Analysis of Distribution Locational Marginal Prices

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***Abstract*—Low-voltage distribution networks are emerging as an increasingly important component of power system operations due to the deployment of distributed renewable resources (e.g., rooftop solar supply) and the need to mobilize the flexibility of consumers that are connected to the low-voltage grid. The pricing of electric power at distribution nodes follows directly from the theory of spot pricing of electricity. However, in contrast to linearized lossless models of transmission networks, an intuitive understanding of prices at the distribution level presents challenges due to voltage limits, reactive power flows, and losses. In this paper, we present three approaches toward understanding distribution locational marginal prices by decomposing them: 1) through a duality analysis of the problem formulated with a global power balance constraint; 2) through a duality analysis of a second-order cone program relaxation; and 3) through an analysis of the impact of marginal losses on price. We discuss the relative strengths and weaknesses of each approach in terms of computation and physical intuition, and demonstrate the concepts on a 15-bus radial distribution network.**

***Index Terms*—Power system economics, power distribution, systems operation.**

I. INTRODUCTION

THE THEORY of spot pricing of electricity [1], [2] has laid the theoretical foundations for the deregulation of electricity markets. Decades of experience with deregulation have led to bid-based, security constrained unit commitment and economic dispatch as the reference paradigm for wholesale electricity market design, closely following the guidelines of the original theory. Despite the progress of wholesale electricity markets, the low-voltage network is largely excluded from short-term (day-ahead and real-time) markets. The current role of distribution systems is passive, in the sense that distributed loads consume power at will and distributed renewable resources inject power as it becomes available, without any coordination with the rest of the system. It is then the responsibility of distribution system operators to ensure that distribution network constraints are respected, and the responsibility of transmission system operators to carry sufficient reserve in order to ensure that aggregate imbalances between real-time power supply and demand are dealt with. Retail and commercial consumers are largely absent from this process, since they are typically exposed to either fixed or time-of-use rates that do not reflect the real-time conditions of the system [3]. The paradigm shift envisioned in recent work focused on the mobilization of distributed resources [4], including the EU SmartNet project, entails a move towards active coordination of low-voltage resources with high-voltage resources through an *integrated scheduling* of the entire system. The basic schemes for TSO-DSO coordination and ancillary service provision envisioned in the SmartNet project are developed in detail by Gerard *et al.* [5].

Correct price signals at the distribution level are essential for providing correct incentives for improving fuel cost efficiency, limiting real power losses over distribution lines, promoting the utilization of renewable resources, preventing the overloading of circuits [6], [7], and enabling the provision of ancillary services by distributed resources [4]. Residential and commercial consumption represents the majority of consumer flexibility [8]. Since these resources are connected to low-voltage distribution grids, pricing energy and services at the distribution level is becoming an increasingly important aspect of electricity market design. The relevance of distributed resources is expected to increase due to the advent of electric vehicles which will require coordinated charging [9], and due to the increase of rooftop solar installations, which may necessitate local consumption in order to respect distribution network constraints [10].

The integration of distributed resources in market clearing raises numerous market design questions that pertain to the interaction of TSOs and DSOs. In contrast to wholesale transmission markets, where linearized lossless models of power flow are deemed acceptable for practical purposes, reactive power flows and voltage need to be accounted for explicitly in distribution networks. The first fundamental question which emerges is what products should be traded, and in which markets. Caramanis *et al.* [4] propose (i) a transmission system market that trades real power and reserves, and (ii) low-voltage markets that clear real power, reactive power and reserves, such that the reserves can be delivered by distributed resources while respecting distribution line limits and voltage limits.

A related question is how distribution locational marginal prices should be computed. In this respect, the common approach adopted in [4], [7], and [11]–[14] is the following: (i) locational marginal prices of high-voltage buses (TLMPs) are determined by clearing transmission markets, and (ii) DSOs clear local markets in the low-voltage grid in order to determine distribution locational marginal prices (DLMPs) for real power at the level of individual distribution nodes, *given* the TLMP of the corresponding transmission buses.

Day-ahead distribution markets are studied by Li *et al.* [12], who propose the following arrangement for day-ahead market

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clearing: (i) aggregators submit demand functions for non-deferrable loads and multi-part bids for deferrable loads that represent the energy requirements and technical constraints of deferrable loads, (ii) the DSO solves a welfare maximizing problem where the marginal cost of power for serving consumers is a forecast of the TLMP of the node at which the distribution feeder is located, and (iii) the DSO uses the DLMPs at the corresponding distribution nodes as market clearing prices for aggregators over a daily horizon. Huang *et al.* [14] extend the model of Li *et al.* [12] in order to address the fact that the price signal announced by the DSO may support multiple optimal dispatch schedules (this is especially the case for deferrable loads such as EVs and heat pumps), which may result in the violation of distribution network constraints. The authors propose that the DSO create a linear approximation of the system supply function around the optimal dispatch level, which results in a unique market clearing solution that respects distribution network constraints. A similar approach for ensuring uniqueness of the DLMPs is adopted in the model of Verzijlbergh *et al.* [13]. Forward dynamic pricing is also considered by Zugno *et al.* [15], who focus on a monopolist retailer who procures electricity at a stochastic price from the spot market, and faces uncertainty in the reaction of consumers to its chosen retail price, as well as an imbalance penalty. Real-time markets for distributed resources are analyzed by Ding *et al.* [11], who present the organization of the real-time market in the context of the EcoGrid EU pilot project.

The motivation for this paper is that being able to interpret the DLMP will be crucial in the policy debate that will surround the evolution of distribution markets, in the same way that understanding the behavior of locational marginal prices was crucial in the policy debate that surrounded wholesale markets [16]. This question has received limited attention in the literature. Compared to transmission systems, distribution networks exhibit a simpler radial topology, which implies that loop flows (which have been a cause for heated debates surrounding TLMPs) are not relevant. In this respect, the analysis of DLMPs is simpler. However, insofar as the nonlinearities of power flow need to be accounted for, the analysis is more complicated than that of TLMPs. In particular, we wish to understand how congestion, voltage constraints, and real power losses affect the formation of DLMPs. For this purpose, we analyze and compare three alternative approaches for decomposing DLMPs: (i) duality analysis of a second order conic programming (SOCP) relaxation of optimal power flow; (ii) duality analysis of a formulation of optimal power flow with a global power balance constraint; (iii) analysis of the contribution of marginal losses on DLMPs.

(i) Duality analysis of SOCP. The analysis of KKT conditions often provides important insights about the behavior of prices. For example, the KKT analysis of the DC optimal power flow based on power transfer distribution factors reveals how TLMPs can be expressed as the weighted sum of a reference price plus the weighted sum of the shadow price of congested lines [1]. This result is intuitive, and allows for a better understanding of TLMP formation. DLMPs have been commonly analyzed in the literature using the KKT conditions of linearized formulations of the optimal power flow problem. Sotkiewicz and Vignolo [7] derive DLMPs for active and reactive power. Line congestion and voltage are ignored in the analysis of the authors. Similarly, Li *et al.* [12] and Huang *et al.* [14] assume a linearized model of power flow which ignores voltage, reactive power and real power losses. Recently a large number of convex relaxations of optimal power flow have emerged, including SDP relaxations [17], and conic relaxations based on the bus injection formulation [18] and branch flow formulation [19]. Although the tightness of such formulations has been studied extensively [20], there has been limited focus on duality analysis, which often provides insights about pricing. In this paper we focus on the duality analysis of the branch flow formulation [19].

(ii) Duality analysis of formulation with global power balance constraint. A particularly interesting decomposition of DLMPs is obtained by considering power flows as implicit functions of nodal net injections. This is the approach adopted by the originators of the spot pricing of electricity [1], [2], and it has been applied recently in the context of radial distribution systems [4]. Although the DLMP nicely decomposes into components that are directly associated to problem constraints (distribution line capacity, voltage limits, ...), we demonstrate that its computation requires numerical estimation.

(iii) Analysis of marginal losses. Finally, we discuss a third alternative approach to decomposing DLMPs, which focuses exclusively on the role of real power losses. We argue that this latter approach is well grounded in terms of physical interpretation, and may therefore provide a more intuitive understanding regarding the behavior of DLMPs.

The paper is structured as follows. In Section II we present two alternative formulations of the AC optimal power flow problem on radial networks. Each of these formulations is used for obtaining three different decompositions of DLMPs in Section III. These alternative decompositions are analyzed and compared on a 15-bus distribution system in Section IV. Conclusions and perspectives for future research are discussed in Section V. The notation used in the paper is summarized in the Appendix. This research has been conducted in the framework of the SmartNet EU project.

II. Optimal Power Flow Formulations

In this section we consider two alternative formulations of the optimal power flow problem. Both of these formulations are used in the subsequent analysis. We focus on a radial distribution network, which is represented as a graph (N, E), where N is the set of nodes and E is the set of edges. This graph is a tree, and we denote the root of the tree as 0 and $N^+ = N - \{0\}$ as the set of all nodes except the root.

We associate with each node i: (i) the following decision variables: real power generation p_i^g, real power consumption p_i^c, reactive power generation q_i^g, reactive power consumption q_i^c, voltage magnitude squared v_i; and (ii) the following parameters: marginal cost of generators C_i^g, marginal benefit of consumers C_i^c, shunt conductance G_i, shunt susceptance B_i, maximum and minimum real power capacity for generators P_i^{g+} and P_i^{g-}, maximum and minimum real power capacity

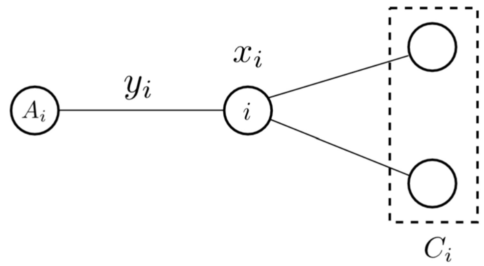

Fig. 1. The radial notation employed in the paper. The node i has a unique ancestor A_i and a set of children nodes C_i. Nodal variables and parameters are indicated in the figure by x_i. Branch variables and parameters are indicated in the figure by y_i, and are associated with the unique branch that is adjacent to i and A_i.

for consumers P_i^{c+} and P_i^{c-}, maximum and minimum reactive power capacity for generators Q_i^{g+} and Q_i^{g-}, maximum and minimum reactive power capacity for consumers Q_i^{c+} and Q_i^{c-}, and maximum and minimum voltage limits V_i^+ and V_i^-.

Since the network is radial, lines $i \in E$ can be indexed by the set N^+. We associate with each edge i an ancestor A_i and a set of children C_i, as shown in Fig. 1. In addition, we associate with each edge i: (i) the following decision variables: real power flow f_i^p from node i to A_i (with flow measured on the side of i), reactive power flow f_i^q from node i to A_i (with flow measured on the side of i), current magnitude squared l_i; and (ii) the following parameters: resistance R_i, reactance X_i, and complex power flow limit S_i.

A. Implicit Function Formulation

This formulation is based on determining a reference node in the network, the voltage magnitude of which is fixed. We set the root node as the reference node. Then the real and reactive power injection at all buses except the reference node, and the voltage magnitude at the root node, fully determine the power flow over the entire network. One can therefore define the following *functions*: $(v_i(\mathbf{y}^p, \mathbf{y}^q, v_0), i \in N^+)$, $(f_i^p(\mathbf{y}^p, \mathbf{y}^q, v_0), i \in E)$, $(f_i^q(\mathbf{y}^p, \mathbf{y}^q, v_0), i \in E)$, $(l_i(\mathbf{y}^p, \mathbf{y}^q, v_0), i \in E)$ where $\mathbf{y}^p = (p_1^g - p_1^c, p_2^g - p_2^c, \ldots, p_{|N|}^g - p_{|N|}^c)$ and $\mathbf{y}^q = (q_1^g - q_1^c, q_2^g - q_2^c, \ldots, q_{|N|}^g - q_{|N|}^c)$. That is to say, the vectors $\mathbf{y}^p$ and $\mathbf{y}^q$ specify the real and reactive net power injections for all nodes except the root. We refer to this formulation as an implicit function formulation, because voltage magnitude, current magnitude, and real and reactive power flows are not directly expressed as decision variables, but instead as nonlinear functions of voltage magnitude at the root, and net power injections at all buses except for the root. The general model can then be written as

$$(I): \max \sum_{i \in N^+} C_i^c p_i^c - \sum_{i \in N} C_i^g p_i^g \quad (1)$$

$$(\eta_i^+): f_i^p(\mathbf{y}^p, \mathbf{y}^q, v_0)^2 + f_i^q(\mathbf{y}^p, \mathbf{y}^q, v_0)^2 \le S_i^2, \; i \in E \quad (2)$$

$$(\eta_i^-): \left(f_i^p(\mathbf{y}^p, \mathbf{y}^q, v_0) - R_i l_i\right)^2 + \left(f_i^q(\mathbf{y}^p, \mathbf{y}^q, v_0) - X_i l_i\right)^2 \le S_i^2, \; i \in E \quad (3)$$

$$\left(\delta_i^{g+}\right): p_i^g \le P_i^{g+}, i \in N \quad (4)$$

$$\left(\delta_i^{g-}\right): P_i^{g-} \le p_i^g, i \in N \quad (5)$$

$$\left(\delta_i^{c+}\right): p_i^c \le P_i^{c+}, i \in N^+ \quad (6)$$

$$\left(\delta_i^{c-}\right): P_i^{c-} \le p_i^c, i \in N^+ \quad (7)$$

$$\left(\theta_i^{g+}\right): q_i^g \le Q_i^{g+}, i \in N \quad (8)$$

$$\left(\theta_i^{g-}\right): Q_i^{g-} \le q_i^g, i \in N \quad (9)$$

$$\left(\theta_i^{c+}\right): q_i^c \le Q_i^{c+}, i \in N^+ \quad (10)$$

$$\left(\theta_i^{c-}\right): Q_i^{c-} \le q_i^c, i \in N^+ \quad (11)$$

$$\left(\sigma_i^+\right): v_i(\mathbf{y}^p, \mathbf{y}^q, v_0) \le V_i^+, i \in N^+ \quad (12)$$

$$\left(\sigma_i^-\right): V_i^- \le v_i(\mathbf{y}^p, \mathbf{y}^q, v_0), i \in N^+ \quad (13)$$

$$(\phi 1): \sum_{i \in N} p_i^g = \sum_{i \in N^+} p_i^c + \sum_{i \in E} R_i l_i(\mathbf{y}^p, \mathbf{y}^q, v_0) \quad (14)$$

$$(\phi 2): \sum_{i \in N} q_i^g = \sum_{i \in N^+} q_i^c + \sum_{i \in E} X_i l_i(\mathbf{y}^p, \mathbf{y}^q, v_0) \quad (15)$$

where the dual multipliers are indicated with Greek letters in the left of the constraints.

The objective of (I) is to maximize social welfare. Constraints (2) and (3) correspond to complex flow constraints on both directions of each line.[1] Constraints (4) and (5) correspond to real power capacity limits of distributed generators, and constraints (6) and (7) are the upper and lower real power limits of consumers. Constraints (8)-(11) impose analogous limits for reactive power to generators and consumers. Voltage limits are represented in constraints (12) and (13). Real and reactive power balance are represented in constraints (14) and (15). We note that limits on line current magnitude can be included in the model of this and the subsequent section in the same way as constraints (2), (3), but are omitted here in order to simplify the exposition.

From a computational standpoint, it makes little sense to formulate the problem as (I) because it is highly non-linear and will lead to divergence of commercial non-linear optimization solvers, for networks containing as few as two nodes. Moreover, it is typically not possible to derive the functions $f_i^p(\cdot), f_i^q(\cdot), v_i(\cdot), l_i(\cdot)$ in closed form, even for networks of relatively small size. Nevertheless, these functions can be defined (even if one cannot derive them in closed form), and one obtains interesting insights about the formation of DLMPs from this formulation, as we show in the next section.

In order to derive distribution locational marginal prices for real power (denoted λ_n) and reactive power (denoted μ_n), we decompose the problem by agent and focus on the subproblem of the consumer at location n. This consists of maximizing consumer surplus $C_n^c p_n^c - \lambda_n p_n^c - \mu_n q_n^c$, subject to consumer constraints (6), (7), (10), (11). The optimality conditions of the consumer subproblem reveal the DLMP for real power as $\lambda_n = C_n^c - \delta_n^{c+} + \delta_n^{c-}$. Analogously, the DLMP for reactive power is expressed as $\mu_n = \theta_n^{c+} - \theta_n^{c-}$.

[1]The model presented in the paper is inspired by a forward-looking scenario where there are non-trivial control decisions to be made at the distribution level that involve local storage of distributed supply, provision of ancillary services by distributed resources for supporting imbalances in different parts of the network [4], or price-based demand response through distributed flexible demand. Distribution line capacity constraints become relevant in this context. Ignoring capacity constraints would simplify the DLMP decomposition presented in the following sections, since terms related to line capacity constraints drop out.

B. Second Order Conic Program Formulation

The following second order cone program corresponds to a convex relaxation of the problem, and is based on Peng and Low [19].

$$(II): \ \max \sum_{i\in N^+} C_i^c p_i^c - \sum_{i\in N} C_i^g p_i^g \tag{16}$$

$$\left(\eta_i^+\right): \ \left(f_i^p\right)^2 + \left(f_i^q\right)^2 \le S_i^2, i \in E \tag{17}$$

$$\left(\eta_i^-\right): \ \left(f_i^p - l_i R_i\right)^2 + \left(f_i^q - l_i X_i\right)^2 \le S_i^2, i \in E \tag{18}$$

$$(\beta_i): \ v_i - 2\left(R_i f_i^p + X_i f_i^q\right) + l_i\left(R_i^2 + X_i^2\right) = v_{A_i}, i \in E \tag{19}$$

$$(\gamma_i): \ \frac{\left(f_i^p\right)^2 + \left(f_i^q\right)^2}{l_i} \le v_i, i \in E \tag{20}$$

$$(\lambda_0): \ -\sum_{j\in C_0}\left(f_j^p - l_j R_j\right) - p_0^g + G_0 v_0 = 0 \tag{21}$$

$$(\lambda_i): \ f_i^p - \sum_{j\in C_i}\left(f_j^p - l_j R_j\right) - p_i^g + p_i^c + G_i v_i = 0, i \in N^+ \tag{22}$$

$$(\mu_0): \ -\sum_{j\in C_0}\left(f_j^q - l_j X_j\right) - q_0^g - B_0 v_0 = 0 \tag{23}$$

$$(\mu_i): \ f_i^q - \sum_{j\in C_i}\left(f_j^q - l_j X_j\right) - q_i^g + q_i^c - B_i v_i = 0, i \in N^+ \tag{24}$$

$$\left(\sigma_i^+\right): \ v_i \le V_i^+, i \in N \tag{25}$$

$$\left(\sigma_i^-\right): \ V_i^- \le v_i, i \in N \tag{26}$$

$$(4) - (11)$$

Constraints (17) and (18) are analogous to constraints (2), (3), with the difference that the flow variables f_i^p and f_i^q are expressed as decision variables of (*II*), rather than functions of decision variables, as is the case in (*I*). Similarly, constraints (25)-(26) are analogous to constraints (12)-(13), and we can set $V_0^- = V_0^+$ in order to fix the root voltage exogenously. Constraints (19)-(24) correspond to a relaxation of the power flow equations, and are derived in [21]. Constraint (20) is a second order cone constraint, and needs to be satisfied as an equality in order for the problem to admit physical interpretation. The SOCP relaxation is exact under the condition that $P_i^{c+} = +\infty, Q_i^{c+} = +\infty$ for all $i \in N^+$, see theorem 1 of [21]. By 'exact', what is meant is that there exists an optimal solution of (*II*) such that the inequality constraint (20) is satisfied as an equality. The advantage of (*II*) over (*I*) is computational tractability. Nevertheless, duality analysis of each of these problems provides a different viewpoint to DLMPs, as we demonstrate in the next section. DLMPs for real power (denoted λ_n) and reactive power (denoted μ_n) correspond to the dual multipliers of the real power balance constraints (21)-(22) and reactive power balance constraints (23)-(24) respectively.

III. DLMP Decomposition

In this section we present three ways of decomposing DLMPs.

A. DLMP Decomposition Based on the Implicit Function Formulation

The implicit function formulation has been employed by the originators of the theory of spot pricing of electricity [1], [2] in order to analyze spot prices. This is also the approach employed for analyzing DLMPs in [4]. After presenting the derivation of the DLMP expression, we provide a simple example of its application, which also highlights that the expression of the DLMP decomposition in closed form becomes challenging even in a system with two nodes.

Proposition 1: The DLMP λ_n at a certain distribution node n can be expressed as the sum of the following terms:

- a term relating to power at the root: $\phi 1$
- a term relating to real power losses:

$$-\phi 1 \sum_{i\in E} R_i \frac{\partial l_i}{\partial y_n^p} \tag{27}$$

- a term relating to reactive power losses:

$$-\phi 2 \sum_{i\in E} X_i \frac{\partial l_i}{\partial y_n^p} \tag{28}$$

- a term relating to voltage constraints:

$$-\sigma_n^+ \frac{\partial v_n}{\partial y_n^p} + \sigma_n^- \frac{\partial v_n}{\partial y_n^p} \tag{29}$$

- and a term relating to transmission constraints:

$$-\sum_{i\in E}\eta_i^+\left(2f_i^p\frac{\partial f_i^p}{\partial y_n^p} + 2f_i^q\frac{\partial f_i^q}{\partial y_n^p}\right) - \sum_{i\in E}\eta_i^-\left(2\left(f_i^p - R_i l_i\right)\left(\frac{\partial f_i^p}{\partial y_n^p} - R_i\frac{\partial l_i}{\partial y_n^p}\right) + 2\left(f_i^q - X_i l_i\right)\left(\frac{\partial f_i^q}{\partial y_n^p} - X_i\frac{\partial l_i}{\partial y_n^p}\right)\right). \tag{30}$$

Proof: Let us first write out the Lagrangian:

$$\begin{aligned}
\mathcal{L} = &\sum_{i\in N} C_i^g p_i^g - \sum_{i\in N^+} C_i^c p_i^c \\
&+ \sum_{i\in E}\eta_i^+\left(f_i^p\left(\mathbf{y}^p,\mathbf{y}^q,v_0\right)^2 + f_i^q\left(\mathbf{y}^p,\mathbf{y}^q,v_0\right)^2 - S_i^2\right) \\
&+ \sum_{i\in E}\eta_i^-\Big(\left(f_i^p\left(\mathbf{y}^p,\mathbf{y}^q,v_0\right) - R_i l_i\left(\mathbf{y}^p,\mathbf{y}^q,v_0\right)\right)^2 \\
&+ \left(f_i^q\left(\mathbf{y}^p,\mathbf{y}^q,v_0\right) - X_i l_i\left(\mathbf{y}^p,\mathbf{y}^q,v_0\right)\right)^2 - S_i^2\Big) \\
&+ \sum_{i\in N}\left(\delta_i^{g+}\left(p_i^g - P_i^{g+}\right) + \delta_i^{g-}\left(P_i^{g-} - p_i^g\right)\right) \\
&+ \sum_{i\in N^+}\left(\delta_i^{c+}\left(p_i^c - P_i^{c+}\right) + \delta_i^{c-}\left(P_i^{c-} - p_i^c\right)\right) \\
&+ \sum_{i\in N^+}\sigma_i^+\left(v_i\left(\mathbf{y}^p,\mathbf{y}^q,v_0\right) - V_i^+\right) \\
&+ \sum_{i\in N^+}\sigma_i^-\left(V_i^- - v_i\left(\mathbf{y}^p,\mathbf{y}^q,v_0\right)\right) \\
&+ \phi 1\left(\sum_{i\in N^+} p_i^c + \sum_{i\in E} R_i l_i\left(\mathbf{y}^p,\mathbf{y}^q,v_0\right) - \sum_{i\in N} p_i^g\right) \\
&+ \phi 2\left(\sum_{i\in N^+} q_i^c + \sum_{i\in E} X_i l_i\left(\mathbf{y}^p,\mathbf{y}^q,v_0\right) - \sum_{i\in N} q_i^g\right).
\end{aligned} \tag{31}$$

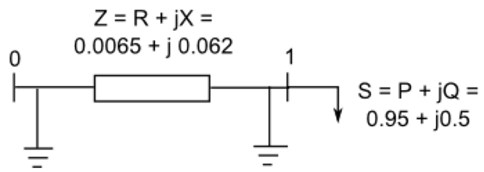

Fig. 2. A two-node system with fixed real and reactive demand at bus 1.

Consider then the stationarity condition of the Lagrangian function with respect to p_n^c, $\frac{\partial \mathcal{L}}{\partial p_n^c} = 0$:

$$\begin{aligned}
&-C_n^c + \delta_n^{c+} - \delta_n^{c-} - \sum_{i \in E} \eta_i^+ \left(2 f_i^p \frac{\partial f_i^p}{\partial y_n^p} + 2 f_i^q \frac{\partial f_i^q}{\partial y_n^p} \right) \\
&\quad - \sum_{i \in E} \eta_i^- \left(2\left(f_i^p - R_i l_i\right) \left(\frac{\partial f_i^p}{\partial y_n^p} - R_i \frac{\partial l_i}{\partial y_n^p} \right) \right. \\
&\quad\quad \left. + 2\left(f_i^q - X_i l_i\right) \left(\frac{\partial f_i^q}{\partial y_n^p} - X_i \frac{\partial l_i}{\partial y_n^p} \right) \right) \\
&\quad - \sigma_n^+ \frac{\partial v_n}{\partial y_n^p} + \sigma_n^- \frac{\partial v_n}{\partial y_n^p} \\
&\quad + \phi 1 \left(1 - \sum_{i \in E} R_i \frac{\partial l_i}{\partial y_n^p} \right) - \phi 2 \sum_{i \in E} X_i \frac{\partial l_i}{\partial y_n^p} = 0 \qquad (32)
\end{aligned}$$

Expressing the DLMP at location n as $\lambda_n = C_n^c - \delta_n^{c+} + \delta_n^{c-}$, we obtain the desired result from the previous first-order condition. ∎

We now consider the application of this approach to a simple network. Consider the two-bus system shown in Fig. 2. We have the following implicit functions:

$$f_1^p\left(y_1^p, y_1^q, v_0\right) = y_1^p \qquad (33)$$

$$f_1^q\left(y_1^p, y_1^q, v_0\right) = y_1^q \qquad (34)$$

$$v_1\left(y_1^p, y_1^q, v_0\right) = \frac{-b + \sqrt{b^2 - 4c}}{2} \qquad (35)$$

$$l_1\left(y_1^p, y_1^q, v_0\right) = \frac{\left(y_1^p\right)^2 + \left(y_1^q\right)^2}{v_1} \qquad (36)$$

where

$$b = -v_0 - 2\left(R_1 y_1^p + X_1 y_1^q\right) \qquad (37)$$

$$c = \left(R_1^2 + X_1^2\right)\left(f_1^p\left(y_1^p, y_1^q, v_0\right)^2 + f_1^q\left(y_1^p, y_1^q, v_0\right)^2\right) \qquad (38)$$

For the specific choice of parameters in Fig. 2, we have

$$\begin{aligned}
&v_0 = 1, v_1 = 0.921, l_1 = 1.252, p_0^g = 0.958, \\
&q_0^g = 0.578, f_1^p = -0.95, f_1^q = -0.5 \qquad (39)
\end{aligned}$$

The DLMP consists of the global power balance contribution, 50 \$/MWh, and the contribution from losses, 0.68 \$/MWh, amounting to a DLMP in node 1 of $\lambda_1 = 50.68$ \$/MWh.

It is clear that the implicit functions derived above are highly non-linear, even for this simple two-bus network.[2] These functions are dependent on the topology of the network and their derivation in closed form seems to be intractable for general networks. In this paper we estimate the partial derivatives numerically, as we discuss in detail in Section IV-C.

[2] KNITRO fails to solve the problem, even when initialized near the optimal solution.

B. DLMP Decomposition Based on the SOCP Formulation

In this section we derive a recursive formula for DLMPs which exploits the radial structure of the network. The formula expresses the DLMP at a certain node as a function of the reactive power price at that same node, the price of real and reactive power at the ancestor node, and the contribution of the capacity constraint of the distribution line that connects a node to its ancestor. This formula is derived by resorting to the KKT conditions of the SOCP relaxation.

Proposition 2: The DLMP λ_i at a certain distribution node i can be expressed as the sum of the following terms:

- The real power price at the ancestor node:

$$A_1\left(f_i^p, f_i^q, l_i\right) \cdot \lambda_{A_i} \qquad (40)$$

- The reactive power price at the current node:

$$A_2\left(f_i^p, f_i^q, l_i\right) \cdot \mu_i \qquad (41)$$

- The reactive power price at the ancestor node:

$$A_3\left(f_i^p, f_i^q, l_i\right) \cdot \mu_{A_i} \qquad (42)$$

- The contribution of the first complex power constraint:

$$A_4\left(f_i^p, f_i^q, l_i\right) \cdot \eta_i^+ \qquad (43)$$

- The contribution of the second complex power constraint:

$$A_5\left(f_i^p, f_i^q, l_i\right) \cdot \eta_i^- \qquad (44)$$

where the functions $A_i, i = 1, \ldots, 5$ are non-linear functions of f_i^p, f_i^q, l_i, given by equations (73)-(76) in the Appendix.

Proof: The KKT conditions (apart from the primal feasibility constraints) of formulation (II) can be developed as follows:

$$\left(p_i^g\right): \; C_g^i - \lambda_i + \delta_i^{g+} - \delta_i^{g-} = 0, i \in N \qquad (45)$$

$$\left(p_i^c\right): \; -C_i^c + \lambda_i + \delta_i^{c+} - \delta_i^{c-} = 0, i \in N^+ \qquad (46)$$

$$\left(q_i^g\right): \; -\mu_i + \theta_i^{g+} - \theta_i^{g-} = 0, i \in N \qquad (47)$$

$$\left(q_i^c\right): \; -\mu_i + \theta_i^{c+} - \theta_i^{c-} = 0, i \in N^+ \qquad (48)$$

$$\begin{aligned}
\left(v_i\right): \; &\beta_i - \sum_{j \in C_i} \beta_j - \gamma_i + \sigma_i + G_i \lambda_i \\
&- B_i \mu_i = 0, i \in N^+ \qquad (49)
\end{aligned}$$

$$\left(v_0\right): \; -\sum_{j \in C_0} \beta_j + \sigma_0 + G_0 \lambda_0 - B_0 \mu_0 = 0 \qquad (50)$$

$$\begin{aligned}
\left(l_i\right): \; &\left(R_i^2 + X_i^2\right)\beta_i + \lambda_{A_i} R_i + \mu_{A_i} X_i \\
&- \gamma_i \frac{\left(f_i^p\right)^2 + \left(f_i^q\right)^2}{l_i^2} - 2R_i\left(f_i^p - l_i R_i\right)\eta_i^- \\
&- 2X_i\left(f_i^q - l_i X_i\right)\eta_i^- = 0, i \in E \qquad (51)
\end{aligned}$$

$$\begin{aligned}
\left(f_i^p\right): \; &-2R_i\beta_i + \lambda_i - \lambda_{A_i} + \frac{2 f_i^p}{l_i}\gamma_i \\
&+ 2 f_i^p \eta_i^+ + 2\left(f_i^p - l_i R_i\right)\eta_i^- = 0, i \in E \qquad (52)
\end{aligned}$$

$$\left(f_i^q\right): \ -2X_i\beta_i + \mu_i - \mu_{A_i} + \frac{2f_i^q}{l_i}\gamma_i + 2f_i^q\eta_i^+ + 2\left(f_i^q - l_iX_i\right)\eta_i^- = 0, i \in E \quad (53)$$

$$0 \le \gamma_i \perp v_i - \frac{\left(f_i^p\right)^2 + \left(f_i^q\right)^2}{l_i} \ge 0, i \in E \quad (54)$$

$$0 \le \delta_i^{g+} \perp P_i^{g+} - p_i^g \ge 0, i \in N \quad (55)$$

$$0 \le \delta_i^{g-} \perp p_i^g - P_i^{g-} \ge 0, i \in N \quad (56)$$

$$0 \le \delta_i^{c+} \perp P_i^{c+} - p_i^c \ge 0, i \in N^+ \quad (57)$$

$$0 \le \delta_i^{c-} \perp p_i^c - P_i^{c-} \ge 0, i \in N^+ \quad (58)$$

$$0 \le \theta_i^{g+} \perp Q_i^{g+} - q_i^g \ge 0, i \in N \quad (59)$$

$$0 \le \theta_i^{g-} \perp q_i^g - Q_i^{g-} \ge 0, i \in N \quad (60)$$

$$0 \le \theta_i^{c+} \perp Q_i^{c+} - q_i^c \ge 0, i \in N^+ \quad (61)$$

$$0 \le \theta_i^{c-} \perp q_i^c - Q_i^{c-} \ge 0, i \in N^+ \quad (62)$$

$$0 \le \sigma_i^- \perp v_i - V_i^- \ge 0, i \in N \quad (63)$$

$$0 \le \sigma_i^+ \perp V_i^+ - v_i \ge 0, i \in N \quad (64)$$

$$0 \le \eta_i^+ \perp S_i^2 - \left(f_i^p\right)^2 - \left(f_i^q\right)^2 \ge 0, i \in E \quad (65)$$

$$0 \le \eta_i^- \perp S_i^2 - \left(f_i^p - l_iR_i\right)^2 - \left(f_i^q - l_iX_i\right)^2 \ge 0, i \in E \quad (66)$$

We would like to derive an identity that helps explain how the DLMP at each node is formed. The KKT conditions (52), (53) are appropriate for deriving such a recursive relation, since they link real and reactive power price at a certain node i to the price of the ancestor node. We additionally use KKT condition (51) in order to substitute out β_i and γ_i from condition (52), thereby arriving at the result of the proposition, which expresses the DLMP at node i as a function of the active power price at the ancestor node, the reactive power price at the current and ancestor node, and the price of congestion over the line that is adjacent to i and A_i. ■

Certain dual multipliers admit a natural economic interpretation. From conditions (45) we can interpret $\delta_i^{g+} - \delta_i^{g-}$ as the scarcity rent of a generator in location i, which equals the DLMP in that location minus the marginal cost of the local producer. This condition, combined with conditions (55) and (56), simply states that if a generator at a certain location i is producing strictly within its technical region, then the DLMP at that location is equal to the marginal cost of the local resource, but if the DLMP in that location is different from the marginal cost of the local resource then the local generator is at its operating limits (either upper or lower). Analogous interpretations hold for multipliers $\delta_i^{c,+/-}$ and $\theta_i^{g/c,+/-}$, based on KKT conditions (46)-(48), combined with conditions (57)-(62). Arguing on the basis of sensitivity, the multipliers $\eta_i^{+/-}$ can be interpreted as the marginal value of additional distribution line capacity, while the multipliers $\sigma_i^{+/-}$ can be interpreted as the marginal value of relaxing voltage constraints. It is not straightforward to provide a meaningful interpretation for multipliers β_i and γ_i.

It might be tempting to interpret the recursive condition of proposition 2 physically, as follows: a marginal change in real power injection in node i only requires a marginal change in real power injection of the ancestor node and a marginal change in reactive power of the present and ancestor node, while leaving the rest of the network unaffected. This interpretation is incorrect. One can show that nodes beyond the neighbors of i exhibit a non-zero change in their variables (e.g., voltage) in response to a marginal change in real power injection in location i. This inspires the decomposition approach of the next section, based on marginal losses. Interestingly, there seems to be no recursive relation that involves the DLMPs for real power alone.

C. DLMP Decomposition Based on Marginal Losses

An alternative approach towards understanding DLMPs involves the analysis of the sensitivity of real power losses over branches with respect to marginal changes in nodal injections. This approach provides a physical intuition for understanding DLMPs. In the subsequent proposition we use the following notation for real power losses on a given line $i \in E$:

$$Losses_i = R_il_i. \quad (67)$$

Proposition 3: Suppose that a single marginal resource is re-dispatched in order to optimally balance a marginal change dD_n in net power injection in location n, and denote m as the bus of the marginal resource. Then the DLMP can be expressed as

$$\lambda_n = \lambda_m\left(1 + \sum_{i \in E}\frac{dLosses_i}{dD_n}\right). \quad (68)$$

Proof: Assume that the marginal resource which is re-dispatched is a generator (the proof follows analogously if the marginal resource is a consumer). Consider the global power balance equation:

$$p_m^g = \sum_{i \in N^+} p_i^c + \sum_{i \in E} Losses_i - \sum_{i \in N-\{m\}} p_i^g, \quad (69)$$

where m is the marginal generator. Substituting into the objective function,

$$\sum_{i \in N} C_i^g p_i^g - \sum_{i \in N^+} C_i^c p_i^c = \sum_{i \in N-\{m\}} C_i^g p_i^g - \sum_{i \in N^+} C_i^c p_i^c + C_m^g\left(\sum_{i \in N^+} p_i^c + \sum_{i \in E} Losses_i - \sum_{i \in N-\{m\}} p_i^g\right). \quad (70)$$

Differentiating the objective function with respect to D_n yields the DLMP at location n. On the other hand, we have that $\lambda_m = C_m^g$. Moreover, by definition of the marginal resource, the derivative of p_i^g with respect to D_n is equal to zero for all $i \neq m$ and the derivative of p_i^c with respect to D_n is also zero for all $i \in N^+$. We thus arrive to the following identity:

$$\lambda_n = \lambda_m\left(1 + \sum_{i \in E}\frac{dLosses_i}{dD_n}\right). \quad (71)$$

■

The derived identity demonstrates that the DLMP of a given location cannot be understood by reasoning exclusively in terms of the DLMP of the neighbor, even for a radial network. Instead, the impact on losses and re-dispatch over the entire network needs to be understood.

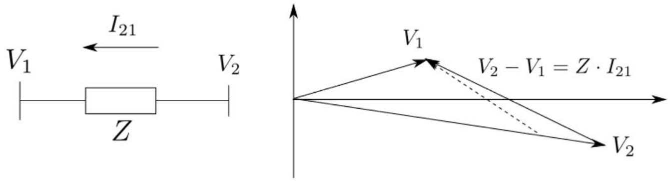

Fig. 3. A reduction in voltage magnitude in a location with high voltage results in a reduction in current magnitude over the line due to Ohm's law: $|I_{21}| = |V_2 - V_1|/|Z|$. In the figure, node 2 is the location with the greater voltage magnitude. The dashed line indicates the voltage magnitude difference when the magnitude of voltage V_2 is slightly decreased.

The sensitivity of losses to nodal injections can be approximated numerically. What we observe in numerical examples, which is in line with physical intuition, is that the sensitivity of losses with respect to nodal injections weakens as we move further away from a node.

One can gain additional physical intuition about the sensitivity of losses to nodal power injections, $dLosses_i/dD_n$, by considering Ohm's law. Provided voltage angles are close across a given line i (which must be true due to stability constraints), decreasing the magnitude of the voltage with the greater magnitude results in a decrease of the voltage difference phasor magnitude. This will therefore decrease the magnitude of the current phasor, as a consequence of Ohm's law, which will reduce losses. The idea is illustrated in figure 3. Given that a net power injection will tend to reduce the voltage magnitude on a given bus, by inspection of the optimal power flow solution one can predict whether a given net power injection will tend to align voltage magnitudes across neighboring buses (and therefore reduce losses along the line), or vice versa.

IV. Numerical Illustration

In this section we analyze the decomposition of DLMPs on a 15-bus example for two scenarios: (i) a case without capacity limits on complex power flows over lines, and (ii) a case with binding capacity limits on complex power flow over lines. We consider the radial system presented in Fig. 4, whose parameters are provided in Table I. Two flexible resources are located at the root node (node 0) and node 11, with $C_0^g =$ 50 \$/MWh and $C_{11}^g = 10$ \$/MWh. The flexible resource at the root has unbounded capacity, $P_0^+ = +\infty$, whereas $P_{11}^+ = 0.4$. Voltage limits are set uniformly to $V_i^- = 0.81$ and $V_i^+ = 1.21$ for all $i \in E$. Demand for real and reactive power, p_i^c and q_i^c, is fixed. Most real power consumption is in bus 1. These parameter values remain valid for both the cases considered in Sections IV-A and IV-B.

A. Without Line Capacity Limits

In this case we assume that $S_i = +\infty$ for all $i \in E$. Since $C_{11}^g = 10$, the maximum possible quantity of real power is injected in node 11. The binding constraints are the voltage magnitude limit and the available capacity of the distributed generator at node 11 (indicated with bold font in Table II). The DLMPs are shown in the last column of Table II. Observe the drop in the DLMP of location 11, which is the location

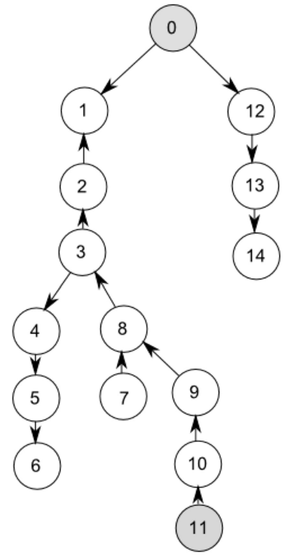

Fig. 4. The 15-bus example of Section IV. Grey nodes indicate locations of flexible resources. The arrows indicate the direction of flow of real power for the optimal solution for both the case with and without line capacity limits.

TABLE I
The Parameters for the Example of Section IV

i	R_i	X_i	S_i	p_i^c	q_i^c	B_i $\cdot 10^{-3}$
0	-	-	-	-	-	0
1	0.001	0.12	2	0.7936	0.1855	1.1
2	0.0883	0.1262	0.256	0	0	2.8
3	0.1384	0.1978	0.256	0.0201	0.0084	2.4
4	0.0191	0.0273	0.256	0.0173	0.0043	0.4
5	0.0175	0.0251	0.256	0.0291	0.0073	0.8
6	0.0482	0.0689	0.256	0.0219	0.0055	0.6
7	0.0523	0.0747	0.256	-0.1969	0.0019	0.6
8	0.0407	0.0582	0.256	0.0235	0.0059	1.2
9	0.01	0.0143	0.256	0.0229	0.0142	0.4
10	0.0241	0.0345	0.256	0.0217	0.0065	0.4
11	0.0103	0.0148	0.256	0.0132	0.0033	0.1
12	0.001	0.12	1	0.6219	0.1291	0.1
13	0.1559	0.1119	0.204	0.0014	0.0008	0.2
14	0.0953	0.0684	0.204	0.0224	0.0083	0.1

of the low-cost producer. The DLMP then increases as one moves away from this node. The reason that the DLMP is not equal to C_{11}^g at location 11 is that the generator at location 11 is utilized up to its full capacity. A marginal increase in demand in location 11 would have to be served by the root node, (which would incur the marginal cost of the root generator) but would result in real and reactive power *savings* (instead of losses) since it would depress the voltage in location 11 and bring it closer to the voltage of its neighbors.

To further illustrate the usefulness of reasoning in terms of losses, consider the prices in locations 7 and 8. One might find it counter-intuitive that the DLMP in location 7 should be lower than that of location 8, since location 7 is further away from the production resources and therefore a marginal increase in real power demand in node 7 might be expected

TABLE II
THE OPTIMAL SOLUTION OF THE NUMERICAL EXAMPLE FOR THE CASE WITHOUT LINE CAPACITY LIMITS, SECTION IV-A

i	p_i^g	q_i^g	$(f_i^p)^2+$ $(f_i^q)^2$	v_i	l_i	λ_i
0	1.063	0.431	-	1	-	50
1	-	-	0.218	0.945	0.231	50.06
2	-	-	0.152	1.009	0.151	46.79
3	-	-	0.169	1.121	0.151	42.04
4	-	-	0.005	1.118	0.004	42.14
5	-	-	0.003	1.116	0.002	42.21
6	-	-	0.001	1.113	0.0005	42.30
7	-	-	0.039	1.188	0.033	39.78
8	-	-	0.262	1.168	0.224	40.49
9	-	-	0.118	1.177	0.101	40.23
10	-	-	0.139	1.199	0.116	39.60
11	**0.4**	0.092	0.158	**1.21**	0.130	39.32
12	-	-	0.436	0.959	0.455	50.07
13	-	-	0.001	0.950	0.001	50.46
14	-	-	0.001	0.944	0.001	50.69

TABLE III
THE OPTIMAL SOLUTION OF THE NUMERICAL EXAMPLE FOR THE CASE WITH LINE CAPACITY LIMITS, SECTION IV-B

i	p_i^g	q_i^g	$(f_i^p)^2+$ $(f_i^q)^2$	v_i	l_i	λ_i
0	1.282	0.459	-	1	-	50
1	-	-	0.447	0.942	0.475	50.08
2	-	-	0.026	0.964	0.027	48.68
3	-	-	0.028	1.000	0.028	46.51
4	-	-	0.005	0.997	0.005	46.64
5	-	-	0.003	0.994	0.003	46.73
6	-	-	0.001	0.992	0.001	46.83
7	-	-	0.039	1.041	0.037	9.89
8	-	-	**0.066**	1.021	0.064	10.09
9	-	-	0.007	1.023	0.007	10.08
10	-	-	0.012	1.031	0.012	10.03
11	0.143	0.039	0.018	1.034	0.017	10
12	-	-	0.436	0.959	0.455	50.07
13	-	-	0.001	0.950	0.001	50.46
14	-	-	0.001	0.944	0.001	50.69

to increase losses. An explanation that stems from reasoning in terms of Fig. 3 is developed as follows: in node 7 we have $v_7 = 1.188$, which is in fact larger than the voltage magnitude of the neighboring node 8, $v_8 = 1.168$. A marginal increase in demand in location 7 tends to depress voltage in node 7. However, lowering the voltage in node 7 brings it closer, in magnitude, to the voltage in node 8, which tends to reduce the current magnitude flowing over the line. Therefore, one can expect that a marginal increase in demand in location 7 should result in reduced losses in line 8. This is indeed validated by the numerical solution of the model, which gives $dLosses_8/dD_7 = -0.017$.

B. With Capacity Limits

We now consider the case where S_i is finite and given in Table I. The direction of flows is shown in Fig. 4 and the solution is indicated in Table III. Note that the transmission capacity limit is binding on line 8 (indicated by bold font in the table), and a load pocket is created in the subset of buses 7-11.

1) Implicit Function Formulation: The components of the DLMP according to this price decomposition are shown in Table IV. Slight differences between the DLMP and the sum of the components may occur due to numerical errors and rounding. The term $\phi 1$ is the marginal cost of providing an extra MW at the root node, and is common for all nodes. The voltage contribution is zero for all nodes, since there are no voltage limits that are binding at the optimal solution. The negative sign of the transmission component in the load pocket (nodes 7-11) can be understood physically as follows: a marginal increase in real power demand in any node within this load pocket results in a slight de-congestion of link 8. This implies that demand in other parts of the network can be met by an increase in the output of the low-cost generator in location 11, which results in cost savings.

TABLE IV
THE BREAKDOWN OF DLMPS ACCORDING TO THE IMPLICIT FUNCTION METHOD FOR THE CASE WITH CAPACITY LIMITS, SECTION IV-B

i	λ_i	$\phi 1$	Loss Eq. (27)	Voltage Eq. (29)	Transmission Eq. (30)
0	50	-	-	-	-
1	50.08	50	0.08	0	-0.002
2	48.68	50	-1.31	0	-0.02
3	46.51	50	-3.46	0	-0.04
4	46.64	50	-3.33	0	-0.04
5	46.73	50	-3.25	0	-0.04
6	46.83	50	-3.15	0	-0.04
7	9.89	50	-5.34	0	-34.78
8	10.09	50	-4.42	0	-35.50
9	10.08	50	-4.50	0	-35.44
10	10.03	50	-4.73	0	-35.25
11	10	50	-4.85	0	-35.16
12	50.07	50	0.07	0	0
13	50.46	50	0.45	0	0
14	50.69	50	0.68	0	0

2) SOCP Formulation: The decomposition of DLMPs according to this approach is shown in Table V. The DLMP at a given node is mostly driven by the λ_A component, i.e., the contribution of the DLMP for real power from the ancestor node. The term related to the η_i^+ multiplier is non-zero for node 8, which is located under a congested line. Reactive power prices appear to have a minor influence on the formation of DLMPs.

3) Marginal Losses: The results for the method based on marginal losses are shown in Table VI for a subset of nodes. Note that the loss component that contributes most to the formation of the price is typically in the neighborhood of the node whose DLMP we are considering, and there is a tendency for the contributions to diminish the further away we move from the node in question. For example, when considering

TABLE V
THE BREAKDOWN OF DLMPs ACCORDING TO THE SOCP FORMULATION FOR THE CASE WITH CAPACITY LIMITS, SECTION IV-B

i	λ_i	Term λ_{A_i} Eq. (40)	Term μ_i Eq. (41)	Term μ_{A_i} Eq. (42)	Term η_i^+ Eq. (43)
0	50	-	0	-	0
1	50.08	50.07	0.01	0	0
2	48.68	48.47	0.31	-0.10	0
3	46.51	46.29	0.56	-0.34	0
4	46.64	46.62	0.63	-0.61	0
5	46.73	46.71	0.64	-0.63	0
6	46.83	46.81	0.66	-0.64	0
7	9.89	9.89	0.02	-0.02	0
8	10.09	45.58	0.02	-0.61	-34.89
9	10.08	10.08	0.01	-0.02	0
10	10.03	10.04	0.005	-0.01	0
11	10	10	0	-0.005	0
12	50.07	50.07	0.002	0	0
13	50.46	50.26	0.24	-0.03	0
14	50.69	50.57	0.35	-0.24	0

TABLE VI
THE BREAKDOWN OF DLMPs ACCORDING TO THE MARGINAL LOSS METHOD FOR A SELECTION OF NODES, SECTION IV-B. λ_m IS THE DLMP OF THE MARGINAL BUS CORRESPONDING TO NODE n

	$n=1$	$n=4$	$n=7$	$n=10$	$n=13$
λ_m	50	50	10	10	50
$\lambda_m \frac{dR_1 l_1}{dD_n}$	0.068	0.061	0	0	0
$\lambda_m \frac{dR_2 l_2}{dD_n}$	0.003	-1.399	0	0	0
$\lambda_m \frac{dR_3 l_3}{dD_n}$	0.004	-2.194	0	0	0
$\lambda_m \frac{dR_4 l_4}{dD_n}$	0	0.134	0	0	0
$\lambda_m \frac{dR_5 l_5}{dD_n}$	0	0.001	0	0	0
$\lambda_m \frac{dR_6 l_6}{dD_n}$	0	0.001	0	0	0
$\lambda_m \frac{dR_7 l_7}{dD_n}$	0.002	0.040	-0.195	0	0
$\lambda_m \frac{dR_8 l_8}{dD_n}$	0.003	0.054	0	0	0
$\lambda_m \frac{dR_9 l_9}{dD_n}$	0	0.001	0.016	0	0
$\lambda_m \frac{dR_{10} l_{10}}{dD_n}$	0	0.005	0.050	0	0
$\lambda_m \frac{dR_{11} l_{11}}{dD_n}$	0	0.003	0.025	0.026	0
$\lambda_m \frac{dR_{12} l_{12}}{dD_n}$	0	0	0	0	0.069
$\lambda_m \frac{dR_{13} l_{13}}{dD_n}$	0	0	0	0	0.402
$\lambda_m \frac{dR_{14} l_{14}}{dD_n}$	0	0	0	0	0.001
λ_n	50.08	46.64	9.89	10.03	50.46

the DLMP of node 1 there is tendency for the components to diminish as we move further away from node 1.

C. Discussion

The DLMP decomposition based on the KKT analysis of the SOCP formulation (Section III-B) is simplest from a computational standpoint. The reason is that there is no need for numerical approximation: the DLMP decomposition for all nodes N^+ can be obtained by solving model (II) once. The DLMP decomposition based on the implicit function formulation (Section III-A) requires additional computational effort, because it is necessary to approximate the partial derivatives of primal decision variables numerically, and this is done by solving the optimal power flow for a reference level of net demand and a slightly perturbed level of net demand. Thus, in order to evaluate the DLMP decomposition for all nodes N^+, we need to solve $|N^+| + 1$ optimal power flow problems (once for the reference case, and once for each $n \in N^+$ at a slightly perturbed level of net demand). The DLMP decomposition based on marginal losses (Section III-C) requires the same amount of computation as the approach based on the implicit function formulation.

We note that as long as the SOCP relaxation remains tractable for large-scale radial networks, the proposed DLMP decompositions remain tractable, even if they scale linearly with the size of the network. It is reasonable to expect that SOCP relaxations of optimal power flow problems for radial networks should scale well with respect to network size, given the recent progress that has been achieved in developing distributed algorithms for these problems [19], and the demonstrated performance of distributed methods on problems of massive scale with hundreds of thousands of nodes [22]. Note that the DLMP decomposition of each node can be computed independently of other nodes, thereby enabling further benefits from parallelization.

In terms of physical intuition, the approach based on marginal losses appears to be the most intuitive to interpret. The approach based on the implicit function formulation generalizes the standard models used in the literature by the originators of the theory of the spot pricing of electricity [1], [2], [4], while the approach based on the KKT analysis of the SOCP relaxation provides an alternative recursive relation. However, both approaches are more challenging to interpret than the approach based on marginal losses.

V. CONCLUSION

In this paper we have presented three approaches towards understanding the formation of DLMPs in radial distribution networks. We have demonstrated that these methods are capable of predicting DLMPs within acceptable numerical error on a 15-bus example. Among these methods, the approach based on marginal losses appears to be better grounded in terms of physical intuition. In future work it will be interesting to explore the KKT conditions of alternative convex relaxations that may provide further insight regarding the behavior of DLMPs, and the impact of reserve on energy price formation. At a higher level, the transition towards mobilizing distributed resources will require the clarification of the interactions between TSOs, DSOs and aggregators, and addressing the computational challenges that arise from the vast number of distributed resources and the non-linearities of the distribution network.

APPENDIX

A. Nomenclature

Sets

N: set of distribution network nodes
N^+: set of distribution network nodes except the root

E: set of distribution network edges.

Primal Variables

p_i^g/q_i^g: real/reactive power generation of node i
p_i^c/q_i^c: real/reactive power consumption of node i
v_i: voltage magnitude squared of node i
f_i^p/f_i^q: real/reactive power flow from i to A_i
l_i: current magnitude squared on branch i.

Parameters

C_i^g: marginal cost of generation at node i
C_i^c: marginal benefit of consumption at node i
P_i^{g+}/P_i^{g-}: maximum/minimum real power capacity for generation at node i
P_i^{c+}/P_i^{c-}: maximum/minimum real power capacity for consumption at node i
Q_i^{g+}/Q_i^{g-}: maximum/minimum reactive power capacity for generation at node i
Q_i^{c+}, Q_i^{c-}: maximum/minimum reactive power capacity for consumption at node i
V_i^+, V_i^-: maximum/minimum voltage limits at node i
G_i/B_i: shunt conductance/susceptance of line i
R_i/X_i: resistance / reactance of branch i
S_i: complex power flow limit of branch i
$\mathbf{y}^p = (p_1^g - p_1^c, p_2^g - p_2^c, \ldots, p_{|N|}^g - p_{|N|}^c)$
$\mathbf{y}^q = (q_1^g - q_1^c, q_2^g - q_2^c, \ldots, q_{|N|}^g - q_{|N|}^c)$.

Dual Variables

β_i: multiplier of voltage change along branch i
γ_i: multiplier of conic relaxation along branch i
$\delta_i^{g+}/\delta_i^{g-}/\delta_i^{c+}/\delta_i^{c-}$: multipliers of real power flow upper/lower bounds of generator/consumer i
η_i^+/η_i^-: multipliers of complex power flow upper/lower limit on branch i
$\theta_i^{g+}/\theta_i^{g-}/\theta_i^{c+}/\theta_i^{c-}$: multipliers of reactive power flow upper/lower bounds of generator/consumer i
λ_i: multiplier of real power balance of node i (model (II)), also the real power DLMP of node i
μ_i: multiplier of reactive power balance of node i (model (II)), also the reactive power DLMP of node i
σ_i^+/σ_i^-: multipliers of voltage upper/lower limit of node i
$\phi 1/2$: multipliers of global real / reactive power balance (model (I)).

B. Functional Expressions in the Statement of Proposition 2

The functional of proposition 2 are as follows:

$$A_1(f_i^p, f_i^q, l_i) = \frac{\left((f_i^p)^2 + (f_i^q)^2\right)X_i + l_i f_i^q(R_i^2 - X_i^2) - 2l_i f_i^p R_i X_i}{\left((f_i^p)^2 + (f_i^q)^2\right)X_i - l_i f_i^q(R_i^2 + X_i^2)} \quad (72)$$

$$A_2(f_i^p, f_i^q, l_i) = \frac{\left((f_i^p)^2 + (f_i^q)^2\right)R_i - l_i f_i^p(R_i^2 + X_i^2)}{\left((f_i^p)^2 + (f_i^q)^2\right)X_i - l_i f_i^q(R_i^2 + X_i^2)} \quad (73)$$

$$A_3(f_i^p, f_i^q, l_i) = \frac{-\left((f_i^p)^2 + (f_i^q)^2\right)R_i + l_i f_i^p(R_i^2 - X_i^2) + 2l_i f_i^q R_i X_i}{\left((f_i^p)^2 + (f_i^q)^2\right)X_i - l_i f_i^q(R_i^2 + X_i^2)} \quad (74)$$

$$A_4(f_i^p, f_i^q, l_i) = \frac{2\left((f_i^q)^3 R_i - (f_i^p)^3 X_i\right) + 2f_i^p f_i^q(f_i^p R_i - f_i^q X_i)}{\left((f_i^p)^2 + (f_i^q)^2\right)X_i - l_i f_i^q(R_i^2 + X_i^2)} \quad (75)$$

$$\begin{aligned} A_5(f_i^p, f_i^q, l_i) &= \frac{2\left((f_i^q)^3 R_i - 2(f_i^p)^3 X_i\right) + 2f_i^p f_i^q(f_i^p R_i - f_i^q X_i)}{\left((f_i^p)^2 + (f_i^q)^2\right)X_i - l_i f_i^q(R_i^2 + X_i^2)} \\ &+ \frac{2l_i^2(f_i^q R_i^3 - f_i^p X_i^3) - 4l_i f_i^p f_i^q(R_i^2 - X_i^2)}{\left((f_i^p)^2 + (f_i^q)^2\right)X_i - l_i f_i^q(R_i^2 + X_i^2)} \\ &+ \frac{4l_i R_i X_i\left((f_i^p)^2 - (f_i^q)^2\right)}{\left((f_i^p)^2 + (f_i^q)^2\right)X_i - l_i f_i^q(R_i^2 + X_i^2)} \\ &+ \frac{-2l_i^2 R_i X_i(f_i^p R_i - f_i^q X_i)}{\left((f_i^p)^2 + (f_i^q)^2\right)X_i - l_i f_i^q(R_i^2 + X_i^2)}. \end{aligned} \quad (76)$$

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