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Evaluation and default time for companies with uncertain cash flows

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ABSTRACT

In this study, we propose a modelling framework for evaluating companies financed by random liabilities, such as insurance companies or commercial banks. In this approach, earnings and costs are driven by double exponential jump–diffusion processes and bankruptcy is declared when the income falls below a default threshold, which is proportional to the charges. A change of numeraire, under the Esscher risk neutral measure, is used to reduce the dimension. A closed form expression for the value of equity is obtained in terms of the expected present value operators, with and without disinvestment delay. In both cases, we determine the default threshold that maximizes the shareholder's equity. Subsequently, the probabilities of default are obtained by inverting the Laplace transform of the bankruptcy time. In numerical applications of the proposed model, we apply a procedure for calibration based on market and accounting data to explain the behaviour of shares for two real-world examples of insurance companies.

1. Introduction

Evaluations of companies and determining the optimal stopping time for an activity are both central issues in corporate finance. Leland (1994) and Leland and Toft (1996) investigated these topics for a company that maintains a constant debt profile and by adjusting the criteria for bankruptcy endogenously to maximize the value of the equity. They showed that a company's value depends greatly on its capital structure. This approach is related closely to the structural models of Merton (1974) and Black and Cox (1976), where default occurs when the assets first fall below a threshold. Other studies, including Longstaff and Schwartz (1995) and Collin-Dufresne and Goldstein (2001), used stochastic interest rates in their models. Duffie and Lando (2001) and Jarrow and Protter (2004) showed that structural models under incomplete information can be viewed as intensity models, which are competing approaches for default risk. Hilberink and Rogers (2002) extended the framework of Leland and Toft (1996) by including jumps in the dynamics of the assets. Similar models were considered by Le Courtois and Quittard-Pinon (2006) and by Dao and Jeanblanc (2012) where the assets return was driven by jump–diffusion. Le Courtois and Quittard-Pinon (2008) later employed α-stable processes. Boyarchenko and Levendorskii (2007) also developed a general method based on expected present value operators to optimize the entry or exit times in a non-Brownian setting.

However, models that assume a constant profile for debts, such as those considered in most of the previous studies mentioned above, are not applicable to companies financed by uncertain liabilities, including insurance companies or banks. In exchange for premiums, insurers commit to compensating their customers for claims but their charges can be very volatile. Insurance companies also invest temporary premiums in financial assets that are exposed to the fluctuations of the market. Banks also experience a similar asset–liability risk:return on investments, and the related cost of funding is both uncertain and subject to serious perturbations. Furthermore, an empirical analysis by Eom et al. (2004) emphasized that Brownian structural models systematically underestimate credit spreads. Motivated by these observations, we propose a model where earnings and company charges are driven by double exponential jump–diffusion processes (DEJDs). These modelling processes, which were used for option pricing by Lipton (2002) and by Kou and Wang (2003, 2004), can replicate the sudden and extreme shocks caused by major insurance claims, as well as credit losses or crises on financial markets. The current study is a continuation of research by Saa-Requejo and Santa-Clara (1999) and by Gerber and Shiu (1996), except a jump–diffusion framework is applied.

Because the market related to these companies is incomplete in nature, several equivalent risk neutral measures exist for evaluating stock. In the present study, this evaluation is performed under the Esscher risk neutral measure. This method is popular in the field of actuarial sciences and it was promoted by Gerber and Shiu (1994) for appraising liabilities. It provides a general, transparent

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and unambiguous framework that preserves the fundamental features of the processes that rule assets and liabilities. Using this methodology, company equity is valued as the integral of the expected cash flows until default, which is discounted at the risk-free rate. Bankruptcy is assumed to occur when the income first falls below a certain fraction, which is called the "default threshold" of charges.

In this study, we first use the technique of a change of numeraire to reduce the number of state variables, as suggested by Margrabe (1978) and by Gerber and Shiu (1996) for pricing exchange options. The equity value is then expressed in terms of expected present value operators, as described by Boyarchenko and Levendorskii (2007). These operators, which are closely related to Wiener–Hopf factorization, comprise an elegant method for solving the problems of timing. Furthermore, it is possible to obtain a closed form expression for the value of equity and for the default threshold that maximizes the equity value. We also analyse the impact on stock prices of a random delay between the decision to declare bankruptcy and the actual closure of the company, and we propose an analytical formula for the Laplace transform of the default time (with and without disinvestment delay). We provide numerical illustrations, which include a procedure for calibration based on stock prices and accounting information. The proposed method is illustrated by studying the movements of the share prices of two insurance companies: Generali and Axa.

2. The proposed company model

The set of companies considered in this study are assumed to receive a continuous income, which is denoted by a_t, from their assets and they pay continuous charges, denoted by l_t, for their liabilities. The growth rates of these cash flows are modelled by two DEJDs, (X_t^A, X_t^L), on a probability space $(\Omega, \mathcal{F}, \{\mathcal{F}\}_t, P)$. Their dynamics are governed by the following SDEs:

$$dX_t^A = \mu_A dt + \sigma_A dW_t^A + Y^A dN_t^A \tag{2.1}$$

$$dX_t^L = \mu_L dt + \sigma_{AL} dW_t^A + \sigma_L dW_t^L + Y^L dN_t^L, \tag{2.2}$$

where $\mu_A, \sigma_A, \mu_L, \sigma_L, \sigma_{AL}$ are constant. N_t^A, N_t^L are Poisson processes with constant intensities, which are denoted by λ_A and λ_L. The initial values of X_t^A and X_t^L are zero. The jumps that hit the income and charges (Y^A and Y^L, respectively) are double-exponential random variables. The density functions of Y^A and Y^L are given by:

$$f_{Y^A}(y) = p_A \eta_A^1 e^{-\eta_A^1 y} 1_{\{y\geq 0\}} + q_A \eta_A^2 e^{\eta_A^2 y} 1_{\{y<0\}},$$

$$f_{Y^L}(y) = p_L \eta_L^1 e^{-\eta_L^1 y} 1_{\{y\geq 0\}} + q_L \eta_L^2 e^{\eta_L^2 y} 1_{\{y<0\}},$$

where p_A, q_A, η_A^1, η_A^2, p_L, q_L, η_L^1, η_L^2 are positive constants. The parameters $p_{A,L}$ and $q_{A,L}$ satisfy the relation: $p_{A,L} + q_{A,L} = 1$, and they represent the probability of observing upward and downward exponential jumps, respectively. The expectations for $Y^{A,L}$ under P are:

$$\mathbb{E}^P(Y^A) = p_A \frac{1}{\eta_A^1} - q_A \frac{1}{\eta_A^2},$$

$$\mathbb{E}^P(Y^L) = p_L \frac{1}{\eta_L^1} - q_L \frac{1}{\eta_L^2}.$$

Furthermore, η_L^1 is assumed to be greater than one ($\eta_L^1 > 1$). This assumption means that the positive jumps that hit the growth rates of the liabilities are less than 100% on average. The reasons for this assumption are explained in Section 5. Thus, the cash flows for incomes and charges are:

$$a_t = a_0 e^{X_t^A} = a_0 e^{\mu_A t + \sigma_A W_t^A + \sum_{j=1}^{N_t^A} Y_j^A}, \tag{2.3}$$

$$l_t = l_0 e^{X_t^L} = l_0 e^{\mu_L t + \sigma_{AL} W_t^A + \sigma_L W_t^L + \sum_{j=1}^{N_t^L} Y_j^L} \tag{2.4}$$

which have the following geometric dynamics:

$$\frac{da_t}{a_t} = \left(\mu_A + \frac{1}{2}\sigma_A^2\right) dt + \sigma_A dW_t^A + \left(e^{Y^A} - 1\right) dN_t^A$$

$$\frac{dl_t}{l_t} = \left(\mu_L + \frac{1}{2}\sigma_L^2 + \frac{1}{2}\sigma_{AL}^2\right) dt + \sigma_{AL} dW_t^A + \sigma_L dW_t^L + \left(e^{Y^L} - 1\right) dN_t^L.$$

In the proposed model, the Laplace transforms and characteristic exponents of X_t^A and X_t^L are needed. Given that jumps are independent of diffusion, the Laplace transforms are the products of diffusion and the jumps transforms. The Laplace functions of the jump processes are given by the following expressions (Schreve, 2004, p. 486):

$$\mathbb{E}^P\left(\exp\left(z\sum_{j=1}^{N_t^A} Y_j^A\right)\right) = \exp\left(\lambda_A t\left(\phi_{Y^A}(z) - 1\right)\right)$$

$$\mathbb{E}^P\left(\exp\left(z\sum_{j=1}^{N_t^L} Y_j^L\right)\right) = \exp\left(\lambda_L t\left(\phi_{Y^L}(z) - 1\right)\right),$$

where $\phi_{Y^A}(u)$ and $\phi_{Y^L}(u)$ are the Laplace functions of Y^A and Y^L:

$$\phi_{Y^A}(z) = p_A \frac{\eta_A^1}{\eta_A^1 - z} + q_A \frac{\eta_A^2}{\eta_A^2 + z}$$

$$\phi_{Y^L}(z) = p_L \frac{\eta_L^1}{\eta_L^1 - z} + q_L \frac{\eta_L^2}{\eta_L^2 + z}.$$

In addition, the Laplace transforms of X_t^A and X_t^L are defined in terms of their related characteristic exponents $\psi^A(z)$ and $\psi^L(z)$, as follows: $\mathbb{E}^P\left(e^{zX_t^{A,L}}\right) = e^{t\psi^{A,L}(z)}$, where the values of $\psi^{A,L}(z)$ are such that:

$$\psi^A(z) = \mu_A z + \frac{1}{2}z^2\sigma_A^2 + \lambda_A\left(\phi_{Y^A}(z) - 1\right) \tag{2.5}$$

$$\psi^L(z) = \mu_L z + \frac{1}{2}z^2\sigma_{AL}^2 + \frac{1}{2}z^2\sigma_L^2 + \lambda_L\left(\phi_{Y^L}(z) - 1\right). \tag{2.6}$$

3. Evaluation of the equity under the Esscher measure

For all companies with shares that are traded in financial markets, the evaluation has to be performed under a risk neutral measure to avoid any arbitrage. However, the market is incomplete in nature, so there is no unique risk neutral measure. Instead, many suitable equivalent measures can be suggested, including those based on distance minimization such as the relative entropy or Kullback–Leibler distance. However, we prefer to use the Esscher Risk Neutral measure. This measure, which was recommended by Gerber and Shiu (1994), provides a general, transparent and unambiguous framework for evaluation. The Esscher risk neutral measure, denoted by Q, is defined by two parameters $k := (k_A, k_L)$ and its Radon Nikodym density is equal to:

$$\left(\frac{dQ}{dP}\right)_t = \frac{e^{k_A X_t^A + k_L X_t^L}}{\mathbb{E}^P\left(e^{k_A X_t^A + k_L X_t^L}\right)}.$$

By construction, $\left(\frac{dQ}{dP}\right)_t$ is a martingale under P. Furthermore, the sum $k_A X_t^A + k_L X_t^L$ is equal to

$$k_A X_t^A + k_L X_t^L = (k_A\mu_A + k_L\mu_L)\, t + (k_A\sigma_A + k_L\sigma_{AL})\, W_t^A + k_L\sigma_L W_t^L + k_A\sum_{j=1}^{N_t^A} Y_j^A + k_L\sum_{j=1}^{N_t^L} Y_j^L$$

and its Laplace transform, under the real measure, is defined as follows

$$\mathbb{E}^P\left(e^{z\left(k_A X_t^A + k_L X_t^L\right)}\right) = e^{t\,\psi^{k_A,k_L}(z)}$$

where $\psi^{k_A k_L}(z)$ is the characteristic exponent:

$$\begin{aligned}\psi^{k_A k_L}(z) &= (k_A\mu_A + k_L\mu_L)\,z \\ &\quad + \frac{1}{2}z^2\left(k_L^2\sigma_L^2 + (k_A\sigma_A + k_L\sigma_{AL})^2\right) \\ &\quad + \lambda_A\left(\phi_{Y^A}(zk_A) - 1\right) + \lambda_L\left(\phi_{Y^L}(zk_L) - 1\right).\end{aligned}$$

If the cash flows of assets and liabilities come from traded securities, their market values are denoted by A_t and L_t, respectively. Their prices are equal to the expected discounted value of the cash flows under the Esscher measure, which are given by the following proposition.

Proposition 3.1. *The market values of the incomes and charges cash flows, A_t and L_t, are equal to*

$$A_t = \frac{1}{r + \psi^{k_A k_L}(1) - \psi^{1+k_A,k_L}(1)}a_t \tag{3.1}$$

$$L_t = \frac{1}{r + \psi^{k_A k_L}(1) - \psi^{k_A,1+k_L}(1)}l_t \tag{3.2}$$

under the constraints that $r + \psi^{k_A k_L}(1) - \psi^{1+k_A,k_L}(1)$ and $r + \psi^{k_A k_L}(1) - \psi^{k_A,1+k_L}(1)$ are strictly positive.

Proof. These results are obtained by direct integration. Thus, for income cash flows [A1], it can be shown directly that

$$\begin{aligned}A_t &= \mathbb{E}^Q\left(\int_t^\infty e^{-r(s-t)}a_s ds \mid \mathcal{F}_t\right) \\ &= \int_t^\infty e^{-r(s-t)}a_t \frac{\mathbb{E}^P\left(e^{(1+k_A)X_{s-t}^A + k_L X_{s-t}^L} \mid \mathcal{F}_t\right)}{e^{(s-t)\,\psi^{k_A k_L}(1)}}ds \\ &= \int_t^\infty e^{-(s-t)\left(r+\psi^{k_A k_L}(1)-\psi^{1+k_A,k_L}(1)\right)}a_t ds. \quad\square\end{aligned}$$

A_t and L_t are comparable to the market values of the assets and liabilities. They are valued as perpetual annuities, which pay an increasing cash flow, and they are discounted at the risk-free rate. This finding reflects that given by Gordon and Shapiro (first described by Gordon and Myron (1959)). We can also check that under the risk neutral measure, the prices of assets or liabilities are indeed martingales, e.g., $A_0 = \mathbb{E}^Q\left(e^{-rt}A_t + \int_0^t e^{-rs}a_s ds \mid \mathcal{F}_0\right)$. This ensures that the market is arbitrage-free. If the market values of assets and liabilities are available, the values of k_A and k_L satisfy Eqs. (3.1) and (3.2). In the numerical application, k_L is assumed to be null whereas k_A is fitted to best explain the history of the share prices. We assume that $k_L = 0$ is equivalent to considering that liabilities have the same behaviour under the real or risk neutral measures. This assumption is common among actuaries, e.g., for calculating the net asset value [A2], as defined by the Solvency II regulation. To conclude this section, the next proposition shows that the dynamics of a_t and l_t are preserved under the Esscher measure. Le Courtois and Quittard-Pinon (2006) obtained a similar result for one JEDC [A3] process.

Proposition 3.2. *When evaluating under the Esscher measure, a_t and l_t are still DEJD processes with the following parameters.*

Proof. The Laplace transforms of X_t^A and X_t^L developed by applying the risk neutral measure are transforms of DJED processes obtained using the parameters provided in this proposition, e.g.,

$$\begin{aligned}\mathbb{E}^Q\left(e^{zX_t^L}\right) &= \mathbb{E}^P\left(\frac{e^{k_A X_t^A + (z+k_L)X_t^L}}{\mathbb{E}^P\left(e^{k_A X_t^A + k_L X_t^L}\right)}\right) \\ &= e^{t\left(\psi^{k_A,z+,k_L}(1) - \psi^{k_A,k_L}(1)\right)}\end{aligned}$$

and direct calculations lead to the following equality

$$e^{t\left(\psi^{k_A,z+,k_L}(1) - \psi^{k_A,k_L}(1)\right)} = e^{t\psi^{L,Q}(z)},$$

where $\psi^{L,Q}(z)$ is provided by expression (2.6), in which the parameters are replaced by those given in Table 3.1. $\square$

Table 3.1
DEJD parameters under Q.

μ_A^Q	$\mu_A + \sigma_A\,(k_A\sigma_A + k_L\sigma_{AL})$	μ_L^Q	$\mu_L + \left(k_L\sigma_L^2 + k_A\sigma_A\sigma_{AL} + k_L\sigma_{AL}^2\right)$
σ_A^Q	σ_A	σ_L^Q	$\sqrt{\sigma_{AL}^2 + \sigma_L^2}$
η_A^{1Q}	$\eta_A^1 - k_A$	η_L^{1Q}	$\eta_L^1 - k_L$
η_A^{2Q}	$\eta_A^2 + k_A$	η_L^{2Q}	$\eta_L^2 + k_L$
p_A^Q	$p_A\frac{\eta_A^1}{\xi_A\left(\eta_A^1 - k_A\right)}$	p_L^Q	$p_L\frac{\eta_L^1}{\xi_L\left(\eta_L^1 - k_L\right)}$
λ_A^Q	$\lambda_A\xi_A$	λ_L^Q	$\lambda_L\xi_L$
ξ_A	$p_A\frac{\eta_A^1}{\eta_A^1 - k_A} + q_A\frac{\eta_A^2}{\eta_A^2 + k_A}$	ξ_L	$p_L\frac{\eta_L^1}{\eta_L^1 - k_L} + q_L\frac{\eta_L^2}{\eta_L^2 + k_L}$

It should be noted that by construction, the dynamics of L_t are equal to

$$\begin{aligned}dL_t &= L_t\left(\mu_L^Q + \frac{1}{2}\sigma_L^{Q\,2} + \frac{1}{2}\sigma_{AL}^{Q\,2}\right)dt + L_t\sigma_{AL}^Q dW_t^{A,Q} \\ &\quad + L_t\sigma_L^Q dW_t^{L,Q} + L_t\left(e^{Y^{L,Q}} - 1\right)dN_t^{L,Q}\end{aligned}$$

and according to Proposition 3.1, the expected growth rate of liabilities is lower than the risk-free rate

$$\begin{aligned}\mathbb{E}\left(\frac{dL_t}{L_t} \mid \mathcal{F}_t\right) &= \left(\mu_L^Q + \frac{1}{2}\sigma_L^{Q\,2} + \frac{1}{2}\sigma_{AL}^{Q\,2} + \lambda^Q(\phi_{Y^A}^Q(1) - 1)\right)dt \\ &= \left(\psi^{k_A,1+k_L}(1) - \psi^{k_A k_L}(1)\right)dt < r\,dt.\end{aligned} \tag{3.3}$$

If the latter condition is not satisfied, the market value of L_t would be infinite. Because all of the following developments are performed under the risk neutral measure, the index Q in the terms μ_A^Q, σ_A^Q, η_A^{1Q}, η_A^{2Q}, p_A^Q, λ_A^Q, μ_L^Q, σ_L^Q, η_L^{1Q}, η_L^{2Q}, p_L^Q and λ_L^Q is omitted intentionally to simplify the notations. The characteristic exponents of X_t^A and X_t^L under the risk neutral measure are provided by expressions (2.5) and (2.6), where the parameters are replaced by their equivalents under Q. Thus, they are denoted by $\psi^A(z)$ and $\psi^L(z)$ in the following sections.

Now, having defined the risk neutral measure that we apply, the market value of the company's equity is the sum of the future expected cash flows, which are discounted at the risk-free rate until default. The time to default is denoted by τ, which is the stopping time on the filtration $\mathcal{F}_t$. In the first example, it is assumed that the company is closed immediately after the decision to declare bankruptcy. In the next section, we assume that there is a delay between filing for bankruptcy and the effective cessation of activity. If shareholders intend to maximize the market value of their investment, the value of the equity (denoted by V_t at time t) is equal to:

$$V_t(a_t, l_t) = \max_\tau \mathbb{E}^Q\left(\int_t^\tau e^{-r(s-t)}\,(a_s - l_s)\,ds \mid \mathcal{F}_t\right). \tag{3.4}$$

This can also be rewritten as a function of the market value of the investments and liabilities. From the definition of A_t and L_t, it can be stated that:

$$V(A_t, L_t) = (A_t - L_t) + \max_\tau \mathbb{E}^Q\left(-e^{-r(\tau-t)}(A_\tau - L_\tau) \mid \mathcal{F}_t\right)$$
$$= (A_t - L_t) - \min_\tau \mathbb{E}^Q\left(e^{-r(\tau-t)}(A_\tau - L_\tau) \mid \mathcal{F}_t\right). \tag{3.5}$$

The latter equation emphasizes that valuing the equity using this modelling framework is equivalent to assessing a perpetual exchange option between assets and liabilities and determining the stopping time that minimizes this option, which is referred to as the "option to default". In the remainder of this study, all of the model developments that follow from Eq. (3.4) are used to appraise the company equity. However, the dimension of this stopping time problem is first reduced by a change of numeraire.

4. Reducing the number of state variables

The time to default, which is denoted by τ, is a stopping time on the filtration $\mathcal{F}_t$ and it is decided by shareholders when the income falls below a predetermined level. This threshold is a percentage of charges denoted by h and bankruptcy is declared when a_t falls below the threshold, denoted by $h\,l_t$. By applying the definitions of assets and liabilities, the latter is equivalent to assuming that the default is triggered when A_t falls below the following minimum:

$$A_t \le h\left(\frac{r + \psi^{k_A k_L}(1) - \psi^{k_A, 1+k_L}(1)}{r + \psi^{k_A k_L}(1) - \psi^{1+k_A, k_L}(1)}\right) L_t.$$

This equation emphasizes the relationship between the approach proposed in the present study and other structural models. At this point, h is assumed to be known and thus τ is defined as $\inf\{t \mid a_t \le h\,l_t\, t \ge 0\}$. The value of the equity for a given threshold, h, is given by

$$V_t^h(a_t, l_t) = \mathbb{E}^Q\left(\int_t^\tau e^{-r(s-t)}\,(a_s - l_s)\,ds \mid \mathcal{F}_t\right). \tag{4.1}$$

In order to assess this expectation, the number of state variables must be reduced, thereby implying a transformation of measure. Let us denote $\tilde{Q}$ as a new measure of probability, which is defined by the following change of numeraire:

$$\left.\frac{d\tilde{Q}}{dQ}\right|_t = e^{-\underbrace{\left(\mu_L + \frac{1}{2}\sigma_{AL}^2 + \frac{1}{2}\sigma_L^2 + \lambda_L\left(\phi_{Y^L}(1) - 1\right)\right)}_{\delta} t}\,\frac{l_t}{l_0}$$
$$= e^{-\delta t}\frac{l_t}{l_0}, \tag{4.2}$$

where $\delta = \psi^L(1)$ is the growth rate of the average liabilities such that $\mathbb{E}^Q(l_t \mid \mathcal{F}_0) = l_0 e^{\delta t}$. As explained previously, by construction, the risk-free rate r will be greater than δ (see Eq. (3.3)). If this were not the case, the market value of the liabilities would be infinite. By definition, the Radon Nikodym derivative (4.2) is a martingale under Q and its expectation is equal to 1. Furthermore, for any fixed time T, we have

$$\mathbb{E}^{\tilde{Q}}\left(\int_t^T e^{-(r-\delta)(s-t)}\left(\frac{a_s}{l_s} - 1\right) ds \mid \mathcal{F}_t\right)$$
$$= \int_t^T \mathbb{E}^{\tilde{Q}}\left[e^{-(r-\delta)(s-t)}\left(\frac{a_s}{l_s} - 1\right) \mid \mathcal{F}_t\right] ds$$
$$= \int_t^T \frac{\mathbb{E}^Q\left(e^{-(r-\delta)(s-t)}\frac{e^{-\delta s}}{l_0}(a_s - l_s) | \mathcal{F}_t\right)}{\mathbb{E}^Q\left(\left.\frac{d\tilde{Q}}{dQ}\right|_s | \mathcal{F}_t\right)} ds$$
$$= \frac{1}{l_t}\mathbb{E}^Q\left(\int_t^T e^{-r(s-t)}(a_s - l_s) ds \mid \mathcal{F}_t\right).$$

By applying the optional stopping theorem, the value of the equity defined by Eq. (4.1) can be reformulated as follows:

$$V_t^h(a_t, l_t) = l_t \mathbb{E}^{\tilde{Q}}\left(\int_t^\tau e^{-(r-\delta)(s-t)}\left(\frac{a_s}{l_s} - 1\right) ds \mid \mathcal{F}_t\right). \tag{4.3}$$

Before determining this expectation, we specify and explain the dynamics of $\frac{a_s}{l_s}$, under both the original and the risk neutral measures (Q and $\tilde{Q}$). For this purpose, we define the process X_t^S, which is equal to the difference between X_t^A and X_t^L. The ratio $\frac{a_s}{l_s}$ for $s \ge t$ is reformulated as a function of this process, $\frac{a_s}{l_s} = \frac{a_t}{l_t} e^{X_{s-t}^S}$, where

$$X_{s-t}^S = (\mu_A - \mu_L)(s - t) + (\sigma_A - \sigma_{AL}) W_{s-t}^A - \sigma_L W_{s-t}^L$$
$$+ \sum_{j=1}^{N_{s-t}^A} Y_j^A - \sum_{j=1}^{N_{s-t}^L} Y_j^L \tag{4.4}$$

and with the initial value $X_0^S = 0$. The Laplace transform of X_{s-t}^S under the risk neutral measure Q is such that

$$\mathbb{E}^Q\left(e^{zX_{s-t}^S} \mid \mathcal{F}_t\right) = e^{(s-t)\psi^S(z)}$$

with the following characteristic exponent:

$$\psi^S(z) = (\mu_A - \mu_L) z + \frac{1}{2}z^2\left((\sigma_A - \sigma_{AL})^2 + \sigma_L^2\right)$$
$$+ \lambda_A\left(\phi_{Y^A}(z) - 1\right) + \lambda_L\left(\phi_{Y^L}(-z) - 1\right). \tag{4.5}$$

The Laplace transform of X_{s-t}^S under $\tilde{Q}$ and its exponent $\psi^{\tilde{S}}(z)$ are obtained by a change of measure:

$$\mathbb{E}^{\tilde{Q}}\left(e^{zX_{s-t}^S} \mid \mathcal{F}_t\right) = \mathbb{E}^Q\left(e^{-\delta(s-t)}\frac{l_s}{l_t}e^{zX_{s-t}^S} \mid \mathcal{F}_t\right)$$
$$= e^{(s-t)\psi^{\tilde{S}}(z)},$$

where

$$\psi^{\tilde{S}}(z) = [(\mu_L - \delta) + z(\mu_A - \mu_L)] + \frac{1}{2}[\sigma_{AL} + z(\sigma_A - \sigma_{AL})]^2$$
$$+ \frac{1}{2}(\sigma_L - z\sigma_L)^2 + \lambda_A\left(\phi_{Y^A}(z) - 1\right)$$
$$+ \lambda_L\left(\phi_{Y^L}(1 - z) - 1\right). \tag{4.6}$$

We use this result to determine the Wiener–Hopf factorization of X_t^S when using the new measure $\tilde{Q}$.

5. Wiener–Hopf factorization under $\tilde{Q}$

In this section, we review the basic features of the Wiener–Hopf factorization and of the expected present value operators. Under the condition that $r - \delta > 0$, the following result is obtained by direct integration:

$$\mathbb{E}^{\tilde{Q}}\left(\int_t^\infty e^{-(r-\delta)(s-t)}e^{zX_{s-t}^S}ds \mid \mathcal{F}_t\right) = \frac{1}{(r-\delta) - \psi^{\tilde{S}}(z)}. \tag{5.1}$$

However, if a random exponential time Γ is introduced with an intensity equal to $r - \delta$, we obtain the following Wiener–Hopf factorization for the left-hand side of Eq. (5.1):

$$(r-\delta)\mathbb{E}^{\tilde{Q}}\left(\int_t^\infty e^{-(r-\delta)(s-t)}e^{zX_{s-t}^S}ds \mid \mathcal{F}_t\right)$$
$$= \mathbb{E}^{\tilde{Q}}\left(e^{zX_{t+\Gamma}^S} \mid \mathcal{F}_t\right)$$
$$= \mathbb{E}^{\tilde{Q}}\left(e^{z\bar{X}_{t+\Gamma}^S} \mid \mathcal{F}_t\right)\mathbb{E}^{\tilde{Q}}\left(e^{z\underline{X}_{t+\Gamma}^S} \mid \mathcal{F}_t\right)$$
$$:= \kappa_{(r-\delta)}^+(z)\kappa_{(r-\delta)}^-(z), \tag{5.2}$$

where $\bar{X}^S_{t+\Gamma}$ and $\underline{X}^S_{t+\Gamma}$ are the maximum and minimum, respectively, of the process X^S_{s-t} on the time interval $[t, t+\Gamma]$. This relation is derived from the fact that $X^S_T = \bar{X}^S_t + X^S_T - \bar{X}^S_t$. Because $\bar{X}^S_t$ and $X^S_T - \bar{X}^S_t$ are mutually independent and $X^S_T - \bar{X}^S_t$ has a similar distribution to $\underline{X}^S_t$, then Eq. (5.2) is deduced. The remaining calculations are based on the expected present value (EPV) operators described by Boyarchenko and Levendorskii (2007, see Chapter 11). For any function $g(.)$ defined on $\mathbb{C}$, three EPV operators are defined as follows:

$$
\begin{aligned}
(\mathcal{E}g)(x) &= (r-\delta)\mathbb{E}^{\tilde{Q}}\left(\int_t^\infty e^{-(r-\delta)(s-t)} g(x + X^S_{s-t})\, ds\right) \\
\left(\mathcal{E}^+ g\right)(x) &= (r-\delta)\mathbb{E}^{\tilde{Q}}\left(\int_t^\infty e^{-(r-\delta)(s-t)} g(x + \bar{X}^S_{s-t})\, ds\right) \qquad (5.3)\\
\left(\mathcal{E}^- g\right)(x) &= (r-\delta)\mathbb{E}^{\tilde{Q}}\left(\int_t^\infty e^{-(r-\delta)(s-t)} g(x + \underline{X}^S_{s-t}) ds\right).
\end{aligned}
$$

The Wiener–Hopf factors $\kappa^+_{(r-\delta)}(z)$ and $\kappa^-_{(r-\delta)}(z)$ given above (Eq. (5.2)) are closely related to EPV operators. Indeed, if $g(.) = e^{z\cdot}$,

$$
\begin{aligned}
\left(\mathcal{E}e^{z\cdot}\right)(x) &= \frac{(r-\delta)}{(r-\delta) - \psi^{\tilde{S}}(z)} e^{zx} \\
\left(\mathcal{E}^+ e^{z\cdot}\right)(x) &= (r-\delta)e^{zx}\kappa^+_{(r-\delta)}(z) \qquad (5.4)\\
\left(\mathcal{E}^- e^{z\cdot}\right)(x) &= (r-\delta)e^{zx}\kappa^-_{(r-\delta)}(z)
\end{aligned}
$$

and the relationships given by Eqs. (5.1) and (5.2) lead to $(\mathcal{E}e^{z\cdot}) = \left(\mathcal{E}^+ e^{z\cdot}\right)\left(\mathcal{E}^- e^{z\cdot}\right)$. Boyarchenko and Levendorskii extended this result to cover all functions $g \in \mathcal{L}_\infty(\mathbb{R})$. $\mathcal{E}$ is also the inverse of the operator $(r-\delta)^{-1}\left((r-\delta) - \mathcal{L}\right)$, where $\mathcal{L}$ is the infinitesimal generator of the process X^S_t. Furthermore, $\mathcal{E}^{-1} = \left(\mathcal{E}^+\right)^{-1}\left(\mathcal{E}^-\right)^{-1}$ or $\mathcal{E}^{-1} = \left(\mathcal{E}^-\right)^{-1}\left(\mathcal{E}^+\right)^{-1}$. These properties are used in further developments of our model to evaluate the equity of the company. In general, these Wiener–Hopf factors do not have closed form formulae, excepted for DEJD processes. Boyarchenko and Levendorskii (2007, Lemma 11.2.1 p. 197) showed that $\psi^{\tilde{S}}(z) - (r-\delta)$ is the ratio of the two polynomials $P(z)$ and $Q(z)$,

$$
\psi^{\tilde{S}}(z) - (r-\delta) = \frac{P(z)}{Q(z)}, \qquad (5.5)
$$

where the numerator $P(z)$ in this case is a polynomial of degree 6,

$$
\begin{aligned}
P(z) = &\left[\mu_L - r - \lambda_A - \lambda_L + \frac{1}{2}\sigma^2_{AL} + \frac{1}{2}\sigma^2_L\right] Q(z) \\
&+ z\left[(\mu_A - \mu_L) + \sigma_{AL}(\sigma_A - \sigma_{AL}) - \sigma^2_L\right] Q(z) \\
&+ z^2\left[\frac{1}{2}(\sigma_A - \sigma_{AL})^2 + \frac{1}{2}\sigma^2_L\right] Q(z) \\
&+ \lambda_A\left[p_A\eta^1_A\left(\eta^2_A + z\right) + q_A\eta^2_A\left(\eta^1_A - z\right)\right] \\
&\times \left(\eta^1_L - 1 + z\right)\left(\eta^2_L + 1 - z\right) \\
&+ \lambda_L\left[p_L\eta^1_L\left(\eta^2_L + 1 - z\right)\right. \\
&\left. + \ q_L\eta^2_L\left(\eta^1_L - 1 + z\right)\right]\left(\eta^1_A - z\right)\left(\eta^2_A + z\right),
\end{aligned}
$$

whereas the denominator $Q(z)$ is the product:

$$
Q(z) = \left(\eta^1_A - z\right)\left(\eta^2_A + z\right)\left(\eta^1_L - 1 + z\right)\left(\eta^2_L + 1 - z\right).
$$

Analysing the variation shows that the ratio $(P/Q)(z)$ has four asymptotes, which are the roots of $Q(z)$. Two are found in the left half-plane and the two others are in the right half-plane (under the condition stated earlier that $\eta^1_L > 1$). Furthermore, $P(z)$ reaches a maximum at around zero and $\psi^{\tilde{S}}(z) \to \infty$ as $z \to \pm\infty$. Thus, $P(z)$ crosses the zero axis six times and it has three positive and three negative roots, which are denoted by β^+_k and β^-_k, $k = 1, 2, 3$. The two positive and two negative roots of $Q(z)$ are denoted by λ^+_j and λ^-_j, respectively, $j = 1, 2$. The roots of $P(z)$ and $Q(z)$ follow the order,

$$
\beta^-_3 < \lambda^-_2 < \beta^-_2 < \lambda^-_1 < \beta^-_1 < 0 < \beta^+_1 < \lambda^+_1 < \beta^+_2 < \lambda^+_2 < \beta^+_3.
$$

The Wiener–Hopf factors are:

$$
\kappa^+_{(r-\delta)}(z) = \prod_{j=1}^{2}\frac{\lambda^+_j - z}{\lambda^+_j}\prod_{k=1}^{3}\frac{\beta^+_k}{\beta^+_k - z} \qquad (5.6)
$$

$$
\kappa^-_{(r-\delta)}(z) = \prod_{j=1}^{2}\frac{\lambda^-_j - z}{\lambda^-_j}\prod_{k=1}^{3}\frac{\beta^-_k}{\beta^-_k - z}, \qquad (5.7)
$$

which can be restated as the following sums:

$$
\kappa^\pm_{(r-\delta)}(z) = a^\pm_1\frac{\beta^\pm_1}{\beta^\pm_1 - z} + a^\pm_2\frac{\beta^\pm_2}{\beta^\pm_2 - z} + a^\pm_3\frac{\beta^\pm_3}{\beta^\pm_3 - z}, \qquad (5.8)
$$

where

$$
a^\pm_1 = \frac{\beta_2\beta_3}{\lambda_1\lambda_2}\frac{(\beta_1 - \lambda_1)(\beta_1 - \lambda_2)}{(\beta_1 - \beta_2)(\beta_1 - \beta_3)} \qquad (5.9)
$$

$$
a^\pm_2 = \frac{\beta_1\beta_3(\beta_2 - \lambda_1)(\beta_2 - \lambda_2)}{\lambda_1\lambda_2(\beta_2 - \beta_1)(\beta_2 - \beta_3)} \qquad (5.10)
$$

$$
a^\pm_3 = \frac{\beta_1\beta_2}{\lambda_1\lambda_2}\frac{(\beta_3 - \lambda_1)(\beta_3 - \lambda_2)}{(\beta_3 - \beta_2)(\beta_3 - \beta_1)}. \qquad (5.11)
$$

We note that the symbol $\pm$ has been removed from the terms on the RHS to simplify the equation. Boyarchenko and Levendorskii (2007) showed that $\mathcal{E}^+$ and $\mathcal{E}^-$ act on bounded measurable functions $g(.)$ as the following integral operators:

$$
(\mathcal{E}^+ g)(x) = \sum_{j=1}^{3} a^+_j \int_0^{+\infty} \beta^+_j e^{-\beta^+_j y} g(x+y)dy \qquad (5.12)
$$

$$
(\mathcal{E}^- g)(x) = \sum_{j=1}^{3} a^-_j \int_{-\infty}^{0} (-\beta^-_j) e^{-\beta^-_j y} g(x+y)dy. \qquad (5.13)
$$

It is easy to show that this formula is true for all exponential functions of the form $g(x) = e^{zx}$ and for any linear combination of exponential functions. We use expressions (5.12) and (5.13) later to evaluate the equity.

6. Evaluation of the company equity

The value of the equity when the bankruptcy is triggered immediately (if a_s falls below hl_s) is given by Eq. (4.3). This can be rewritten as:

$$
V^h_t(a_t, l_t) = a_t\,\mathbb{E}^{\tilde{Q}}\left(\int_t^\tau e^{-(r-\delta)(s-t)}\left(e^{X^S_{s-t}} - \frac{l_t}{a_t}\right) ds \mid \mathcal{F}_t\right). \qquad (6.1)
$$

It is possible to restate this last expectation in terms of EPV operators in Proposition 6.1, as follows.

Proposition 6.1. *f $X^S_0 = x$, the value of the company equity is equal to:*

$$
V^h_t(x) = a_t\,(r-\delta)^{-1}\left(\mathcal{E}^-\mathbf{1}_{[b,\infty)}\mathcal{E}^+ g\right)(x), \qquad (6.2)
$$

where $b = \ln\left(\frac{hl_t}{a_t}\right)$ and the function $g(.)$ is defined as

$$
g(x) = \left(e^x - \frac{l_t}{a_t}\right). \qquad (6.3)
$$

Proof. If the infinitesimal generator of X_t^S under $\tilde{Q}$ is denoted by $\mathcal{L}$, the function $\frac{1}{a_t}V_t^h$ is a solution of the following system:

$$\begin{cases} ((r-\delta)-\mathcal{L})\left(\frac{1}{a_t}V_t^h\right)=g(x) & \text{if } x>\ln\left(\frac{hl_t}{a_t}\right) \\ \frac{1}{a_t}V_t^h=0 & \text{if } x\leq \ln\left(\frac{hl_t}{a_t}\right). \end{cases} \tag{6.4}$$

Given that $\mathcal{E}^{-1}=(r-\delta)\left((r-\delta)-\mathcal{L}\right)$, system (6.4) can be rewritten as

$$\mathcal{E}^{-1}\frac{1}{a_t}V_t^h(x)=(r-\delta)^{-1}g(x)+g^-(x),$$

where $g^-(x):=\mathcal{E}^{-1}\frac{1}{a_t}V_t^h(x)-(r-\delta)^{-1}g(x)$ is a function that vanishes on $x>\ln\left(\frac{hl_t}{a_t}\right)$. Because $\mathcal{E}^{-1}=\left(\mathcal{E}^+\right)^{-1}\left(\mathcal{E}^-\right)^{-1}$, the previous equation is equivalent to

$$\left(\mathcal{E}^-\right)^{-1}\frac{1}{a_t}V_t^h(x)=(r-\delta)^{-1}\mathcal{E}^+g(x)+\mathcal{E}^+g^-(x).$$

Implicit in this construction, $\mathcal{E}^+g^-(x)$ and V_t^h are null above and below $\ln\left(\frac{hl_t}{a_t}\right)$, respectively, and this completes the proof. □

Given that the EPV operator of $e^{z.}$ is related to the Wiener–Hopf factorization, the following relationship exists:

$$\begin{aligned}\left(\mathcal{E}^+g\right)(x) &= (r-\delta)\mathbb{E}^{\tilde{Q}}\left(\int_t^\infty e^{-(r-\delta)(s-t)}\left(e^{x+\bar{X}^S_{s-t}}-\frac{l_t}{a_t}\right)ds\right)\\ &= e^x(r-\delta)\,\mathbb{E}^{\tilde{Q}}\left(\int_t^\infty e^{-(r-\delta)(s-t)}e^{\bar{X}^S_{s-t}}\,ds\right)-\frac{l_t}{a_t}\\ &= e^x\kappa^+_{r-\delta}(1)-\frac{l_t}{a_t}\end{aligned} \tag{6.5}$$

and

$$\begin{aligned}&\left(\mathcal{E}^-\mathbf{1}_{[b,\infty)}\mathcal{E}^+g\right)(x_t^S)\\ &=(r-\delta)\mathbb{E}^{\tilde{Q}}\left(\int_t^{+\infty}e^{-(r-\delta)(s-t)}\left[e^{x+\underline{X}^S_{s-t}}\kappa^+_{r-\delta}(1)-\frac{l_t}{a_t}\right]\right.\\ &\quad\left.\times\ 1_{\{x+\underline{X}_{s-t}>b\}}ds\right).\end{aligned} \tag{6.6}$$

From this equation, we can deduce the closed-form expressions for the equity and optimal threshold.

Corollary 6.2. *The value of the company equity is equal to:*

$$\begin{aligned}V_t^h(a_t,l_t) &= \frac{a_t}{r-\delta}\sum_{j=1}^3 a_j^-\kappa^+_{r-\delta}(1)\frac{\beta_j^-}{1-\beta_j^-}\left(\left(\frac{hl_t}{a_t}\right)^{(1-\beta_j^-)}-1\right)\\ &\quad-\frac{l_t}{r-\delta}\sum_{j=1}^3 a_j^-\left(1-\left(\frac{hl_t}{a_t}\right)^{-\beta_j^-}\right),\end{aligned} \tag{6.7}$$

where the coefficients a_j^- for $j=1,2,3$ are defined by Eqs. (5.9)–(5.11).

Proof. This result is an immediate consequence of expressions (5.13) and (6.5):

$$\begin{aligned}V_t^h(x) &= \frac{a_t}{r-\delta}\sum_{j=1}^3 a_j^-\kappa^+_{r-\delta}(1)e^x\frac{\beta_j^-}{1-\beta_j^-}\left(e^{(1-\beta_j^-)\left(\ln\left(\frac{hl_t}{a_t}\right)-x\right)}-1\right)\\ &\quad-\frac{a_t}{r-\delta}\sum_{j=1}^3 a_j^-\frac{l_t}{a_t}\left(1-e^{-\beta_j^-\left(\ln\left(\frac{hl_t}{a_t}\right)-x\right)}\right).\end{aligned} \tag{6.8}$$

To conclude, it is sufficient to recall that $X_0^S=x=0$. □

Corollary 6.3. *In order to maximize the present value of their investment, shareholders should close the company when the income a_t falls below h^*l_t, where*

$$h^*=\frac{1}{\kappa^+_{r-\delta}(1)}. \tag{6.9}$$

Proof. According to Eq. (6.2), the value of the equity is directly proportional to the quantity (6.6). Then, the constant h^* that maximizes the shareholder's equity is such that the integrand $\left(\mathcal{E}^+g\right)(x)=e^x\kappa^+_{r-\delta}(1)-\frac{l_t}{a_t}$ is null on the boundary $x=b=\ln\left(\frac{hl_t}{a_t}\right)$. Another way to prove this relation is to set the derivative of Eq. (5.13) with respect to h as zero. □

We test these results numerically in Section 9. First, we study the impact of a disinvestment delay on the equity value and the optimal threshold.

7. The period between bankruptcy and the cessation of activity

In practice, there is a period of time between filing for bankruptcy and the cessation of a company's activity. The length of this time is variable and it depends on many concurrent factors such as negotiating with labour unions or with eventual prospective buyers. In the remainder of this section, this period of disinvestment is denoted by Δ and it is assumed to be an exponential random variable with the parameter γ. The average delay and its density under Q are equal to $\mathbb{E}^Q(\Delta)=\frac{1}{\gamma}$ and $f_\Delta(t)=\gamma e^{-\gamma t}$, respectively. The time to default is denoted by τ. To recap, bankruptcy is declared when a_t falls below the threshold $h\,l_t$. Thus, the value of the equity for a given h when considering the disinvestment period is now

$$\begin{aligned}V_t^h(a_t,l_t) &= \mathbb{E}^Q\left(\int_t^{\tau+\Delta}e^{-r(s-t)}\left(a_s-l_s\right)ds\mid\mathcal{F}_t\right).\\ &= \mathbb{E}^Q\left(\int_0^\infty\int_t^{\tau+\epsilon}e^{-r(s-t)}\left(a_s-l_s\right)ds\,\gamma e^{-\gamma\epsilon}\,d\epsilon\mid\mathcal{F}_t\right).\end{aligned} \tag{7.1}$$

Proposition 7.1 develops $V_t^h(a_t,l_t)$ in terms of EPV operators, as follows.

Proposition 7.1. *The three negative roots of the numerator of $\psi^{\tilde{S}}(z)-(r+\gamma-\delta)$ are $\beta_k'^-$ for $k=1,2,3$, and*

$$\kappa^+_{(r+\gamma-\delta)}(1)=\prod_{j=1}^2\frac{\lambda_j^+-1}{\lambda_j^+}\prod_{k=1}^3\frac{\beta_k'^+}{\beta_k'^+-1}.$$

The term $a_j'^-$ is defined for $j=1,2,3$ by expressions (5.9)–(5.11), *where $\beta_k'^-$ is substituted with β_k^-. The value of the equity for a disinvestment delay is given by the following expression.*

$$\begin{aligned}V_t^h(a_t,l_t) &= \frac{a_t}{(r+\gamma)-\psi^A(1)}\\ &\quad+\frac{a_t}{r-\delta}\sum_{j=1}^3 a_j^-\kappa^+_{r-\delta}(1)\frac{\beta_j^-}{1-\beta_j^-}\left(\left(\frac{hl_t}{a_t}\right)^{(1-\beta_j^-)}-1\right)\\ &\quad-\frac{a_t}{r+\gamma-\delta}\sum_{j=1}^3 a_j'^-\kappa^+_{r+\gamma-\delta}(1)\frac{\beta_j'^-}{1-\beta_j'^-}\left(\left(\frac{hl_t}{a_t}\right)^{(1-\beta_j'^-)}-1\right)\\ &\quad-\frac{l_t}{(r+\gamma)-\psi^L(1)}+\frac{l_t}{r+\gamma-\delta}\sum_{j=1}^3 a_j'^-\left(1-\left(\frac{hl_t}{a_t}\right)^{-\beta_j'^-}\right)\\ &\quad-\frac{l_t}{r-\delta}\sum_{j=1}^3 a_j^-\left(1-\left(\frac{hl_t}{a_t}\right)^{-\beta_j^-}\right).\end{aligned}$$

Proof. Using Fubini's theorem, the equity value (7.1) can be restated as follows:

$$V_t^h(a_t, l_t) = \mathbb{E}^Q\left(\int_t^{\tau} e^{-r(s-t)}(a_s - l_s)\,ds \mid \mathcal{F}_t\right) + \mathbb{E}^Q\left(\int_{\tau}^{\infty} e^{-(r+\gamma)(s-t)}(a_s - l_s)\,ds \mid \mathcal{F}_t\right). \quad (7.2)$$

The first term of (7.2) is provided by Eq. (6.7) in Corollary 6.2. The second expectation is the difference between the residual values of the assets and liabilities. This residue is denoted by $R_t^h(a_t, l_t)$ and it is split as follows:

$$R_t^h(a_t, l_t) = \mathbb{E}^Q\left(\int_t^{\infty} e^{-(r+\gamma)(s-t)}(a_s - l_s)\,ds \mid \mathcal{F}_t\right) - \mathbb{E}^Q\left(\int_t^{\tau} e^{-(r+\gamma)(s-t)}(a_s - l_s)\,ds \mid \mathcal{F}_t\right). \quad (7.3)$$

Direct manipulation yields the next expression for the first term of (7.3):

$$\mathbb{E}^Q\left(\int_t^{\infty} e^{-(r+\gamma)(s-t)}(a_s - l_s)\,ds \mid \mathcal{F}_t\right) = \frac{1}{(r+\gamma) - \psi^A(1)}a_t - \frac{1}{(r+\gamma) - \psi^L(1)}l_t. \quad (7.4)$$

The second term is obtained by replacing r with $r + \gamma$ in Corollary 6.2 from Section 6. □

The optimal threshold is found by cancelling the derivative of the equity with respect to h.

Corollary 7.2. *The threshold h that maximizes the shareholder's equity is the solution of the following non-linear equation.*

$$0 = \frac{1}{r-\delta}\sum_{j=1}^{3} a_j^- \beta_j^- \left(\frac{hl_t}{a_t}\right)^{-\beta_j^-}\left(h\kappa_{r-\delta}^+(1) - 1\right) - \frac{1}{r+\gamma-\delta}\sum_{j=1}^{3} a_j'^- \beta_j'^- \left(\frac{hl_t}{a_t}\right)^{-\beta_j'^-}\left(h\kappa_{r+\gamma-\delta}^+(1) - 1\right). \quad (7.5)$$

The next section introduces a method for calculating the probabilities of bankruptcy, when applied with and without a disinvestment delay.

8. Estimating the probabilities of default

In this section, we propose a method for determining the probability that a given company enters bankruptcy in a certain period of time, with and without disinvestment delay. This is applied under the risk neutral measure. However, the application of this method under P does not require major modifications and it provides useful information for risk management. The approach employed is based on the inversion of the Laplace transform of the hitting time τ. By definition, for a given constant α, the Laplace transform of τ is such that

$$\mathbb{E}^Q\left(e^{-\alpha\tau} \mid \mathcal{F}_t\right) = \alpha\int_t^{+\infty} e^{-\alpha s} Q(\tau \le s \mid \mathcal{F}_t)ds = \alpha\mathcal{L}_{\alpha}(Q(\tau \le s \mid \mathcal{F}_t)), \quad (8.1)$$

where $\mathcal{L}_{\alpha}$ is the Laplace operator. The probability of default is then obtained by inverting this transform:

$$Q(\tau \le s \mid \mathcal{F}_t) = \mathcal{L}_{\alpha}^{-1}\left(\frac{1}{\alpha}\mathbb{E}^Q\left(e^{-\alpha\tau} \mid \mathcal{F}_t\right)\right) = \frac{1}{2\pi i}\lim_{T\to\infty}\int_{\gamma-iT}^{\gamma+iT} e^{\alpha s}\frac{1}{\alpha}\mathbb{E}^Q\left(e^{-\alpha\tau} \mid \mathcal{F}_t\right)d\alpha,$$

where γ is greater than the real part of all the singularities of $\mathbb{E}^Q\left(e^{-\alpha\tau} \mid \mathcal{F}_t\right)$. Using a similar method to that described in Section 5, it is possible to show that for any given positive α, the equation

$$\psi^S(z) - \alpha = 0 \quad (8.2)$$

has exactly six roots, i.e., three negative and three positive roots denoted by $\beta_{1,\alpha}^-$, $\beta_{2,\alpha}^-$, $\beta_{3,\alpha}^-$ and $\beta_{1,\alpha}^+$, $\beta_{2,\alpha}^+$, $\beta_{3,\alpha}^+$, respectively. The Laplace transform of the time to bankruptcy derives from Proposition 8.1 as follows.

Proposition 8.1. *The Laplace transform of the default time is*

$$\mathbb{E}^Q\left(e^{-\alpha\tau} \mid \mathcal{F}_t\right) = A_1 e^{-\ln\left(\frac{hl_t}{a_t}\right)\beta_{1,\alpha}^-} + A_2 e^{-\ln\left(\frac{hl_t}{a_t}\right)\beta_{2,\alpha}^-} + A_3 e^{-\ln\left(\frac{hl_t}{a_t}\right)\beta_{3,\alpha}^-}, \quad (8.3)$$

where

$$A_1 = \frac{\beta_{2,\alpha}^-\beta_{3,\alpha}^-}{\eta_A^2\eta_L^1}\left(\frac{\left(\eta_A^2 + \beta_{1,\alpha}^-\right)\left(\eta_L^1 + \beta_{1,\alpha}^-\right)}{\left(\beta_{1,\alpha}^- - \beta_{2,\alpha}^-\right)\left(\beta_{1,\alpha}^- - \beta_{3,\alpha}^-\right)}\right) \quad (8.4)$$

$$A_2 = \frac{\beta_{1,\alpha}^-\beta_{3,\alpha}^-}{\eta_A^2\eta_L^1}\left(\frac{\left(\eta_A^2 + \beta_{2,\alpha}^-\right)\left(\eta_L^1 + \beta_{2,\alpha}^-\right)}{\left(\beta_{2,\alpha}^- - \beta_{1,\alpha}^-\right)\left(\beta_{2,\alpha}^- - \beta_{3,\alpha}^-\right)}\right) \quad (8.5)$$

$$A_3 = \frac{\beta_{1,\alpha}^-\beta_{2,\alpha}^-}{\eta_A^2\eta_L^1}\left(\frac{\left(\eta_A^2 + \beta_{3,\alpha}^-\right)\left(\eta_L^1 + \beta_{3,\alpha}^-\right)}{\left(\beta_{3,\alpha}^- - \beta_{1,\alpha}^-\right)\left(\beta_{3,\alpha}^- - \beta_{2,\alpha}^-\right)}\right). \quad (8.6)$$

Proof. The Laplace transform is a function of X_t^S:

$$\mathbb{E}^Q\left(e^{-\alpha\tau} \mid \mathcal{F}_t\right) := u(X_t^S)$$

and if the infinitesimal generator of X_t^S under P is denoted by $\mathcal{L}$,

$$\mathcal{L}u(x) = (\mu_A - \mu_L)\frac{\partial u}{\partial x} + \frac{1}{2}\left(\left(\sigma_A - \sigma_{AL}^2\right)^2 + \sigma_L^2\right)\frac{\partial^2 u}{\partial x^2} + \lambda_A\int_{-\infty}^{+\infty} u(x+y) - u(x)f_{Y^A}(y)dy + \lambda_L\int_{-\infty}^{+\infty} u(x-y) - u(x)f_{Y^L}(y)dy,$$

then the function $u(x)$ is the solution of the following system [A4]:

$$\begin{cases} (\mathcal{L} - \alpha)\,u(x) = 0 & \text{if } x > \ln\left(\frac{hl_t}{a_t}\right) \\ u(x) = 1 & \text{if } x \le \ln\left(\frac{hl_t}{a_t}\right). \end{cases} \quad (8.7)$$

For any level $b = \ln\left(\frac{hl_t}{a_t}\right)$, we can test a solution of the form

$$u(x) = \begin{cases} A_1 e^{(x-b)\beta_{1,\alpha}^-} + A_2 e^{(x-b)\beta_{2,\alpha}^-} + A_3 e^{(x-b)\beta_{3,\alpha}^-} & x > b \\ 1 & x \le b, \end{cases} \quad (8.8)$$

where A_1, A_2 and A_3 must be such that

$$A_1 + A_2 + A_3 = 1. \quad (8.9)$$

Implicitly, $0 \le u(x) \le 1$ for all $x \in (-\infty, +\infty)$, given that $\beta_{j,\alpha}^-$ is negative. By substituting this form of $u(.)$ and integrating in two regions, $\int_{-\infty}^{+\infty} = \int_{-\infty}^{b-x} + \int_{b-x}^{+\infty}$, for all values of $x > b$ yields

$$(\mathcal{L} - \alpha)\,u(x) = \sum A_i e^{(x-b)\beta_{i,\alpha}^-}\left(-\alpha + \psi^S(\beta_{i,\alpha}^-)\right) + \lambda^A q^A e^{\eta_A^2(b-x)}\left(\sum_{i=1}^{3} A_i\frac{\beta_{i,\alpha}^-}{\eta_A^2 + \beta_{i,\alpha}^-}\right)$$

Table 9.1
Parameters defining the dynamics of a_t, fitted by log-likelihood maximization to the daily returns of a benchmark portfolio (comprising 65% of Eurostoxx 600 and 35% Bofa Merill Lynch Index 7–10 years EUR).

	Parameters	Standard errors
μ_A	0.0896	0.0012
σ_A	0.02029	0.0004
p_A	0.35739	0.0037
η_A^1	663.92257	0.7231
η_A^2	572.09112	0.9463
λ_A	93.50947	0.5982

$$+\lambda^L q^L e^{\eta_L^1(b-x)}\left(\sum_{i=1}^{3} A_i \frac{\beta_{i,\alpha}^-}{\eta_L^1+\beta_{i,\alpha}^-}\right),$$

because $-\alpha+\psi^S(\beta_{i,\alpha}^-)=0$, provided that the following relations are satisfied:

$$\sum_{i=1}^{3} A_i \frac{\beta_{i,\alpha}^-}{\eta_A^2+\beta_{i,\alpha}^-}=0 \tag{8.10}$$

$$\sum_{i=1}^{3} A_i \frac{\beta_{i,\alpha}^-}{\eta_L^1+\beta_{i,\alpha}^-}=0. \tag{8.11}$$

It is clear that $(\mathcal{L}-\alpha)\,u(x)=0$ for $x>b$. Solving the system of Eqs. (8.9) (8.10) and (8.11) leads to expressions (8.4) (8.5) and (8.6) for A_i. The function $u(x)$ defined by Eq. (8.8) is not C^1 around the boundary $x=b$. However, as demonstrated by Kou and Wang (2003), it is possible to build a sequence of smooth functions $u_n(x)$ that also converge towards $u(x)$. $\square$

Because the Laplace transform of the default time is known, the Gaver–Stehfest algorithm is used to invert it numerically. This approach was described by Davies (2002, Chapter 19) and by Usabel (1999). Finally, we note that according to the Markov inequality, the asymptotic probability of default is bounded by the following limit:

$$Q(\tau\le\infty\mid\mathcal{F}_t)\le\lim_{\alpha\to 0}\mathbb{E}^Q\left(e^{-\alpha\tau}\mid\mathcal{F}_t\right). \tag{8.12}$$

This boundary can be used as a risk measure to compare the credit risk of several companies.

9. Numerical application of the model

In this section, we illustrates how the proposed model can explain movements in the share prices of two insurances companies: Axa and Generali. Accounting figures from 31/8/2009 to 29/8/2014 indicate that both companies had a similar investment strategy over the study period. On average, 65% of their portfolio was invested in state or corporate bonds and the remaining 35% was invested in stocks or assimilated risky assets. Thus, the income was assumed to have the same dynamics as a benchmark index made up of the Eurostoxx 600 (35%) and the Bofa Merrill Lynch Index 7–10 years EUR (65%). The Merrill Lynch index tracks the total performance of corporate debts (investment grade, extending between 7 and 10 years).

The parameters used to define a_t were subsequently calibrated to reflect the daily return of this benchmark using log-likelihood maximization. As the density function of a_t has no closed form expression, it was computed numerically by inverting its Fourier transform. Details of this procedure were provided by Hainaut and Deelstra (2014). Table 9.1 shows the parameters obtained using this approach. Jumps introduce asymmetry and leptokurticity in returns, which are observed often in financial markets. Thus, the quality of the fit (measured by the log-likelihood = 6321.6) was better than that obtained using Brownian motion (log-likelihood = 6121.6).

Table 9.2
Parameters of the liabilities, which were obtained by minimizing the squared errors between the actual daily stock prices and the prices predicted by the model. The discount rate was set to the 10-year risk-free rates in France (1.268%) and Italy (2.545%), on 16/9/2014. The Esscher parameter for liabilities, k_L, was null.

	Generali		Axa
μ_L	−0.0443	μ_L	−0.0122
σ_L	0.2181	σ_L	0.1736
σ_{AL}	−0.3605	σ_{AL}	−0.2108
p_L	0.3309	p_L	0.4099
η_L^1	448.8088	η_L^1	788.6129
η_L^2	808.5012	η_L^2	447.4008
λ_L	216.8481	λ_L	104.9724
h^*	0.5923	h^*	0.7185
δ	0.0252	δ	0.0124
k_A	−139.0857	k_A	−124.6903

The next step was to determine the parameters driving the companies' liabilities. Detailed information about claims is not usually disclosed, so liabilities cannot be calibrated directly by log-likelihood maximization. Instead, we used an alternative approach based on historical share prices and accounting information. In this method, the parameters of l_t were inferred by minimizing the summed squared errors between the daily values of shares and the share prices predicted by the model (Eq. (6.7)). The model inputs comprised earnings and charges, which were retrieved from standardized income statements, such as those reported by Bloomberg (see Table A.1). To separate financial incomes from liabilities, a_t was assumed to be equal to the total revenue (year to date), which was marked down by the net earned premiums per stock. The total charge per stock, l_t, was the sum of claims (year to date) and all related costs, which were decreased by the net earned premiums. The cash flows a_t and l_t were updated on a semi-annual basis (see the Appendix). The discount rate was the 10-year risk-free rate in France (1.268%) and Italy (2.545%) on 16/9/2014. The Esscher parameter for liabilities, k_L, was null and thus the liabilities had the same dynamics under the real and risk neutral measures. This assumption is common among actuaries when calculating the net asset value in Solvency II. Table 9.2 shows the parameters obtained using this method and Fig. 9.1 presents the quotes for the stocks and prices predicted by the model. The figures produced by the model should be viewed as target prices, similar to those reported by financial analysts based on a fundamental analysis of companies. Over the 10-year study period, the model followed the market prices reasonably well, if trading noise is not considered. These results were obtained under an assumption that the default trigger h maximized the stock value (see Eq. (6.9)).

The liabilities of both companies exhibited comparable volatility, which was negatively correlated with their income. The frequency of jumps, λ_L, for Generali was twice that of Axa, but the probabilities of upward jumps, p_L, were similar. The parameter δ, which was defined previously as the growth rate of average liabilities, was positive and close to the risk-free rate chosen for the evaluation of each company. The Esscher parameter k_A was negative, so the return of assets under Q was lower than that under P. Shareholders in Generali and AXA would have optimized their investment if activities stopped at a point when income dropped below 59% and 72% of the charges, respectively.

The probabilities of default are shown in the left panel of Fig. 9.1. A comparison between the 10-year probabilities of default bootstrapped by credit default swaps (CDS) on 16/9/2014 (around 21% for Generali and 19% for Axa) suggests that a lower trigger rate of 15% (all of the other parameters were identical) should be used to assess the bankruptcy risk. The discrepancy in these probabilities can be explained by the difference between the risk neutral measures used by financial analysts for stock valuation and those used

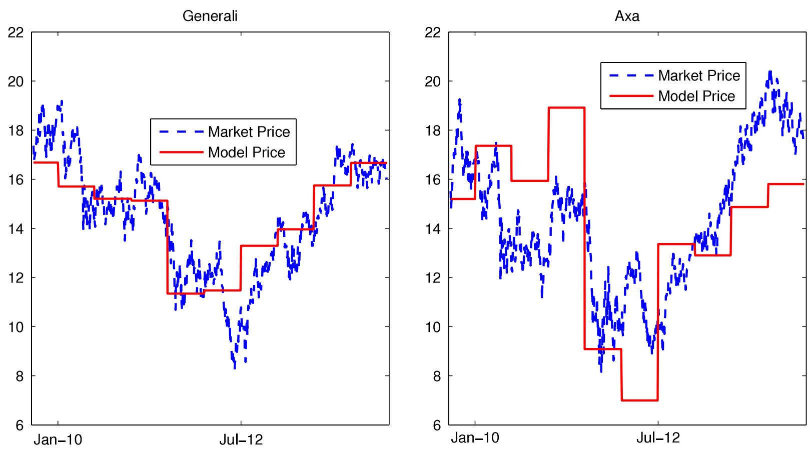

Fig. 9.1. Comparison of stock prices predicted by the model and the true market quotes for the period from 31/8/2009 to 29/8/2014.

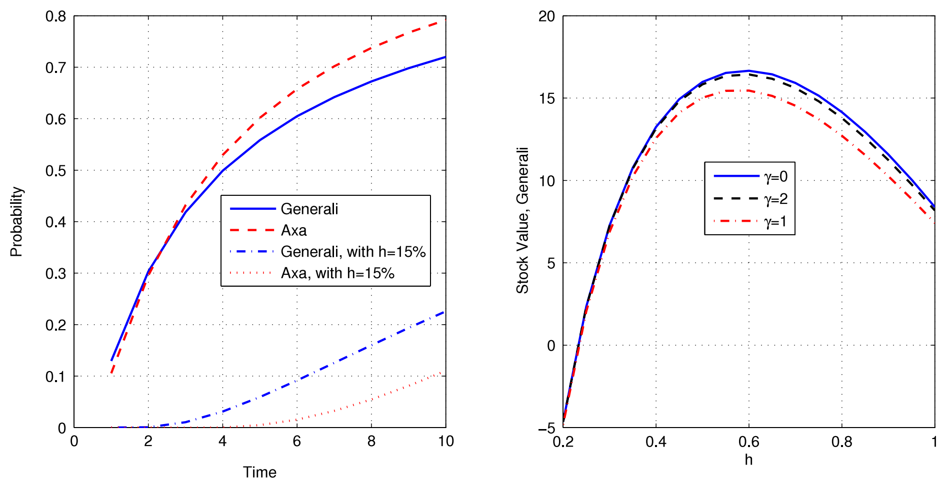

Fig. 9.2. Left graph: Probabilities of default based on parameters obtained from the stock market and the probabilities of default with a lower trigger $h = 15\%$. Right graph: Stock value of Generali for different triggers, h, and different delays.

by credit analysts for CDS pricing. This intuition can be confirmed by comparing the model results with the multiples of valuation, $M = \frac{\textit{stock price}}{\textit{Operating income}}$. Multiples are very popular among financial analysts. Under the assumption that the operating income, $(a_t - l_t)$, is constant, the target stock price is estimated multiple times as the product of the last operating income (e.g., see Vernimmen et al., 2014, Chapter 35). Therefore, this multiple amounts to the sum of the discount factors weighted by the probability of survival: $M \approx \sum_{t=1}^{\infty} Q(\tau \geq t)e^{-rt}$. If our proposed model is reliable, then the observed multiples should be comparable with this weighted sum, which was the case. The multiples for Generali and Axa on 9/8/2014 were $M^{Generali} = 6.25$ and $M^{Axa} = 5.45$, which are close to $\sum_{t=1}^{40} Q^{Generali}(\tau \geq t)e^{-rt} = 6.27$ and $\sum_{t=1}^{40} Q^{Axa}(\tau \geq t)e^{-rt} = 5.82$. The sums calculated with survival probabilities bootstrapped by CDS quotes on 16/9/2014 are three times higher. The Generali stock values for different triggers and for three different average periods between bankruptcy and closure (no delay, and 6 months or 1 year) are shown in Fig. 9.2 (right-hand side). The stock prices as a function of a given threshold, h, follow a concave curve. Extending the delay between the decision of bankruptcy and the closure of the company reduced the stock price.

10. Conclusions

The current study extends the endogenous structural model initially introduced by Leland (1994), where we include companies financed by stochastic liabilities. This framework is applicable to companies that face uncertainty in their investments and also in their costs of funding. The types of companies that belong to this category are typically insurance companies and commercial banks. Based on Wiener–Hopf factorization, we established closed-form expressions for the equity value, for the optimal default threshold and the Laplace transform of the default time, where we applied the Esscher risk neutral measure. We extended these results to the case where the closure of the company occurs after the decision of bankruptcy. The numerical application of the model employed a calibration procedure based on both the market and fundamental analysis. The proposed method was applied to two examples of companies, i.e., Generali and Axa, and it explained the main movements of their stock prices. However, the probabilities of company default obtained with these parameters were higher than those used for pricing CDSs. This suggests that financial analysts do not use the same risk neutral measure as credit risk analysts.

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Table A.1
All of the figures are in millions of US dollars (except $a(t)$ and $b(t)$).
Source: Biannual standardized income statements for Generali and Axa and Bloomberg database.

Semester	Revenue-net earned premium	Insurance claims, others and underwriting costs-net earned premium	Basic weighted avg shares	$a(t)$	$l(t)$
GENERALI					
S2 2009	21 754.50	18 763.00	1 414.07	16.16	13.94
S1 2010	22 715.60	19 235.80	1 540.84	14.59	12.36
S2 2010	21 345.60	17 837.50	1 540.85	13.71	11.46
S1 2011	18 827.60	14 946.20	1 540.87	12.22	9.70
S2 2011	10 743.30	7 319.10	1 540.88	6.97	4.75
S1 2012	12 363.90	9 090.40	1 540.88	8.02	5.90
S2 2012	18 209.00	15 126.00	1 540.80	11.82	9.82
S1 2013	18 874.40	15 515.90	1 540.79	12.13	9.97
S2 2013	20 887.00	17 006.00	1 547.20	13.42	10.93
S1 2014	22 681.00	18 692.00	1 555.98	14.57	12.01
AXA					
S2 2009	41 086.00	34 773.00	2 193.20	18.66	15.79
S1 2010	44 869.00	37 104.00	2 263.00	19.76	16.34
S2 2010	37 390.00	29 863.00	2 293.53	16.30	13.02
S1 2011	41 344.00	32 181.00	2 298.00	17.95	13.97
S2 2011	21 387.00	17 078.00	2 301.00	9.28	7.41
S1 2012	22 987.00	20 203.00	2 340.00	9.81	8.62
S2 2012	35 763.00	29 583.00	2 343.00	15.22	12.59
S1 2013	34 524.00	28 417.00	2 380.60	14.46	11.90
S2 2013	39 375.00	32 256.00	2 388.00	16.43	13.46
S1 2014	41 141.00	33 365.00	2 417.90	16.91	13.71

Appendix

See Table A.1.

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