathrm{Aff}(\mathbf{R})$ are the same. This is easily obtained by comparing [BT72, Proposition 10.1.28 (iii)] with our Definition 2.20.

2.B A review of the theory of CSA over local fields

Let D be a central division algebra of degree d over a local field K (recall that the degree of D over K is the square root of the dimension of the K-vector space D). It is well known that such division algebras are classified (up to isomorphism) by elements of $(\mathbf{Z}/d\mathbf{Z})^{\times}$.

To be explicit, for $r \in (\mathbf{Z}/d\mathbf{Z})^{\times}$, the corresponding division algebra is the cyclic algebra $(E/K, \sigma^r, \pi_K)$ where

- E is the unramified extension of K of dimension d.
- $\sigma \in \mathrm{Gal}(E/K)$ is the element in $\mathrm{Gal}(E/K)$ inducing the Frobenius automorphism on $\overline{E}$.

For the reader's convenience, we recall the definition of a cyclic algebra.

Definition 2.89. Let K be a field and let E/K be a cyclic extension of degree d. Let σ be a generator of $\mathrm{Gal}(E/K)$, and let $a \in K^\times$. The cyclic algebra $(E/K, \sigma, a)$ is defined as follows:

- $(E/K, \sigma, a) = \bigoplus_{i=0}^{d-1} u^i E$
- $u^{-1}xu = \sigma(x)$, for all $x \in E$
- $u^d = a$

Definition 2.90. As in [dT15], for a finite central division algebra D of degree d over K, we call the corresponding element $r \in (\mathbf{Z}/d\mathbf{Z})^\times$ the Hasse invariant of D.

An important fact about such a division algebra D is that it splits over E. It is important for us to describe explicitly the embedding of D inside $M_d(E)$, the algebra of $d \times d$ matrices with coefficients in E.

Definition 2.91. Let D be a division algebra isomorphic to the cyclic algebra $(E/K, \sigma^r, \pi_K)$ of degree d over K. Consider the isomorphism of (right) E-vector spaces

$$f\colon E^n \to D\colon v = (v_1, \ldots, v_n) \mapsto \sum_{i=0}^{d-1} u^i v_{i+1}$$

Let $\varphi\colon D \to M_n(E)\colon x \mapsto (v \mapsto f^{-1}(x.f(v)))$. More explicitly,

$$\varphi(\sum_{i=0}^{d-1} u^i x_{i+1}) = \begin{pmatrix} x_1 & \pi_K\sigma^r(x_d) & \pi_K\sigma^{2r}(x_{d-1}) & \ldots & \pi_K\sigma^{(d-1)r}(x_2) \\ x_2 & \sigma^r(x_1) & \pi_K\sigma^{2r}(x_d) & \ldots & \pi_K\sigma^{(d-1)r}(x_3) \\ x_3 & \sigma^r(x_2) & \sigma^{2r}(x_1) & \ldots & \pi_K\sigma^{(d-1)r}(x_4) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_d & \sigma^r(x_{d-1}) & \sigma^{2r}(x_{d-1}) & \ldots & \sigma^{(d-1)r}(x_1) \end{pmatrix}$$

We can now properly spell out the definition of the reduced norm.

Definition 2.92. Let D be a division algebra isomorphic to the cyclic algebra $(E/K, \sigma^r, \pi_K)$ of degree d over K. We define the reduced norm Nrd as follows:

$$\mathrm{Nrd}\colon M_n(D) \to K\colon g \to \det(\varphi(g_{ij}))$$

where $\varphi(g_{ij})$ is seen as a $(dn)^2$ matrix with coefficient in E.

We end this discussion by an analysis of the ring of integers D.

Lemma 2.93. *Let D be a central division algebra over K of degree d, and let $r \in \mathbf{N} \cup \{\infty\}$ (with the convention that $\mathfrak{m}^\infty = (0)$). Since E is unramified, $\mathcal{O}_E/\mathfrak{m}_E^r \cong \mathcal{O}_K/\mathfrak{m}_K^r \oplus \cdots \oplus \mathcal{O}_K/\mathfrak{m}_K^r$. Furthermore, $\mathcal{O}_D/\mathfrak{m}_D^{rd} \cong \bigoplus_{i=0}^{d-1} u^i.\mathcal{O}_E/\mathfrak{m}_E^r$. Otherwise stated, $\mathcal{O}_D/\mathfrak{m}_D^{rd}$ is a free $\mathcal{O}_E/\mathfrak{m}_E^r$-module, and we can define a map $\overline{\varphi}\colon \mathcal{O}_D/\mathfrak{m}_D^{rd} \hookrightarrow M_d(\mathcal{O}_E/\mathfrak{m}_E^r)$, which is compatible with the map φ of Definition 2.91, in the sense that the following diagram commutes*

$$\begin{array}{ccc} \mathcal{O}_D & \hookrightarrow & M_d(\mathcal{O}_E) \\ \downarrow & & \downarrow \\ \mathcal{O}_D/\mathfrak{m}_D^{rd} & \hookrightarrow & M_d(\mathcal{O}_E/\mathfrak{m}_E^r) \end{array}$$

Proof. This is straightforward from the definitions. $\square$

2.C An integral model of $\mathrm{SL}_2(D)$

Recall that the group $\mathrm{SL}_2(D)$ consists of the 2×2 matrices with coefficient in D having reduced norm 1 (Definition 2.10). Recall the definition of the embedding $\varphi\colon D \to M_n(E)$ given in Definition 2.91. In view of the definition of the reduced norm (Definition 2.92), we arrive at the following explicit definition of $\mathrm{SL}_2(D)$.

Definition 2.94. $\mathrm{SL}_2(D) = \{\left(\begin{smallmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{smallmatrix}\right) \mid g_{ij} \in D,\ \det(\varphi(g_{ij})) = 1\}$

Mimicking this definition, we can define a similar group over $\mathcal{O}_D/\mathfrak{m}_D^{rd}$.

Definition 2.95. Let D be a central division algebra over K of degree d, and let $r \in \mathbf{N} \cup \{\infty\}$. Keeping the notations of Lemma 2.93, we define

$$\mathrm{SL}_2(\mathcal{O}_D/\mathfrak{m}_D^{rd}) = \{\left(\begin{smallmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{smallmatrix}\right) \mid g_{ij} \in \mathcal{O}_D/\mathfrak{m}_D^{rd},\ \det\left(\begin{pmatrix} \overline{\varphi}(g_{11}) & \overline{\varphi}(g_{12}) \\ \overline{\varphi}(g_{21}) & \overline{\varphi}(g_{22}) \end{pmatrix}\right) = 1\}$$

Let us now discuss the underlying algebraic group of $\mathrm{SL}_2(D)$. Let $M_2(D)$ be the algebra of 2×2 matrices with coefficient in D. Using the embedding $D \hookrightarrow M_d(E)$, we can identify $M_2(D)$ with a K-linear subspace of $M_{2d}(E)$. Now, $\mathrm{SL}_2(D)$ is the closed subspace of $M_2(D) \cong \mathbf{A}_K^{(2d)^2}$ cut out by the polynomial equation $\mathrm{Nrd} = 1$. We can now mimic this situation over the ring of integers to define an integral model of $\mathrm{SL}_2(D)$.

Definition 2.96. Let D be a central division algebra of degree d over K, and let $M_2(\mathcal{O}_D)$ be the $\mathcal{O}_K$-algebra of 2×2 matrices with coefficient in $\mathcal{O}_D$. Using the embedding $\mathcal{O}_D \hookrightarrow M_d(\mathcal{O}_E)$, where E is the unramified extension of K of degree d, we can identify $M_2(\mathcal{O}_D)$ with a free $\mathcal{O}_K$-submodule of $M_{2d}(\mathcal{O}_E)$. We define the $\mathcal{O}_K$-scheme $\underline{\mathrm{SL}}_{2,D}$ to be the closed subscheme of $M_2(\mathcal{O}_D) \cong \mathbf{A}_{\mathcal{O}_K}^{(2d)^2}$ cut out by the polynomial equation $\mathrm{Nrd} = 1$.

Of course, the crucial point is to check that $\underline{\mathrm{SL}}_{2,D}$ is in fact smooth.

Theorem 2.97. *$\underline{\mathrm{SL}}_{2,D}$ is a smooth $\mathcal{O}_K$-scheme.*

Proof. This is one of the main results in [BT84b]. Let us explain how to extract it from there. Let φ be the valuation of $GL_2(D)$ defined in [BT84b, 2.2, display (4)]. The valuation φ is thus a point of the enlarged apartment A_1. The associated norm is defined as $\alpha_\varphi(e_1x_1 + e_2x_2) = \inf\{\omega(x_1), \omega(x_2)\}$ (following the definition in [BT84b, 2.8, display (9)]). The corresponding order $\mathscr{M}_{\alpha_\varphi}$ of $M_2(D)$ defined in [BT84b, 1.17] is $\{\left(\begin{smallmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{smallmatrix}\right) \in M_2(D) \mid \omega(g_{ij}) \geq 0\}$ (this is easily computed using the description of $\operatorname{End}\alpha(u)$ in [BT84b, 1.11, display (17)]). Note that $\mathscr{M}_{\alpha_\varphi}$ is isomorphic to the affine space $\mathbf{A}_{\mathcal{O}_K}^{(2d)^2}$ (being a free $\mathcal{O}_K$-module). Finally, following [BT84b, 3.6], let $\mathfrak{G}_\varphi$ be the (principal) open subscheme of the affine space $\mathscr{M}_{\alpha_\varphi}$ defined by the non-vanishing of the reduced norm (see also [BT84b, 3.2]).

$\mathfrak{G}_\varphi$ is actually an integral model for $GL_2(D)$, and in [BT84b, §5], the $\mathrm{SL}_2(D)$ case is then treated. Let $\mathfrak{G}_{1,\varphi}$ be the schematic adherence of $\mathrm{SL}_2(D)$ in $\mathfrak{G}_\varphi$ (following the definition in [BT84b, 5.3]). It is mentioned in [BT84b, 5.5] that the group $\mathfrak{G}_{1,\varphi}$ is the closed subgroup of $\mathfrak{G}_\varphi$ defined by the equation $\mathrm{Nrd} = 1$, and hence it coincides with our group $\underline{SL}_{2,D}$. But by [BT84b, 5.5], $\mathfrak{G}_{1,\varphi}$ is smooth over $\mathcal{O}_K$, concluding the proof. Note that to apply [BT84b, 5.5], we should check that a finite unramified extension of a local field is étale in the sense of [BT84b]. But this is clear in view of [BT84a, 1.6.1 (f) and Definition 1.6.2]. □

We conclude our study of the $\mathrm{SL}_2(D)$ case by identifying the rational points of $\underline{\mathrm{SL}}_{2,D}$.

Lemma 2.98. *Let D be a central division algebra over K of degree d, and let $r \in \mathbf{N} \cup \{\infty\}$. Then $\underline{\mathrm{SL}}_{2,D}(\mathcal{O}_K/\mathfrak{m}_K^r) \cong \mathrm{SL}_2(\mathcal{O}_D/\mathfrak{m}_D^{rd})$ (where by convention, $\mathfrak{m}^\infty = (0)$).*

Proof. Because the diagram appearing in Lemma 2.93 is commutative, we have

$$\begin{aligned}\underline{\mathrm{SL}}_{2,D}(\mathcal{O}_K/\mathfrak{m}_K^r) &= \{\left(\begin{smallmatrix} g_{11} & g_{12} \\ g_{21} & g_{22}\end{smallmatrix}\right) \in M_2(\mathcal{O}_D/\mathfrak{m}_D^{rd}) \mid \mathrm{Nrd}(g) = 1\} \\ &\cong \{\left(\begin{smallmatrix} g_{11} & g_{12} \\ g_{21} & g_{22}\end{smallmatrix}\right) \in M_2(\mathcal{O}_D/\mathfrak{m}_D^{rd}) \mid \det(\begin{pmatrix} \overline{\varphi}(g_{11}) & \overline{\varphi}(g_{12}) \\ \overline{\varphi}(g_{21}) & \overline{\varphi}(g_{22})\end{pmatrix}) = 1\}\end{aligned}$$

as wanted. □

Chapter 3

On the semilinear automorphism group of a semisimple algebraic group

Let G be an algebraic group over a field k, and consider the group of semilinear automorphisms $\mathrm{Aut}(G \to \operatorname{Spec} k)$, which consists of algebraic automorphisms of G over automorphisms of k. We study the splitting of the exact sequence $1 \to (\mathrm{Aut}\, G)(k) \to \mathrm{Aut}(G \to \operatorname{Spec} k) \to \mathrm{Aut}_G(k) \to 1$. If G is a reductive group, we can consider another short exact sequence $1 \to (\mathrm{Aut}\, \mathrm{Dyn}G)(k) \to \mathrm{Aut}(\mathrm{Dyn}G \to \operatorname{Spec} k) \to \mathrm{Aut}_{\mathrm{Dyn}G}(k) \to 1$ describing semilinear automorphisms of the scheme of Dynkin diagrams $\mathrm{Dyn}G$ of G over automorphisms of k. We show that if G is semisimple, simply connected and quasi-split, the former exact sequence splits if and only if the latter exact sequence splits, and we explain why the splitting of the latter exact sequence is easy to check in practice. As a corollary, we get lots of examples of algebraic groups G over k whose group of abstract automorphisms does not decompose as the semidirect product of $(\mathrm{Aut}\, G)(k)$ with $\mathrm{Aut}_G(k)$. We also study the same question for inner forms of SL_n, restricting ourselves to the case when the ground field is a local field.

3.1 Introduction

In their famous article [BT73], Armand Borel and Jacques Tits describe the "abstract" automorphisms of the group of rational points of an absolutely simple algebraic group. To wit, here is one of their results:

Theorem 3.1 ([BT73, Corollaire 8.13])**.** *Let k be an infinite field, and let G be an absolutely simple algebraic group over k. Assume that G is either simply connected or adjoint, and that $G(k)$ is generated by the groups $U(k)$, where U runs through the set of unipotent algebraic k-subgroups of G that are split over k. Furthermore, if k is of characteristic 2 or 3, assume that k is not perfect. Let α be an automorphism of $G(k)$, considered as an abstract group. Then there exists a unique automorphism $\varphi\colon k \to k$ and a unique semilinear automorphism $f\colon G \to G$ over φ such that for $g \in G(k) = \mathrm{Hom}_{k\text{-schemes}}(\operatorname{Spec} k, G)$, we have $\alpha(g) = f_\varphi \circ g \circ \operatorname{Spec} \varphi^{-1}$.*

By a **semilinear automorphism** f_φ of a k-group scheme G over an automorphism $\varphi\colon k \to k$, we mean that we have the following commutative diagram in the category of group schemes

$$\begin{array}{ccc} G & \xrightarrow{f_\varphi} & G \\ \downarrow & & \downarrow \\ \operatorname{Spec} k & \xrightarrow{\operatorname{Spec}\varphi} & \operatorname{Spec} k \end{array}$$

where the vertical arrows are the structural morphisms of the k-scheme G. Let $\mathrm{Aut}(G \to \operatorname{Spec} k)$ denotes the group of semilinear automorphisms of G. We can then rephrase Theorem 3.1 as saying that under the assumptions of the theorem, the group of abstract automorphisms of $G(k)$ is isomorphic to $\mathrm{Aut}(G \to \operatorname{Spec} k)$.

Given a k-group scheme G, we have a homomorphism $\mathrm{Aut}(G \to \operatorname{Spec} k) \to \mathrm{Aut}(k)\colon f_\varphi \mapsto \varphi^{-1}$. Let $\mathrm{Aut}_G(k)$ denotes the image of this homomorphism, and let $(\mathrm{Aut}\, G)(k)$ denotes the group of k-algebraic automorphisms of G, or in other words the kernel of $\mathrm{Aut}(G \to \operatorname{Spec} k) \to \mathrm{Aut}(k)$. Those groups fit in the short exact sequence $1 \to (\mathrm{Aut}\, G)(k) \to \mathrm{Aut}(G \to \operatorname{Spec} k) \to \mathrm{Aut}_G(k) \to 1$.

It is well-known that the short exact sequence $1 \to (\mathrm{Aut}\, G)(k) \to \mathrm{Aut}(G \to \operatorname{Spec} k) \to \mathrm{Aut}_G(k) \to 1$ splits when G is a split algebraic group (a statement already made in [Tit74, Corollary 5.10]), and it is then natural to wonder if this remains the case in general. The aim of this article is to show that this is not the case. Furthermore, examples of non-splitting already occur when the algebraic group is quasi-split. Our main theorem gives a necessary and sufficient condition for $\mathrm{Aut}(G \to \mathrm{Spec}\, k)$ to be a split extension of $\mathrm{Aut}_G(k)$ when G is a semisimple, simply connected (or adjoint) quasi-split algebraic group over a field k.

Theorem 3.2 (The bowtie theorem)**.** *Let G be a semisimple, simply connected* (*or adjoint*) *quasi-split algebraic group over k, and let* $\mathrm{Dyn}G$ *be the scheme of Dynkin diagrams of G. Then the short exact sequence*

$$1 \to (\mathrm{Aut}\, G)(k) \to \mathrm{Aut}(G \to \mathrm{Spec}\, k) \to \mathrm{Aut}_G(k) \to 1$$

splits if and only if the short exact sequence

$$1 \to (\mathrm{Aut}\,\mathrm{Dyn}G)(k) \to \mathrm{Aut}(\mathrm{Dyn}G \to \mathrm{Spec}\, k) \to \mathrm{Aut}_{\mathrm{Dyn}G}(k) \to 1$$

splits.

The bowtie theorem (whose name is due to the diagram appearing in its proof) uses the notion of a scheme of Dynkin diagrams $\mathrm{Dyn}G$ (of a reductive k-group G), which was introduced in [ABD$^+$64]. In Section 3.3, we discuss in detail the definition of $\mathrm{Dyn}G$, and we state explicitly the classification of k-schemes isomorphic to a Dynkin's scheme $\mathrm{Dyn}G$, for G an absolutely simple k-group. We also use the notation $\mathrm{Aut}(\mathrm{Dyn}G \to \mathrm{Spec}\, k)$ for the group of semilinear automorphisms of $\mathrm{Dyn}G$, see Definition 3.12.

We can identify the short exact sequence $1 \to (\mathrm{Aut}\,\mathrm{Dyn}G)(k) \to \mathrm{Aut}(\mathrm{Dyn}G \to \mathrm{Spec}\, k) \to \mathrm{Aut}_{\mathrm{Dyn}G}(k) \to 1$ as a sequence involving various automorphism groups of fields associated with G (see Corollary 3.29). Using this description, we give in Corollary 3.32 many explicit examples for which $\mathrm{Aut}(G \to \mathrm{Spec}\, k)$ is not a split extension of $\mathrm{Aut}_G(k)$.

In the last section of the chapter, we also explore the question when G is an inner form of SL_n over a field k. The first step in finding conditions for non-splitting is to prove that $\mathrm{Aut}_G(k)$ is non-trivial. By exhibiting explicitly automorphisms of a cyclic division algebra, we can prove that over a local field K, $\mathrm{Aut}_G(K) = \mathrm{Aut}(K)$.

Theorem 3.3. *Let K be a non-archimedean local field, let D be a division algebra of degree d over K, and consider the algebraic group $G = \mathrm{SL}_n(D)$. Then $\mathrm{Aut}_G(K) = \mathrm{Aut}(K)$.*

We prove this theorem in Corollary 3.37. As we note in Remark 3.38, this was essentially already known when K is of characteristic 0, but we are not aware of such a result in positive characteristic.

By the theory of Galois descent, if k is a finite Galois extension of k', then giving a homomorphism $\mathrm{Gal}(k/k') \to \mathrm{Aut}(G \to \mathrm{Spec}\, k)$ whose composition with $\mathrm{Aut}(G \to \mathrm{Spec}\, k) \to \mathrm{Aut}_G(k)$ is the identity on $\mathrm{Gal}(k/k')$ is equivalent to give a descent datum (from k to k') for G (we recall this fact in Theorem 3.7). Hence, working with forms of algebraic groups, we are able to deduce the following theorem:

Theorem 3.4. *Let K be a non-archimedean local field, let D be a division algebra of degree d over K, and consider the algebraic group $G = \mathrm{SL}_n(D)$.*

1. *The short exact sequence $1 \to (\mathrm{Aut}\, G)(K) \to \mathrm{Aut}(G \to \mathrm{Spec}\, K) \to \mathrm{Aut}_G(K) \to 1$ does not split if there exists a subfield $K' \leq K$ such that K/K' is a finite Galois extension and $\gcd(nd, [K : K'])$ does not divide n.*

2. *If K is of characteristic 0, the converse holds, i.e. if for all subfields $K' \leq K$ such that K/K' is finite Galois, $\gcd(nd, [K : K'])$ divides n, then the short exact sequence $1 \to (\mathrm{Aut}\, G)(K) \to \mathrm{Aut}(G \to \mathrm{Spec}\, K) \to \mathrm{Aut}_G(K) \to 1$ splits.*

3. *If K is of characteristic $p > 0$, then K is isomorphic to $\mathbf{F}_{p^i}(\!(T)\!)$. Assume that for all subfields $K' \leq K$ such that K/K' is finite Galois, we have that $\gcd(nd, [K : K'])$ divides n. Further assume that $\gcd(d, i) = 1$. Then the short exact sequence $1 \to (\mathrm{Aut}\, G)(K) \to \mathrm{Aut}(G \to \mathrm{Spec}\, K) \to \mathrm{Aut}_G(K) \to 1$ splits.*

As we prove in Proposition 3.58, for $K = \mathbf{F}_{p^i}(\!(T)\!)$, the condition "for all subfield $K' \leq K$ such that K/K' is finite Galois, we have that gcd($nd, [K : K']$) divides n" is equivalent to requiring that $\gcd(d,p) = 1$ and that $\gcd(nd, i(p^i - 1))$ divides n. Hence, in positive characteristic, Theorem 3.4 leaves open the question of the existence of a splitting in a few cases. We explain in Appendix 3.B why our strategy is not able to cover those cases.

3.2 Semilinear automorphisms and descent

In this section, we work with a general (finite type, affine) group scheme G defined over a field k. We already gave in the introduction the definition of a semilinear automorphism of G. The vocabulary "semilinear automorphism" is already used in the literature (see for example [FSS98, Section 1.2]). It has the same meaning than our usage, except that it only applies when k is assumed to be the separable closure of a base field k_0, and when the underlying isomorphism acts trivially on k_0. In Section 3.6, we consider the case $k = \mathbf{F}_p(\!(T)\!)$, which is a more general situation.

The concept is also used in various later articles, see for examples [BKLR12, Section 3.2] and references therein. In those references, the notation $\mathrm{SAut}(G_{k_s})$ is used for the group of semilinear automorphisms. We prefer to use the notation $\mathrm{Aut}(G \to \operatorname{Spec} k)$ so that the ground field explicitly appears in the notation. Let us continue by recalling some standard vocabulary.

Definition 3.5. Let k' be a subfield of k and let H be a k'-group scheme.

1. The group of automorphisms of k whose restriction to k' is trivial is denoted $\mathrm{Aut}(k/k')$.

2. We denote by H_k the base change of H along $\operatorname{Spec} k \to \operatorname{Spec} k'$. If H_k is isomorphic to G (as a k-group scheme), we say that H is a k/k'-**form of** G (or just a form of G if the field extension is understood from the context). If there exists a k/k'-form of G, we say that G is **defined over** k'.

3. For l a field and $\varphi\colon k \to l$ a field homomorphism, for G' a k-group scheme and $f\colon G \to G'$ a homomorphism of k-group schemes, we

denote by $\varphi_* f\colon \varphi_* G \to \varphi_* G'$ the base change of f along $\operatorname{Spec}\varphi$.

The following elementary observation plays a fundamental role in this chapter.

Lemma 3.6. *Let k' be a subfield of k, and assume that G is defined over k'. Then there exists a homomorphism $\operatorname{Aut}(k/k') \to \operatorname{Aut}(G \to \operatorname{Spec} k)$ whose composition with $\operatorname{Aut}(G \to \operatorname{Spec} k) \to \operatorname{Aut}_G(k)$ is the identity on $\operatorname{Aut}(k/k')$. In particular, $\operatorname{Aut}_G(k)$ contains $\operatorname{Aut}(k/k')$.*

Proof. Let H be a k/k'-form of G. For $\varphi \in \operatorname{Aut}(k/k')$, we define

$$f_{\varphi^{-1}} = \mathrm{Id}_H \times \operatorname{Spec}\varphi^{-1}\colon H \times_{\operatorname{Spec} k'} \operatorname{Spec} k \to H \times_{\operatorname{Spec} k'} \operatorname{Spec} k.$$

The map $\operatorname{Aut}(k/k') \to \operatorname{Aut}(G \to \operatorname{Spec} k)\colon \varphi \mapsto f_{\varphi^{-1}}$ is a homomorphism. Furthermore, its composition with $\operatorname{Aut}(G \to \operatorname{Spec} k) \to \operatorname{Aut}_G(k)$ is the identity on $\operatorname{Aut}(k/k')$, as wanted. $\square$

In fact, if the field extension k/k' appearing in Lemma 3.6 is finite Galois, then we have a converse to Lemma 3.6 by the theory of Galois descent.

Theorem 3.7 (Galois descent)**.** *Let k' be a subfield of k such that k/k' is a finite Galois extension. If there exists a homomorphism $\operatorname{Aut}(k/k') \to \operatorname{Aut}(G \to \operatorname{Spec} k)$ whose composition with $\operatorname{Aut}(G \to \operatorname{Spec} k) \to \operatorname{Aut}_G(k)$ is the identity on $\operatorname{Aut}(k/k')$, then G is defined over k'.*

Proof. This is a classical result from descent theory, see [Poo10, Section 4.4]. Note that giving such a homomorphism is the same as giving a descent datum on G by [Poo10, Proposition 4.4.2], so that the result holds by [Poo10, Corollary 4.4.6]. $\square$

Note that for a field isomorphism $\varphi\colon k \to k$, giving a semilinear automorphism f_φ of G over φ amounts to give an isomorphism of k-group schemes $G \cong \varphi_* G$, as one can see from the following diagram:

$$\begin{tikzcd}
G \arrow[drr, "f_\varphi"] \arrow[dr] \arrow[ddr] & & \\
& \varphi_* G \arrow[r] \arrow[d] & G \arrow[d] \\
& \operatorname{Spec} k \arrow[r, "\operatorname{Spec}\varphi"'] & \operatorname{Spec} k
\end{tikzcd}$$

Hence, the group $\mathrm{Aut}(G \to \operatorname{Spec} k)$ can be described as

$$\{\mathrm{Isom}_{k\text{-group schemes}}(G, \varphi_* G) \mid \varphi \in \mathrm{Aut}(k)\}.$$

Let k_s be a separable closure of k, and let G_{k_s} be the base change of G over $\operatorname{Spec} k_s \to \operatorname{Spec} k$. We go on to describe semilinear automorphisms of G_{k_s}, and give their relation to semilinear automorphisms of a k_s/k-form of G_{k_s}.

Lemma 3.8. *Let H be a k_s/k-form of G_{k_s}, corresponding to the cocycle $c\colon \mathrm{Gal}(k_s/k) \to \mathrm{Aut}(G_{k_s})$. Let l be a field with separable closure l_s, and let $\alpha\colon k \to l$ be an isomorphism of fields. We choose an extension of α to $k_s \to l_s$ that we also denote α. Then $\alpha_* H$ is a l_s/l-form of $\alpha_*(G_{k_s})$, whose corresponding cocycle is given by $\alpha_* c\colon \mathrm{Gal}(l_s/l) \to \mathrm{Aut}(\alpha_*(G_{k_s}))\colon \lambda \mapsto \alpha_*(c_{\alpha^{-1}\lambda\alpha})$.*

Proof. Let $f\colon H_{k_s} \to G_{k_s}$ be a chosen isomorphism of k_s-group scheme. In view of [Con14, Lemma 7.1.1], for $\gamma \in \mathrm{Gal}(k_s/k)$, the automorphism c_γ arising from the choice of f is the following composition:

$$c_\gamma\colon G_{k_s} \cong \gamma_*(G_{k_s}) \overset{\gamma_* f^{-1}}{\cong} \gamma_*(H_{k_s}) \cong H_{k_s} \overset{f}{\cong} G_{k_s}\ .$$

Now, we choose $\alpha_* f\colon \alpha_*(H_{k_s}) \to \alpha_*(G_{k_s})$ as our isomorphism of l_s-group scheme. An element of $\mathrm{Gal}(l_s/l)$ is of the form $\alpha\gamma\alpha^{-1}$ for some $\gamma \in \mathrm{Gal}(k_s/k)$. For such an element of $\mathrm{Gal}(l_s/l)$, the automorphism $(\alpha_* c)_{\alpha\gamma\alpha^{-1}}$ of $\alpha_*(G_{k_s})$ arising from the choice of $\alpha_* f$ is:

$$\begin{aligned}(\alpha_* c)_{\alpha\gamma\alpha^{-1}}\colon \alpha_*(G_{k_s}) &\cong (\alpha\gamma\alpha^{-1})_*(\alpha_*(G_{k_s})) \overset{(\alpha\gamma)_* f^{-1}}{\cong} (\alpha\gamma\alpha^{-1})_*(\alpha_*(H_{k_s}))\\ &\cong \alpha_*(H_{k_s}) \overset{\alpha_* f}{\cong} \alpha_*(G_{k_s})\ ,\end{aligned}$$

so that indeed, $(\alpha_* c)_{\alpha\gamma\alpha^{-1}} = \alpha_*(c_\gamma)$. □

Lemma 3.9. *Let H be a k_s/k-form of G_{k_s}, corresponding to the cocycle $c\colon \mathrm{Gal}(k_s/k) \to \mathrm{Aut}(G_{k_s})$. Let $\alpha\colon k \to k$ be an automorphism of k. We choose an extension of α to $k_s \to k_s$ that we also denote α. Let $f\colon G_{k_s} \to \alpha_* G_{k_s}$ be a morphism of k_s-group scheme. For $\gamma \in \mathrm{Gal}(k_s/k)$, let $t_\gamma\colon G_{k_s} \to \gamma_* G_{k_s}$ be the isomorphism coming from the k-structure G*

on G_{k_s}. The morphism f descends to a morphism of $H \to \alpha_* H$ if and only if $\gamma_* f = \alpha_*(t_{\alpha^{-1}\gamma\alpha} c_{\alpha^{-1}\gamma\alpha}^{-1}) f c_\gamma t_\gamma^{-1}$ for all $\gamma \in \mathrm{Gal}(k_s/k)$, where $\gamma_* f$ is seen as a morphism from $\gamma_* G_{k_s}$ to $(\gamma\alpha)_* G_{k_s} = \alpha_*(\alpha^{-1}\gamma\alpha)_* G_{k_s}$.

Proof. Let us treat the case of a general k-scheme. Let X (respectively Y) be a k-scheme, and let X' (respectively Y') be a k_s/k-form of X_{k_s} (respectively Y_{k_s}), with corresponding cocycle $c^X\colon \mathrm{Gal}(k_s/k) \to \mathrm{Aut}(X_{k_s})$ (respectively $c^Y\colon \mathrm{Gal}(k_s/k) \to \mathrm{Aut}(X_{k_s})$). For $\gamma \in \mathrm{Gal}(k_s/k)$, let $t_\gamma^X\colon X_{k_s} \to \gamma_* X_{k_s}$ (respectively $t_\gamma^Y\colon Y_{k_s} \to \gamma_* Y_{k_s}$) be the isomorphism coming from the k-structure X (respectively Y) on X_{k_s} (respectively Y_{k_s}). From descend theory, we have that a morphism $f\colon X_{k_s} \to Y_{k_s}$ descends to $X' \to Y'$ if and only if $\gamma_* f = t_\gamma^Y (c_\gamma^Y)^{-1} f c_\gamma^X (t_\gamma^X)^{-1}$ for all $\gamma \in \mathrm{Gal}(k_s/k)$. Hence the result follows from Lemma 3.8. □

3.3 Scheme of Dynkin diagrams

In [ABD+64, Exposé 24, section 3] (for which one can consult the wonderful reissue [GP11]), the authors define what they call a "Dynkin's scheme" of a reductive group G. The strategy is to first define this Dynkin's scheme for split reductive groups, and then to use descent. Since this notion is not so widely known, let us review its definition.

Recall that a Dynkin diagram of a root system Φ is a graph (V, E) whose vertices V are labelled with $\{1, 2, 3\}$. The vertices are in $1-1$ correspondence with a set of simple roots in Φ. Two vertices are adjacent if and only if the corresponding simple roots are not orthogonal. Finally, the label of a vertex gives the square length of the corresponding root, taking the shortest root in each irreducible component to be of length 1. This motivates the following definition, where for k a field and for a set X, the notation X_k stands for the disjoint union $\coprod_{i\in X} \operatorname{Spec} k$ (in the category of schemes).

Definition 3.10. Let D be a Dynkin diagram with underlying set of vertices V. The **split scheme of Dynkin diagrams corresponding to** D over a field k is a triple (Δ, E, λ), where $\Delta = V_k$, E is a closed subscheme of $\Delta \times_k \Delta$, and λ is a k-morphism $\lambda\colon \Delta \to \{1, 2, 3\}_k$. Furthermore, the triple (Δ, E, λ) satisfies the following properties:

1. A pair in $\Delta \times_k \Delta$ belongs to E if and only if the corresponding vertices are adjacent in D.

2. For $v \in \Delta$ and $i \in \{1, 2, 3\}$, $\lambda(v) = i$ if and only if the corresponding vertex has label i in D.

A **split scheme of Dynkin diagrams** is a split scheme of Dynkin diagrams corresponding to D for some Dynkin diagram D, and a **scheme of Dynkin diagrams** is a form of a split scheme of Dynkin diagrams.

Remark 3.11. Let k_s be a separable closure of a field k. It follows from Definition 3.10 that a scheme of Dynkin diagrams over k splitting over k_s consists of the spectrum of two etale algebras Δ and E, together with a labelling map λ. Hence, following [KMRT98, Theorem 18.4], we can describe a scheme of Dynkin diagrams using the category of $\mathrm{Gal}(k_s/k)$-sets. Under this correspondence, Δ corresponds to a finite $\mathrm{Gal}(k_s/k)$-sets $\tilde{\Delta}$, E corresponds to a $\mathrm{Gal}(k_s/k)$ invariant subset $\tilde{E}$ of $\tilde{\Delta} \times \tilde{\Delta}$ (endowed with the diagonal action), and λ correspond to a $\mathrm{Gal}(k_s/k)$-equivariant map $\tilde{\Delta} \to \{1, 2, 3\}$ (the latter set being endowed with the trivial $\mathrm{Gal}(k_s/k)$ action). We use this correspondence to give an explicit description of scheme of Dynkin diagrams in the classification appearing in Corollary 3.25.

Morphisms of schemes of Dynkin diagrams are defined in the obvious way. Let us spell out the definition of semilinear automorphisms of schemes of Dynkin diagrams, mimicking our definition for group schemes.

Definition 3.12. Let k be a field and let Dyn be a scheme of Dynkin diagrams over k. Let Δ be the underlying set of vertices of Dyn (i.e. Δ is a form over k of $\{1, \ldots, n\}_l$ for some field extension l/k). A **semilinear automorphism** f_φ of Dyn over an automorphism $\varphi\colon k \to k$ is an automorphism $f_\varphi\colon \mathrm{Dyn} \to \mathrm{Dyn}$ in the category of schemes of Dynkin diagrams such that the following diagram in the category of schemes commutes

$$\begin{array}{ccc} \Delta & \xrightarrow{(f_\varphi)_{|\Delta}} & \Delta \\ \downarrow & & \downarrow \\ \operatorname{Spec} k & \xrightarrow{\operatorname{Spec} \varphi} & \operatorname{Spec} k \end{array}$$

We let $\mathrm{Aut}(\mathrm{Dyn} \to \operatorname{Spec} k)$ denotes the group of semilinear automorphisms of Dyn.

As in the case of k-group schemes, for Dyn a scheme of Dynkin diagrams over k, we have a homomorphism $\mathrm{Aut}(\mathrm{Dyn} \to \operatorname{Spec} k) \to \mathrm{Aut}(k)\colon f_\varphi \mapsto \varphi^{-1}$. We let $\mathrm{Aut}_{\mathrm{Dyn}}(k)$ be the image of this homomorphism. Furthermore denoting the k-automorphisms of the Dynkin diagram Dyn by $(\mathrm{Aut}\,\mathrm{Dyn})(k)$, we get a short exact sequence $1 \to (\mathrm{Aut}\,\mathrm{Dyn})(k) \to \mathrm{Aut}(\mathrm{Dyn} \to \operatorname{Spec} k) \to \mathrm{Aut}_{\mathrm{Dyn}}(k) \to 1$.

We now discuss how to associate a canonical scheme of Dynkin diagrams over k to a given reductive group over k. The following account follows closely [Con14, Remark 7.1.2]. Recall that in the classical setting, the Dynkin diagram is constructed after a choice of a maximal torus and of a Borel subgroup containing it. In a sense, we would like to remember that different choices are possible, but still have a canonical scheme of Dynkin diagrams associated to a reductive group. We achieve this by taking an inductive limit.

Definition 3.13. Let G be a split reductive group over k. For T a maximal torus and B a borel subgroup containing T, let $\mathcal{E} = (T, B)$ and set $\Delta(\mathcal{E})$ to be the (split) scheme of Dynkin diagrams corresponding to the classical Dynkin diagram associated to the pair (T, B). For another pair $\tilde{\mathcal{E}} = (\tilde{T}, \tilde{B})$, there exists $g \in G(k)$ such that g conjugates T (respectively B) to $\tilde{T}$ (respectively $\tilde{B}$), so that we get an isomorphism $a_{\mathcal{E},\tilde{\mathcal{E}}}\colon \Delta(\mathcal{E}) \to \Delta(\tilde{\mathcal{E}})$. Furthermore, the choice of g is unique up to $N_G(B)(k) \cap N_G(T)(k) = T(k)$ (see [ABD+64, Exposé 22, 5.6.1] for the latter equality), and thus the isomorphism $a_{\mathcal{E},\tilde{\mathcal{E}}}\colon \Delta(\mathcal{E}) \to \Delta(\tilde{\mathcal{E}})$ does not depend on g. Hence, the $a_{\mathcal{E},\tilde{\mathcal{E}}}$ form an inductive system, and we denote the limit by $\mathrm{Dyn}G$ (or also $\mathrm{Dyn}(G)$ if a parenthesis is needed) and call it the (split) **scheme of Dynkin diagrams of** G.

When G is not split, there exists a finite Galois extension l of k such that G splits over l. Now, $\mathrm{Aut}(l/k)$ acts on $\mathrm{Dyn}(G_l)$ in the following way. For $\sigma \in \mathrm{Aut}(l/k)$, let f_σ be the induced automorphism of G_l over σ. We have

$$\mathrm{Dyn}(G_l) = \varinjlim \Delta((T, B)) \cong \varinjlim \Delta(f_\sigma(T), f_\sigma(B)) = \mathrm{Dyn}(G_l)$$

This action might be non-trivial because the isomorphism, induced by σ, $\Delta((T,B)) \to \Delta(f_\sigma(T), f_\sigma(B))$ might not coincide with the effect of conjugation by an element that carries (T,B) to $(f_\sigma(T), f_\sigma(B))$.

With this action, for all $\sigma \in \mathrm{Aut}(l/k)$, we get an automorphism $\tilde{f_\sigma}$ of $\mathrm{Dyn}(G_l)$ over σ, and the $\tilde{f_\sigma}$'s satisfy the condition to be a descent datum. Furthermore, the descended k-scheme of Dynkin diagrams does not depend on the choice of the splitting field.

Definition 3.14. Let G be a reductive group over k, and let l be a finite Galois extension splitting G. Then $\mathrm{Dyn}(G_l)$ together with its descent datum defines a scheme over k that we call the **scheme of Dynkin diagrams of** G, and that we denote $\mathrm{Dyn}G$ (or also $\mathrm{Dyn}(G)$ if a parenthesis is needed).

Example 3.15. Here are examples of schemes of Dynkin diagrams. Consider $G = SU^{l/k}_{2n+1}(h)$, where

- l is a quadratic separable extension of k
- $n \geq 1$
- σ is the nontrivial element of $\mathrm{Aut}(l/k)$, whose action by conjugation on l is denoted $x \mapsto \bar{x}$
- h is the hermitian form of l^{2n+1} given by

$$((x_{-n}, \dots, x_n), (y_{-n}, \dots, y_n)) \mapsto \bar{x}_{-n} y_n + ... + \bar{x}_n y_{-n}$$

As is well-known, G is a form of SL_{2n+1}, and we now describe the corresponding action of σ on SL_{2n+1}. Consider the transposition along the anti-diagonal ${}^S(.)\colon \mathrm{SL}_{2n+1}(L) \to \mathrm{SL}_{2n+1}(L)\colon g \mapsto {}^S g$. More explicitly, $({}^S g)_{-j,-i} = g_{ij}$, for $i, j \in \{-n, \dots, 0, \dots, n\}$. Note that as for the transposition, ${}^S g\ {}^S h = {}^S(hg)$. In particular, inversion and the map $g \mapsto {}^S g$ commute. The automorphism f_σ over σ given by $f_\sigma(g) = {}^S(\bar{g}^{-1})$ has order 2, and hence is a descent datum of SL_{2n+1}, which defines the k-form $\mathrm{SU}^{l/k}_{2n+1}(h)$.

Let us now describe the corresponding schemes of Dynkin diagrams. Let T be the torus in SL_{2n+1} consisting of diagonal matrices having determinant 1, and let B be the Borel subgroup of SL_{2n+1} consisting of upper triangular matrices having determinant 1. Let A_{2n} be the classical Dynkin diagram associated to the pair (T, B). The scheme of Dynkin diagrams $\mathrm{Dyn}(\mathrm{SL}_{2n+1})$ over l is the triple (Δ, E, λ), where

1. $\Delta = \coprod_{i=-n}^{-1} x_{i,(i+1)} \sqcup \coprod_{i=1}^{n} x_{(i-1),i}$ where for all i, $x_{i,(i+1)} = \mathrm{Spec}_l\, l$.
2. $E = \{(x_{(i-1),i}; x_{i,(i+1)})$ or $(x_{(i+1),i}; x_{i,(i-1)}) \mid i = -n+1, \dots, n-1\} \subset \Delta \times \Delta$
3. $\lambda\colon \Delta \to \{1,2,3\}_l$ is the projection on 1.

Now, the action of σ on this scheme of Dynkin diagrams is readily seen to correspond to the inversion of A_{2n}. Hence, the scheme of Dynkin diagrams $\mathrm{Dyn}(\mathrm{SU}^{l/k}_{2n+1}(h)) = (\Delta', E', \lambda')$, which is by definition the twisted form of the schematic A_{2n} along inversion, can be described as follows

1. $\Delta' = \coprod_{i=-n}^{-1} x_i$, where for all i, $x_i = \mathrm{Spec}_k\, l$
2. $E' = \{(x_i; x_{(i+1)})$ or $(x_{(i+1)}; x_i) \mid i = -n, \dots, -2\} \cup \{(x_{-1}; x^{\sigma}_{-1})\} \subset \Delta' \times \Delta'$
3. $\lambda'\colon \Delta' \to \{1,2,3\}_k$ is the projection on the first component.

Remark 3.16. In the above example, $\mathrm{Spec}_k\, l$ denotes the scheme $\mathrm{Spec}\, l$ considered as a k-scheme. Equivalently, if $l \cong k[X]/(f)$, then $\mathrm{Spec}_k\, l \cong \mathrm{Spec}\, k[X]/(f)$. It is worth to keep in mind that $(\mathrm{Spec}_k\, l)_l$ is thus isomorphic to $\mathrm{Spec}\, l[X]/(f)$. But l contains the roots of f, hence $l[X]/(f) \cong l[X]/(X-\alpha)(X-\beta)$, so that $(\mathrm{Spec}_k\, l)_l \cong \mathrm{Spec}_l\, l \coprod \mathrm{Spec}_l\, l$, with Galois action exchanging the two points.

Remark 3.17. It is not hard to classify schemes of Dynkin diagram. We do this in Corollary 3.25.

We end the section with an elementary observation.

Lemma 3.18. *Let G be a reductive group over a field k and let l/k be a field extension. Then* $\mathrm{Dyn}(G_l) \cong (\mathrm{Dyn} G)_l$

Proof. This directly follows from the definition □

This result allows us to make an abuse of notation and to spare a parenthesis in writing $\mathrm{Dyn}G_l$, since the way to place the parenthesis does not matter. We freely use this abuse of notation in the next section.

3.4 Semilinear automorphisms of quasi-split algebraic groups

Let G be a split reductive algebraic group over k. A pinning of G gives rise to a splitting of the exact sequence $1 \to \operatorname{Inn} G \to \operatorname{Aut} G \to \operatorname{Out} G \to 1$. Furthermore, $\operatorname{Out} G$ is isomorphic to the (constant group scheme associated to the) group of automorphisms of the based root datum given by the pinning (see [Con14, Theorem 7.1.9]). If we furthermore assume that G is semisimple and simply connected (or adjoint), then this latter group is just the automorphism group of the Dynkin diagram.

This decomposition of $\operatorname{Aut} G$ as the semidirect product $\operatorname{Inn} G \rtimes \operatorname{Out} G$ is preserved when G is quasi-split. On the other hand, as we outlined in the introduction, the short exact sequence $1 \to (\operatorname{Aut} G)(k) \to \operatorname{Aut}(G \to \operatorname{Spec} k) \to \operatorname{Aut}_G(k)$ does not always split. The purpose of this section is to show that we have another semidirect product decomposition for the semilinear automorphism group when G is quasi-split, namely that $\operatorname{Aut}(G \to \operatorname{Spec} k) \cong \operatorname{Inn} G \rtimes \operatorname{Aut}(\operatorname{Dyn} G \to \operatorname{Spec} k)$ (notice the strong parallel with $\operatorname{Aut} G \cong \operatorname{Inn} G \rtimes \operatorname{Out} G$).

We begin by recalling the classification of quasi-split groups. Our description is based on [Con14, Section 7], but the result we give is already present in [ABD+64, Exposé 24, Théorème 3.10]. Let k_s denote a separable closure of k. Recall that k-forms of G are classified by the Galois cohomology set $H^1(k_s/k, \operatorname{Aut}(G_{k_s}))$ (see [Con14, Theorem 7.1.1]).

Lemma 3.19. *Let G be a split semisimple simply connected* (*or adjoint*) *group over k, and let k_s denote a separable closure of k. Choosing a pinning of G, let $\operatorname{Aut}(\mathrm{Dyn}G) \hookrightarrow \operatorname{Aut} G$ be the corresponding embedding. The induced map $H^1(k_s/k, \operatorname{Aut}(\mathrm{Dyn}G_{k_s})) \to H^1(k_s/k, \operatorname{Aut}(G_{k_s}))$ is injective, and its image gives all the k_s/k forms of G_{k_s} that are quasi-split over k.*

Proof. Since G is simply connected (or adjoint), the automorphism group of the based root datum (corresponding to the chosen pinning of G) is simply the automorphism group of the classical Dynkin diagram of G, which in turn is the group $\mathrm{Aut}(\mathrm{Dyn}G_{k_s})$. In summary, $\mathrm{Aut}(G_{k_s}) \cong (G_{k_s}/Z_{G_{k_s}})(k_s) \rtimes \mathrm{Aut}(\mathrm{Dyn}G_{k_s})$. This directly implies that the map from $H^1(k_s/k, \mathrm{Aut}(\mathrm{Dyn}G_{k_s}))$ to $H^1(k_s/k, \mathrm{Aut}(G_{k_s}))$ is injective, since it has a right inverse.

By definition, automorphisms in $\mathrm{Aut}(\mathrm{Dyn}\, G_{k_s}) \leq \mathrm{Aut}(G_{k_s})$ consists of automorphisms preserving the Borel subgroup given by the pinning. Hence, forms of G corresponding to classes in the image of the map $H^1(k_s/k, \mathrm{Aut}(\mathrm{Dyn}G_{k_s})) \to H^1(k_s/k, \mathrm{Aut}(G_{k_s}))$ are quasi-split.

To conclude, we use the fact that given a set of classes that are inner forms of each other in $H^1(k_s/k, \mathrm{Aut}(G_{k_s}))$, there is a unique quasi-split groups among the corresponding k-forms of G, by [Con14, Proposition 7.2.12]. But the fibres of $H^1(k_s/k, \mathrm{Aut}(G_{k_s})) \to H^1(k_s/k, \mathrm{Out}(G_{k_s}))$ are precisely the classes that are inner forms of each other, in view of [Con14, Theorem 7.2.2], so that the theorem follows. □

The key result towards the proof of Theorem 3.2 is a semidirect decomposition of the group of semilinear automorphisms of a quasi-split group.

Theorem 3.20. *Let G be a quasi-split semisimple simply connected (or adjoint) group over k. The group* $\mathrm{Aut}(G \to \mathrm{Spec}\, k)$ *decomposes as* $G/Z_G(k) \rtimes \mathrm{Aut}(\mathrm{Dyn}G \to k)$. *Furthermore, the projection* $\mathrm{Aut}(G \to \mathrm{Spec}\, k) \to \mathrm{Aut}(\mathrm{Dyn}G \to k)$ *preserves the underlying field automorphism, so that* $\mathrm{Aut}_G(k) = \mathrm{Aut}_{\mathrm{Dyn}G}(k)$.

Proof. Let k_s be a separable closure of k. Recall that G_{k_s} is a split group, and let G_0 be the split reductive $\mathbf{Z}$-group scheme having same root datum as G_{k_s}. To ease the notations, we set $(G_0)_{k_s} = \overline{G_0}$. Since split groups are determined by their root datum, G is a k_s/k-form of $\overline{G_0}$.

Let $c\colon \mathrm{Gal}(k_s/k) \to \mathrm{Aut}(\mathrm{Dyn}\overline{G_0})$ be the cocycle corresponding to $\mathrm{Dyn}G$. By Lemma 3.19, we can and do assume that the composition $\mathrm{Gal}(k_s/k) \to \mathrm{Aut}(\mathrm{Dyn}\overline{G_0}) \to \overline{G_0}/Z_{\overline{G_0}}(k_s) \rtimes \mathrm{Aut}(\mathrm{Dyn}\overline{G_0})$ is the cocycle corresponding to G. We also denote this cocycle by c.

In view of Lemma 3.9, we can study $\mathrm{Aut}(G \to \operatorname{Spec} k)$ by studying $\mathrm{Aut}(\overline{G_0} \to \operatorname{Spec} k_s)$. But this latter group is isomorphic to $\mathrm{Aut}(k_s) \ltimes (\mathrm{Aut}(\mathrm{Dyn}\overline{G_0}) \ltimes (\overline{G_0}/Z_{\overline{G_0}})(k_s))$, which is in turn isomorphic to the semi-direct product $\mathrm{Aut}(\mathrm{Dyn}\overline{G_0} \to \operatorname{Spec} k_s) \ltimes (\overline{G_0}/Z_{\overline{G_0}})(k_s)$.

Claim 1. *Let f_α be a semilinear automorphism of $\mathrm{Dyn}\overline{G_0}$ over an automorphism $\alpha \in \mathrm{Aut}(k_s)$ such that $\alpha(k) = k$, and let $g \in (\overline{G_0}/Z_{\overline{G_0}})(k_s)$. Then $f_\alpha g \in \mathrm{Aut}(\overline{G_0} \to \operatorname{Spec} k_s)$ descends to $\mathrm{Aut}(G \to \operatorname{Spec} k)$ if and only if f_α descends to $\mathrm{Aut}(\mathrm{Dyn}G \to \operatorname{Spec} k)$ and g descends to $(G/Z_G)(k)$.*

Proof of the claim: We decompose $\mathrm{Aut}(\mathrm{Dyn}\overline{G_0} \to \operatorname{Spec} k_s)$ as $\mathrm{Aut}(k_s) \times \mathrm{Aut}(\mathrm{Dyn}\overline{G_0})$, using the $\mathbf{Z}$-structure on $\mathrm{Dyn}\overline{G_0}$. For $\beta \in \mathrm{Aut}(k_s)$, let $\mathrm{Id}_\beta \in \mathrm{Aut}(k_s) \leq \mathrm{Aut}(\mathrm{Dyn}\overline{G_0} \to \operatorname{Spec} k_s)$ be the semilinear automorphism over β given by the $\mathbf{Z}$-structure of $\mathrm{Dyn}\overline{G_0}$. With those notations, f_α decomposes as $\mathrm{Id}_\alpha f \in \mathrm{Aut}(k_s) \times \mathrm{Aut}(\mathrm{Dyn}\overline{G_0})$. By Lemma 3.9, the element $\mathrm{Id}_\alpha fg$ descends to $\mathrm{Aut}(G \to \operatorname{Spec} k)$ if and only if for all $\gamma \in \mathrm{Gal}(k_s/k)$, we have $\gamma_*(\mathrm{Id}_\alpha fg) = \alpha_*(t_{\alpha^{-1}\gamma\alpha} c^{-1}_{\alpha^{-1}\gamma\alpha}) \mathrm{Id}_\alpha fgc_\gamma t_\gamma^{-1}$.

Since the image of c consists of automorphisms defined over $\mathbf{Z}$, we have $\alpha_*(c^{-1}_{\alpha^{-1}\gamma\alpha}) \mathrm{Id}_\alpha = \mathrm{Id}_\alpha c^{-1}_{\alpha^{-1}\gamma\alpha}$. Furthermore, since f is also defined over $\mathbf{Z}$, we have $t_\gamma c^{-1}_{\alpha^{-1}\gamma\alpha} f c_\gamma t_\gamma^{-1} = \gamma_*(c^{-1}_{\alpha^{-1}\gamma\alpha} f c_\gamma)$. Thus,

$$\begin{aligned}
&\alpha_*(t_{\alpha^{-1}\gamma\alpha} c^{-1}_{\alpha^{-1}\gamma\alpha}) \mathrm{Id}_\alpha fgc_\gamma t_\gamma^{-1} \\
&\qquad\qquad = \alpha_*(t_{\alpha^{-1}\gamma\alpha}) \mathrm{Id}_\alpha t_\gamma^{-1} t_\gamma c^{-1}_{\alpha^{-1}\gamma\alpha} f c_\gamma t_\gamma^{-1} t_\gamma c_\gamma^{-1} g c_\gamma t_\gamma^{-1} \\
&\qquad\qquad = \gamma_*(\mathrm{Id}_\alpha)\gamma_*(c^{-1}_{\alpha^{-1}\gamma\alpha} f c_\gamma)\gamma_*(c_\gamma^{-1}(g)).
\end{aligned}$$

Now, in view of the decomposition of $\mathrm{Aut}(\overline{G_0} \to \operatorname{Spec} k_s)$ into $\mathrm{Aut}(k_s) \ltimes (\mathrm{Aut}(\mathrm{Dyn}\overline{G_0}) \ltimes \overline{G_0}/Z_{\overline{G_0}})$, $\gamma_*(\mathrm{Id}_\alpha fg) = \alpha_*(t_{\alpha^{-1}\gamma\alpha} c^{-1}_{\alpha^{-1}\gamma\alpha}) \mathrm{Id}_\alpha fgc_\gamma t_\gamma^{-1}$ holds if and only if $\gamma_*(\mathrm{Id}_\alpha f)$ is equal to $\gamma_*(\mathrm{Id}_\alpha)\gamma_*(c^{-1}_{\alpha^{-1}\gamma\alpha} f c_\gamma)$ and $\gamma_*(g) = \gamma_*(c_\gamma^{-1}(g))$. Going through the displayed equalities in reverse order, this is indeed equivalent to requiring that $\mathrm{Id}_\alpha f$ descends to $\mathrm{Aut}(\mathrm{Dyn}G \to \operatorname{Spec} k)$ and that g descends to $(G/Z_G)(k)$, as wanted. ■

The conclusion of the theorem readily follows. Indeed, in view of Claim 1, we have

$$\begin{aligned}
\mathrm{Aut}(G \to \operatorname{Spec} k) = \{&(g, f_\alpha) \in (\overline{G_0}/Z_{\overline{G_0}})(k_s) \rtimes \mathrm{Aut}(\mathrm{Dyn}\overline{G_0} \to \operatorname{Spec} k_s) \mid \\
&g \in (G/Z_G)(k) \text{ and } f_\alpha \in \mathrm{Aut}(\mathrm{Dyn}G \to \operatorname{Spec} k)\}
\end{aligned} \qquad \square$$

Remark 3.21. For G a simple group which is not quasi-split, the decomposition of $\operatorname{Aut} G$ as a semidirect product is usually destroyed. Similarly, one should not expect to obtain a semidirect decomposition of $\operatorname{Aut}(G \to \operatorname{Spec} k)$ for a general simple algebraic group. Investigating a possible semidirect decomposition of the group of semilinear automorphisms of simple algebraic groups is an entirely different matter when G is not quasi-split, as is illustrated by our treatment of the $\mathrm{SL}_n(D)$ case in Section 3.6.

As a corollary of Theorem 3.20, we obtain a proof of Theorem 3.2.

Proof of Theorem 3.2. $\operatorname{Aut}(G \to \operatorname{Spec} k) \cong (\operatorname{Inn} G)(k) \rtimes \operatorname{Aut}(\mathrm{Dyn}G \to \operatorname{Spec} k)$ by Theorem 3.20. This shows in particular that $\operatorname{Aut}_{\mathrm{Dyn}G}(k) = \operatorname{Aut}_G(k)$. We thus obtain the following commutative diagram:

$$
\begin{tikzcd}
1 \arrow[dr] & 1 \arrow[d] & & 1 & 1 \\
& (\operatorname{Inn} G)(k) \arrow[dd] \arrow[dr] & & \operatorname{Aut}_{\mathrm{Dyn}G}(k) \arrow[u] \arrow[ur] & \\
& & \operatorname{Aut}(G \to \operatorname{Spec} k) \arrow[ur] \arrow[dr, shift right, "\pi_2"'] & & \\
& (\operatorname{Aut} G)(k) \arrow[ur] \arrow[d] & & \operatorname{Aut}(\mathrm{Dyn}G \to \operatorname{Spec} k) \arrow[uu] \arrow[ul, shift right, "\iota"'] \arrow[dr] & \\
1 \arrow[ur] & (\operatorname{Out} G)(k) \arrow[d] & & (\operatorname{Aut} \mathrm{Dyn}G)(k) \arrow[u] & 1 \\
& 1 & & 1 \arrow[u] &
\end{tikzcd}
$$

where all diagonal lines and vertical lines are exact. Here, π_2 denotes the projection of $(\operatorname{Inn} G)(k) \rtimes \operatorname{Aut}(\mathrm{Dyn}G \to \operatorname{Spec} k)$ onto its second component, and ι is a section of π_2.

We thus conclude that the short exact sequence $1 \to (\operatorname{Aut} G)(k) \to \operatorname{Aut}(G \to \operatorname{Spec} k) \to \operatorname{Aut}_G(k) \to 1$ splits if and only if the short exact sequence involving schemes of Dynkin diagrams $1 \to (\operatorname{Aut} \mathrm{Dyn}G)(k) \to \operatorname{Aut}(\mathrm{Dyn}G \to \operatorname{Spec} k) \to \operatorname{Aut}_{\mathrm{Dyn}G}(k) \to 1$ does, as was to be shown. $\square$

3.5 Semilinear automorphisms of the scheme of Dynkin diagrams

We aim to give an explicit description of the group of semilinear automorphisms of Dynkin diagrams. One way to do that is to first base change to a separable closure, and then push a little further the computation appearing in the proof of Theorem 3.20. Instead, we prefer to do it in a more down-to-earth manner, by looking explicitly at forms of schemes of Dynkin diagrams.

Note that for Dyn a split scheme of Dynkin diagrams, the $\mathrm{Gal}(k_s/k)$-action on $\mathrm{Aut}(\mathrm{Dyn}_{k_s})$ is trivial, so that $H^1(k_s/k, \mathrm{Aut}(\mathrm{Dyn}_{k_s}))$ is isomorphic to the set of continuous homomorphisms $\mathrm{Hom}(\mathrm{Gal}(k_s/k), \mathrm{Aut}(D))$ modulo conjugation (where D is the corresponding classical Dynkin diagram).

Definition 3.22. Let Dyn be a split scheme of Dynkin diagrams over k corresponding to the Dynkin diagram D, and let Dyn' be a k_s/k form of Dyn_{k_s}. Let $N \trianglelefteq \mathrm{Gal}(k_s/k)$ be the kernel of $\mathrm{Gal}(k_s/k) \to \mathrm{Aut}(D)$ classifying the form Dyn', and let l be the Galois extension of k fixed by N. We call l the classifying field of Dyn'. Once a separable closure of k has been fixed, the classifying field of Dyn' is uniquely determined by Dyn'.

We say that a split scheme of Dynkin diagrams is connected if the corresponding Dynkin diagram is connected, and that a scheme of Dynkin diagrams is connected if it is a form of a connected split scheme of Dynkin diagrams (so connected really means "geometrically connected" or "absolutely connected"). For the rest of the section, we focus on connected schemes of Dynkin diagrams. At the level of algebraic groups, this amounts to require G to be absolutely simple.

Lemma 3.23. *Let* Dyn *be a split scheme of Dynkin diagrams over k corresponding to a connected Dynkin diagram D. Let k_s be a separable closure of k. The map which associates to a k_s/k-form of* Dyn_{k_s} *its classifying field is a bijection between scheme of Dynkin diagrams over k (up to k-isomorphism) and subfields $l \leq k_s$ such that (l is Galois over k and)* $\mathrm{Gal}(l/k)$ *is isomorphic to a subgroup of* $\mathrm{Aut}(D)$.

Proof. Since D is connected, $\mathrm{Aut}(D)$ is either trivial, $\mathbf{Z}/2\mathbf{Z}$ or S_3. Hence, if two subgroups of $\mathrm{Aut}(D)$ are isomorphic, they are actually conjugate. The result follows from the fact that the Galois action on $\mathrm{Aut}(\mathrm{Dyn}_{k_s})$ is trivial, and hence $H^1(k_s/k, \mathrm{Aut}(\mathrm{Dyn}_{k_s}))$ is isomorphic to the set of continuous homomorphisms $\mathrm{Hom}(\mathrm{Gal}(k_s/k), \mathrm{Aut}(D))$ modulo conjugation. □

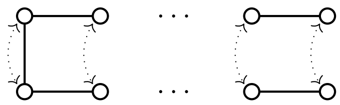
Definition 3.24. Let Dyn be a connected scheme of Dynkin diagrams over k, and let k_s be a separable closure of k. We define the index of Dyn to be ${}^gX_{n,l}$ where

1. $l \leq k_s$ is the classifying field of Dyn (and hence, a finite Galois extension of k).
2. X_n is the label of the Dynkin diagram associated to $(\mathrm{Dyn})_l$.
3. g is the order of the Galois group $\mathrm{Gal}(l/k)$.

As explained in Remark 3.11, we can describe a scheme of Dynkin diagrams using the category of $\mathrm{Gal}(k_s/k)$-sets. Under this correspondence, a scheme of Dynkin diagram (Δ, E, λ) becomes a triple $(\tilde{\Delta}, \tilde{E}, \tilde{\lambda})$, where $\tilde{\Delta}$ is a finite $\mathrm{Gal}(k_s/k)$-sets, $\tilde{E}$ is a $\mathrm{Gal}(k_s/k)$-invariant subset of $\tilde{\Delta} \times \tilde{\Delta}$ (this latter product being endowed with the diagonal action), and $\tilde{\lambda} \colon \tilde{\Delta} \to \{1, 2, 3\}$ is a $\mathrm{Gal}(k_s/k)$-equivariant map ($\{1, 2, 3\}$ being endowed with the trivial $\mathrm{Gal}(k_s/k)$ action). We do so to ease the description of scheme of Dynkin diagrams in the following classification.

Corollary 3.25. *In view of Lemma 3.23, once we fix a field k together with a separable closure k_s, a connected scheme of Dynkin diagrams over k is entirely determined by its index. Here is a description of all possible connected scheme of Dynkin diagrams over a field k.*

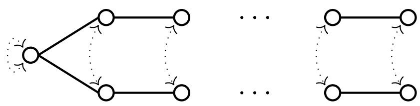
1. *$l = k$. Hence the scheme of Dynkin diagrams is split, and split schemes of Dynkin diagrams were described explicitly in Definition 3.10.*
2. *${}^2A_{2n,l}$ $(n \geq 1)$. The triple $(\tilde{\Delta}, \tilde{E}, \tilde{\lambda})$ in the category of $\mathrm{Gal}(k_s/k)$-sets corresponding to this scheme of Dynkin diagrams can be described as follows:*



(*$2n$ vertices*)

The presence of an edge linking nodes i and j indicates that the pairs (i,j) and (j,i) belong to $\tilde{E}$. Here, the $\mathrm{Gal}(k_s/k)$ *action factors through the action of* $\mathrm{Gal}(l/k) \cong \mathbf{Z}/2\,\mathbf{Z}$ *depicted by the dotted arrows. Finally, $\tilde{\lambda}$ is the projection onto* 1.

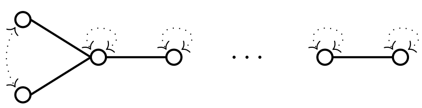
3. ${}^2A_{2n+1,l}$ *($n \geq 0$). The triple $(\tilde{\Delta}, \tilde{E}, \tilde{\lambda})$ in the category of* $\mathrm{Gal}(k_s/k)$*-sets corresponding to this scheme of Dynkin diagrams can be described as follows:*



(*$2n+1$ vertices*)

The presence of an edge linking nodes i and j indicates that the pairs (i,j) and (j,i) belong to $\tilde{E}$. Here, the $\mathrm{Gal}(k_s/k)$ *action factors through the action of* $\mathrm{Gal}(l/k) \cong \mathbf{Z}/2\,\mathbf{Z}$ *depicted by the dotted arrows. Finally, $\tilde{\lambda}$ is the projection onto* 1.

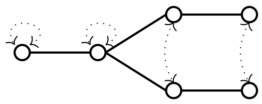
4. ${}^2D_{n,l}$ *($n \geq 4$). The triple $(\tilde{\Delta}, \tilde{E}, \tilde{\lambda})$ in the category of* $\mathrm{Gal}(k_s/k)$*-sets corresponding to this scheme of Dynkin diagrams can be described as follows:*



(*n vertices*)

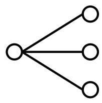
The presence of an edge linking nodes i and j indicates that the pairs (i,j) and (j,i) belong to $\tilde{E}$. Here, the $\mathrm{Gal}(k_s/k)$ *action factors through the action of* $\mathrm{Gal}(l/k) \cong \mathbf{Z}/2\,\mathbf{Z}$ *depicted by the dotted arrows. Finally, $\tilde{\lambda}$ is the projection onto* 1.

5. ${}^2E_{6,l}$. *The triple $(\tilde{\Delta}, \tilde{E}, \tilde{\lambda})$ in the category of* $\mathrm{Gal}(k_s/k)$*-sets corresponding to this scheme of Dynkin diagrams can be described as follows:*



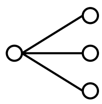
The presence of an edge linking nodes i and j indicates that the pairs (i,j) and (j,i) belong to $\tilde{E}$. Here, the $\mathrm{Gal}(k_s/k)$ *action factors through the action of* $\mathrm{Gal}(l/k) \cong \mathbf{Z}/2\,\mathbf{Z}$ *depicted by the dotted arrows. Finally, $\tilde{\lambda}$ is the projection onto* 1.

6. ${}^3D_{4,l}$. *The triple $(\tilde{\Delta}, \tilde{E}, \tilde{\lambda})$ in the category of* $\mathrm{Gal}(k_s/k)$*-sets corresponding to this scheme of Dynkin diagrams can be described as follows:*



The presence of an edge linking nodes i and j indicates that the pairs (i,j) and (j,i) belong to $\tilde{E}$. Here, the $\mathrm{Gal}(k_s/k)$ *action factors through the action of* $\mathrm{Gal}(l/k) \cong \mathbf{Z}/3\,\mathbf{Z}$ *acting cyclically on the three rightmost nodes. Finally, $\tilde{\lambda}$ is the projection onto* 1.

7. ${}^6D_{4,l}$. *The triple $(\tilde{\Delta}, \tilde{E}, \tilde{\lambda})$ in the category of* $\mathrm{Gal}(k_s/k)$*-sets corresponding to this scheme of Dynkin diagrams can be described as follows:*



The presence of an edge linking nodes i and j indicates that the pairs (i,j) and (j,i) belong to $\tilde{E}$. Here, the $\mathrm{Gal}(k_s/k)$ *action factors through the action of* $\mathrm{Gal}(l/k) \cong \mathrm{Sym}\,3$ *acting freely on the three rightmost nodes. Finally, $\tilde{\lambda}$ is the projection onto* 1.

Lemma 3.26. *Let* Dyn *be a connected scheme of Dynkin diagrams over k with index ${}^gX_{n,l}$. For H a group, we denote its opposite group by H^{op}.*

1. *If $g = 1$,* $\mathrm{Aut}(\mathrm{Dyn} \to \mathrm{Spec}\,k) \cong \mathrm{Aut}(k)^{op} \times \mathrm{Aut}(D)$ (*where D is the Dynkin diagram labelled by X_n*). *Furthermore,* $\mathrm{Aut}(D)$ *is identified with* $(\mathrm{Aut}\,\mathrm{Dyn})(k)$.

2. *If $g = 2$ or $g = 3$,* $\mathrm{Aut}(\mathrm{Dyn} \to \mathrm{Spec}\, k) \cong \{\alpha \in Aut(l) \mid \alpha(k) = k\}^{op}$*, while* $(\mathrm{Aut}\,\mathrm{Dyn})(k) \cong \mathrm{Gal}(l/k)$*.*

3. *If $g = 6$,* $\mathrm{Aut}(\mathrm{Dyn} \to \mathrm{Spec}\, k) \cong \{\alpha \in Aut(l_3) \mid \alpha(k) = k\}^{op}$*, where l_3 is any non-normal cubic subextension of l/k.*

Proof.

1. The case $g = 1$ means that Dyn is a split scheme of Dynkin diagrams. Since D is connected, $(\mathrm{Aut}\,\mathrm{Dyn})(k) \cong \mathrm{Aut}(D)$. Furthermore $\mathrm{Aut}_{\mathrm{Dyn}}(k) \cong \mathrm{Aut}(k)$ and the short exact sequence $1 \to \mathrm{Aut}(D) \to \mathrm{Aut}(\mathrm{Dyn} \to \mathrm{Spec}\, k) \to \mathrm{Aut}(k) \to 1$ introduced below Definition 3.12 splits. Furthermore $\mathrm{Aut}(k)$ acts trivially on $\mathrm{Aut}(D)$, so that the result follows.

2. An isomorphism of $\mathrm{Dyn} = (\Delta, E, \lambda)$ over an isomorphism of k is in particular an edge preserving bijection φ of the set of points of the topological space underlying the scheme Δ. In Corollary 3.25, the scheme Δ has been described as a $\mathrm{Gal}(k_s/k)$-set $\tilde{\Delta}$. Recall that Δ is just the spectrum of the etale algebra corresponding to $\tilde{\Delta}$, and that for example, the points of the topological space underlying the scheme Δ are in bijection with the orbits of $\tilde{\Delta}$.

   A case-by-case analysis of Corollary 3.25 readily reveals that the bijection φ must be the identity. Hence, we are left with a set of automorphisms of $\mathrm{Spec}\, l$ over an automorphism of k. But since Dyn is connected, an element of $\mathrm{Aut}(\mathrm{Dyn} \to \mathrm{Spec}\, k)$ is actually determined by one automorphism of $\mathrm{Spec}\, l$ over an automorphism of k. The description of $\mathrm{Aut}(\mathrm{Dyn} \to \mathrm{Spec}\, k)$ follows. Finally, the algebraic automorphisms are just the ones which act trivially on k, so that $(\mathrm{Aut}\,\mathrm{Dyn})(k)$ is indeed isomorphic to $\mathrm{Aut}(l/k)$.

3. Again, the result is clear from the picture given in Corollary 3.25.

□

Corollary 3.27. *Let* Dyn *be a connected scheme of Dynkin diagrams over k with classifying field l. If* $\mathrm{Aut}(l/k) \not\cong S_3$*, then* $\mathrm{Aut}_{\mathrm{Dyn}}(k) \cong \{\alpha \in \mathrm{Aut}(k) \mid$ *there exists* $\tilde{\alpha} \in \mathrm{Aut}(l)$ *extending* $\alpha\}$*. While if* $\mathrm{Aut}(l/k) \cong S_3$*, then* $\mathrm{Aut}_{\mathrm{Dyn}}(k) \cong \{\alpha \in \mathrm{Aut}(k) \mid$ *there exists* $\tilde{\alpha} \in \mathrm{Aut}(l_3)$ *extending* $\alpha\}$*, where l_3 is a chosen non-normal cubic subextension of l/k.*

Proof. This follows from the surjectivity of $\mathrm{Aut}(\mathrm{Dyn} \to \mathrm{Spec}\, k) \to \mathrm{Aut}_{\mathrm{Dyn}}(k)$ and from the description of $\mathrm{Aut}(\mathrm{Dyn} \to \mathrm{Spec}\, k)$ contained in Lemma 3.26. □

In view of Lemma 3.26 and Corollary 3.27, it is useful to introduce the following notations.

Definition 3.28. Let k be a field, and let l be a field extension of k. We denote by $\mathrm{Aut}(l \geq k)$ the group of automorphisms of l preserving k, i.e. $\mathrm{Aut}(l \geq k) = \{\alpha \in \mathrm{Aut}(l) \mid \alpha(k) = k\}$. Also, we denote by $\mathrm{Aut}_l(k)$ the group of automorphisms of k that extends to an automorphism of l, i.e. $\mathrm{Aut}_l(k) = \{\alpha \in \mathrm{Aut}(k) \mid \text{ there exists } \tilde{\alpha} \in \mathrm{Aut}(l) \text{ extending } \alpha\}$.

Using the identifications we made in Lemma 3.26 and Corollary 3.27, we can rewrite in a very explicit form the short exact sequence $1 \to (\mathrm{Aut}\,\mathrm{Dyn}G)(k) \to \mathrm{Aut}(\mathrm{Dyn}G \to \mathrm{Spec}\, k) \to \mathrm{Aut}_{\mathrm{Dyn}G}(k) \to 1$.

Proposition 3.29. *Let* Dyn *be a connected scheme of Dynkin diagrams over k with index ${}^gX_{n,l}$.*

1. *If $g = 1$, let D be the Dynkin diagram labelled by X_n. The short exact sequence $1 \to (\mathrm{Aut}\,\mathrm{Dyn}G)(k) \to \mathrm{Aut}(\mathrm{Dyn}G \to \mathrm{Spec}\, k) \to \mathrm{Aut}_{\mathrm{Dyn}G}(k) \to 1$ is isomorphic to the short exact sequence $1 \to \mathrm{Aut}(D) \to \mathrm{Aut}(D) \times \mathrm{Aut}(k)^{op} \to \mathrm{Aut}(k) \to 1$. In particular, it always splits.*

2. *If $g = 2$ or $g = 3$, the short exact sequence $1 \to (\mathrm{Aut}\,\mathrm{Dyn}G)(k) \to \mathrm{Aut}(\mathrm{Dyn}G \to \mathrm{Spec}\, k) \to \mathrm{Aut}_{\mathrm{Dyn}G}(k) \to 1$ is isomorphic to $1 \to \mathrm{Gal}(l/k)^{op} \to \mathrm{Aut}(l \geq k)^{op} \to \mathrm{Aut}_l(k) \to 1$.*

3. *If $g = 6$, let l_3 be a (non normal) cubic subextension of l/k. The short exact sequence $1 \to (\mathrm{Aut}\,\mathrm{Dyn}G)(k) \to \mathrm{Aut}(\mathrm{Dyn}G \to \mathrm{Spec}\, k) \to \mathrm{Aut}_{\mathrm{Dyn}G}(k) \to 1$ is isomorphic to $1 \to 1 \to \mathrm{Aut}(l_3 \geq k)^{op} \to \mathrm{Aut}_{l_3}(k) \to 1$. In particular, it always splits.*

Proof. This is a direct consequence of Lemma 3.26 and Corollary 3.27. Note that in each case, the map $\mathrm{Aut}(l \geq k)^{op} \to \mathrm{Aut}_l(k)$ is given by restriction to k followed by inversion. Also note that when $g = 6$, and since l_3 is a non normal cubic extension of k, the group $\mathrm{Aut}(l_3/k)$ is trivial, and $\mathrm{Aut}(l_3 \geq k) \cong \mathrm{Aut}_{l_3}(k)$. □

Remark 3.30. Note that for example, the short exact sequence $1 \to \mathrm{Gal}(l/k)^{op} \to \mathrm{Aut}(l \geq k)^{op} \to \mathrm{Aut}_l(k) \to 1$ appearing in Proposition 3.29 is also isomorphic to $1 \to \mathrm{Gal}(l/k) \to \mathrm{Aut}(l \geq k) \to \mathrm{Aut}_l(k) \to 1$. We use this simpler form in Corollary 3.32.

We end this discussion with examples where the short exact sequence $1 \to (\mathrm{Aut}\,\mathrm{Dyn}G)(k) \to \mathrm{Aut}(\mathrm{Dyn}G \to \mathrm{Spec}\,k) \to \mathrm{Aut}_{\mathrm{Dyn}G}(k) \to 1$ does not split.

Definition 3.31. A field k is called **rigid** if every automorphism of k is trivial, and **strongly rigid** if for any finite extension l of k, every automorphisms of l acts trivially on k.

Examples of strongly rigid fields include finite fields of prime order, the field $\mathbf{Q}$ and p-adic fields $\mathbf{Q}_p$ for any prime p. On the other hand, $\mathbf{R}$ is an example of a rigid field which is not strongly rigid.

Corollary 3.32. *Let k be a finite Galois extension of a strongly rigid field k_0. Let G be an absolutely simple, simply connected (or adjoint) quasi-split k-group such that* $\mathrm{Dyn}G$ *has index ${}^g X_{n,l}$, with $g = 2$ or $g = 3$. Further assume that l is a Galois extension of k_0. Then the short exact sequence* $1 \to (\mathrm{Aut}\,G)(k) \to \mathrm{Aut}(G \to \mathrm{Spec}\,k) \to \mathrm{Aut}_G(k) \to 1$ *splits if and only if* $1 \to \mathrm{Gal}(l/k) \to \mathrm{Gal}(l/k_0) \to \mathrm{Gal}(k/k_0) \to 1$ *splits.*

Proof. In view of Theorem 3.2, Proposition 3.29 and Remark 3.30, the short exact sequence $1 \to (\mathrm{Aut}\,G)(k) \to \mathrm{Aut}(G \to \mathrm{Spec}\,k) \to \mathrm{Aut}_G(k) \to 1$ splits if and only if the short exact sequence $1 \to \mathrm{Gal}(l/k) \to \mathrm{Aut}(l \geq k) \to \mathrm{Aut}_l(k) \to 1$ splits. Since k_0 is strongly rigid and k is a normal extension, $\mathrm{Aut}(l \geq k) = \mathrm{Gal}(l/k_0)$. Furthermore, $\mathrm{Aut}(k) = \mathrm{Gal}(k/k_0)$, and since l/k_0 is Galois, every element of $\mathrm{Gal}(k/k_0)$ extends to $\mathrm{Gal}(l/k_0)$. Hence $\mathrm{Aut}_l(k) = \mathrm{Gal}(k/k_0)$, and we get the result. □

3.6 The $\mathrm{SL}_n(D)$ case

3.6.1 Outer automorphisms of finite dimensional central simple algebras over local fields

We now explore the same question for algebraic groups of the form $\mathrm{SL}_n(D)$. We will restrict ourselves to the case of a local field. Let us begin by recalling the classification of central simple algebras over local fields.

Definition 3.33. Let K be a local field and let $d, r \in \mathbf{N}$ with $d \geq 1$. Let K_d be the unramified extension of K of degree d, let $\sigma \in \mathrm{Gal}(K_d/K)$ be the Frobenius automorphism (i.e. the automorphism inducing the Frobenius automorphism on $\mathrm{Gal}(\overline{K_d}/\overline{K})$), and let π be a uniformiser of K. We define $A(d,r)$ to be the central simple algebra isomorphic to the cyclic algebra $(K_d/K, \sigma, \pi^r)$. More explicitly, $A(d,r)$ is described as $\bigoplus_{i=0}^{d-1} u^i K_d$, with the relations $u^{-1}au = \sigma(a)$ for all $a \in K_d$, and $u^d = \pi^r$. Note that up to isomorphism, $A(d,r)$ does not depend on the choice of π (in fact, given two uniformisers π and $\tilde{\pi}$, an explicit isomorphism $(K_d/K, \sigma, \pi^r) \cong (K_d/K, \sigma, \tilde{\pi}^r)$ having the same form as the one appearing in Lemma 3.34 can be given).

Lemma 3.34. *Let K be a local field. Let $A = A(d,r)$ and K_d, σ, π be as in Definition 3.33. Let α be an automorphism of K_d such that $\alpha(K) = K$, and assume that there exists an element x in K_d such that $N_{K_d/K}(x) = \frac{\alpha(\pi^r)}{\pi^r}$. Then the map $\phi(\alpha, x)\colon A \to A\colon \sum_{i=0}^{d-1} u^i a_i \mapsto \sum_{i=0}^{d-1} (ux)^i \alpha(a_i)$ is a ring automorphism of A.*

Proof. We view A as a quotient of the twisted polynomial ring $K_d[u;\sigma]$ (see [Jac96, Section 1.1] for the definition of a twisted polynomial ring) modulo the relation $u^d = \pi^r$. Given an automorphism α in $\mathrm{Aut}(K_d)$, we can define a map $f_\alpha\colon K_d[u;\sigma] \to K_d[u;\sigma]\colon \begin{cases} u \mapsto ux \\ a \mapsto \alpha(a) \text{ for all } a \in K_d \end{cases}$.
By [Jac96, Proposition 4.6.20], f_α is a ring automorphism as soon as $\alpha\sigma = \sigma\alpha$. Recall that by assumption, $\alpha(K) = K$. Hence $\sigma^{-1}\alpha\sigma\alpha^{-1}$ belongs to $\mathrm{Gal}(K_d/K)$, and its induced automorphism on the residue field $\overline{K_d}$ is a commutator in $\mathrm{Aut}(\overline{K_d})$, thus trivial (note that since every automorphism of a local field is continuous, it always induces an automorphism of the residue field). We conclude that $\sigma^{-1}\alpha\sigma\alpha^{-1}$ itself was trivial, by [Ser79, Chapter III, §5, Theorem 3]. Hence, f_α is indeed a ring automorphism.

Furthermore, if it passes to the quotient, f_α induces the automorphism $\phi(\alpha, x)$. Hence it suffices to check that f_α preserves the relation. But we have $f_\alpha(u^d - \pi^r) = (ux)^d - \alpha(\pi^r) = u^d N_{K_d/K}(x) - \alpha(\pi^r) = (u^d - \pi^r)\frac{\alpha(\pi^r)}{\pi^r}$, as wanted. □

For α an automorphism of a (non-necessarily commutative) ring R, we denote by $\tilde{\alpha}$ the corresponding automorphism of $M_n(R)$ (the algebra of $n \times n$ matrices with coefficient in R) obtained by applying α coefficient by coefficient. Also, for A a finite dimensional central simple algebra over a field k, we denote by $\mathrm{Nrd}\colon A \to k$ its reduced norm.

Lemma 3.35. *Let k be a field and let A be a central simple k-algebra of degree d. Let l be a field extension of k splitting A. Let α be a ring automorphism of A such that $\alpha(k) = k$ and $\alpha(l) = l$. For $x \in A$, $\mathrm{Nrd}(\alpha(x)) = \alpha(\mathrm{Nrd}(x))$.*

Proof. Let $f\colon A \to M_d(l)$ be the representation of A given by the fact that l splits A. Let $\tilde{\alpha}\colon M_d(l) \to M_d(l)$ be the automorphism corresponding to $\alpha\colon l \to l$. We consider the following non commutative diagram

$$\begin{array}{ccccc} A & \xrightarrow{f} & M_d(l) & \xrightarrow{\det} & l \\ \alpha\big\downarrow & & \tilde{\alpha}\big\downarrow & & \big\downarrow\alpha \\ A & \xrightarrow{f} & M_d(l) & \xrightarrow{\det} & l \end{array}$$

In the above diagram, the square on the left does not commute, but the square on the right does commute. By definition, for $x \in A$, $\mathrm{Nrd}(x) = (\det \circ f)(x)$. Also note that since $\tilde{\alpha} \circ f \circ \alpha^{-1}$ is another representation of A, we have $\det \circ \tilde{\alpha} \circ f \circ \alpha^{-1} = \det \circ f$, by [Pie82, Chapter 16, §1, Corollary a]. Hence $\mathrm{Nrd}(\alpha(x)) = (\det \circ f \circ \alpha)(x) = (\det \circ \tilde{\alpha} \circ f)(x) = (\alpha \circ \det \circ f)(x) = \alpha(\mathrm{Nrd}(x))$, as wanted. □

We set some notations that we use for the rest of the chapter.

Definition 3.36. Let K be a local field. Let $A(d, r)$ and K_d, σ, π be as in Definition 3.33. Let α be an automorphism of K_d such that $\alpha(K) = K$, and assume that there exists an element x in K_d such that $N_{K_d/K}(x) = \frac{\alpha(\pi^r)}{\pi^r}$. The map $\tilde{\phi}(\alpha, x)\colon M_n(A) \to M_n(A)$, corresponding to the automorphism $\phi(\alpha, x)\colon A \to A$ from Lemma 3.34, preserves elements of reduced norm 1 by Lemma 3.35. We again denote its restriction to $\mathrm{SL}_n(A)$ by $\tilde{\phi}(\alpha, x)$.

Corollary 3.37. *Let A be a finite dimensional central simple algebra over a local field K. Every automorphism of K extends to an automorphism of A. Furthermore,* $\mathrm{Aut}_{\mathrm{SL}_n(A)}(K) = \mathrm{Aut}(K)$.

Proof. By Theorem 3.59, the central simple algebra A is an algebra of the form $A(d, r)$, i.e. a cyclic algebra of the form $(K_d/K, \sigma, \pi^r)$ with K_d, σ, π as in Definition 3.33.

Let $\alpha \in \mathrm{Aut}(K)$. Since K_d/K is Galois, there exists $\tilde{\alpha} \in \mathrm{Aut}(K_d)$ whose restriction to K is α. On the other hand, by [Ser79, Chapter V,§2, Corollary], $N_{K_d/K}$ is surjective on $\mathcal{O}_K^\times$. Furthermore, as we recall in the proof of Lemma 1.54, any automorphism of a local field preserves the valuation. Hence there exists $x \in K_d$ such that $N_{K_d/K}(x) = \frac{\alpha(\pi^r)}{\pi^r}$. Then the automorphism $\phi(\alpha, x)$ defined in Lemma 3.34 is an extension of α to A. Furthermore, $\tilde{\phi}(\alpha, x)$ from Definition 3.36 is defined over α^{-1}, so that the last claim follows. □

Remark 3.38. If $\alpha \in \mathrm{Aut}(K)$ is of finite order, the result in Corollary 3.37 asserting that α extends to an automorphism of D can be found in the literature. Indeed, using Lemma 3.63, it is a direct consequence of [EM48, Corollary 7.3]. See also [Han07, Theorem 5.6]. This already settle the question in characteristic 0, but we are not aware of such a result in positive characteristic.

3.6.2 Sufficient condition for the exact sequence not to split

Let us introduce another notation for a subgroup of the group of semilinear automorphisms, which allow us to introduce a "ground field".

Definition 3.39. Let k be a field and let G be a (finite type, affine) reductive group over k. Let k' be a subfield of k. We denote by $\mathrm{Aut}(G \to \operatorname{Spec} k/k')$ the subgroup of $\mathrm{Aut}(G \to \operatorname{Spec} k)$ consisting of semilinear automorphisms over an automorphism α belonging to $\mathrm{Aut}(k/k')$. Furthermore, we denote by $\mathrm{Aut}_G(k/k')$ the image of $\mathrm{Aut}(G \to \operatorname{Spec} k/k')$ under the map $\mathrm{Aut}(G \to \operatorname{Spec} k) \to \mathrm{Aut}_G(k)$.

Theorem 3.40. *Let D be a division algebra of degree d over a local field K and let $G = \mathrm{SL}_n(D)$. Let K' be a subfield of K such that K/K' is a finite Galois extension. Then the short exact sequence $1 \to (\operatorname{Aut} G)(K) \to \mathrm{Aut}(G \to \operatorname{Spec} K/K') \to \mathrm{Aut}_G(K/K') \to 1$ splits if and only if $\gcd(nd, [K:K'])$ divides n.*

Proof. By Corollary 3.37, $Aut_G(K) = Aut(K)$. Hence, since $\mathrm{Gal}(K/K')$ is contained in $Aut_G(K)$, the short exact sequence splits if and only if G is defined over K' (see Theorem 3.7). Let H be this hypothetical form of G over K'.

The case $d = 1$ being obviously true, let us assume that $d \geq 2$. Now, by the classification of simple groups over local fields (see [Tit79, Section 4.2 and 4.3]), the Tits index of H is of the form ${}^1A^{(d')}$ or ${}^2A^{(1)}$, since these are the only groups of type A over local fields. Note that a distinguished orbit has to remain distinguished after scalar extension, because a non-trivial root remains non-trivial after scalar extension. Hence H cannot be of type ${}^2A^{(1)}$, because groups of type ${}^2A^{(1)}$ have extremal roots that are distinguished, whereas G has undistinguished extremal roots when $d \geq 2$. But the only groups of type ${}^1A^{(d')}$ are groups of the form $\mathrm{SL}_{n'}(D')$ where $n' \geq 1$ and D' is a division algebra over K'. So we conclude that H is of this form.

We use the notation inv for the map classifying division algebras over local fields (see Theorem 3.59 for a precise definition of inv). Let d' be the degree of D' over K', and let r' be such that $[\frac{r'}{d'}] = \mathrm{inv}([D'])$ in $\mathbf{Q}/\mathbf{Z}$. Also, let $a = \gcd(d', [K:K'])$. The base change of $\mathrm{SL}_{n'}(D')$ from K' to K is the algebraic group $\mathrm{SL}_{an'}(A(\frac{d'}{a}, \frac{[K:K']}{a} r'))$ by Proposition 3.64. Since H is isomorphic to G over K, $an' = n$ and $ad = d'$. Hence, $a = \gcd(ad, [K:K'])$, which implies that $\gcd(adn', [K:K'])$ divides an'. Now, the equation $an' = n$ already proves that if H exists, then $\gcd(nd, [K:K'])$ divides n.

Conversely, let $a = \gcd(nd, [K:K'])$, and assume that a divides n. We then set $n' = \frac{n}{a}$, $d' = ad$ and r' such that $\frac{[K:K']}{a} r' - r \in d\mathbf{Z}$ (such an r' exists because $\frac{[K:K']}{a}$ is prime to d). With those parameters, the algebraic group $\mathrm{SL}_{n'}(A(d',r'))$ is a form of G over K', as wanted. □

Remark 3.41. The condition that $\gcd(nd,[K:K'])$ divides n is equivalent to require that for all primes p dividing d, the p-adic valuation of $[K:K']$ is less than or equal to the p-adic valuation of n.

Corollary 3.42. *Let D be a division algebra of degree d over a local field K and let $G = \mathrm{SL}_n(D)$. The short exact sequence $1 \to (\operatorname{Aut} G)(K) \to \operatorname{Aut}(G \to \operatorname{Spec} K) \to \operatorname{Aut}_G(K) \to 1$ does not split if there exists a subfield $K' \leq K$ such that K/K' is finite Galois and $\gcd(nd,[K:K'])$ does not divide n.*

Proof. $1 \to (\operatorname{Aut} G)(K) \to \operatorname{Aut}(G \to \operatorname{Spec} K/K') \to \operatorname{Aut}_G(K/K') \to 1$ does not split by Theorem 3.40, hence neither does $1 \to (\operatorname{Aut} G)(K) \to \operatorname{Aut}(G \to \operatorname{Spec} K) \to \operatorname{Aut}_G(K) \to 1$. □

3.6.3 Sufficient condition for the exact sequence to split

In characteristic 0, it is actually straightforward to prove the converse to Corollary 3.42.

Theorem 3.43. *Let D be a division algebra of degree d over a local field K of characteristic 0 and let $G = \mathrm{SL}_n(D)$. The short exact sequence $1 \to (\operatorname{Aut} G)(K) \to \operatorname{Aut}(G \to \operatorname{Spec} K) \to \operatorname{Aut}_G(K) \to 1$ does not split only if there exists a subfield $K' \leq K$ such that K/K' is finite Galois and $\gcd(nd,[K:K'])$ does not divide n.*

Proof. By Corollary 3.37, $Aut_G(K) = Aut(K)$. Since K is of characteristic 0, it is a finite extension of $\mathbf{Q}_p$ for some prime p. But every automorphism of K is continuous and fixes $\mathbf{Q} \leq K$, so that every automorphism acts trivially on $\mathbf{Q}_p$. Hence, by Galois theory, $\operatorname{Aut}(K)$ is a finite group. Furthermore, letting $K^{\operatorname{Aut}(K)}$ be the subfield of K fixed by $\operatorname{Aut}(K)$, the extension $K/K^{\operatorname{Aut}(K)}$ is Galois with Galois group $\operatorname{Aut}(K)$.

Let $a = \gcd(nd,[K:K^{\operatorname{Aut}(K)}])$. Assuming that there does not exist a subfield $K' \leq K$ such that K/K' is finite Galois and such that $\gcd(nd,[K:K'])$ does not divide n, we have in particular that a divides n. Also, let $r \in \mathbf{N}$ be such that $[\frac{r}{d}] = \operatorname{inv}([D])$. Since $\frac{[K:K^{\operatorname{Aut}(K)}]}{a}$ is prime to d, there exists $r' \in \mathbf{N}$ such that $\frac{[K:K^{\mathrm{Aut}(K)}]}{a} r' - r \in d\mathbf{Z}$. Hence, by Proposition 3.64, the algebraic group $\mathrm{SL}_{\frac{n}{a}}(A(ad, r'))$ is a form of G over $K^{\mathrm{Aut}(K)}$, because $\gcd(ad, [K : K^{\mathrm{Aut}(K)}]) = a$. But in view of Lemma 3.6, this implies that the homomorphism $\mathrm{Aut}(G \to \operatorname{Spec} K) \to \mathrm{Aut}(K) = \mathrm{Gal}(K/K^{\mathrm{Aut}(K)})$ has a section, as wanted. □

We now aim to prove an analogue of Theorem 3.43 but in positive characteristic. When K is of positive characteristic, the fixed field $K^{\mathrm{Aut}(K)}$ is finite and $K/K^{\mathrm{Aut}(K)}$ is not Galois. Thus we cannot use the same method than in characteristic 0.

Instead, the strategy goes as follows: we decompose $\mathrm{Aut}(K)$ in various pieces, we give a section of $\mathrm{Aut}(\mathrm{SL}_n(D) \to \operatorname{Spec} K) \to \mathrm{Aut}(K)$ separately for each pieces and then we check that everything can be glued. Let us begin by decomposing $\mathrm{Aut}(K)$.

Lemma 3.44. *Let $K = \mathbf{F}_{p^i}(\!(T)\!)$. Since $\mathbb{F}_{p^i}$ is the algebraic closure of the prime field in K, $\mathbf{F}_{p^i}$ is preserved by any automorphism of K. Let $N(K) = \{\alpha \in \mathrm{Aut}(K) \mid \alpha$ acts trivially on $\mathbf{F}_{p^i}\}$. We have $\mathrm{Aut}(K) \cong N(K) \rtimes \mathrm{Gal}(K/\mathbf{F}_p(\!(T)\!))$.*

Proof. We want to show that the short exact sequence $1 \to N(K) \to \mathrm{Aut}(K) \xrightarrow{f} \mathrm{Gal}(\mathbf{F}_{p^i}/\mathbf{F}_p) \to 1$ splits. But by [Ser79, Chapter III, §5, Theorem 3], f maps $\mathrm{Gal}(K/\mathbf{F}_p(\!(T)\!))$ isomorphically onto $\mathrm{Gal}(\mathbf{F}_{p^i}/\mathbf{F}_p)$, hence the result. □

We furthermore decompose the group $N(K)$. Since automorphisms of K are continuous, an element α of $N(K)$ is therefore defined by its action on T, and is of the form $\alpha(T) = \sum_{j=1}^{\infty} a_j T^j$, where $a_j \in \mathbf{F}_{p^i}$.

Definition 3.45. Let $J(K) = \{\alpha \in N \mid \alpha(T) = T + \sum_{j=2}^{\infty} a_j T^j,\ a_j \in \mathbf{F}_{p^i}\}$ and let $C_{p^i-1} = \{\alpha \in N \mid \alpha(T) = aT,\ a \in \mathbf{F}_{p^i}^{\times}\}$. With those notations, the group $N(K)$ is isomorphic to $J(K) \rtimes C_{p^i-1}$. For $a \in \mathbf{F}_{p^i}^{\times}$, we denote by $\mathrm{ev}(aT)$ the corresponding element of $\mathrm{Aut}(K)$.

In summary, we have decomposed $\mathrm{Aut}(K)$ as the group $(J(K) \rtimes C_{p^i-1}) \rtimes \mathrm{Gal}(K/\mathbf{F}_p(\!(T)\!))$. We go on by giving a section to $\mathrm{Aut}(\mathrm{SL}_n(D) \to$

$\operatorname{Spec} K) \to \operatorname{Aut}(K)$ for each component of $\operatorname{Aut}(K)$, one at a time. Let us first set some notations.

Definition 3.46. Let $K = \mathbf{F}_{p^i}(\!(T)\!)$ and let E be a finite unramified extension of K. For $\alpha \in N(K)$ we define its extension α_E to $\operatorname{Aut}(E)$ as follows: α_E acts trivially on the residue field, while $\alpha_E(T) = \alpha(T)$. We thus get an injective homomorphism $N(K) \to N(E) \colon \alpha \mapsto \alpha_E$. In Proposition 3.47 and Proposition 3.50, we abuse notations and also denote α_E as α.

Proposition 3.47. *Let $K = \mathbf{F}_{p^i}(\!(T)\!)$ and let D be a division algebra of degree d over K. Using the notations of Definition 3.33, D is the cyclic algebra $(K_d/K, \sigma, \pi^r)$ for some $r \in \mathbf{N}$. Assume that* $\gcd(p, d) = 1$. *For $\alpha \in J(K)$, there exists a unique x_α in $1 + T\mathbf{F}_{p^i}[\![T]\!]$ such that $x_\alpha^d = \frac{\alpha(T^r)}{T^r}$. Let $G = \mathrm{SL}_n(D)$ and recall the notation introduced in Definition 3.36. The map*

$$\begin{aligned} f_{J(K)} \colon J(K) &\to \operatorname{Aut}(G \to \operatorname{Spec} K) \\ \alpha &\mapsto \tilde{\phi}(\alpha, x_\alpha) \end{aligned}$$

is a homomorphism whose composition with the map $\operatorname{Aut}(G \to \operatorname{Spec} K) \to \operatorname{Aut}_G(K)$ *is the identity on $J(K)$.*

Proof. For $\alpha \in J(K)$, the uniqueness of x_α in $1+T\mathbf{F}_{p^i}[\![T]\!]$ such that $x_\alpha^d = \frac{\alpha(T^r)}{T^r}$ follows directly from Hensel's lemma, since $\gcd(d, p) = 1$. We claim that for $\alpha, \beta \in J(K)$, $x_{\beta\circ\alpha} = x_\beta . \beta(x_\alpha)$. By uniqueness, this equation holds if and only if $\frac{(\beta\circ\alpha)(T^r)}{T^r} = [x_\beta . \beta(x_\alpha)]^d$. But the right hand side is equal to $\frac{\beta(T^r)}{T^r} . \beta(\frac{\alpha(T^r)}{T^r})$, which is indeed equal to $\frac{(\beta\circ\alpha)(T^r)}{T^r}$. Checking that $f_{J(K)}$ is a homomorphism is now straightforward: $\tilde{\phi}(\beta, x_\beta) \circ \tilde{\phi}(\alpha, x_\alpha) = \tilde{\phi}(\beta \circ \alpha, x_\beta . \beta(x_\alpha)) = \tilde{\phi}(\beta \circ \alpha, x_{\beta\circ\alpha})$. $\square$

We can also prove a converse to Proposition 3.47.

Proposition 3.48. *Let $K = \mathbf{F}_{p^i}(\!(T)\!)$, let D be a division algebra of degree d over K and let $G = \mathrm{SL}_n(D)$. If* $\gcd(p, d) \neq 1$, *there does not exist a homomorphism $f_{J(K)} \colon J(K) \to \operatorname{Aut}(G \to \operatorname{Spec} K)$ whose composition with* $\operatorname{Aut}(G \to \operatorname{Spec} K) \to \operatorname{Aut}_G(K)$ *is the identity on $J(K)$.*

Proof. By Theorem 3.40, it suffices to prove that for all $n \in \mathbf{N}$, there exists $K' \leq K$ such that K/K' is finite Galois with $\mathrm{Gal}(K/K') \leq J(K)$ and $\gcd(nd, [K : K'])$ does not divide n. Let G be a group of order p^n. By [Cam97, Theorem 3], there exists an injective homomorphism $G \hookrightarrow J(\mathbf{F}_p(\!(T)\!))$. Also note that $J(\mathbf{F}_p(\!(T)\!))$ can be seen as a subgroup of $J(K)$ in a natural way, so that $J(K)$ has a subgroup of order p^n, that we again denote by G. Now, let $K' = K^G = \{x \in K \mid \alpha(x) = x \text{ for all } \alpha \in G\}$. Hence, K/K' is a Galois extension with $\mathrm{Gal}(K/K') = G \leq J(K)$ and $\gcd(nd, [K : K']) = \gcd(nd, p^n)$ does not divide n because $\gcd(p, d) \neq 1$, as wanted. □

We now construct a section of $\mathrm{Aut}(G \to \mathrm{Spec}\, K) \to \mathrm{Aut}(K)$ for C_{p^i-1}. In fact, using the same line of argument as for Theorem 3.40, we know that a section for C_{p^i-1} exists if and only if $\gcd(nd, p^i - 1)$ divides n (where d and n appear in the form of $G = \mathrm{SL}_n(D)$, d denoting as usual the degree of D). But we need to have an explicit formula, since we want to ensure that it glues well with the map $f_{J(K)}$ constructed in Proposition 3.47. Let us furthermore decompose C_{p^i-1} according to the degree of the division algebra.

Definition 3.49. Let $d \in \mathbf{N}$ (in practice, d is the degree of the division algebra appearing in $G = \mathrm{SL}_n(D)$). Let $k(d), l(d) \in \mathbf{N}$ be such that $k(d).l(d) = p^i - 1$, with $\gcd(d^{p^i-1}, p^i - 1) = \gcd(d^{k(d)}, k(d)) = k(d)$ and $\gcd(d, l(d)) = 1$. Hence, $\gcd(l(d), k(d)) = 1$, so that $C_{p^i-1} \cong C_{k(d)} \times C_{l(d)}$.

Proposition 3.50. *Let $K = \mathbf{F}_{p^i}(\!(T)\!)$, let D be a division algebra of degree d over K, and let $C_{l(d)} \leq C_{p^i-1} \cong \mathbf{F}_{p^i}^\times$ be as in Definition 3.49. Every element a in $\mathbf{F}_{p^i}^\times \cap C_{l(d)}$ has a unique d-th root $\sqrt[d]{a}$ in $\mathbf{F}_{p^i}^\times \cap C_{l(d)}$. Using the notations of Definition 3.33, D is the cyclic algebra $(K_d/K, \sigma, \pi^r)$ for some $r \in \mathbf{N}$. Let $G = \mathrm{SL}_n(D)$ and recall the notation introduced in Definition 3.36. The map*

$$\begin{aligned} f_{C_{l(d)}} : C_{l(d)} &\to \mathrm{Aut}(G \to \mathrm{Spec}\, K) \\ \mathrm{ev}(aT) &\mapsto \tilde{\phi}(\mathrm{ev}(aT), \sqrt[d]{a^r}) \end{aligned}$$

is a homomorphism whose composition with the map $\mathrm{Aut}(G \to \operatorname{Spec} K) \to \mathrm{Aut}_G(K)$ *is the identity on* $C_{l(d)}$.

Proof. The existence and uniqueness of $\sqrt[d]{a} \in C_{l(d)}$ follows from the fact that $\gcd(d, l(d)) = 1$. Note that $\frac{\mathrm{ev}(aT)(T^r)}{T^r} = a^r = N_{K_d/K}(\sqrt[d]{a^r})$, so that we can indeed use Definition 3.36. Furthermore, by uniqueness of d-th root, $f_{C_{l(d)}}$ is indeed a group homomorphism. □

Before going on and defining a section for $C_{k(d)}$, we need more notations.

Definition 3.51. 1. Let $\mathbf{F}_p^{\mathrm{alg}}$ be the algebraic closure of $\mathbf{F}_p$. We denote by F_p (or simply F when p is clear from the context) the Frobenius automorphism of $\mathbf{F}_p^{\mathrm{alg}}(\!(T)\!)$. For any finite extension (respectively finite unramified) extension E of $\mathbf{F}_p$ (respectively $\mathbf{F}_p(\!(T)\!)$), we also denote by F the restriction of F to E.

2. For G a group and $g \in G$, we denote by $\mathrm{int}(g)$ the automorphism by conjugation of g on G, i.e. $\mathrm{int}(g)\colon G \to G\colon x \mapsto gxg^{-1}$.

Proposition 3.52. *Let* $K = \mathbf{F}_{p^i}(\!(T)\!)$*, let* D *be a division algebra of degree* d *over* K*.Using the notations of Definition 3.33,* D *is the cyclic algebra* $(K_d/K, \sigma, \pi^r)$ *for some* $r \in \mathbf{N}$*. Let* $G = \mathrm{SL}_n(D)$ *and let* $C_{k(d)} \leq C_{p^i-1} \cong \mathbf{F}_{p^i}^{\times}$ *be as in Definition 3.49. Let* ζ *be a generator of* $C_{k(d)}$*. There exists an element* $x \in \mathbf{F}_{p^{idk(d)}}$ *such that* $\frac{F^{id}(x)}{x} = \zeta^r$*. Choosing a* $\mathbf{F}_{p^{id}}$*-basis of* $\mathbf{F}_{p^{idn}}$*, we obtain an embedding* $\varphi\colon \mathbf{F}_{p^{idn}} \to M_n(\mathbf{F}_{p^{id}})$*. Assume that* $k(d)$ *divides* n*. Thus* $\mathbf{F}_{p^{idk(d)}}$ *is a subfield of* $\mathbf{F}_{p^{idn}}$*, and let* g *be the image of* x^{-1} *under the embedding* φ*. Recalling the notation introduced in Definition 3.36, the map*

$$\begin{aligned} f_{C_{k(d)}} \colon C_{k(d)} &\to \mathrm{Aut}(G \to \operatorname{Spec} K) \\ \mathrm{ev}(\zeta^j T) &\mapsto \mathrm{int}(g^j) \circ \tilde{\phi}(\mathrm{ev}(\zeta^j T), (F^i(x)x^{-1})^j) \end{aligned}$$

is a homomorphism whose composition with the map $\mathrm{Aut}(G \to \operatorname{Spec} K) \to \mathrm{Aut}_G(K)$ *is the identity on* $C_{k(d)}$.

Proof. Since d is fixed for the rest of the proof, let us denote $k(d)$ simply by k. We begin by proving the existence of $x \in \mathbf{F}_{p^{idk}}$ such that $\frac{F^{id}(x)}{x} =$

ζ^r. Since ζ belongs to C_k, $\zeta^{rk} = 1$. In other words, $N_{\mathbf{F}_{p^{idk}}/\mathbf{F}_{p^{id}}}(\zeta^r) = 1$. Also note that the extension $\mathbf{F}_{p^{idk}}/\mathbf{F}_{p^{id}}$ is Galois cyclic, and that F^{id} generates its Galois group. Hence, by Hilbert's Theorem 90, there indeed exists $x \in \mathbf{F}_{p^{idk}}$ such that $\frac{F^{id}(x)}{x} = \zeta^r$. For the rest of the proof, we choose such an x.

From $\frac{F^{id}(x)}{x} = \zeta^r$, it readily follows that $F^i(x)x^{-1}$ and x^k belongs to $\mathbf{F}_{p^{id}}$, since they are both invariant under F^{id}. Note that $\frac{\mathrm{ev}(\zeta^j T)(T^r)}{T^r} = \zeta^{jr} = N_{K_d/K}((F^i(x)x^{-1})^j)$, so that we can indeed use Definition 3.36.

It remains to check that f_{C_k} is a homomorphism. Note that $\mathrm{int}(g^j) \circ \tilde{\phi}(\mathrm{ev}(\zeta^j T), (F^i(x)x^{-1})^j)$ followed by $\mathrm{int}(g^{j'}) \circ \tilde{\phi}(\mathrm{ev}(\zeta^{j'} T), (F^i(x)x^{-1})^{j'})$ is just $\mathrm{int}(g^{j'+j}) \circ \tilde{\phi}(\mathrm{ev}(\zeta^{j'+j} T), (F^i(x)x^{-1})^{j'+j})$. Hence, it suffices to check that $\mathrm{int}(g^k) \circ \tilde{\phi}(\mathrm{ev}(\zeta^k T), (F^i(x)x^{-1})^k)$ is the identity on $\mathrm{SL}_n(D)$. But since $x^k \in \mathbf{F}_{p^{id}}$, g^k is diagonal and by definition of g, $x^{-k}g^{-k} = 1$. Furthermore, since σ (respectively F) is the Frobenius element of $\mathbf{F}_{p^i}(\!(T)\!)$ (respectively $\mathbf{F}_p^{\mathrm{alg}}(\!(T)\!)$), $\sigma = F^i$, so that for $u \in D$ as in Definition 3.33, $ug^k u^{-1} = F^i(g^k)$. This concludes the proof. $\square$

Remark 3.53. The automorphism $\mathrm{int}(g^j) \circ \tilde{\phi}(\mathrm{ev}(\zeta^j T), (F^i(x)x^{-1})^j)$ of Proposition 3.52 does not depend on the choice of x. This can be seen by a direct computation, similar to the one we carry out in the proof of Theorem 3.57, Item 5.

As before, we can also prove a converse to Proposition 3.52.

Proposition 3.54. *Let $K = \mathbf{F}_{p^i}(\!(T)\!)$, let D be a division algebra of degree d over K and let $G = \mathrm{SL}_n(D)$. If $k(d)$ does not divide n, there does not exist a homomorphism $f_{C_{k(d)}} : C_{k(d)} \to \mathrm{Aut}(G \to \mathrm{Spec}\, K)$ whose composition with $\mathrm{Aut}(G \to \mathrm{Spec}\, K) \to \mathrm{Aut}_G(K)$ is the identity on $J(K)$.*

Proof. By Theorem 3.40, it suffices to prove that there exists $K' \leq K$ such that K/K' is finite Galois with $\mathrm{Gal}(K/K') \leq C_{k(d)}$ and such that $\gcd(nd, [K : K'])$ does not divide n. Recall from Definition 3.49 that $l(d)$ is prime to $k(d)$ with $k(d)l(d) = p^i - 1$. Hence K has a $k(d)$-th primitive root of unity ζ. Hence, $K' = \mathbf{F}_{p^i}(\!(T^{k(d)})\!)$ is such that K/K' is Galois of degree $k(d)$, and $\mathrm{Gal}(K/K')$ is generated by the automorphism of K sending T to ζT, so that $\mathrm{Gal}(K/K') \leq C_{k(d)}$. Finally, $\gcd(nd, [K :$

$K'])=\gcd(nd,k(d))$ does not divide n because by definition $k(d)=\gcd(d^{k(d)},k(d))$ and $k(d)$ does not divide n. $\square$

Finally, we construct a section to $\mathrm{Aut}(G\to\operatorname{Spec}K)\to\mathrm{Aut}(K)$ for $\mathrm{Gal}(\mathbf{F}_{p^i}(\!(T)\!)/\mathbf{F}_p(\!(T)\!))$.

Proposition 3.55. *Let $K=\mathbf{F}_{p^i}(\!(T)\!)$ and let D be a division algebra of degree d over K. Assume that $\gcd(d,i)=1$. Let $m\in\mathbf{Z}$ be such that $mi+1\in d\mathbf{Z}$. Let $G=\mathrm{SL}_n(D)$ and recall the notation introduced in Definition 3.36. The map*

$$\begin{aligned} f_{\mathrm{Gal}}\colon \mathrm{Gal}(\mathbf{F}_{p^i}(\!(T)\!)/\mathbf{F}_p(\!(T)\!)) &\to \mathrm{Aut}(G\to\operatorname{Spec}K)\\ F^j &\mapsto \tilde{\phi}(F^{j(mi+1)},1)\end{aligned}$$

is a homomorphism. Furthermore, its composition with the map $\mathrm{Aut}(G\to\operatorname{Spec}K)\to\mathrm{Aut}_G(K)$ *is the identity on* $\mathrm{Gal}(\mathbf{F}_{p^i}(\!(T)\!)/\mathbf{F}_p(\!(T)\!))$.

Proof. First, note that m exists because $\gcd(d,i)=1$. Since F^{jmi} acts trivially on $\mathbf{F}_{p^i}(\!(T)\!)$, $F^{j(mi+1)}$ is indeed an extension of F^j (seen as restricted to $\mathbf{F}_{p^i}(\!(T)\!)$) to $\mathbf{F}_{p^{id}}(\!(T)\!)$. Furthermore, $F^{i(mi+1)}$ acts trivially on $\mathbf{F}_{p^{id}}(\!(T)\!)$, because $i(mi+1)\in id\mathbf{Z}$ by definition of m. Hence, f_{Gal} is indeed a homomorphism. $\square$

Remark 3.56. A section $f_{\mathrm{Gal}}\colon \mathrm{Gal}(\mathbf{F}_{p^i}(\!(T)\!)/\mathbf{F}_p(\!(T)\!))\to\mathrm{Aut}(G\to\operatorname{Spec}K)$ exists if and only if $\gcd(nd,i)$ divides n. In Proposition 3.65, we give an explicit formula for f_{Gal} assuming only that $\gcd(nd,i)$ divides n. Unfortunately, this formula does not glue well with the formula for $f_{J(K)}$ (given in Proposition 3.47) when $\gcd(d,i)$ is not equal to 1.

We can finally glue all the previous constructions to obtain a global splitting of the initial short exact sequence.

Theorem 3.57. *Let D be a division algebra of degree d over the local field $K=\mathbf{F}_{p^i}(\!(T)\!)$ and let $G=\mathrm{SL}_n(D)$. Assume that for all subfields $K'\leq K$ such that K/K' is finite Galois,* $\gcd(nd,[K:K'])$ *divides n. Also assume that $\gcd(d,i)=1$. Then the short exact sequence* $1\to(\mathrm{Aut}\,G)(K)\to\mathrm{Aut}(G\to\operatorname{Spec}K)\to\mathrm{Aut}_G(K)\to 1$ *splits.*

Proof. In view of Proposition 3.58, the hypotheses imply that $\gcd(d,p) = 1$, $\gcd(nd, p^i - 1)$ divides n and $\gcd(d,i) = 1$. Hence we can apply Propositions 3.47, 3.50, 3.52 and 3.55. For the rest of the proof, we strictly adhere to the notations that are introduced in the statements of those propositions.

Recall that $\mathrm{Aut}_G(K) = \mathrm{Aut}(K)$ (Corollary 3.37). Also recall that we decompose $\mathrm{Aut}(K)$ as $(J(K) \rtimes (C_{l(d)} \times C_{k(d)})) \rtimes \mathrm{Gal}(\mathbf{F}_{p^i}(\!(T)\!)/\mathbf{F}_p(\!(T)\!))$. We define a map

$$\begin{aligned} f\colon (J(K) \rtimes (C_{l(d)} \times C_{k(d)})) \rtimes \mathrm{Gal}(\mathbf{F}_{p^i}(\!(T)\!)/\mathbf{F}_p(\!(T)\!)) &\to \mathrm{Aut}(G \to \operatorname{Spec} K) \\ (\alpha, \mathrm{ev}(aT), \mathrm{ev}(\zeta^j T), F^{j'}) &\mapsto f_{J(K)}(\alpha) f_{C_{l(d)}}(\mathrm{ev}(aT)) f_{C_{k(d)}}(\mathrm{ev}(\zeta^j T)) f_{\mathrm{Gal}}(F^{j'}) \end{aligned}$$

We claim that f is a homomorphism. To prove this claim, it suffices to compute various commutators in $\mathrm{Aut}(G \to \operatorname{Spec} K)$.

1. The images of $f_{C_{l(d)}}$ and $f_{C_{k(d)}}$ commute. Indeed, $\tilde{\phi}(\mathrm{ev}(aT), \sqrt[d]{a^r})$ readily commutes with $\mathrm{int}(g^j)\tilde{\phi}(\mathrm{ev}(\zeta^j T), (F(x)x^{-1})^j)$.

2. We claim that $f_{C_{l(d)}}(\mathrm{ev}(a^{-1}T)) f_{J(K)}(\alpha) f_{C_{l(d)}}(\mathrm{ev}(aT))$ is equal to $f_{J(K)}(\mathrm{ev}(a^{-1}T) \circ \alpha \circ \mathrm{ev}(aT))$.

   Indeed, $\tilde{\phi}(\mathrm{ev}(a^{-1}T), \sqrt[d]{a^{-r}})\tilde{\phi}(\alpha, x_\alpha)\tilde{\phi}(\mathrm{ev}(aT), \sqrt[d]{a^r}) = \tilde{\phi}(\mathrm{ev}(a^{-1}T) \circ \alpha \circ \mathrm{ev}(aT), \mathrm{ev}(a^{-1}T)(x_\alpha))$. But $\mathrm{ev}(a^{-1}T)(x_\alpha) = x_{\mathrm{ev}(a^{-1}T)\circ\alpha\circ\mathrm{ev}(aT)}$, because $\mathrm{ev}(a^{-1}T)(x_\alpha)$ belongs to $1 + T\mathbf{F}_{p^i}[\![T]\!]$, and

   $$\begin{aligned} \mathrm{ev}(a^{-1}T)(x_\alpha)^d &= \mathrm{ev}(a^{-1}T)(\frac{\alpha(T^r)}{T^r}) \\ &= a^r . \frac{\mathrm{ev}(a^{-1}T)\alpha(T^r)}{T^r} \\ &= \frac{\mathrm{ev}(a^{-1}T) \circ \alpha \circ \mathrm{ev}(aT)(T^r)}{T^r} \end{aligned}$$

   The claimed equality follows.

3. $f_{C_{k(d)}}(\mathrm{ev}(\zeta^{-j}T)) f_{J(K)}(\alpha) f_{C_{k(d)}}(\mathrm{ev}(\zeta^j T)) = f_{J(K)}(\mathrm{ev}(\zeta^{-j}T) \circ \alpha \circ \mathrm{ev}(\zeta^j T))$. This follows from the same kind of computation than in the previous case.

4. $f_{\mathrm{Gal}}(F^{-j'}) f_{J(K)}(\alpha) f_{\mathrm{Gal}}(F^{j'}) = f_{J(K)}(F^{-j'} \alpha F^{j'})$.

Indeed, $F^{-j'(mi+1)}(x_\alpha)$ belongs to $1+T\mathbf{F}_{p^i}[\![T]\!]$, and furthermore $F^{-j'(mi+1)}(x_\alpha)^d = F^{-j'(mi+1)}(\frac{\alpha(T^r)}{T^r}) = \dfrac{F^{-j'(mi+1)}\alpha F^{j'(mi+1)}(T^r)}{T^r}$. Hence

$$\begin{aligned}\tilde{\phi}(F^{-j'(mi+1)},1)\tilde{\phi}(\alpha,x_\alpha)\tilde{\phi}(F^{j'(mi+1)},1) =&\\ =&\tilde{\phi}(F^{-j'(mi+1)}\alpha F^{j'(mi+1)}, F^{-j'(mi+1)}(x_\alpha))\\ =&\tilde{\phi}(F^{-j'(mi+1)}\alpha F^{j'(mi+1)}, x_{F^{-j'(mi+1)}\alpha F^{j'(mi+1)}}))\\ =&\tilde{\phi}(F^{-j'}\alpha F^{j'}, x_{F^{-j'}\alpha F^{j'}}))\end{aligned}$$

as wanted.

5. We readily check the equality $f_{\mathrm{Gal}}(F^{j'})f_{l(d)}(\mathrm{ev}(aT))f_{\mathrm{Gal}}(F^{-j'}) = f_{l(d)}(F^{j'}\,\mathrm{ev}(aT)F^{-j'})$. Let us prove that

$$f_{\mathrm{Gal}}(F^{j'})f_{k(d)}(\mathrm{ev}(\zeta^j T))f_{\mathrm{Gal}}(F^{-j'}) = f_{k(d)}(F^{j'}\,\mathrm{ev}(\zeta^j T)F^{-j'}).$$

The left hand side is equal to

$$\tilde{\phi}(F^{j'(mi+1)},1)\,\mathrm{int}(g^j)\tilde{\phi}(\mathrm{ev}(\zeta^j T),(F^i(x)x^{-1})^j)\tilde{\phi}(F^{-j'(mi+1)},1),$$

which in turn is equal to the automorphism $f_1 =$

$$\mathrm{int}(F^{j'(mi+1)}(g^j))\tilde{\phi}(\mathrm{ev}(F^{j'(mi+1)}(\zeta^j)T), F^{j'(mi+1)}(F^i(x)x^{-1})^j)$$

We claim that the automorphism f_1 is equal to

$$f_2 = \mathrm{int}(F^{j'}(g^j))\tilde{\phi}(\mathrm{ev}(F^{j'}(\zeta^j)T), F^{j'}(F^i(x)x^{-1})^j)$$

To prove the claim, we write an element of $M_n(D)$ as $\sum_{i=0}^{d-1} u^i M_i$, with $M_i \in M_n(\mathbf{F}_{p^{id}}(\!(T)\!))$. Recall that the action of F on $\mathbf{F}_{p^{id}}$ is just given by $y \mapsto y^p$, and that for $M \in M_n(\mathbf{F}_{p^{id}}(\!(T)\!))$, $u^{-1}Mu = \sigma(M) = F^i(M)$. Hence,

$$\begin{aligned} f_1(u) &= uF^{j'(mi+1)+i}(g^j)F^{j'(mi+1)}(F^i(x)x^{-1})^j F^{j'(mi+1)}(g^{-j})\\ &= uF^{j'(mi+1)}(F^i(gx)(gx)^{-1})^j)\\ &= uF^{j'(mi+1)}(gx)^{j(p^i-1)} = uF^{j'(mi+1)}(gx)^{jk(d)l(d)} = u\end{aligned}$$

where for the last equality, we used the fact that $x^{k(d)}g^{k(d)} = 1$. Doing the same computation for f_2, we find that $f_2(u) = u$ as well. There just remains to show that for $M \in M_n(\mathbf{F}_{p^{id}}(\!(T)\!))$, $f_1(M) = f_2(M)$. Since $\zeta \in \mathbf{F}_{p^i}$, $F^{j'(mi+1)}(\zeta^j) = F^{j'}(\zeta^j)$. On the other hand, $F^{j'(mi+1)}(g^j) = g^{jp^{j'(mi+1)}} = g^{jp^{j'}}g^{jp^{j'}(p^{j'mi}-1)}$. Recalling that $k(d)$ divides $p^i - 1$ and that $g^{k(d)}$ is a scalar matrix, this shows that $F^{j'(mi+1)}(g^j)$ is equal to $F^{j'}(g^j)$ up to a scalar matrix, so that $f_1(M) = \mathrm{int}(F^{j'(mi+1)}(g^j))(M) = \mathrm{int}(F^{j'}(g^j))(M) = f_2(M)$.

We conclude that f is indeed a homomorphism. The fact that f is a splitting of the short exact sequence in the statement of the proposition follows from the fact that the restriction of f to each component is locally a section of $\mathrm{Aut}(G \to \operatorname{Spec} K) \to \mathrm{Aut}_G(K)$. $\square$

Since existence of Galois subfield plays a crucial role, we end this section by characterising the existence of a Galois subfield K' of K whose degree is divisible by a prime power.

Proposition 3.58. *Let $K = \mathbf{F}_{p^i}(\!(T)\!)$, let q be a prime number and let $a \in \mathbf{N}$. There exists a subfield K' such that K/K' is finite Galois and q^a divides $[K : K']$ if and only if $q = p$ or q^a divides $i(p^i - 1)$.*

Proof. Assume that such a K' exists. Since K/K' is Galois and q^a divides $[K : K']$, there exists $\tilde{K}$ such that $K/\tilde{K}$ is Galois and $[K : \tilde{K}] = q^a$. Up to replacing K' by $\tilde{K}$, we can thus assume that $[K : K'] = q^a$. Let also K'_{ur} be the maximal unramified extension of K' inside K.

Note that K' and K'_{ur} are local fields, so that in particular $K' \cong \mathbf{F}_{p^k}(\!(T)\!)$ and $K'_{ur} \cong \mathbf{F}_{p^i}(\!(T)\!)$. Since $[K'_{ur} : K']$ divides q^a, there exists a_1 such that $q^{a_1} = \frac{i}{k}$. Letting $a_2 = a - a_1$, we have that K/K'_{ur} is a totally ramified extension of degree q^{a_2}.

If $p = q$, the proposition is proved, hence there just remains to investigate the case $p \neq q$. In this case, K is a tamely totally ramified extension of K'_{ur}. Thus, K is isomorphic to $K'_{ur}[X]/(X^{q^{a_2}} - \pi)$ for some uniformiser $\pi \in \mathbf{F}_{p^i}(\!(T)\!)$. But K is a Galois extension, and hence this implies that $\mathbf{F}_{p^i}(\!(T)\!)$ has a primitive q^{a_2}-th root of unity, so that q^{a_2} divides $p^i - 1$, as wanted.

To prove the converse, we use a classical fact from local class field theory: there exists an extension K_π of K which is Galois and totally ramified, and such that $\mathrm{Gal}(K_\pi/K)$ is isomorphic to the group of invertible elements $\mathbf{F}_{p^i}[\![T]\!]^\times$ of $\mathbf{F}_{p^i}[\![T]\!]$ (see for example [Iwa86, Section 5.3]). Note that the degree of $\mathbf{F}_{p^i}^\times + T^{a+1}\mathbf{F}_{p^i}[\![T]\!]$ in $\mathbf{F}_{p^i}[\![T]\!]^\times$ is equal to p^a. Let L_1 be the Galois extension of K corresponding to $\mathbf{F}_{p^i}^\times + T^{a+1}\mathbf{F}_{p^i}[\![T]\!]$. Let also L_2 be the splitting field of $X^{p^i-1} - T$ over $\mathbf{F}_p(\!(T)\!)$. For $k = 1$ or 2, L_k is totally ramified of finite degree over K, so that there exists an isomorphism $\phi_k \colon K \to L_k$. Hence $K_1 = \phi_1^{-1}(K)$ (respectively $K_2 = \phi_2^{-1}(\mathbf{F}_p(\!(T)\!))$) is such that K/K_1 (respectively K/K_2) is Galois, and $[K : K_1] = p^a$ (respectively $[K : K_2] = i(p^i - 1)$), which concludes the proof. □

3.A Base change of the algebraic group $\mathrm{SL}_n(D)$

We begin by recalling some classical facts about finite dimensional central simple algebras over local fields.

Theorem 3.59. *Let K be a local field. Every central simple algebra over K is isomorphic to an algebra of the form $A(d,r)$ as in Definition 3.33. Furthermore, the map $\mathrm{inv} \colon Br(K) \to \mathbf{Q}/\mathbf{Z} \colon [A(d,r)] \mapsto [\frac{r}{d}]$ is an isomorphism of groups.*

Proof. See for example [Mor97, Theorem 8] for the first assertion, while the second is precisely the content of [Pie82, Chapter 17, §10, Theorem]. □

Corollary 3.60. *Let K be a local field and let $d, r \in \mathbf{N}$ with $d \geq 1$. Let $a = \gcd(d,r)$. Then $A(d,r)$ is a division algebra if and only if $a = 1$, and $A(d,r) \cong M_a(A(\frac{d}{a}, \frac{r}{a}))$.*

Proof. The central simple algebra $A(d,r)$ is a division algebra if and only if all central simple algebras over K in the same Brauer class have a higher degree. In view of Theorem 3.59, it readily implies that $A(d,r)$ is a division algebra if and only if $a = 1$. Furthermore, by Wedderburn's theorem, $A(d,r)$ is isomorphic to $M_n(D)$ for some division algebra D and some $1 \leq n \in \mathbf{N}$, and by definition of the Brauer group, $[D] =$

$[A(d,r)]$. Hence, using the first part of the Theorem, $D \cong A(\frac{d}{a}, \frac{r}{a})$. Now, comparing degrees readily imply that $n = a$, and the result is proved. $\square$

We now study the base change of the algebraic group $SL_n(A)$.

Lemma 3.61. *Let A be a central simple algebra over a field k, and let $\mathrm{SL}_{1,A}$ be the algebraic group over k defined as the kernel of the reduced norm, so that $\mathrm{SL}_{1,A}(k) = \{x \in A \mid \mathrm{Nrd}(x) = 1\} = \mathrm{SL}_1(A)$. For k' a field extension of k, $(\mathrm{SL}_{1,A})_{k'} = \mathrm{SL}_{1,A\otimes_k k'}$.*

Proof. Let $\overline{k'}$ be the algebraic closure of k'. Since $\overline{k'}$ splits A, the reduced norm is the map $f\colon A \to A \otimes_k \overline{k'} \cong M_n(\overline{k'}) \xrightarrow{\det} \overline{k'}$. Let φ denotes the isomorphism $A \otimes_k \overline{k'} \cong M_n(\overline{k'})$. If we take a k-basis of A to get coordinates on $A \otimes_k \overline{k'}$, the map $\det \circ \varphi$ is actually a polynomial map on $A \otimes_k \overline{k'}$ with coefficients in k, by [Bou73, Chapitre VIII, §12, Proposition 11]. Hence, $f_{\overline{k'}} = \det \circ \varphi$. This implies that $f_{k'}\colon A \otimes_k k' \to k'$ is just the composition $A \otimes_k k' \to A \otimes_k \overline{k'} \cong M_n(\overline{k'}) \xrightarrow{\det} \overline{k'}$, i.e. $f_{k'}$ is the reduced norm map of the algebra $A \otimes_k k'$, as wanted. $\square$

Remark 3.62. To avoid having too many subscripts, we denote the algebraic group $\mathrm{SL}_{1,A}$ by its group of rational points, i.e. $\mathrm{SL}_1(A)$. Recall also that by definition, $\mathrm{SL}_{1,M_n(A)} = \mathrm{SL}_{n,A}$. We also denote this equality at the level of rational points, i.e. $\mathrm{SL}_1(M_n(A)) = \mathrm{SL}_n(A)$.

Before giving the formula for a base change of $\mathrm{SL}_n(A)$, we recall the effect of extending scalars for central simple algebras over local fields.

Lemma 3.63. *Let K be a local field and let $A(d,r)$ be the central simple algebra over K defined in Definition 3.33. Let L be a finite extension of K. Then $A(d,r) \otimes_K L \cong A(d, r[L:K])$.*

Proof. By Wedderburn's theorem, a central simple algebra over a field is uniquely determined by its degree and its Brauer class. By [Pie82, Chapter 17, Section 17.10, Proposition], we have $\mathrm{inv}([A(d,r) \otimes_K L]) = [L:K].\mathrm{inv}([A(d,r)])$. Hence $A(d, r[L:K])$ and $A(d,r) \otimes_K L$ are in the same Brauer class. Since they have the same degree as well, this concludes the proof. $\square$

Proposition 3.64. *Let $A(d', r')$ be a division algebra over a local field K' as in Definition 3.33. Let K/K' be a finite field extension and let $a = \gcd(d', [K, K'])$. Then the base change of $\mathrm{SL}_{n'}(A(d', r'))$ to K is isomorphic to $SL_{an'}(A(\frac{d'}{a}, \frac{[K:K']}{a} r'))$.*

Proof. The base change of $\mathrm{SL}_{n'}(A(d', r')) = \mathrm{SL}_1(M_{n'}(A(d', r')))$ to K is isomorphic to $\mathrm{SL}_1(M_{n'}(A(d', r')) \otimes_{K'} K) \cong \mathrm{SL}_{n'}(A(d', r') \otimes_{K'} K)$ by Lemma 3.61. But by Corollary 3.60 and Lemma 3.63, $A(d', r') \otimes_{K'} K \cong M_a(A(\frac{d'}{a}, \frac{[K:K']}{a} r'))$. To conclude, note that for any central simple algebra A, $\mathrm{SL}_{n'}(M_a(A)) \cong SL_{an'}(A)$. □

3.B More automorphisms of $\mathrm{SL}_n(D)$

The following proposition gives explicitly a "partial splitting" of the homomorphism $\mathrm{Aut}(\mathrm{SL}_n(D) \to \mathrm{Spec}\, K) \to \mathrm{Aut}_G(K)$ for the subgroup $\mathrm{Gal}(\mathbf{F}_{p^i}(\!(T)\!)/\mathbf{F}_p(\!(T)\!))$ when $\gcd(nd, i)$ divides n (d being as usual the degree of D over K). Unfortunately, we were not able to include this case in Theorem 3.57 because the formula we found does not glue well with the map $f_{J(K)}$ constructed in Proposition 3.47. Yet, this explicit formula could be of interest, so that we include it here.

Proposition 3.65. *Let $K = \mathbf{F}_{p^i}(\!(T)\!)$, let D be a division algebra of degree d over K and let $G = \mathrm{SL}_n(D)$. There exists a homomorphism $f_{\mathrm{Gal}} \colon \mathrm{Gal}(\mathbf{F}_{p^i}(\!(T)\!)/\mathbf{F}_p(\!(T)\!)) \to \mathrm{Aut}(G \to \mathrm{Spec}\, K)$ whose composition with $\mathrm{Aut}(G \to \mathrm{Spec}\, K) \to \mathrm{Aut}_G(K)$ is the identity on the subgroup $\mathrm{Gal}(\mathbf{F}_{p^i}(\!(T)\!)/\mathbf{F}_p(\!(T)\!))$ if and only if $\gcd(nd, i)$ divides n.*

Proof. First assume that $\gcd(nd, i)$ divides n. We begin by setting some notations. Let $r \in \mathbf{N}$ be such that $\mathrm{inv}([D]) = [\frac{r}{d}]$. By Corollary 3.60, r is prime to d and D is the cyclic algebra $(E/K, \sigma, \pi^r)$.

Recall that $\mathrm{Gal}(\mathbf{F}_{p^i}(\!(T)\!)/\mathbf{F}_p(\!(T)\!))$ is just (isomorphic to) $\mathrm{Gal}(\mathbf{F}_{p^i}/\mathbf{F}_p)$, which is isomorphic to the cyclic group C_i of order i. Let $k, l \in \mathbf{N}$ be such that $kl = i$, with $\gcd(d^i, i) = \gcd(d^k, k) = k$ and $\gcd(d, l) = 1$. Note that since $\gcd(nd, i)$ divides n, it follows that $\gcd(d^i, i) = k$ divides n as well. Furthermore, $\gcd(l, k) = 1$, so that

$$\mathrm{Gal}(\mathbf{F}_{p^i}(\!(T)\!)/\mathbf{F}_p(\!(T)\!)) \cong C_i \cong C_k \times C_l$$

$$\cong \mathrm{Gal}(\mathbf{F}_{p^i}(\!(T)\!)/\mathbf{F}_{p^l}(\!(T)\!)) \times \mathrm{Gal}(\mathbf{F}_{p^i}(\!(T)\!)/\mathbf{F}_{p^k}(\!(T)\!))$$

For the rest of the proof, γ denotes the Frobenius automorphism on $\mathbf{F}_{p^i}(\!(T)\!)$, while $\tilde{\gamma}$ denotes the Frobenius automorphism on $\mathbf{F}_{p^{di}}(\!(T)\!)$. In particular, γ^k generates $C_l \cong \mathrm{Gal}(\mathbf{F}_{p^i}(\!(T)\!)/\mathbf{F}_{p^k}(\!(T)\!))$, while γ^l generates $C_k \cong \mathrm{Gal}(\mathbf{F}_{p^i}(\!(T)\!)/\mathbf{F}_{p^l}(\!(T)\!))$. Furthermore, $\tilde{\gamma}^s$ extends γ^t if and only if $s - t \in i\mathbf{Z}$.

We are going to define f_{Gal} on C_l and C_k separately, beginning with C_l. Let $m \in \mathbf{Z}$ be such that $ml+1 \in d\mathbf{Z}$ (which exists since $\gcd(d,l) = 1$). Recall that γ^k generates C_l. For $j \in \mathbf{Z}$, and using the notation introduced in Definition 3.36, we define the map

$$f_{C_l}\colon C_l \to \mathrm{Aut}(\mathrm{SL}_n(D) \to \mathrm{Spec}\, K)\colon \gamma^{jk} \mapsto \tilde{\varphi}(\tilde{\gamma}^{j(mi+k)}, 1).$$

It follows from the definition of m that $f_{C_l}(\gamma^{lk})$ is the identity, so that f_{C_l} is indeed a homomorphism.

We now define a homomorphism $f_{C_k}\colon C_k \to \mathrm{Aut}(\mathrm{SL}_n(D) \to \mathrm{Spec}\, K)$. Recall that γ^l generates C_k. Since k divides n, let $a \in \mathbf{N}$ be such that $ak = n$. Let also $b \in \mathbf{N}$ be such that $bk = d^i$, which exist since $\gcd(d^i, i) = k$. Recall that K_d is the field $\mathbf{F}_{p^{di}}(\!(T)\!)$. We define an automorphism $\tilde{\alpha}(\gamma^l)$ of K_d^n as follows:

$$\tilde{\alpha}(\gamma^l)\colon K_d^n \to K_d^n\colon (e_{sa+t})_{\substack{s\in\{0,\dots,k-1\}\\ t\in\{1,\dots,a\}}} \mapsto (x_{sa+t})_{\substack{s\in\{0,\dots,k-1\}\\ t\in\{1,\dots,a\}}}$$

$$x_{sa+t} = \begin{cases} \tilde{\gamma}^{bi+l}(e_{(s-1)a+t}) \text{ if } s \geq 1 \\ \tilde{\gamma}^{(b-1)i+l}(e_{(k-1)a+t}) \text{ if } s = 0 \end{cases}$$

We claim that $\tilde{\alpha}(\gamma^l)^k$ is the identity on K_d^n. Indeed, after k iterations of $\tilde{\alpha}(\gamma^l)$, a coefficient is shifted by $ka = n$ places modulo n, hence is fixed. Furthermore, each coefficient is hit by $\tilde{\gamma}^{(bi+l)(k-1)+(b-1)i+l} = \tilde{\gamma}^{bik+lk-i} = (\tilde{\gamma}^{id})^{d^{i-1}}$, which is the identity on K_d, so that the claim holds.

We denote an element of D as $\sum_{i=0}^{d-1} u^i e_i$ with $e_i \in K_d$. Let w be the following element of $M_n(D)$:

$$w = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & u \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix}$$

Let us also identify K_d^n with the algebra of $n \times n$ diagonal matrices with coefficient in K_d via the embedding

$$K_d^n \to M_n(K_d) \colon (e_1, e_2, e_3, \dots, e_n) \mapsto \begin{pmatrix} e_1 & 0 & 0 & \dots & 0 \\ 0 & e_2 & 0 & \dots & 0 \\ 0 & 0 & e_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & e_n \end{pmatrix}$$

With those notations, an element x in $M_n(D)$ is written uniquely in the form $\sum_{i=0}^{nd-1} w^i x_i$, for $x_i \in K_d^n$. We define the following map on $M_n(D)$:

$$f_{C_k}(\gamma^l) \colon M_n(D) \to M_n(D) \colon \sum_{i=0}^{nd-1} w^i x_i \mapsto \sum_{i=0}^{nd-1} w^i \tilde{\alpha}(\gamma^l)(x_i).$$

Claim 2. *The map $f_{C_k}(\gamma^l)$ is a ring automorphism which preserves elements of reduced norm* 1.

Proof of the claim: The proof is a straightforward adaptation of the proof of Lemma 3.34. Let us do it for the ease of the reader.

Let $K_d^n[w;\phi]$ be a twisted polynomial ring (see [Jac96, Section 1.1] for the definition of a twisted polynomial ring), where $\phi \colon K_d^n \to K_d^n$ is the ring automorphism sending $(e_1, \dots, e_n)$ to $(\sigma^{-1}(e_n), e_1, \dots, e_{n-1})$. The algebra $M_n(D)$ is isomorphic to $K_d^n[w;\phi]$ modulo the relations $w^{nd} = \pi^r$. Given an automorphism α in $\mathrm{Aut}(K_d)$, we can define a map $f_\alpha \colon K_d^n[w;\phi] \to K_d^n[w;\phi] \colon \begin{cases} w \mapsto w \\ x \mapsto \alpha(x) \text{ for all } x \in K_d^n \end{cases}$. By [Jac96, Proposition 4.6.20], f_α is a ring automorphism as soon as $\alpha\phi = \phi\alpha$. For $\alpha = \tilde{\alpha}(\gamma^l)$, this readily follows from the definition of $\tilde{\alpha}(\gamma^l)$, recalling that σ is the Frobenius element of $\mathrm{Gal}(K_d/K)$, and hence is equal to $\tilde{\gamma}^i$. Hence, $f_{\tilde{\alpha}(\gamma^l)}$ is indeed a ring automorphism of $K_d^n[w;\phi]$.

Furthermore, if it passes to the quotient, $f_{\tilde{\alpha}(\gamma^l)}$ induces the automorphism f_{C_k}. Hence it suffices to check that $f_{\tilde{\alpha}(\gamma^l)}$ preserves the relation. But we have $f_{\tilde{\alpha}(\gamma^l)}(w^{nd} - \pi^r) = w^{nd} - \pi^r$, so that the claim is proved.

It remains to check that $f_{C_k}(\gamma^l)$ preserves elements of reduced norm 1. In fact, we prove that for $g \in M_n(D)$, $\mathrm{Nrd}(f_{C_k}(\gamma^l)(g)) = \gamma^l(\mathrm{Nrd}(g))$. Let $f\colon M_n(D) \to M_{nd^2}(K)$ be a representation of $M_n(D)$. Furthermore, let $\tilde{\gamma}^l\colon M_{nd^2}(K) \to M_{nd^2}(K)$ be the automorphism defined by applying γ^l coefficient by coefficient. Let us look at the non-commutative diagram

$$\begin{array}{ccccc}
M_n(D) & \xrightarrow{f} & M_{nd^2}(K) & \xrightarrow{\det} & K \\
{\scriptstyle f_{C_k}(\gamma^l)}\downarrow & & \tilde{\gamma}^l\downarrow & & \downarrow\gamma^l \\
M_n(D) & \xrightarrow{f} & M_{nd^2}(K) & \xrightarrow{\det} & K
\end{array}$$

In the above diagram, the square on the left does not commute, but the square on the right does commute. By [Pie82, Chapter 16, §1, Corollary a], for $g \in M_n(D)$, $\mathrm{Nrd}(g) = \sqrt[d]{(\det \circ f)(g)}$. Also note that since $\tilde{\gamma}^l \circ f \circ f_{C_k}(\gamma^l)^{-1}$ is another representation of D, we have $\det \circ \tilde{\gamma}^l \circ f \circ f_{C_k}(\gamma^l)^{-1} = \det \circ f$, again by [Pie82, Chapter 16, §1, Corollary a]. Hence $\mathrm{Nrd}(f_{C_k}(\gamma^l)(g)) = (\det \circ f \circ f_{C_k}(\gamma^l))(g) = (\det \circ \tilde{\gamma}^l \circ f)(g) = (\gamma^l \circ \det \circ f)(g) = \gamma^l(\mathrm{Nrd}(g))$, as wanted. ■

We also denote by $f_{C_k}(\gamma^l))$ the restriction of $f_{C_k}(\gamma^l))$ to $\mathrm{SL}_n(D)$. The homomorphism $f_{\mathrm{Gal}}\colon \mathrm{Gal}(\mathbf{F}_{p^i}(\!(T)\!)/\mathbf{F}_{p^l}(\!(T)\!)) \to \mathrm{Aut}(G \to \mathrm{Spec}\, K)$ is then obtained by glueing f_{C_l} and f_{C_k}. Namely,

$$f_{\mathrm{Gal}}\colon C_l \times C_k \to \mathrm{Aut}(G \to \mathrm{Spec}\, K)\colon (a_1, a_2) \mapsto f_{C_l}(a_1) \circ f_{C_k}(a_2)$$

Since the images of f_{C_l} and f_{C_k} commute, this is indeed a homomorphism. Finally, the fact that the composition of f_{Gal} with $\mathrm{Aut}(G \to \mathrm{Spec}\, K) \to \mathrm{Aut}(K)$ is the identity follows directly from the construction.

For the converse, assume that $\gcd(nd, i)$ does not divide n. By Theorem 3.40, it suffices to prove that there exists $K' \leq K$ such that K/K' is finite Galois and $\gcd(nd, [K : K'])$ does not divide n. But $K' = \mathbf{F}_p(\!(T)\!)$ is such a subfield. $\square$

Bibliography

[ABD+64] Michael Artin, Jean-Etienne Bertin, Michel Demazure, Peter Gabriel, Alexander Grothendieck, Michel Raynaud, and Jean-Pierre Serre. *Schémas en groupes. Fasc. 7: Exposés 23 à 26.* Séminaire de Géométrie Algébrique de l'Institut des Hautes Études Scientifiques. Institut des Hautes Études Scientifiques, Paris, 1963/64.

[BCn] BCnrd. Structure on $X(k)$ for separated finite type alg. space X, for complete valued k. URL:https://mathoverflow.net/q/34631 (version: 2010-08-09).

[BEW11] Yiftach Barnea, Mikhail Ershov, and Thomas Weigel. Abstract commensurators of profinite groups. *Trans. Amer. Math. Soc.*, 363(10):5381–5417, 2011.

[BKLR12] Mikhail Borovoi, Boris Kunyavskii, Nicole Lemire, and Zinovy Reichstein. Stably Cayley groups in characteristic zero. 2012. Preprint: arXiv:1207.1329.

[BLR90] Siegfried Bosch, Werner Lütkebohmert, and Michel Raynaud. *Néron models*, volume 21 of *Ergeb. Math. Grenzgeb. (3)*. Springer-Verlag, Berlin, 1990.

[BM96] Marc Burger and Shahar Mozes. CAT(-1)-spaces, divergence groups and their commensurators. *J. Amer. Math. Soc.*, 9(1):57–93, 1996.

[Bor01] Armand Borel. *Essays in the history of Lie groups and algebraic groups*, volume 21 of *Hist. Math.* American Mathematical Society, Providence, RI; London Mathematical Society, Cambridge, 2001.

[Bou72] N. Bourbaki. *Éléments de mathématique. Fasc. XXXVII. Groupes et algèbres de Lie. Chapitre II: Algèbres de Lie libres. Chapitre III: Groupes de Lie.* Hermann, Paris, 1972. Actualités Scientifiques et Industrielles, No. 1349.

[Bou73] Nicolas Bourbaki. *Éléments de mathématique, Fasc. XXIII, Algèbre, Chapitre 8: Modules et anneaux semi-simples.* Hermann, Paris, 1973. Nouveau tirage de l'édition de 1958; Actualités Scientifiques et Industrielles, No. 1261.

[BT72] François Bruhat and Jacques Tits. Groupes réductifs sur un corps local. *Inst. Hautes Études Sci. Publ. Math.*, 41:5–251, 1972.

[BT73] Armand Borel and Jacques Tits. Homomorphismes "abstraits" de groupes algébriques simples. *Ann. of Math.*, 97(3):499–571, 1973.

[BT84a] François Bruhat and Jacques Tits. Groupes réductifs sur un corps local. II. Schémas en groupes. Existence d'une donnée radicielle valuée. *Inst. Hautes Études Sci. Publ. Math.*, (60), 1984.

[BT84b] François Bruhat and Jacques Tits. Schémas en groupes et immeubles des groupes classiques sur un corps local. *Bull. Soc. Math. France*, 112(2):259–301, 1984.

[BT87] François Bruhat and Jacques Tits. Schémas en groupes et immeubles des groupes classiques sur un corps local. II. Groupes unitaires. *Bull. Soc. Math. France*, 115(2):141–195, 1987.

[Buz] Kevin Buzzard. How did birch and swinnerton dyer arrive at their conjecture? URL: https://mathoverflow.net/q/66636 (version 2011-06-01).

[Cam97] Rachel Camina. Subgroups of the Nottingham group. *J. Algebra*, 196(1):101–113, 1997.

[Cam00] Rachel Camina. The Nottingham group. In *New horizons in pro-p groups*, volume 184 of *Progr. Math.*, pages 205–221. Birkhäuser Boston, Boston, MA, 2000.

[Cap16] Pierre-Emmanuel Caprace. Non-discrete simple locally compact groups. 2016. To appear in the Proceedings of the 7th European Congress of Mathematics. Preprint: http://perso.uclouvain.be/pierre-emmanuel.caprace/papers_pdf/Simple7ECM.pdf.

[Che05] Claude Chevalley. *Classification des groupes algébriques semi-simples*. Springer-Verlag, Berlin, 2005. Collected works. Vol. 3, Edited and with a preface by P. Cartier, With the collaboration of P. Cartier, A. Grothendieck and M. Lazard.

[CM11] Pierre-Emmanuel Caprace and Nicolas Monod. Decomposing locally compact groups into simple pieces. *Math. Proc. Cambridge Philos. Soc.*, 150(1):97–128, 2011.

[Con14] Brian Conrad. Reductive group schemes. In *Autour des schémas en groupes. Vol. I*, volume 42/43 of *Panor. Synthèses*, pages 93–444. Soc. Math. France, Paris, 2014.

[CR09] Pierre-Emmanuel Caprace and Bertrand Rémy. Simplicity and superrigidity of twin building lattices. *Invent. Math.*, 176(1):169–221, 2009.

[CR13] Pierre-Emmanuel Caprace and Bertrand Rémy. Simplicity of twin tree lattices with non-trivial commutation relations. 2013. Preprint: arXiv:1209.5372.

[CR14] Inna Capdeboscq and Bertrand Rémy. On some pro-p groups from infinite-dimensional Lie theory. *Math. Z.*, 278(1-2):39–54, 2014.

[CR16] Pierre-Emmanuel Caprace and Nicolas Radu. Chabauty limits of simple groups acting on trees. 2016. Preprint: arXiv:1608.00461.

[CRW13] Pierre-Emmanuel Caprace, Colin D. Reid, and George A. Willis. Locally normal subgroups of totally disconnected groups. Part I: General theory. 2013. Preprint: arXiv:1304.5144.

[CRW14] Pierre-Emmanuel Caprace, Colin D. Reid, and George A. Willis. Locally normal subgroups of totally disconnected groups. Part II: Compactly generated simple groups. 2014. Preprint: arXiv:1401.3142.

[CS15] Pierre-Emmanuel Caprace and Thierry Stulemeijer. Totally disconnected locally compact groups with a linear open subgroup. *Int. Math. Res. Not.*, 2015(24):13800–13829, 2015.

[DdSMS91] John D. Dixon, Marcus P. F. du Sautoy, Avinoam Mann, and Dan Segal. *Analytic pro-p-groups*, volume 157 of *London Math. Soc. Lecture Note Ser.* Cambridge University Press, Cambridge, 1991.

[Del84] Pierre Deligne. Les corps locaux de caractéristique p, limites de corps locaux de caractéristique 0. pages 119–157, 1984.

[DG70] Michel Demazure and Pierre Gabriel. *Groupes algébriques. Tome I: Géométrie algébrique, généralités, groupes commutatifs.* Masson & Cie, Éditeur, Paris; North-Holland Publishing Co., Amsterdam, 1970. Avec un appendice *Corps de classes local* par Michiel Hazewinkel.

[dlH08] P. de la Harpe. Spaces of closed subgroups of locally compact groups. 2008. URL: `http://adsabs.harvard.edu/abs/2008arXiv0807.2030D` (retrieved 2017-08-16).

[dT15] Mikael de la Salle and Romain Tessera. Characterizing a vertex-transitive graph by a large ball. 2015. Preprint: arXiv:1508.02247.

[Ele] Georges Elencwajg. Surjective map of rings with same dimension. URL: `http://math.stackexchange.com/q/604091` (version 2013-12-12).

[EM48] Samuel Eilenberg and Saunders MacLane. Cohomology and Galois theory. I. Normality of algebras and Teichmüller's cocycle. *Trans. Amer. Math. Soc.*, 64(1):1–20, 1948.

[FSS98] Yuval Z. Flicker, Claus Scheiderer, and R. Sujatha. Grothendieck's theorem on non-abelian H^2 and local-global principles. *J. Amer. Math. Soc.*, 11(3):731–750, 1998.

[FV02] Ivan B. Fesenko and Sergei V. Vostokov. *Local fields and their extensions*, volume 121 of *Transl. Math. Monogr.* American Mathematical Society, Providence, RI, 2 edition, 2002. With a foreword by Igor R. Shafarevich.

[GP11] Philippe Gille and Patrick Polo. Réédition de SGA3, 2011. URL: `https://webusers.imj-prg.fr/~patrick.polo/SGA3/` (retrieved 2017-06-21).

[Gro67] Alexandre Grothendieck. éléments de géométrie algébrique. IV. étude locale des schémas et des morphismes de schémas. *Inst. Hautes Études Sci. Publ. Math.*, 32:361, 1967.

[Han07] Timo Hanke. Computation of outer automorphisms of central-simple algebras, 2007. URL: `http://www.math.rwth-aachen.de/~Timo.Hanke/thanke_outer-automorphisms.pdf` (retrieved 2017-03-30).

[Heg01] Pál Hegedűs. The Nottingham gap for $p = 2$. *J. Algebra*, 246(1):55–69, 2001.

[HR79] Edwin Hewitt and Kenneth A. Ross. *Abstract harmonic analysis. Vol. I*, volume 115 of *Grundlehren Math. Wiss.* Springer-Verlag, Berlin-New York, 2 edition, 1979.

[Iwa86] Kenkichi Iwasawa. *Local class field theory.* Oxford Math. Monogr. Clarendon Press, Oxford University Press, New York, 1986.

[Jac96] Nathan Jacobson. *Finite-dimensional division algebras over fields.* Springer-Verlag, Berlin, 1996.

[Kaz86] David Kazhdan. Representations of groups over close local fields. *J. Analyse Math.*, 47:175–179, 1986.

[KK44] Shizuo Kakutani and Kunihiko Kodaira. Über das Haarsche Mass in der lokal bikompakten Gruppe. *Proc. Imp. Acad. Tokyo*, 20(7):444–450, 1944.

[KMRT98] Max-Albert Knus, Alexander Merkurjev, Markus Rost, and Jean-Pierre Tignol. *The book of involutions*, volume 44 of *Amer. Math. Soc. Colloq. Publ.* American Mathematical Society, Providence, RI, 1998. With a preface in French by J. Tits.

[Lan94] Serge Lang. *Algebraic number theory*, volume 110 of *Grad. Texts in Math.* Springer-Verlag, New York, 2 edition, 1994.

[Lee13] John M. Lee. *Introduction to smooth manifolds*, volume 218 of *Graduate Texts in Mathematics.* Springer, New York, second edition, 2013.

[LS03] Alexander Lubotzky and Dan Segal. *Subgroup growth*, volume 212 of *Progr. Math.* Birkhäuser Verlag, Basel, 2003.

[Mar91] G. A. Margulis. *Discrete subgroups of semisimple Lie groups*, volume 17 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3).* Springer-Verlag, Berlin, 1991.

[Mar14] Timothée Marquis. Abstract simplicity of locally compact Kac-Moody groups. *Compos. Math.*, 150(4):713–728, 2014.

[Mor97] Patrick J. Morandi. Division algebras over local fields, 1997. URL: `http://sierra.nmsu.edu/morandi/notes/localfields.pdf` (retrieved 2017-03-02).

[MP94] Allen Moy and Gopal Prasad. Unrefined minimal k-types for p-adic groups. *Invent. Math.*, 116(1-3):393–408, 1994.

[MPW15] Bernhard Mühlherr, Holger P. Petersson, and Richard M. Weiss. *Descent in buildings*, volume 190 of *Ann. of Math. Stud.* Princeton University Press, Princeton, NJ, 2015.

[MS20] Stefan Mazurkiewicz and Waclaw Sierpinski. Contribution à la topologie des ensembles dénombrables. *Fund. Math.*, 1(1):17–27, 1920.

[Pet89] Vassilij M. Petechuk. Homomorphisms of linear groups over commutative rings. *Mat. Zametki*, 46(5):50–61, 103, 1989.

[Pie82] Richard S. Pierce. *Associative algebras*, volume 88 of *Grad. Texts in Math.* Springer-Verlag, New York-Berlin, 1982.

[Pin98] Richard Pink. Compact subgroups of linear algebraic groups. *J. Algebra*, 206(2):438–504, 1998.

[Poo10] Bjorn Poonen. Rational points on varieties, 2010. URL: `http://www-math.mit.edu/~poonen/papers/Qpoints.pdf` (retrieved 2017-03-22).

[PR94] Vladimir Platonov and Andrei Rapinchuk. *Algebraic groups and number theory*, volume 139 of *Pure Appl. Math.* Academic Press, Inc., Boston, MA, 1994. Translated from the 1991 Russian original by Rachel Rowen.

[Pra82] Gopal Prasad. Elementary proof of a theorem of Bruhat-Tits-Rousseau and of a theorem of Tits. *Bull. Soc. Math. France*, 110(2):197–202, 1982.

[Rad15] Nicolas Radu. A classification theorem for boundary 2-transitive automorphism groups of trees. 2015. Preprint: arXiv:1509.04913.

[Rie70a] Carl Riehm. The congruence subgroup problem over local fields. *Amer. J. Math.*, 92(3):771–778, 1970.

[Rie70b] Carl Riehm. The norm 1 group of a $\mathfrak{P}$-adic division algebra. *Amer. J. Math.*, 92(2):499–523, 1970.

[Sch33] Friedrich Karl Schmidt. Mehrfach perfekte körper. *Math. Ann.*, 108(1):1–25, 1933.

[Ser79] Jean-Pierre Serre. *Local fields*, volume 67 of *Grad. Texts in Math.* Springer-Verlag, New York-Berlin, 1979. Translated from the French by Marvin Jay Greenberg.

[Ser92] Jean-Pierre Serre. *Lie algebras and Lie groups*, volume 1500 of *Lecture Notes in Mathematics.* Springer-Verlag, Berlin, second edition, 1992. 1964 lectures given at Harvard University.

[Sta17] The Stacks Project Authors. Stacks Project. URL: `http://stacks.math.columbia.edu`, 2017.

[Stua] Thierry Stulemeijer. Form over $\mathbb{Z}$ of non-split simple algebraic groups over non-archimedean local fields. URL: `http://mathoverflow.net/q/217779` (retrieved 2017-03-29).

[Stub] Thierry Stulemeijer. Reference for hensel's lemma in algebraic geometry. URL: `http://mathoverflow.net/q/234709` (version: 2016-03-28).

[Stu16] Thierry Stulemeijer. Chabauty limits of algebraic groups acting on trees. 2016. Preprint: arXiv:1610.08454.

[Tao14] Terence Tao. *Hilbert's fifth problem and related topics*, volume 153 of *Grad. Stud. Math.* American Mathematical Society, Providence, RI, 2014.

[Tit64] J. Tits. Algebraic and abstract simple groups. *Ann. of Math.*, 80(2):313–329, 1964.

[Tit66] Jacques Tits. Classification of algebraic semisimple groups. In *Algebraic Groups and Discontinuous Subgroups (Proc. Sympos. Pure Math., Boulder, Colo. 1965)*, pages 33–62. Amer. Math. Soc., Providence, R.I., 1966.

[Tit74] Jacques Tits. *Buildings of spherical type and finite BN-pairs*, volume 386 of *Lecture Notes in Math.* Springer-Verlag, Berlin-New York, 1974.

[Tit79] Jacques Tits. Reductive groups over local fields. In *Automorphic forms, representations and L-functions (Oregon State Univ., Corvallis, Ore. 1977)*, Proc. Sympos. Pure Math., XXXIII, pages 29–69. Amer. Math. Soc., Providence, R.I., 1979.

[Tit86] Jacques Tits. Immeubles de type affine. In *Buildings and the geometry of diagrams (Como 1984)*, volume 1181 of *Lecture Notes in Math.*, pages 159–190. Springer, Berlin, 1986.

[Vak17] Ravi Vakil. The rising sea, foundations of algebraic geometry, 2017. URL: `http://math.stanford.edu/~vakil/216blog/FOAGfeb0717public.pdf` (retrieved 2017-05-21).

[Wei09] Richard M. Weiss. *The structure of affine buildings*, volume 168 of *Ann. of Math. Stud.* Princeton University Press, Princeton, NJ, 2009.

[Wes14] Phillip Wesolek. A note on relative amenability. 2014. Preprint: arXiv:1406.2974.

[Wes15a] Phillip Wesolek. Elementary totally disconnected locally compact groups. *Proc. Lond. Math. Soc.*, 110(6):1387–1434, 2015.

[Wes15b] Phillip Wesolek. Totally disconnected locally compact groups locally of finite rank. *Math. Proc. Cambridge Philos. Soc.*, 158(3):505–530, 2015.

[Wil07] George A. Willis. Compact open subgroups in simple totally disconnected groups. *J. Algebra*, 312(1):405–417, 2007.