Tight Bounds for Deciding Convergence of Consensus Systems ☆

P.-Y. Chevalier[a,*], J. M. Hendrickx[a], R. M. Jungers[a]

[a]*ICTEAM Institute, Université catholique de Louvain, 4 avenue Georges Lemaître, B-1348 Louvain-la-Neuve, Belgium*

Abstract

We analyze the asymptotic convergence of all infinite products of matrices taken in a given finite set by looking only at *finite* or *periodic* products. It is known that when the matrices of the set have a common nonincreasing polyhedral norm, all infinite products converge to zero if and only if all infinite *periodic* products with periods smaller than a certain value converge to zero. Moreover, bounds on that value are available [1].

We provide a stronger bound that holds for both polyhedral norms and polyhedral seminorms. In the latter case, the matrix products do not necessarily converge to 0, but all trajectories of the associated system converge to a common invariant subspace. We prove that our bound is tight for all seminorms.

Our work is motivated by problems in *consensus systems*, where the matrices are *stochastic* (nonnegative with rows summing to one), and hence always share a same common nonincreasing polyhedral seminorm. In that case, we also improve existing results.

Keywords: Stability of Matrix Sets, Stochastic Matrices, Consensus

☆R. M. Jungers is an F.R.S.-FNRS research associate. This work is also supported by the Belgian Network DYSCO, funded by the Belgian government and the Concerted Research Action (ARC) of the French Community of Belgium.

*Corresponding author

Email addresses: pierre-yves.chevalier@uclouvain.be (P.-Y. Chevalier), julien.hendrickx@uclouvain.be (J. M. Hendrickx), raphael.jungers@uclouvain.be (R. M. Jungers)

1. Introduction

We consider the problem of determining the stability of matrix sets, that is, determining whether or not all infinite products of matrices from a given set converge to zero, or more generally to a common invariant subspace. This problem appears in several different situations in control engineering, computer science, and applied mathematics. For instance, the stability of matrix sets characterizes the stability of switching dynamical systems [2], which have numerous application in control [2, 3, 4]. Stability of matrix sets is instrumental in proving the continuity of certain wavelet functions [2, 5]. Somewhat surprisingly, it also helped establishing the best known asymptotic bounds on the number of α-power-free binary words of length n, a central problem in combinatorics on words [6, 7].

Deciding the stability of a matrix set is notoriously difficult and the decidability of this problem is not known. The related problem of the existence of an infinite product whose norm diverges is undecidable [8]. However, it is possible to decide stability when the set has the *finiteness property*, that is, when there is a bound p such that the existence of an infinite nonconverging product[1] implies the existence of an infinite nonconverging *periodic* product with period smaller than or equal to p. Indeed, checking the stability of the set can be done by checking the stability of all products whose length is smaller than or equal to p. In this work, we look for the smallest valid bound p.

A similar question is particularly relevant in the context of consensus problems. These systems are models for groups of agents trying to agree on some common value by an iterative process. Each agent has a value x_i which it updates by computing the weighted average of values of agents with which it can communicate. Consensus systems have attracted considerable attention due to their applications in control of vehicle formations [9], flocking [10, 11] or distributed sensing [12, 13]. They typically have time-varying communication

[1] i.e., an infinite product that does not converge to zero.

networks due to e.g. communication failures, or to the movements of the agents. This leads to systems whose (linear) dynamics may switch at each time-step. When a set of possible linear dynamics is known, one fundamental question is whether the system converges for any switching sequence [14].

Consensus systems can be modeled by discrete-time linear switching systems, $x(t+1) = A_t x(t)$, where the transition matrices A_t are *stochastic* (nonnegative matrices whose rows sum to 1) because the agents always compute weighted averages. In this case, the products certainly do not converge to zero, since products of stochastic matrices remain stochastic. The central question is whether the agents asymptotically converge to the same value. Deciding whether a consensus system converges for any sequence of transition matrices and any initial condition corresponds to determining whether all left-infinite products of matrices taken from a set converge to a rank one matrix. Indeed, a stochastic matrix is rank one if and only if all its rows are the same, and this situation corresponds to consensus. This particularization to stochastic matrices has other applications, including inhomogeneous Markov chains, and probabilistic automata [15].

Stochastic matrices share a *nonincreasing polyhedral seminorm* and this property provides important information on the asymptotic convergence of products of these matrices. Indeed, for sets of matrices sharing a common nonincreasing polyhedral seminorm, a bound p as discussed above is available. This was first established by Lagarias and Wang [1]. The authors also give an explicit value for p (namely half the number of faces of the unit ball of the norm). This result can easily be extended from norms to seminorms and we do so in the proof of Theorem 1.

The case of stochastic matrices has been analyzed earlier in the context of inhomogeneous Markov chains [5, 15, 16, 17] and later in the context of consensus systems [14]. A finiteness result has been known since Paz [15], who proved that all left-infinite products converge to a rank one matrix if and only if a certain condition on all products of length $B = \frac{1}{2}(3^n - 2^{n+1} + 1)$ is satisfied. In our recent paper [18], we showed that this bound can be derived from a generalization of the result of Lagarias and Wang applied to a particular seminorm.

Our Contribution

In this article, we consider a general problem that includes these particular cases: we study matrix sets for which there exists a polyhedral seminorm which is nonincreasing for all matrices in the given set, and we wonder whether long products of these matrices are asymptotically contractive. We improve all the bounds previously known in the particular cases, and prove that our bound is tight. Our analysis relies on the fact that the convergence of the dynamical system can be encapsulated in a *discrete* representation by a dynamical system on the face lattice of the polyhedral (semi)norm. Our results then rely on a careful study of the combinatorial structure of the trajectories in this discrete structure.

The improvement over the previously known bound depends on the seminorm. In the case of stochastic matrices, the improvement is a multiplicative factor of about $\frac{3}{2\sqrt{\pi n}}$.

2. Problem Setting

Let $\Sigma = \{A_1, \ldots, A_m\}$ be a set of matrices and σ an infinite sequence of indices. We say that the product $\ldots A_{\sigma(2)} A_{\sigma(1)}$ is *periodic* if the sequence σ is periodic. We recall that a *seminorm* on $\mathbb{R}^n$ is an application $\|.\|$ with the following properties:

- $\forall x \in \mathbb{R}^n,\ a \in \mathbb{R},\ \|ax\| = |a|\|x\|$
- $\forall x, y \in \mathbb{R}^n,\ \|x + y\| \leq \|x\| + \|y\|$.

We call a *polyhedral seminorm* a seminorm whose unit ball is a *polyhedron*, that is, a set that can be defined by a finite set of linear inequalities

$$\{x : ||x|| \leq 1\} = \{x : \forall i,\ b_i^\top x \leq c_i\}.$$

We say that a seminorm $\|.\|$ is *nonincreasing* with respect to a matrix A if

$$\forall x \in \mathbb{R}^n,\ \|Ax\| \leq \|x\|.$$

Geometrically, this corresponds to its unit ball being invariant

$$A\{x : ||x|| \leq 1\} \subseteq \{x : ||x|| \leq 1\}.$$

We say that a seminorm is nonincreasing with respect to a set Σ of matrices if it is nonincreasing with respect to each of the matrices in Σ. We say that a matrix A *contracts* a seminorm $\|.\|$ if $\forall x \in \mathbb{R}^n,\ \|Ax\| < \|x\|$. We say that an infinite product $\ldots A_{\sigma(2)}A_{\sigma(1)}$ contracts a seminorm $\|.\|$ if there is a t such that

$$A_{\sigma(t)} \ldots A_{\sigma(2)}A_{\sigma(1)}\{x : ||x|| \leq 1\} \subset \mathrm{int}(\{x : ||x|| \leq 1\}).$$

One can easily verify that if there is a p such that all products of length p of matrices in Σ contract a seminorm $\|.\|$, then all trajectories $x(t)$ of the corresponding switching system $x(t+1) = A_{\sigma(t)}x(t)$ asymptotically approach the set $\{x : \|x\| = 0\}$, and that their distance to that set decays exponentially as t increases. In particular, if $\|.\|$ is a norm, x converges exponentially to 0. In addition, if $\|.\|$ is the seminorm $\|x\|_{\mathcal{P}} = \frac{1}{2}(\max_i x_i - \min_i x_i)$ – a seminorm that is nonincreasing for stochastic matrices – then x approaches the *consensus space* $\{\alpha \mathbf{1}\}$. We have proved in previous work [18] that each trajectory actually converges in that case to a specific (but possibly different) point in that set, as opposed to just approaching the set. For these reasons, we will investigate contraction of seminorms, keeping in mind that this question is intimately related to that of convergence.

Question 1. *Let $\|.\|$ be a polyhedral seminorm in $\mathbb{R}^n$ for some fixed n; what is the smallest p such that for any set Σ for which $\|.\|$ is nonincreasing, the existence of an infinite noncontracting product implies the existence of an infinite* periodic *noncontracting product with period smaller than or equal to p?*

3. The General Case

In this section, we answer Question 1. We start by recalling some definitions (see [19] for more details). A *partially ordered set* or *poset* is a set P with a binary relation $\preceq$ that is transitive, antisymmetric and reflexive. We also note $x \prec y$ for the relation $x \preceq y$ and $x \neq y$. A poset $(P, \preceq)$ is called *graded* if it can be equipped with a rank function $r : P \mapsto \mathbb{N}$ such that $x \preceq y \Rightarrow r(x) \leq r(y)$ and $(y \prec x \text{ and } \nexists z,\ y \prec z \prec x) \Rightarrow r(x) = r(y) + 1$. The set of all elements of a given rank is called a rank level. A poset is called a *lattice* if any pair of elements has a unique infimum and a unique supremum.

Intuitively, a *face* is the generalization of a vertex (or an edge, or a facet) to an arbitrary dimension. The formal definition is the following.

Definition 1 (Faces of a Polyhedron). *A nonempty subset F of an n-dimensional polyhedron $\mathcal{Q}$ is called a* face *or* closed face *if one of the following holds:*

- $F = \mathcal{Q}$,
- $F = \varnothing$
- *or F can be represented as $F = \mathcal{Q} \cap \{x : b^\top x = c\}$ where $b \in \mathbb{R}^n$, $c \in \mathbb{R}$ are such that*

$$\forall x \in \mathcal{Q},\ b^\top x \leq c.$$

If the face contains exactly $d + 1$ affinely independent points[2]*, we call d the* dimension *of the face. A* proper face *is a face that is neither the polyhedron itself nor the empty face. An* open face *is the relative interior of a face. Finally, a* facet *is a face of dimension $n - 1$.*

It is well known that faces of any dimension are intersections of facets and their number is therefore finite. It is also known that any polyhedron decomposes into a disjoint union of open faces.

We use the term *double-face* to denote the set $F \cup -F$, for some proper face F. A double-face is called open if the face F is open, and closed otherwise.

Definition 2 (lattice of double-faces). *Given a centrally symmetric polyhedron $\mathcal{Q}$ (i.e., a polyhedron $\mathcal{Q} = -\mathcal{Q}$), we call* lattice of double-faces *the poset $(P, \subseteq)$ where $\subseteq$ is the inclusion relation and P is a set whose members are*

[2]The points $u_0, u_1, \ldots, u_d$ are called affinely independent if $u_1 - u_0, u_2 - u_0, \ldots$ are linearly independent.

- *double-faces of $\mathcal{Q}$ ($r =$ dimension of the face)*
- *$\mathcal{Q}$ ($r = n$)*
- *$\varnothing$ ($r = d_{\min} - 1$, where $d_{\min}$ is the lowest dimension of faces of $\mathcal{Q}$).*

It can be verified that this poset is a lattice and that it is graded; a rank function is given between brackets.

Definition 3 (Antichain)**.** *Let $(P, \preceq)$ be a poset. An* antichain *is a subset $S \subseteq P$ whose elements are not comparable:*

$$\forall S_1, S_2 \in S, \quad S_1 \npreceq S_2.$$

For instance, a set of double-faces that are not included in one another form an antichain in the lattice of double-faces.

Example 1. *The unit ball of the seminorm $\|x\|_{\mathcal{P}} = \frac{1}{2}(\max_i x_i - \min_i x_i)$ in dimension 3 is represented in Figure 1. We will study this seminorm and its relation to stochastic matrices in detail in the next section. The lattice of double-faces of this unit ball and its largest antichain are represented in Figure 2.*

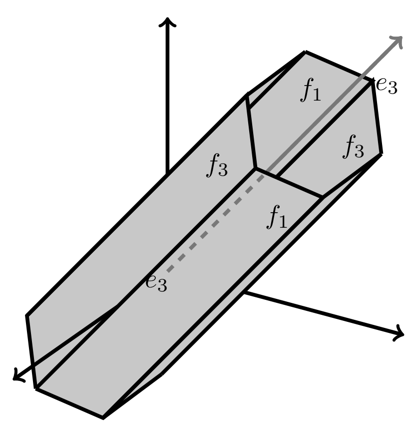

Figure 1: The polyhedron $\mathcal{P}$ for $n = 3$. The gray arrow indicates the direction $a\mathbf{1}$. The polyhedron has 6 facets, one for each constraint of the form $\frac{1}{2}(x_i - x_j) \leq 1$. The sets f_1, f_3 and e_3 are double-faces.

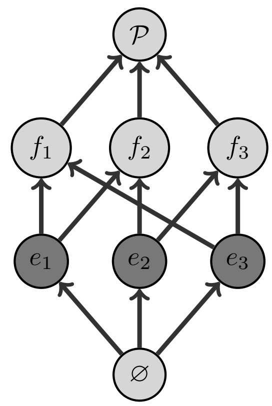

Figure 2: The lattice of double-faces of the polyhedron $\mathcal{P}$ for $n = 3$. The elements f_1, f_2 and f_3 represent the three pairs of opposite facets while e_1, e_2 and e_3 represent the three pairs of opposite edges. In dark gray, a largest antichain in this lattice.

Definition 4 (Width of a Poset)**.** *We call the width $W(P)$ of a poset P the number of elements of the largest antichain of P. We also write $W(\mathcal{Q})$ for the width of the lattice of double-faces of a given centrally symmetric polyhedron $\mathcal{Q}$.*

The following lemma by Lagarias and Wang allows abstracting Question 1 as a combinatorial problem, as it shows that matrices in Σ can be completely abstracted (for our purpose) as functions mapping each face of the invariant polyhedron into another one.

Lemma 1. *Let Σ be a finite set of matrices having a common invariant polyhedron $\mathcal{Q}$. Then, for any $A \in \Sigma$ and any double-face O_1 of $\mathcal{Q}$, there exists exactly one double-face O_2 (possibly int$(\mathcal{Q})$) such that*

$$AO_1 \subseteq O_2.$$

Proof. The result is established in [1, Claim in the proof of Theorem 4.1] for faces instead of double-faces. It is clear that the open faces O_1, O_2 satisfy $AO_1 \subseteq O_2$ if and only if the open double-faces $O_1 \cup -O_1$ and $O_2 \cup -O_2$ satisfy $A(O_1 \cup -O_1) \subseteq (O_2 \cup -O_2)$. The result therefore extends to double-faces. $\square$

The next theorem is an improvement of [1, Theorem 4.1]. We extend it to seminorms and we provide a stronger bound.

Theorem 1. *Let Σ be a set of matrices and let $\|.\|$ be a polyhedral seminorm that is nonincreasing for Σ. If there is a left-infinite product of matrices from Σ that does not contract $\|.\|$, there is one that is periodic with a period p not larger than*

$$p^* = W(\mathcal{B}), \text{ with } \mathcal{B} = \{x : ||x|| \leq 1\}.$$

Proof. We first prove that p is finite. Suppose there exists an infinite noncontracting product $\dots A_{\sigma(2)}A_{\sigma(1)}$ and therefore a point x_0 such that

$$\forall i,\ A_{\sigma(i)} \dots A_{\sigma(1)}x_0 \notin \mathrm{int}(\mathcal{B}).$$

Since the number of faces is finite, there is an open double-face O and indices $i < j$ such that

$$A_{\sigma(i)} \dots A_{\sigma(1)}x_0 \in O \text{ and } A_{\sigma(j)} \dots A_{\sigma(1)}x_0 \in O.$$

By Lemma 1, we have

$$A_{\sigma(j)} \dots A_{\sigma(i+1)}O \subseteq O.$$

Therefore, the infinite power of $A_{\sigma(j)} \dots A_{\sigma(i+1)}$ is an infinite *periodic* noncontracting product, proving that the theorem is true for some finite period $p = j-i$ smaller than the number of double-faces.

We now prove the full theorem. Let P be such that $\dots PPP$ is an infinite noncontracting product with the smallest period p and

$$P = A_{\sigma(p)} \dots A_{\sigma(1)}.$$

Let O_1 be a double-face such that

$$\forall t \geq 0,\ (P)^t O_1 \not\subseteq \mathrm{int}(\mathcal{B})$$

(such a face exists due to Lemma 1 and the fact that $\ldots PPP$ is noncontracting), let O_2 be the double-face containing $A_{\sigma(1)}O_1$ (by Lemma 1, there is exactly one such double-face), O_3 containing $A_{\sigma(2)}A_{\sigma(1)}O_1$ up to O_p containing $A_{\sigma(p-1)}\ldots A_{\sigma(1)}O_1$. Let also $F_1 = \mathrm{cl}(O_1)$, $\ldots$, $F_p = \mathrm{cl}(O_p)$.

We now prove that $\{F_1,\ldots,F_p\}$ in an antichain in the lattice of double-faces. Suppose, to obtain a contradiction, that for some i,j with $i > j$, $F_i \subseteq F_j$. Then,

$$A_{\sigma(i-1)}\ldots A_{\sigma(j)}F_j \subseteq F_i \subseteq F_j,$$

and thus

$$\forall t \geq 0,\ (A_{\sigma(i-1)}\ldots A_{\sigma(j)})^t F_j \subseteq F_j.$$

This contradicts the assumption that $\ldots PPP$ is the infinite periodic noncontracting product with the smallest period. Similarly, if for some i,j with $i < j$, $F_i \subseteq F_j$, then

$$\forall t \geq 0,\ (A_{\sigma(i-1)}\ldots A_{\sigma(1)}A_{\sigma(p)}\ldots A_{\sigma(j)})^t F_j \subseteq F_j,$$

and again we have a contradiction.

□

The bound p^* of Theorem 1 cannot be decreased: it is tight for any polyhedron, as we show next.

Theorem 2. *Let $\|.\|$ be a polyhedral seminorm. There is a set Σ for which $\|.\|$ is non increasing and such that*

- *all infinite periodic products with periods smaller than p^* are contracting,*
- *not all products are contracting.*

Proof. Let again $\mathcal{B} = \{x : ||x|| \leq 1\}$. We construct a set of matrices such that the infinite noncontracting product that has the smallest period has a period equal to $p^* = W(\mathcal{B})$. Let $X = \{F_1,\ldots,F_{p^*}\}$ be the largest antichain in the lattice of double-faces and let $O_1,\ldots,O_{p^*}$ be the corresponding open double-faces.

By definition, each double-face F_i is the union of two opposite proper faces $G_i, -G_i$ and the proper face G_i is the intersection of $\mathcal{B}$ with a hyperplane

$$G_i = \mathcal{B} \cap \{x : b_i^\top x = c_i\}$$

such that $\mathcal{B}$ is in one halfspace defined by the hyperplane:

$$\mathcal{B} \subseteq \{x : b_i^\top x \leq c_i\}.$$

We also have $c_i \neq 0$. Indeed, if $c_i = 0$, then $\mathcal{B} \subseteq \{x : b_i^\top x \leq 0\}$ and because , $\mathcal{B}$ is the unit ball of a seminorm, $\mathcal{B} = -\mathcal{B}$, and $\mathcal{B} \subseteq \{x : -b_i^\top x \leq 0\}$ and this implies $G_i = \mathcal{B} \cap \{x : b_i^\top x = 0\} = \mathcal{B}$ and G_i is not a proper face. Therefore, $c_i \neq 0$ and we can scale b_i and c_i to have $\forall i,\ c_i = 1$. Finally, $F_i = G_i \cup -G_i = \mathcal{B} \cap \{x : b_i^\top x = \pm 1\}$.

By taking any v_i in the open double-face $O_{(i \bmod p^*)+1}$ and defining

$$A_i = v_i b_i^\top \text{ and } \Sigma = \{A_1, \ldots, A_{p^*}\},$$

we have

$$\begin{aligned} \forall i,\ A_i F_i &= A_i(\mathcal{B} \cap \{x : b_i^\top x = \pm 1\}) \\ &\subseteq A_i\{x : b_i^\top x = \pm 1\} \\ &= \{A_i x : b_i^\top x = \pm 1\} \\ &= \{v_i b_i^\top x : b_i^\top x = \pm 1\} \\ &= \{\pm v_i\} \\ &\subseteq O_{(i \bmod p^*)+1}. \end{aligned} \tag{1}$$

We have as well

$$\begin{aligned} \forall i,\ A_i(\mathcal{B} \backslash F_i) &= A_i(\mathcal{B} \cap \{x : -1 < b_i^\top x < 1\}) \\ &\subseteq \{v_i b_i^\top x : -1 < b_i^\top x < 1\} \\ &= \{\lambda v_i : -1 < \lambda < 1\} \\ &\subseteq \{\lambda y : -1 < \lambda < 1,\ y \in \mathcal{B}\} \\ &= \operatorname{int}(\mathcal{B}). \end{aligned} \tag{2}$$

By (1) and (2), for any $j \neq (i \bmod p^*) + 1$ and any subset S of $\mathcal{B}$,

$$\begin{aligned} A_j A_i S &\subseteq A_j \left(\mathrm{int}(\mathcal{B}) \cup O_{(i \bmod p^*)+1}\right) \\ &= A_j \mathrm{int}(\mathcal{B}) \cup A_j O_{(i \bmod p^*)+1} \subseteq \mathrm{int}(\mathcal{B}). \end{aligned}$$

Therefore,

$$\ldots A_{(h+2 \bmod p^*)+1} A_{(h+1 \bmod p^*)+1} A_{(h \bmod p^*)+1} A_h$$

is the only infinite noncontracting product starting with A_h. For any h, this product has a period of p^* (because the matrices $A_1, \ldots, A_{p^*}$ are all different). We conclude that all infinite periodic products with periods smaller than $m = p^*$ are contracting and the theorem is proven. □

Giving an explicit value to the size of the largest antichain may prove difficult in some cases. However, since a set of double-faces of same dimension always constitute an antichain, the largest antichain has at least $\max_i f_i$ elements, and we have the following lower bound

$$p^* = W(\mathcal{B}) \geq \max_i f_i,$$

where f_i is the number of faces of dimension i. If the equality holds, the exact value of p^* can be known. This is the case when the lattice of double-faces of Q has the *Sperner property*:

Definition 5 (Sperner Property [20])**.** *A graded poset is said to have the Sperner property if the largest antichain is equal to the largest rank level.*

4. Stochastic Matrices

We now investigate sets of stochastic matrices, with respect to which the following seminorm is always nonincreasing

$$\|x\|_{\mathcal{P}} = \frac{1}{2}(\max_i x_i - \min_i x_i).$$

The (polyhedral) unit ball of that seminorm:

$$\mathcal{P} = \left\{ x \;:\; \frac{1}{2}(\max_i x_i - \min_i x_i) \leq 1 \right\},$$

is thus invariant under multiplication by any stochastic matrix.

Example 2. *Suppose one wants to know whether all products made of the following two matrices converge.*

$$A_1 = \begin{pmatrix} .5 & 0 & .5 \\ 1 & 0 & 0 \\ 0 & .5 & .5 \end{pmatrix}, \; A_2 = \begin{pmatrix} 0 & 1 & 0 \\ .5 & 0 & .5 \\ 1 & 0 & 0 \end{pmatrix}.$$

Since the matrices are stochastic, the seminorm $\|.\|_{\mathcal{P}}$ is nonincreasing under multiplication by these matrices, as can be seen in Figure 3. In this section, we

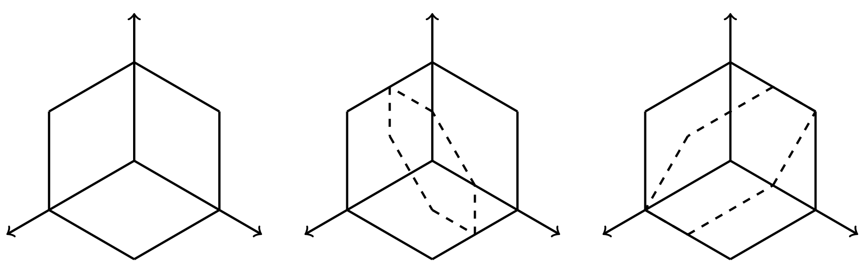

Figure 3: The cross-sections of polyhedra $\mathcal{P}$ (left), $A_1\mathcal{P}$ (center, dashed) and $A_2\mathcal{P}$ (right, dashed). The three polyhedra are infinite in the direction in the direction $\mathbf{1}$.

will see (Theorem 4) that any infinite product of these two matrices converges to a rank one matrix if and only if any infinite periodic product, with period ≤ 3 converges to a rank one matrix.

We prove that the lattice of double-faces of this polyhedron has the Sperner property, allowing us to compute an explicit value for our bound p^*.

Definition 6 (Upper and Lower Shadow [20])**.** *Let $(P, \preceq)$ be a graded poset and let $S \subseteq P$ be such that $\exists k,\ \forall x \in S,\ rank(x) = k$. We call the* upper shadow

$$\nabla(S) = \{x \in P : \exists y \in S,\ y \preceq x,\ rank(x) = k+1\}.$$

Similarly, we define the lower shadow

$$\Delta(S) = \{x \in P : \exists y \in S,\ x \preceq y,\ \mathit{rank}(x) = k-1\}\,.$$

We now describe the structure of the polyhedron $\mathcal{P}$: it has no face of dimension 0 because $\forall x \in \mathcal{P}, a \in \mathbb{R},\ x + a\mathbf{1} \in \mathcal{P}$. The face of dimension n is equal to $\mathcal{P}$ itself. Each double-face of dimension $1 \leq d \leq n-1$ of $\mathcal{P}$ can be written as

$$F = \{x \in \mathcal{P} : \forall i \in S_1, j \in S_2,\ x_i = 2 - x_j\} \tag{3}$$

for some disjoint nonempty sets $S_1, S_2 \subset \{1, \dots, n\}$ with $|S_1 \cup S_2| = n-(d-1)$.

Therefore, the lower shadow of each single double-face F of $\mathcal{P}$ of dimension $2 \leq d \leq n-1$ contains

$$|\Delta(\{F\})| = 2(d-1)$$

elements (the double-faces obtained by adding an element to either S_1 or S_2) and the upper shadow has

$$|\nabla(\{F\})| = n-d+1 \text{ or } |\nabla(\{F\})| = n-d$$

elements (the double-faces obtained by removing an element from either S_1 or S_2, while keeping them both nonempty).

Theorem 3. *The lattice of double-faces of $\mathcal{P}$ has the Sperner property. Its largest antichain is the set of double-faces of dimension $d^* = \lfloor n/3 \rfloor + 1$.*

Proof. Let $S = \{F_1, \dots F_{|S|}\}$ be any set of double-faces of $\mathcal{P}$ of the same dimension d, that is, a subset of a rank level in the lattice of double-faces. Let E_+ be the set of pairs of double-faces of respectively S and $\nabla(S)$ being neighbors to each other:

$$E_+ = \{(F_1, F_2) : F_1 \in S,\ F_2 \in \nabla(\{F_1\})\}.$$

Since the upper shadow of each element of S has at least $n-d$ elements, we have

$$|E_+| \geq |S|(n-d).$$

Since the lower shadow of each element of ∇S contains exactly $2d$ elements – not all of which belonging to S –, we have

$$|E_+| \leq |\nabla(S)|2d.$$

Combining the two inequalities, we obtain $|\nabla(S)| \geq |S|\frac{n-d}{2d}$ and

$$\forall d \leq \frac{n}{3}, \quad |\nabla(S)| \geq |S|. \tag{4}$$

By a similar reasoning, we obtain $|\Delta(S)| \geq |S|\frac{2(d-1)}{n-d+2}$ and

$$\forall d \geq \frac{n+4}{3}, \quad |\Delta(S)| \geq |S|. \tag{5}$$

Let now X be the largest antichain, let d^- be the smallest dimension of an element in $X$and let S^- be the intersection of the antichain with the level d^-. If $d^- \leq \frac{n}{3}$, Equation (4) tells us that the antichain

$$(X \backslash S^-) \cup \nabla(S^-)$$

has at least as many elements as X. We can repeat this process until the antichain contains only faces of dimension strictly larger than $\frac{n}{3}$. Similarly we use (5) to obtain an antichain with at least as many elements of rank strictly smaller than $\frac{n+4}{3}$. Since

$$\frac{n}{3} < d < \frac{n+4}{3}$$

has a unique integer solution $d^* = \lfloor n/3 \rfloor + 1$, the final antichain contains only faces of dimension d^*. □

4.1. A New Finiteness Bound for Consensus

By Theorem 3, the largest antichain in the lattice of double-faces is the set of all double-faces of dimension $d^* = \lfloor n/3 \rfloor + 1$. From Equation (3), one can see that the number of double-faces of dimension d is

$$f_d = \binom{n}{d-1}(2^{n-d} - 1)$$

and the size of the largest antichain is equal to

$$p^* = \binom{n}{\lfloor n/3 \rfloor}(2^{n-\lfloor n/3 \rfloor -1} - 1). \tag{6}$$

Combining this value of p^* with Theorem 1 and [18, Proposition 1.a] yields the next theorem.

Theorem 4. *Let Σ be a set of stochastic matrices. Any left-infinite product of matrices from Σ converges to a rank one matrix if and only if any* periodic *left-infinite product,* with period $\leq p^* = \binom{n}{\lfloor n/3 \rfloor}(2^{n-\lfloor n/3 \rfloor -1} - 1)$*, converges to a rank one matrix.*

As announced in Example 2, if one wants to know if all infinite products of matrices from the set $\{A_1, A_2\}$ converge to a rank one matrix, Theorem 4 implies that it is the case if and only if all infinite products with periods $\leq p^* = \binom{3}{\lfloor 3/3 \rfloor}(2^{3-\lfloor 3/3 \rfloor -1} - 1) = 3$ converge to a rank one matrix.

A finiteness result such as Theorem 4 was known [14, 15] with $B = \frac{1}{2}(3^n - 2^{n+1} + 1)$ instead of p^*. The new value p^* is approximately equal to $\frac{3}{2\sqrt{\pi n}} B$. Moreover, we prove next that Theorem 4 is tight. This is not a consequence of Theorem 2. Indeed, Theorem 2 applied to polyhedron $\mathcal{P}$ guarantees that for any dimension n, there is a set of matrices such that Theorem 1 is tight for $\mathcal{P}$. However, the matrices in this set are not necessarily stochastic.

Theorem 5. *For any $n \geq 2$, there is a set of stochastic matrices such that:*

- *There is a product of length p^* whose powers do not converge to a rank one matrix*
- *For any product P of length $\leq p^* - 1$, the sequence of powers converges to a rank one matrix.*

Proof. We will construct *stochastic* matrices that have the two properties:

$$\forall i,\ A_i F_i \subseteq O_{(i \bmod p^*)+1} \tag{7}$$

$$\forall i,\ A_i(\mathcal{P} \backslash (F_i \cup -F_i)) \subseteq \mathrm{int}(\mathcal{P}). \tag{8}$$

Then the same argument as in the proof of Theorem 2 will allow allow us to conclude. Recall that each face can be written as

$$F = \{x \in \partial\mathcal{P} : \forall i \in S_m,\ x_i = \min_j x_j,\ \forall i \in S_M,\ x_i = \max_j x_j\} \tag{9}$$

for certain disjoint nonempty sets $S_m, S_M \subset \{1, \ldots, n\}$. Let F_i be a face such that $S_{mi} = \{1, \ldots, a_i\}$ and $S_{Mi} = \{n - c_i + 1, \ldots, n\}$ for some a_i and c_i and similarly let $F_j = F_{(i \mod p^*)+1}$ be such that $S_{mj} = \{1, \ldots, a_j\}$ and $S_{Mj} = \{n - c_j + 1, \ldots, n\}$ for some a_i and c_i. Let $b_i = n - a_i - c_i$ and $b_j = n - a_j - c_j$. One matrix satisfying properties (7) and (8) is

$$A_i = \begin{pmatrix} +_{a_j \times a_i} & 0 & 0 \\ +_{b_j \times a_i} & +_{b_j \times b_i} & +_{b_j \times c_i} \\ 0 & 0 & +_{c_j \times c_i} \end{pmatrix}$$

where $+$ represents a positive element chosen such that the sum of the elements on each row sum to one. Let us see why property (7) is satisfied. Let $x \in F_i$, we have that the first a_j elements of $A_i x$ are weighted averages of the first a_i elements of x and therefore they are equal to $\min_k x_k$. Similarly, the last c_j elements of $A_i x$ are weighted averages of the last c_i elements of x and therefore, they are equal to $\max_k x_k$. The remaining elements are weighted averages of all elements of x and therefore they are strictly smaller than $\max_k x_k$ and strictly larger than $\min_k x_k$. These three facts imply $A_i x \in O_j$ and since it is the case for any $x \in F_i$, property (7) is satisfied. Property (8) is proved in a similar manner.

Without the assumption on the specific form of the faces F_i and F_j, the matrix A_i is the same up to some permutations of the rows and of the columns. $\square$

Conclusion

Deciding the asymptotic convergence of long matrix products has various applications in engineering and computer science [5, 6]. In this paper, we have studied this problem for the case where the given set of matrices admits a non-increasing polyhedral seminorm, and one wonders whether all long products of these matrices map the state space onto points whose seminorm is equal to zero (the so-called consensus problem is a particular case of this setting). We have significantly improved the available bound by leveraging the combinatorial structure of (an abstraction of) the dynamical system described by these matrices.

We see several further directions for our work: a major tool in our analysis is Lemma 1, derived from Lagarias and Wang's work. In [1], they also provide a similar result when the invariant set is not a polyhedron, but has a more involved algebraic structure (namely, piecewise analytic). We believe that our analysis could be further applied to piecewise analytic seminorms, but it is not clear whether there would be particular relevant applications in that setting.

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