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Revisiting European day-ahead electricity market auctions: MIP models and algorithms

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Doctoral Thesis 2 | 2017

Université catholique de Louvain
LOUVAIN RESEARCH INSTITUTE IN MANAGEMENT AND ORGANIZATIONS

UNIVERSITÉ CATHOLIQUE DE LOUVAIN

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Thèse présentée en vue de l'obtention
du grade de
Docteur en sciences économiques et de gestion

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Louvain-la-Neuve
Mars 2017

À mes parents

Abstract

Large amounts of electricity are traded in so-called day-ahead (spot) markets where market participants can sell or buy electrical energy for each hour of the following day. Sell or buy orders describing operational and economic constraints render the underlying microeconomic optimization problem "non-convex", departing from more classical assumptions in microeconomic theory. Because of these non-convexities, most of the time, no market equilibrium supported by uniform prices exists, where uniform pricing means that in the market outcome, every market participant of a same market segment (location and hour of the day) will pay or receive the same electricity price and no other transfers or payments are considered.

In Europe, the orders are submitted to power exchanges, most of which are integrated at a European level under the Price Coupling of Region project. Uniform prices are computed, at the expense of having some bids "paradoxically rejected" in the market outcome, as for the computed market prices, some bids propose a price which is "good enough" but are yet rejected. It is also at the expense of welfare optimality, as most of the time, no welfare optimal solution can be supported by uniform prices such that no financial losses are incurred.

The present thesis proposes mixed integer programming models and algorithms for such non-convex uniform price auctions. In particular, a new bidding product is proposed which generalizes both block orders used in the Central Western Europe Region (France, Germany, Belgium, the Netherlands, etc) or Nord Pool (coupling Northern countries), and, *mutatis mutandis*, complex orders with a minimum income condition as used in Spain and Portugal. It allows participants describing e.g. their start up costs - which must be recovered if the corresponding bid is accepted - and indivisibilities in production or consumption, yielding mixed integer programming models seemingly more appropriate than current practice both from an economic modelling and a computational viewpoint.

The thesis is structured as follows.

Part I is a preliminary part devoted to presenting the general context in which the work takes place. It ends with an outline of the three articles presented in Part II which form the core of the thesis, emphasizing the continuity between each of these contributions.

Part II is the core of the text and consists in the collection of the three articles containing the main contributions. Two of them have already been published in international peer-reviewed journals, the third one has been submitted, and the texts reproduced here correspond to the accepted manuscript versions of the published papers, and of an updated version of the submitted manuscript of the third one.

Finally, two appendices are also provided, describing in more details - with a few historical notes - Dorn's duality for convex quadratic programs and the notion of spatial price equilibrium presented using an abstract linear transmission model, as both notions are used in the contributions presented in Part II, making the present text more self-contained.

Acknowledgments

First, I would like to warmly thank my advisor Mathieu Van Vyve for all he taught me during these years, his seminal advices, the time devoted for all the discussions we had, and for all his encouragements. This goes far beyond the scope of the matters addressed here later on, and I have both appreciated and had this chance to learn from his approach to many kinds of problems, always looking for excellent, fair and rational solutions, rather than average workarounds. Besides this, I owe to his suggestion the possibility of such a work despite my initial light background in mathematical optimization, and to have presented part of it in many nice places, meeting unique people. Occasions are too scarce to miss the opportunity one has to say all one's gratefulness: these few words are in this respect certainly also too few.

My warm thanks also go to Per Agrell, Bertrand Cornélusse, François Glineur, Anthony Papavasiliou and Carlos Ruiz, for having accepted to be part of the doctoral jury. I have had this pleasure and chance to work with most of them past years, tutoring class projects under their supervision or in administrative instances. What first comes to my mind regarding these years is a lasting idea of commitment to professional excellence, a never ending challenging curiosity, and a strong taste of communicating them to others.

Some parts of the present work have been substantially boosted by the suggestion of François Glineur to present the historical paper of J.F. Benders on Benders decompositions at a graduate reading seminar, a most unexpectedly helpful graduate student exercise. I have also enjoyed interesting discussions on parts of the work with Sophie Marquet (N-Side) and Bertrand Cornélusse (then at N-Side), whom I thank a lot for the major interest they have shown, and in particular for pointing out concrete benefits some contributions may present compared to existing approaches.

I would also like to greatly thank Alain Marien, senior advisor at CREG (the Belgian Federal Regulatory Agency for energy markets), for interesting discussions we have had on related topics, first in Krakow in 2014, and also for providing real order book data of 2015 from the Belgian market used in a co-authored paper we presented in Porto in June 2016. Great thanks also naturally go to Caroline Gras, Joël Hoeksema and Yves Langer from EPEX Spot, first for providing real order book data in 2012 used for computational experiments, and for their kind invitation to present parts of the results of this dissertation in November 2013 and January 2016, with the interesting discussions which have followed.

Thanks also to my colleagues (and friends) for all the good time and laughter we have or have had, which are much too numerous to cite them all here!: yet thank you Alejandro, Tanja, Andris, Abdel, Mathieu, Bertrand, Jorge, Iyad, Samedi, Emmanuël, Sarah, Jean, Olivier, Etienne, Marie, Ignacio, Manuel, Paul, Wenli, Corentin, Arnaud, ...

I would also like to take this unique occasion in order to remedy to a long standing omission, by thanking to the greatest extent Jean Schmets, for many years President of the Department of Mathematics at the University of Liege, for a few kind encouraging words which, seemingly, have had lasting effects.

Last but not least, I deeply thank my parents and my family for their strong support, and Sabine, never far away, for her love and patience all these years.

Le temps mène la vie dure à ceux qui veulent le tuer.

Jacques Prévert, 1966.

Contents

Part I

Context and contributions

Chapter 1

General Introduction

This first chapter introduces the general context in which the three contributions presented in Part II take place, and is structured as follows.

Section 1.1 aims at briefly positioning the work in its historical and economic contexts and in particular synthetically reviews aspects of the transition from vertically integrated monopoly utilities to liberalized markets, the place of day-ahead markets in the whole electricity supply chain, and the recent institutional evolutions in Europe. The interested reader will find much more information on these aspects in the references provided, as they are beyond the scope of the present dissertation.

Section 1.2 aims at positioning the general European approach in the galaxy of pricing rules in non-convex day-ahead electricity markets. It is widely known and illustrated below that market equilibrium with uniform prices in the presence of non-convexities is a mathematical impossibility [99]. Several toy examples complementing those presented in the main contributions are presented, first for positioning the underlying problem, then to provide insights on the main different approaches previously proposed to deal with these non-convexities. It may be argued from some of these toy examples that current European market rules - and IP pricing as historically proposed by O'Neill et al. [71]) - are not the most appropriate way to deal with such non-convexities, though the approach has proved to be a working and intuitive solution for many years now, be it in France, Germany, the Netherlands, Belgium, Nordic countries or Spain and Portugal, without being exhaustive, and obvisouly presents some interesting features.

Finally, Section 1.3 outlines the content of the three contributions presented in Part II which form the main part of the present thesis. Interesting research questions which deserve further attention are occasionally pointed out.

1.1 Restructured Electricity Industries and Day-ahead Markets

1.1.1 From vertically integrated monopolies to liberalized markets

Details on the history of the economic structure of the electricity industry (economic agents and their relations) are given e.g. in [45] and in classical textbooks on power systems economics such as [92] (in particular Chapter 1 on deregulation and Chapter 3 on market architecture), which both inspired the present discussion.

The supply chain in the electricity industry can be decomposed into three main links corresponding to the following distinct activities: generation, transport and distribution. Generation refers to the transformation of any other source of energy into electrical energy, transportation refers to the transmission of electricity over long distances using high-voltage cables, while distribution aims at serving the end user and generally uses low-voltage cables. Transmission is operated by so-called Transmission System Operators (TSO), while distribution is operated by Distribution System Operators (DSO). The restructuring of the industry has also added a fourth link with retail competition: retailers buy large amounts of electrical energy to generation companies and sell them to the final consumers. In Belgium for example, bills from retailers clearly specify how much they charge for the commodity, and how much goes to the Distribution System Operator (DSO) per KWh.

Transmission and distribution are seen as natural monopolies where a very high level of coordination is required to maintain the reliability of the system in real time, and where substantial economies of scale are present, which raises complex regulatory issues [1, 47, 48]. This doesn't prevent the existence of market mechanisms for the allocation of transmission resources, and a substantial public debate about the relative merits of Physical Transmission Rights (PTR), Financial Transmission Rights (FTR), or so-called Contracts For Difference (CFD) which are pure financial contracts remunerating the owner according to price differences between bidding/price zones, hence providing hedging instruments to market participants. Regarding Financial Transmission Rights and related issues, the reader may refer to [82].

Regarding some elements of the economic structure of the industry, the following main historical waves are distinguished in [45, p.25], see also [92, Chapter 1]. Until 1920, ownership was private with a very low level of coordination among agents. Substantial public investments were made after 1920 as the necessity of electricity was recognized, but still with many private agents and a "fragmented industry" until World War II.

In the aftermath of World War II, nationalisation occurred in many countries such as France, while others applied stricter economic regulations. For example, nationalisation in France gave birth to Electricité de France (EDF) in 1946, today one of the giant actors of the sector, headed from 1967 to 1987 by Marcel Boiteux to whom is due the so-called Ramsey-Boiteux pricing rule for public monopolies with balanced budget constraints [11, 77]. Further integration of generation and transmission seeking economies of scale was operated in the seventies which also saw the shift to nuclear power production partly due to well-known oil price shocks.

Aferwards, the evolution of available technologies and their costs certainly have had a substantial impact on the economic structure of the industry in particular during the eighties and nineties, as supported in [45, p.23] which further describes other key factors to the wave of liberalization that occured at the time. For example, besides political factors, the liberalization process started in the eighties may have been fostered to a given extent by the availability of CCGT units able to generate power at much smaller scales than nuclear plants say, while still at competitive prices, diminishing the importance of economies of scale favouring monopoly utilities. A detailed discussion of these arguments and many others regarding regulation and deregulation can be found in [92, Chapter 1]. The views of Marcel Boiteux regarding the European liberalization process, as the former CEO of a public monopoly utility (EDF), are described (among other comments) in [12] where he states that "là où les monopoles étaient mal gérés (faible productivité) ou mal régulés (enrichissement abusif), la libéralisation est un moindre mal, si ce n'est un bienfait. Mais tous les monopoles d'électricité n'étaient pas malades." Also, Newbery discusses in [66] conditions for a successful liberalization taking into account the then recent Californian Crisis of 2001, while [8] proposes a history of the liberalization process in California from the early 1990s until 2001.

The liberalization of the electricity industry is directly related to the notion of unbundling, which refers to the separation of generation, transmission, distribution, and end user supply. Unbundling can be functional, legal, managerial or of ownership. The four kinds of unbundling may have different economic effects [46].

Unbundling and the general trends and facts described above implies the need for organized wholesale markets, among which (spot) day-ahead markets which are to some extent and as discussed below related to classical scheduling problems in power systems, with the additional key issue of determining market prices providing adequate price signals to market participants. Day-ahead markets are nowadays coupled at the European level under the Price Coupling of Region project. Though some uncertainty remains regarding the precise timing, a Pan-European intraday market should go live in 2017 based on a system called XBid.

A brief overview of the liberalization stages in Europe - and the role of power exchanges - is given in [96] while more lenghty and detailed accounts and analyses, in particular of the earliest days, are provided in [10, 81]. Regarding market coupling in Europe, the determination of bidding zones representing local markets is an important subject currently debated across Europe, see e.g. the study commissioned by CREG [4] and also [14, 76, 24]. For the years to come, the main European legislation concerning day-ahead and intraday markets is contained in the Capacity Allocation and Congestion Management (CACM) Guidelines published in 2015 [16], and an updated account of related European directives and of Market Coupling from a legal and regulatory perspective can be found among other matters in [58].

1.1.2 Day-ahead markets and unit commitment problems

Wholesale electricity markets where large producers and large consumers or retailers exchange energy vary according to the time horizon and include long-term bilateral contracts, forwards/futures where trading occurs by definition years or months ahead the actual delivery of electricity, and (spot) day-ahead markets (see e.g. a synthetic account in [2]). In the European market structure, intraday markets are used to balance unbalanced positions due e.g. to outages or short-term variability of consumption or production, while balancing/spinning reserves markets managed by Transmission System Operators are used to maintain real-time reliability of the system. Due to the growing importance of renewable energy, substantial debates on capacity remuneration mechanisms and how to take into account the costs of reserve and balancing operations are of main current interest in Europe [41].

On the other hand, the so-called Unit Commitment and Economic dispatch problems are classical optimization problems dating back to the period of centralized monopolistic utility companies and still of the same kind as those considered today by large producers to schedule their production for different time horizons, though they now include forecast spot market prices as part of the models.

Given a set of generation units at hand, Economic dipsatch (ED) refers to the choice of generation levels (power outputs) of these units that minimize the total operating costs while serving demand. Optimal Power Flow problems (OPF) seek the same goal taking into account transmission constraints of the network, which leads to substantially more challenging optimization problems when AC networks are considered, due to the highly non-linear nature of the power flow equations involved in the constraints. A historical account of OPF models is given in [15] where a typology is also presented highlighting key characteristics of the variants (AC and DC OPF, security-constrained OPF, etc).

Unit commitment problems (UC) seek to determine which units to turn on or off - and when - as well as how much units should generate, depending on characteristics such as start-up costs, minimum up and down times, ramping constraints, on top of the marginal costs and operating constraints such as minimum and maximum power outputs, with again the goal of minimizing the total operating costs of serving demand over a given time horizon. Again, it corresponds to a field of research in itself with sophisticated variants to handle stochasticity both in demand and generation, e.g. due to the massive integration of renewable energy sources, see e.g. [73, 74, 75] and the recent review [94]. The acronym UCED (standing for Unit Commitment and Economic Dispatch) is also generally used in the literature, as in [40], emphasizing that the models considered involve both commitment decisions and power output or consumption decisions paired with these commitments.

European day-ahead markets where both demand and offer bids are present (i.e. the demand is elastic), can be seen to some extent as a mean to solve a Unit Commitment and Economic Dispatch problem where characteristics are given to Market Operators (coupled power exchanges) in charge of computing the solution, and determining the corresponding market prices. Regarding demand, let us note that examples of indivisibilities on the demand side providing a rationale for demand block orders in the PCR market are given in [71, p.278]: "Demand as well as supply can have significant nonconvexities. For example, the electricity consumption of an aluminum smelter or a cyclotron may be an all-or-nothing choice."

Regarding prices, they should be compatible with the computed schedule, avoiding incurring losses or opportunity costs. However, as discussed in the next Section, most of the time no market equilibrium exists which is supported by uniform prices. Price signals are also important information that agents take into account e.g. for investment or hedging decisions.

1.2 Pricing rules in non-convex day-ahead electricity markets

Once the format in which market participants can describe their technical constraints and costs or utility structure to the Market Operator has been specified, the remaining question is to determine rules specifying the set of admissible exchanges of quantities (bid selections) and the corresponding payments between market participants and the Market Operator where applicable. Regarding these payments, it is generally argued that the law of one price should hold where possible, which in the present context is called uniform pricing: all payments depend on a single electricity price per location and time period. However, as recalled in Section 1.2.1 with toy examples, uniform prices supporting a market equilibrium often do not exist in the presence of non-convexities such as indivisibilities of production or start up costs in the bids of market participants. (The term "non-convexities" is due to the fact that in both cases, binary variables must be introduced in the underlying microeconomic optimization problems, rendering the corresponding feasible sets non-convex.)

Several pricing rules have been proposed the last two decades, trying to deal in the best way with these non-convexities. We review here the most important propositions using unified and simplified notation (at the expense of a negligible loss of generality in some cases, as for example the proposition in [71] goes beyond the sole scope of electricity auctions and related models).

Exposition here doesn't aim at being exhaustive. It rather seeks to position European market rules - which are the main topic of the present work - among the key pricing rules historically proposed to which most of the other more recent propositions in the academic literature refer, namely IP pricing [71] and Convex Hull pricing [40]. In particular, though this seems not highlighted enough in the literature, European market rules could generally be described as IP pricing plus some additional constraints saying that one only considers market outcomes where no make-whole payments compensating losses are needed.

Let us note that a review of most of the existing pricing rules in the particular case of two producers with different start up costs but no indivisibilities and fixed non-elastic demand is provided in a very recent article [50]. However some important features only appear when elastic demand and larger instances are considered, and the topic deserves further investigation in a near future, especially in the setting of power exchanges facing large-scale instances where demand bids representing elastic demand are present. It is of practical and current interest both in Europe and in the US.

1.2.1 Non-convexities and market equilibrium

Two main kinds of non-convexities can be distinguished. First, technical constraints such as minimum power output or consumption levels, which render the production and consumption sets non-convex. Second, non-convexities can arise from the cost structure such as in the presence of start-up costs of power plants, besides their marginal costs, or utility reduced by a constant term on the demand side. Both types require the introduction of binary variables for modelling purposes, contributing to rendering the underlying optimization problems non-convex.

The "primal" optimization problem usually seeks to optimize welfare (defined as the total utility of consumption minus the total costs of production), and its constraints describe the physical/technical constraints related to production, consumption and transmission.

Welfare is usually expressed as a concave function of the decision variables, though other more general objective functions could be considered. Maximizing welfare is motivated both because of the underlying economic interpretation - maximizing the economic surpluses of market participants - and also because in a classical convex setting, welfare maximization is equivalent to the determination of a market equilibrium, see Appendix B.

The following two toy examples are taken from [52]. They show for each of the two types of non-convexities mentioned above that no market equilibrium supported by uniform prices exists in their presence. The conference paper [52] also briefly discusses - using recent real data from the Belgian day-ahead market - the potential interest of non-uniform pricing rules compared to the current European practice detailed below, with a focus on IP pricing as the alternative, though other non-uniform pricing rules should deserve more attention.

Example 1.1. *Toy examples, with associated to C, respectively a minimum acceptance ratio as described in Table 1.1, or a start up cost as described in Table 1.2. We will refer to them later on as Examples 1.1.1 and 1.1.2 respectively. Both types of non-convexities can obviously be combined.*

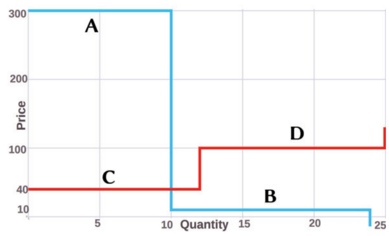

Figure 1.1: Instance with a 'non-convex bid' C - start up cost or min. power output

In the first case of a minimum acceptance ratio, the pure welfare maximizing solution is to accept C at its minimum acceptance level of (11/12), that is accept 11 MW from C, to fully accept A, and to accept the fraction of B needed to match the accepted fraction of C. For a market equilibrium to exist, the market price should be 10 EUR/MW, set by B which is fractionally accepted: otherwise, there would be either some leftover demand

Bids	Quantity (MW)	Limit price (EUR/MW)	Min. Acceptance Ratio
A - Buy bid	10	300	-
B - Buy bid	14	10	-
C - Sell bid 1	12	40	$\frac{11}{12}$
D - Sell bid 2	13	100	-

Table 1.1: Instance with a minimum acceptance ratio (minimum power output level)

Bids	Quantity (MW)	Limit price (EUR/MW)	Start-up costs
A - Buy bid (step 1)	10	300	-
B - Buy bid (step 2)	14	10	-
C - Sell bid 1	12	40	200
D - Sell bid 2	13	100	-

Table 1.2: Instance with start-up costs

from B if the price is below, or B would prefer to be fully rejected if the price is above. However, at this market price, C is loosing $11(40-10) = 330$ EUR and would therefore prefer not to be dispatched. Hence, there is no market equilibrium with uniform prices in the present case.

In the second case of the presence of start up costs, it can be easily checked that the pure welfare maximizing solution is to fully accept A, fully reject B, and accept the fraction of C needed to match A. Any level of acceptance of B would inevitably degrade welfare as the bid price of B is lower than the bid price of any other offer bid, and also, it can be readily checked that discarding C in order to avoid the associated start-up cost would also lead to less welfare (see the discussion of European rules below). The optimal welfare is hence "utiliy of A minus costs of the production by C", that is $10(300)–[10(40) + 200] = 2400$. Here again, if there is any market equilibrium supported by uniform prices, the price is set by fractionally accepted bids, here by C at 40 EUR/MW. However, at such a market price, C doesn't recover its start up costs and would prefer to be rejected: there is no market equilibrium with uniform prices.

For the discussion of pricing rules in the next subsections, we use a stylized welfare maximizing program (SWP) very similar to the "stylized economic unit commitment and dispatch problem" in [78] - Model (1), except that we make the simplifying assumption that all the functions involved are linear and we consider only one location and one time period, as it is enough for the main observations we seek to present using any of the toy examples at hand. We also add the possibility of non-convexities on the demand side (indivisibilities of consumption or utility reduced by a constant term), which simplifies notation as well.

$$(SWP) \qquad \max_{x,u} \sum_c \sum_{ic \in I_c} Q_{ic} P_{ic} x_{ic} - \sum_c F_c u_c$$

s.t.

$$\sum_c \sum_{ic \in I_c} Q_{ic} x_{ic} = 0 \qquad [\pi] \qquad (1.1)$$

$$x_{ic} \leq u_c \qquad \forall ic \in I_c \subseteq I, c \in C \qquad (1.2)$$

$$x_{ic} \geq r_{ic} u_c \qquad \forall ic \in I_c \subseteq I, c \in C \qquad (1.3)$$

$$u_c \leq 1 \qquad (1.4)$$

$$x_{ic}, u_c \geq 0 \qquad (1.5)$$

$$u_c \in \{0, 1\} \qquad \forall c \in C \qquad (1.6)$$

Here, I_c denotes the set of "continuous bids" ic controlled by the binary decision variable u_c. The level of acceptance of the quantity Q_{ic} is determined by x_{ic} which is constrained to lie in a given interval included in $[0, 1]$ if $u_c = 1$, see (1.2)-(1.3). We use the convention according to which $Q_{ic} < 0$ for a sell order, and $Q_{ic} > 0$ for a buy order. Hence, (1.1) is a balance constraint stating that the "market clears", while the objective function represents the welfare in which the P_{ic} are the limit prices representing marginal cost or marginal utility per (sub-)bid ic, and F_c the start up cost or the reduction of utility by a constant term associated to the whole offer or demand bid c which is incurred if the bid is at least partially accepted.

For example, the two toy examples presented above can readily be described as an instance of SWP (the right column corresponds to the instance of Table 1.2). Here, we drop the index i as all the sets I_c involved are singletons.

Example 1.1.1:

$$\max_{x,u} \; (10)(300)x_a + (14)(10)x_b - (12)(40)x_c - (13)(100)x_d$$

$$10x_a + 14x_b - 12x_c - 13x_d = 0 \qquad (1.7)$$

$$x_a \leq u_a \qquad (1.8)$$

$$x_b \leq u_b \qquad (1.9)$$

$$x_c \leq u_c \qquad (1.10)$$

$$x_c \geq (11/12)u_c \qquad (1.11)$$

$$x_d \leq u_d \qquad (1.12)$$

$$u \leq 1 \qquad (1.13)$$

$$x, u \geq 0 \qquad (1.14)$$

$$u \in \{0, 1\}^4 \qquad (1.15)$$

Example 1.1.2:

$$\max_{x,u} \; (10)(300)x_a + (14)(10)x_b - (12)(40)x_c - (13)(100)x_d - 200u_c$$

$$10x_a + 14x_b - 12x_c - 13x_d = 0 \qquad (1.16)$$

$$x_a \leq u_a \qquad (1.17)$$

$$x_b \leq u_b \qquad (1.18)$$

$$x_c \leq u_c \qquad (1.19)$$

$$x_d \leq u_d \qquad (1.20)$$

$$u \leq 1 \qquad (1.21)$$

$$x, u \geq 0 \qquad (1.22)$$

$$u \in \{0, 1\}^4 \qquad (1.23)$$

Obviously, the binary variables u_a, u_b and u_d can readily be set to 1 and removed from the formulation: they are associated to the simple "convex bids" A, B and D and actually not required as it is always optimal to set them to 1.

1.2.2 IP Pricing

The proposition in [71] is to determine prices by using the convex part of the welfare maximization problem: *roughly speaking* "marginal units in the chosen unit commitment and dispatch are setting the price". More precisely, the approach proposed is to (a) maximize welfare, (b) fix all binary variables to the optimal values found, (c) derive commodity (electricity) prices as optimal dual variables of the balance constraints - as usual to determine locational marginal prices, see Appendix B - and start up prices (or commitment prices) as optimal dual variables to the constraints fixing the binary variables to their optimal value. The key contribution is to show that the derived price system supports a market equilibrium *if the market rules specify that payments appropriately depend on both kinds of prices* (Theorem 2 of the original paper).

Because of the "Samuelson principle" (as called in [71] and reviewed in Appendix B), establishing the equivalence between welfare maximization and market equilibrium in well-behaved convex contexts, the fact that the commodity prices are derived as optimal dual variables of the balance constraints in the restricted welfare maximizing problem where integer decisions are fixed implies that these prices are equilibrium prices supporting the values of the continuous primal decision variables, and in particular, that "marginal units" (here whose production or consumption level is partial with regard to their technical capabilities) are setting the price.

For example, with Example 1.1.1 above and considering its optimal solution, the market price is set by the marginal unit B to 10 EUR/MW, and the commitment price associated to the constraint fixing the commitment binary variable $u_c = 1$ is - 330EUR, corresponding here to the incurred loss to unit C at the given commodity market price. With the instance 1.1.2, the market price would be 40 EUR/MW and the commitment price set to (- 200) EUR, again corresponding to the incurred loss. These prices for the commodity and the commitments can readily be derived as the optimal dual variables π and δ (in square brackets) in:

Example 1.1 (continued)

Example 1.1.1 (IP Pricing case):

$$\max_{x,u} (10)(300)x_a + (14)(10)x_b - (12)(40)x_c - (13)(100)x_d$$

$$10x_a + 14x_b - 12x_c - 13x_d = 0 \qquad [\pi^* = 10]$$
$$x_a \leq 1$$
$$x_b \leq 1$$
$$x_c \leq u_c$$
$$x_c \geq (11/12)u_c$$
$$x_d \leq 1$$
$$u_c = 1 \qquad [\delta^* = -330]$$
$$x \geq 0$$

Example 1.1.2 (IP Pricing case):

$$\max_{x,u} (10)(300)x_a + (14)(10)x_b - (12)(40)x_c - (13)(100)x_d - 200u_c$$

$$10x_a + 14x_b - 12x_c - 13x_d = 0 \qquad [\pi^* = 40]$$
$$x_a \leq 1$$
$$x_b \leq 1$$
$$x_c \leq u_c$$
$$x_d \leq 1$$
$$u_c = 1 \qquad [\delta^* = -200]$$
$$x \geq 0$$

Given these prices $(\pi, \delta) = (10, -330)$ or $(40, -200)$ respectively, participant C receives as a payment $\pi(Q_c x_c) - \delta u_c$, here respectively $10(11) - (-330)1 = 440$ or 40(10) - (-200)1 = 600. In each case, it is exactly corresponding to the production costs of C, and the primal decisions (u_c, x_c) are optimal for the market participant C, that is they respectively solve the following profit-maximizing programs:

$$\max_{u_c, x_c} 12(\pi - 40)x_c - \delta u_c \tag{1.24}$$

$$x_c \leq u_c \tag{1.25}$$

$$x_c \geq (11/12)u_c \tag{1.26}$$

$$u_c \in \{0, 1\} \tag{1.27}$$

$$[\pi := 10, \delta := -330]$$

$$\max_{u_c, x_c} [12\pi x_c - \delta u_c] - [12(40)x_c + 200u_c] \tag{1.28}$$

$$x_c \leq u_c \tag{1.29}$$

$$x_c \geq 0 \tag{1.30}$$

$$u_c \in \{0, 1\} \tag{1.31}$$

$$[\pi := 40, \delta := -200]$$

Let us note that in general, with several non-convex bids and when both indivisibilities and start up costs are considered at the same time, the parameter δ in (1.28)-(1.31) corresponding to the commitment price may differ from the start up cost of the market participant present in the input data: the fact that $\delta = (-200)$ is exactly offsetting the start up cost here, yielding 0 as the coefficient of u_c (after rearrangement), is peculiar to the present toy example or related similar situations.

It may obviously happen that the committed units (i.e. such that $u_c = 1$) are profitable at the market price(s) π, in which case the optimal dual variable δ^* to the fixing constraint is positive. (Such observations, as others which follow, can be derived by writing down the dual and complementarity conditions of the welfare maximization programs with the fixing constraints and discussing them in a way similar to what is done in Chapter 4 below.) In such a case, if strictly applied, IP pricing would require a payment $\pi Q_c x_c - \delta u_c$, where $-\delta u_c$ is negative and corresponds to a situation where the market participant gives its marginal rent back to the Market Operator and makes zero profits, similarly to a pay-as-bid scheme. However, as described in the original contribution [71, p.282] about the practice of the New-York Independant System Operator NYISO and Pennsylvania–New Jersey– Maryland Interconnection (PJM), and also in [90], market rules could specify that such profits can be kept by market participants. In such a setting, IP pricing could be described, quite roughly speaking, as "marginal pricing plus make-whole payments" as only losses are compensated, while market participants can keep rents if any at the given market prices. Note that this approach seems also close to the current practice in Ireland [20, pp.40-43].

Let us emphasize that according to the payment scheme proposed whereby seller market participants are paid $\pi x - \delta u$ with π the market price and δ the discretionary start up price, no payment is made to non-committed units since then u=0 and x=0. However, it may happen that rejected bids are profitable at the commodity market prices, in which case δ is positive. In that situation, the term $-\delta u$ in the settlement rule makes the market participant indifferent to being committed or not: if u was switched to one to allow a profitable generation of electricity, a corresponding payment from the market participant to the MO would occur offsetting these potential profits. This is another - maybe surprising - aspect of the underlying idea of Theorem 2 in [71], and the fact that for the obtained price system, optimal primal variables of the welfare program are also solving the market participant's profit-maximizing programs like (1.24) or (1.28).

In the original article [71], the question is asked to know if there are multiple equilibria of this kind, i.e. supported by prices both for the commodity, and for the commitment decisions. See [71, p.283]: "Finally, our results say nothing about the uniqueness of equilibrium prices. In fact, as can be seen in Scarf's example in Section 4, there can be multiple equilibria. (In simple examples, degeneracy of the augmented LP can be a problem leading to multiple dual solutions. However, in larger more complex problems, it is not entirely clear how big a problem a multiplicity of solutions will be)". The question is raised for "efficient equilibria" in the sense of an equilibrium corresponding to a welfare maximizing market outcome, the only outcomes considered in [71]. Otherwise, it can be shown that any arbitrary choice of commitment decisions, i.e. of values for the corresponding binary variables, will lead to an equilibrium as defined in [71]. For example, outcomes under European market rules discussed below are examples of other equilibria of such a kind where all commitment prices are positive or null. Again, for non-convex bids "paradoxically rejected" according to the commodity market prices only which are "good enough", the positive or null commitment prices δ would generate a payment $-\delta$ if u_c was switched from 0 to 1, which would correspond to a transfer from the market participant to the market operator, and these commitment prices δ are determined to offset the corresponding opportunity costs with regard to the commodity market prices, making the market participant indifferent to being committed or not under the payment scheme described above. Most of the time, these "European-like equilibria" are not efficient as they do not correspond to a pure welfare maximizing outcome.

Pursuing in the same direction and considering the bidding products proposed in Europe (so-called block orders), the reference [70] shows that with IP pricing, provided the welfare is positive, a welfare maximizing solution is always such that there is enough welfare to finance compensations paid to bids losing money, so-called "paradoxically accepted block orders", if they are allowed.

One recurring grief made to the IP pricing approach is that it exhibits important commodity price volatility [78, 84], the reason intuitively being that the units which are marginal and hence setting this price, which can have substantially different greater or lower marginal costs, can quickly change with an increase of load. We argue here that it also leads to counter-intuitive market prices, as the following example shows:

Example 1.2. *This example is described in Table 1.3*

Bids	Quantity (MW)	Limit price (EUR/MW)	Min. Acc. Ratio
A - Sell bid	50	30	-
B - Buy bid	50	130	-
C - Sell bid	40	40	-
D - Sell block bid	200	60	1
E - Buy block bid	200	90	1

Table 1.3: Instance with non-intuitive 'IP pricing' outcome

It could be shown (e.g. by solving the corresponding MILP problem) that the welfare maximizing solution is here given by fully accepting A, B, D, E and rejecting C. As C is fully rejected, the commodity market price must be less than or equal to 40 EUR/MW (the marginal cost of C), and as A is fully accepted, the market price must be greater than or equal to 30 EUR/MW. For such a price, D is "paradoxically accepted" with respect to the commodity price and would benefit from a "start up price" corresponding to a make-whole payment as shown above. However, intuitively, one may rather prefer to set the price e.g. at 75 EUR/MW, in between the "marginal costs" of D and E, in which case no make-whole payment is needed. If the bid C is removed from the instance, such an outcome would correspond to a market equilibrium based on the commodity price only.

Indeed, the anomaly here (if judged so) is related to an arbitrary distinction between bids including non-convexities and those which don't, and the fact that convex bids can not be paradoxically rejected, while non-convex bids can be. As a consequence, rejected convex bids if any impose conditions on market prices, while rejected non-convex bids do not. It is also more generally related to the possibility for rejected bids, convex or not, to impact market prices, a property related to Property 4 in [86], namely the possibility for offline generators to set the market price, see Section D therein.

Finally, still concerning Example 1.2, let us recall that fully indivisible bids (so-called block bids in EU markets) could correspond to real technical conditions of power plants as reported in [78], p.9 concerning "combustion turbine units for which the minimum and maximum outputs are the same".

1.2.3 Convex hull pricing

Convex Hull Pricing (CHP) has first been proposed in [40]. Ring has proposed in [78] to minimize so-called uplifts - a formal definition is provided below - made to market participants to compensate them from the actual losses or opportunity costs they face at the computed market prices. The key contribution in [40] has been to show how to compute market prices minimizing the corresponding required uplifts using Lagrangian duality (see [38] on classical Lagrangian duality results). Prices obtained are sometimes also called Extended Locational Marginal Prices (ELMP), see [100]. The approach is of main current interest in the US where several Independent System Operators are considering its implementation, though it is acknowledged that more research on the topic is still needed, see for example the contribution [86] by researchers at the ISO New England.

Let us consider here a program slightly more general than (SWP). (We come back below to the two instances of (SWP) previously used as toy examples.)

$$\max \sum_{c} B_c(u_c, x_c) \tag{1.32}$$

$$\sum_c \sum_{ic} Q_{ic} x_{ic} = 0 \qquad [\pi] \tag{1.33}$$

$$(u_c, x_c) \in X_c \qquad \forall c \in C \tag{1.34}$$

where X_c is the set describing the technical constraints proper to participant $c \in C$, while $B(.)$ represents the costs of production (with $B < 0$), or the utility of consumption (with $B > 0$) corresponding to production levels $Qx < 0$ or consumption levels $Qx > 0$. Here, $x_c \in \mathbb{R}^I$ is a vector whose components x_{ic} correspond to the respective acceptance levels of several bids ic, or several steps of a step-wise bid curve, all controlled by the binary variable u_c.

Given an optimal solution (u^*, x^*) and a market price π, the uplift of participant $c \in C$ is defined as:

$$\left(\max_{(u_c, x_c) \in X_c} \left[B_c(u_c, x_c) - \pi \sum_{ic \in I_c} Q_{ic} x_{ic} \right] \right) - \left(B_c(u_c^*, x_c^*) - \pi \sum_{ic \in I_c} Q_{ic} x_{ic}^* \right) \tag{1.35}$$

The interpretation is straightforward: the uplift is the gap between the maximum surplus participant c could extract facing the market price π by choosing the best option regarding only its own technical constraints, and the surplus obtained with this same market price and the welfare maximizing solution. This gap is trivially always non-negative.

The contribution [40] has shown that market price(s) such that the sum of all these uplifts is minimal can be obtained by solving the Lagrangian dual of the welfare maximizing program (1.32)-(1.34) where only the balance constraint(s) (1.33) have been dualized. Indeed, [40] considers a context where costs of production to serve a given load y should be minimized, but can straighforwardly be adapted to our context of two-sided auctions with both offers and demands. We review here this result, specializing the presentation in [40] to the present context and notation.

Let us consider the following Lagrangian dual where the balance constraint(s) have been dualized:

$$\min_{\pi} \left[\max_{(u_c, x_c) \in X_c, c \in C} \left[\sum_c B_c(u_c, x_c) - \pi \sum_c \sum_{ic \in I_c} Q_{ic} x_{ic} \right] \right] \tag{1.36}$$

As the lower level program is separable in $c \in C$, the dual can equivalently be written as:

$$z^* = \min_{\pi} \left[\sum_c \max_{(u_c, x_c) \in X_c, c \in C} \left[B_c(u_c, x_c) - \pi \sum_{ic \in I_c} Q_{ic} x_{ic} \right] \right] \tag{1.37}$$

Let us observe that under constraint(s) (1.33), we have:

$$\sum_c B_c(u_c, x_c) = \sum_c B(u_c, x_c) - \pi \sum_c \sum_{ic} Q_{ic} x_{ic} \tag{1.38}$$

Hence (1.32)-(1.34) can equivalently be written with an *arbitrary* π as:

$$w^*(\pi) = w^* = \max \left[\sum_c \left[B_c(u_c, x_c) - \pi \sum_{ic \in I_c} Q_{ic} x_{ic} \right] \right] \tag{1.39}$$

$$\sum_c \sum_{ic} Q_{ic} x_{ic} = 0 \tag{1.40}$$

$$(u_c, x_c) \in X_c \qquad \forall c \in C \tag{1.41}$$

By weak duality, $w^* \leq z^*$. Moreover, as now detailed, the duality gap $DG = z^* - w^*$ exactly corresponds to the sum of the uplifts, and solving the Lagrangian dual hence aims at minimizing these. Again, let (u^*, x^*) be a welfare optimal solution, i.e. solving (1.39)-(1.41), then $DG = z^* - w^*$ can be written as:

$$\min_\pi \left[\sum_c \max_{(u_c, x_c) \in X_c, c \in C} \left[B_c(u_c, x_c) - \pi \sum_{ic \in I_c} Q_{ic} x_{ic} \right] - \sum_c \left[B_c(u_c^*, x_c^*) - \pi \sum_{ic \in I_c} Q_{ic} x_{ic}^* \right] \right] \tag{1.42}$$

or equivalently as:

$$\min_\pi \left[\sum_c \left(\max_{(u_c, x_c) \in X_c, c \in C} \left[B_c(u_c, x_c) - \pi \sum_{ic \in I_c} Q_{ic} x_{ic} \right] - \left(B_c(u_c^*, x_c^*) - \pi \sum_{ic \in I_c} Q_{ic} x_{ic}^* \right) \right) \right] \tag{1.43}$$

This shows that solving the Lagrangian dual with the balance constraints dualized provides prices minimizing the sum of the uplifts.

As first observed in [99] and more recently in [86], [44], solving the Lagrangian dual can in certain situations be reduced to solving the continuous relaxation of the primal (1.32)-(1.34). This holds when this continuous relaxation is itself equivalent to the following "equivalent" formulation (under rather mild assumptions requiring the X_c to be *compact* mixed integer linear sets, see [44]) of the Lagrangian dual to consider:

$$\max \sum_c B_{c,X_c}^{**}(u_c, x_c) \tag{1.44}$$

$$\sum_{c}\sum_{ic} Q_{ic}x_{ic} = 0 \qquad [\pi] \qquad (1.45)$$

$$(u_c, x_c) \in conv(X_c) \qquad \forall c \in C \qquad (1.46)$$

where $conv(X_c)$ denotes the convex hull of the feasible set X_c, and B^{**}_{c,X_c} the convex envelope of B_c taken over X_c, i.e. the lowest concave over-estimator of B_c on $conv(X_c)$, see [44, Theorem 1] in a slightly different setting where costs are minimized instead of welfare maximized. See also the underlying results in [29] used therein, or also [38] for equivalent results in a mixer integer linear setting. In such a case, the optimal dual variables π^* related to the constraint(s) (1.33) of the continuous relaxation, which can often be obtained as a by product when solving this continuous relaxation, provide an optimal solution to the Lagrangian dual (1.36).

Still following [44], assuming that the B_c are linear functions (the *marginal* costs/utilities are constant), B_c and B^{**}_{c,X_c} have the same "functional forms" and it is only required to describe appropriately $conv(X_c)$, hence the interest polyhedral studies of the sets X_c could have, a review of which is given in [44] which also considers quadratic cost functions and their convex envelopes over the X_c.

Regarding this, a key contribution in [99] improving on the more recent literature (e.g. polyhedral studies discussed in [44]) is to provide a tight extended formulation of X_c when all of the following aspects are *simultaneously* considered: start-ups and shut-downs with minimum shut-down periods, minimum/maximum power output levels and ramping constraints. This is done with the purpose of computing efficiently the uplift minimizing market prices.

Let us also note that formulations in [84] of the mixed integer linear feasible set of each market participant are good with regard to the present discussion in the sense that its convex hull is exactly given by the continuous relaxation. Moreover, the welfare objective function is linear. Hence, according to the discussion above, minimizing the duality gap between the primal and the dual of the continuous relaxation aims at minimizing the uplifts. This is done in [84] under the additional constraints of revenue adequacy for producers, in the sense that all the costs of producers must be recovered, i.e. both the start up costs and the marginal costs.

Let us now observe the outcome Convex Hull Pricing (CHP) gives on the Examples described above. In the context of Examples 1.1.1 and 1.1.2, the sets X_c are described by $r_c u_c \leq x_c \leq u_c, u_c \in \{0,1\}$, where r_c is respectively (11/12) and 0. It is trivial to verify that in these cases, $conv(X_c)$ is described by its continuous relaxation, i.e. by $r_c u_c \leq x_c \leq u_c, 0 \leq u_c \leq 1$.

Example 1.1 (continued)

Example 1.1.1 (CHP case):

$$\max_{x,u}\ (10)(300)x_a + (14)(10)x_b - (12)(40)x_c - (13)(100)x_d$$

$10x_a + 14x_b - 12x_c - 13x_d = 0 \quad [\pi^* = 40]$
$x_a \leq 1$
$x_b \leq 1$
$x_c \leq u_c$
$x_c \geq (11/12)u_c$
$x_d \leq 1$
$x \geq 0$
$0 \leq u_c \leq 1$

Example 1.1.2 (CHP case):

$$\max_{x,u}\ (10)(300)x_a + (14)(10)x_b - (12)(40)x_c - (13)(100)x_d - 200u_c$$

$10x_a + 14x_b - 12x_c - 13x_d = 0 \quad [\pi^* = 56.6...]$
$x_a \leq 1$
$x_b \leq 1$
$x_c \leq u_c$
$x_d \leq 1$
$x \geq 0$
$0 \leq u_c \leq 1$

Hence, the outcomes are:

Example 1.1.1 (CHP case):

1. Welfare maximizing solution: fully accept A, accept (11/12) of C, accept (1/14) of B, fully reject D.
2. Market price: $\pi = 40$
3. Uplifts: no uplift for A, C, D, while *B* requires an uplift of 30 EUR.

Example 1.1.2 (CHP case):

1. Welfare maximizing solution: fully accept A, accept (10/12) of C, fully reject B, D.
2. Market price: $\pi = 56.6...$
3. Uplifts: no uplift for A, B, D, while *C* requires an uplift of

$$[(12)56.6.. - ((12)40 + 200)] - [(10)56.6.. - ((10)40 + 200)] = 0 - (-33.333..) = 33.333..$$

One can readily check by solving the corresponding LP that the sum of the uplifts, respectively of 30 EUR and 33.333 EUR, correspond to the duality gaps.

Finally, let us go back to Example 1.2 used to identify a non-intuitive outcome when IP Pricing is used. Solving the continuous relaxation, i.e. leaving aside that D and E can only be fully accepted or fully rejected, the optimal dual variable value of the balance constraint gives an uplift minimizing price of 60 EUR/MW. Only C requires an uplift, as at that price the participant would prefer to have the bid fully accepted, with a surplus of $40(60-40) = 800$ instead of 0, the uplift hence being of 800 EUR. Again, though the outcome may be more intuitive than when IP pricing is used, the price is still influenced by bids rejected in the welfare maximizing solution, as the uplifts could correspond either to "opportunity costs" or to "actual losses" incurred.

1.2.4 European rules

European market rules are intimately related to IP pricing proposed in [71]. They can be generally described as IP Pricing plus the constraints that all start up prices (or commitment prices) of committed plants - or more generally accepted non-convex bids - must be positive, meaning that a non-convex bid cannot be paradoxically accepted, while marginal bids are setting the price. Hence no make-whole payments are needed. Also, non-convex bids can be paradoxically rejected and are not compensated for the corresponding opportunity costs, which corresponds to a situation where the optimal dual variable δ_c associated to the constraint of the form $u_c = 0$ rejecting the bid is positive. Let us recall that according to the IP Pricing rule, rejected bids are not compensated, as the payment of the form $\pi Q x_c - \delta_c u_c$ is null if $u_c = 0$, as observed in Section 1.2.2: the term $-\delta_c u_c$ in the objective just makes the participant indifferent to being committed or not at electricity market prices π: there is no real opportunity costs according to the definition of the payment rule.

To a given extent, this view or precision on European market rules may be one contribution of the present work. More precisely, our third contribution presented in Chapter 4 has shown how to model the fact that no losses could be incurred when start up costs are considered in a way which - as we argue for in the contribution - improves on current practice and previous models proposed for these so-called "minimum profit conditions" in uniform price auctions. This new market model essentially relies on the view of "European rules" in a broad sense as IP Pricing plus the additional conditions just described. It seems more natural, more precisely as it relies on one simple principle: roughly speaking, the condition "no losses incurred" corresponds to stating that there is no "shadow cost" at forcing a non-convex bid to be accepted.

With these rules, in each Example 1.1.1 and 1.1.2, the bid C must be rejected. Once rejected, the market price is increased to 100 EUR/MW, and the bid C is paradoxically rejected in both cases.

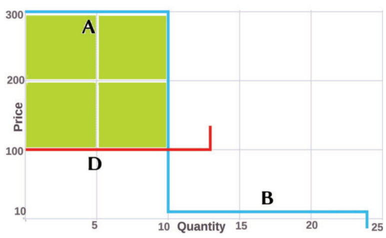

Figure 1.2: A welfare sub-optimal solution satisfying European-like market rules

Let us note that in practice, there are hundreds of non-convex bids and only a limited fraction are paradoxically rejected. However, due to the increase of so-called block orders submitted past years, the number of these paradoxically rejected block orders has substantially increased and is a source of concerns for all stakeholders, see [52].

Let us now consider Example 1.2. The optimal solution under current European market rules is to fully accept A and B, and to reject C, D, E. The market price must then lie in the interval [30; 40] as A is fully accepted while C is a convex bid which is fully rejected and must hence be out-of-the money or at-the-money (i.e. the bid price must be "not good enough").

This can straightforwardly be shown using the following heuristic arguments. First, note that due to the bid quantities at hand, D is accepted if and only if E is accepted as well. However, if both are accepted, as no losses could be incurred, the market price must lie in the interval [60; 90]. In this case, A, B and C are all strictly in-the-money and should be fully accepted, which leads to a contradiction as the balance constraint would be violated. So D and E must be both rejected which sets no particular condition on the outcome (as they are block orders which can be paradoxically rejected), and it is then direct to check that given the bids A, B and C only, the market outcome is the one just described above.

These market rules lead to particularly interesting modelling and algorithmic issues related to peculiar MPEC models, and are the main topic of the present work.

1.3 Outline of the contributions

Part II of the present work presents three standalone articles, i.e. which can be read independently from each other. Each article provides distinct key new results. However, they all rest on a common technique that we now briefly discuss before moving to the summary of the contributions themselves.

As mentioned above, the European rule according to which the convex part of the problem must be at equilibrium (or roughly "marginal units are setting the price"), and which is common to IP Pricing and the European market rules, can be modelled by requiring optimality of the continuous decision variables for the welfare maximization problem where the binary commitment decisions have been arbitrarily fixed. This leads to a simple bilevel programming view on both IP Pricing and the European rules. In the European market rules however, not all arbitrary commitment decisions lead to prices such that no compensations are needed to cover the financial losses of some plants or consumers with indivisibilities or fixed costs. Hence, one must determine commitment decisions (upper level binary decision variable values) such that no losses are incurred at the prices obtained as optimal dual variables of a lower level convex welfare maximizing program with these upper level decision variable values given.

The class of bilevel programs where the lower level program parametrized by binary upper decision variables is convex and for which strong duality holds is not of the most difficult kind to handle, yet several modelling variants can be proposed. The modelling trick used in the articles and now exposed in a slightly more general setting turns out to be of interest both from an optimization and economic interpretation point of view, and seems to improve on previous bilevel programming models proposed to address related pricing problems in day-ahead electricity markets (see references reviewed below).

Let us consider the following bilevel program of the kind just described:

$$\max_{x,u} c_1^T x + d_1^T u \tag{1.47}$$

s.t.

$$A_1x + B_1u \leq b_1 \tag{1.48}$$
$$x \in arg\max_x\{c_2^Tx + d_2^Tu | A_2x + B_2u = b_2, x \geq 0\} \tag{1.49}$$
$$x \geq 0, \tag{1.50}$$
$$x \in \mathbb{R}^I, u \in \{0,1\}^K \tag{1.51}$$

A classical approach would be to use strong duality to write optimality conditions of the lower level program considering u as a parameter in the constraints $A_2x = b_2 - B_2u$. If π denotes the corresponding dual variables, this would first lead to conditions where non-convex quadratic terms appear, corresponding to products of the form $\pi_{l,t}u_c$, where $\pi_{l,t}$ is to interpret in our context as the market price at location l in period t. These products can be linearized exactly using a well-known "McCormick convexification technique", requiring the introduction of many auxiliary continuous variables and constraints with "big M's" (one variable and four constraints per product $\pi_{l,t}u_c$), and other ad hoc linear constraints can then ensure that the upper level decisions u are consistent with the market prices π in the sense that no financial losses are incurred in case of acceptance, i.e. when $u_c = 1$. This is the general approach in [104, 32, 30, 31].

Instead, let us consider a partition $K_r \cup K_a = K$ (r stands for "rejected" and a for "accepted") of the indices of u. In the lower level program (LLP), u acts as a parameter. However, we keep it as a vector of variables and add constraints fixing these variables to some arbitrary values according to the partition, providing the following Restricted LLP (RLLP):

RLLP

$$obj = \max c_2^Tx + d_2^Tu \tag{1.52}$$

s.t.

$$A_2x + B_2u = b_2 \qquad [\pi] \tag{1.53}$$
$$u_{k_r} \leq 0 \qquad \forall k_r \in K_r[\delta^r_{k_r}] \tag{1.54}$$
$$-u_{k_a} \leq -1 \qquad \forall k_a \in K_a[\delta^a_{k_a}] \tag{1.55}$$
$$u \leq 1 \tag{1.56}$$
$$x, u \geq 0 \tag{1.57}$$

The dual of RLLP is well-defined:

RLLP-DUAL

$$dualobj = \min_{\pi,\delta^a,\delta^r} b_2^T\pi - \sum_{k_a} \delta^a_{k_a} \tag{1.58}$$

s.t.

$$
\begin{aligned}
A_2^T \pi &\geq c_2 && && (1.59)\\
(B_2^{k_r})^T \pi + \delta_{k_r}^r &\geq d_2^{k_r} && \forall k_r \in K_r[y_{k_r}] && (1.60)\\
(B_2^{k_a})^T \pi - \delta_{k_a}^a &\geq d_2^{k_a} && \forall k_a \in K_a[u_{k_a}] && (1.61)\\
\delta^a, \delta^r &\geq 0 && && (1.62)
\end{aligned}
$$

Optimality conditions for RLLP are conditions (1.53)-(1.57), (1.59)-(1.62) as well as the condition $obj \geq dualobj$.

The key point is that we obviously do not know the values u in advance, as they should be determined with respect to the upper level objective value and other ad hoc constraints relating u to the prices π where applicable. Yet it is possible to overcome this issue and to reuse optimality conditions for RLLP, with the following simple "modelling trick".

Let us underline here the similarity of the present approach to the idea in [71] of fixing the binary commitment variables to their optimal value to derive optimal dual variable values used to form contracts. Here, the approach is very similar, but the "modelling trick" which follows allows considering *arbitrary* binary variable values determined by an arbitrary "upper level program objective" and the corresponding optimal dual variables to the *restricted* welfare maximization problem where these binary variables are fixed to their value (this restricted welfare maximization program being the "lower level program").

The "modelling trick" is as follows. Let us assume that we know bounds M_k on the possible values of δ_k^a, δ_k^r valid for every partition $K_a \cup K_r$, and consider the following conditions:

BLPFS (Bilevel program feasible set conditions)

$$
\begin{aligned}
A_1 x + B_1 u &\leq b_1 && && (1.63)\\
A_2 x + B_2 u &= b_2 && [\pi] && (1.64)\\
A_2^T \pi &\geq c_2 && [x] && (1.65)\\
(B_2^k)^T \pi + \delta_k^r - \delta_k^a &\geq d_2^k && \forall k \in K[u_k] && (1.66)\\
\delta_k^r &\leq M_k (1 - u_k) && && (1.67)\\
\delta_k^a &\leq M_k u_k && && (1.68)\\
c_2^T x + d_2^T u &\geq b_2^T \pi - \sum_k \delta_k^a && && (1.69)\\
x, u, \delta^a, \delta^r &\geq 0 && && (1.70)\\
x \in \mathbb{R}^I, \pi \in \mathbb{R}^N, u \in \{0,1\}^K, \delta^a, \delta^r &\in \mathbb{R}^K, && && (1.71)
\end{aligned}
$$

It is then not too hard to check that every feasible point in BLPFS will provide a point (u, x) feasible for (1.48)-(1.51) and that any point feasible for (1.48)-(1.51) will provide a point feasible for BLPFS. The general idea is that a point is feasible for BLPFS if and only if it provides a partition $K_a \cup K_r$ determined by the values of the u_c (more precisely, $K_a := \{c|u_c = 1\}$, $K_r := \{c|u_c = 0\}$), such that the optimality conditions for the lower level program in (1.49) hold for this partition, which are given by (1.53)-(1.57), (1.59)-(1.62) and $obj \geq dualobj$ as described above. Values of u_c are determined according to some objective function to choose, e.g. the objective (1.47) if the goal is to solve (1.47)-(1.51).

The reason is essentially that conditions (1.66)-(1.68) are enforcing (1.60)-(1.61) according to this determined partition, and conversely that they are easily shown to hold given a partition and conditions (1.60)-(1.61). Given these dual conditions, the condition (1.69), enforcing equality of the primal and dual objective values, is equivalent to requiring optimality for the lower level program in (1.49).

Theorem 3.1 in Chapter 3 or Theorem 4.2 in Chapter 4 are particular cases of this approach, on which also relies the proof of Theorem 2.2 in Chapter 2 (Theorem 2.3 of the same Chapter uses in the same way strong duality for convex quadratic programs, as described in Appendix A).

More precisely, taking the context previously described of pricing rules in day-ahead markets where marginal pricing is used, optimal dual variable values to the fixing constraints, the values of the variables δ^a, which we call the shadow cost of acceptance, and δ^r, the shadow cost of rejection, are shown in the following chapters to respectively correspond to upper bounds on losses of accepted non-convex bids, and to upper bounds on the opportunity costs of rejected non-convex bids. European-like market rules can hence be specified by simply removing the variables δ^a from the formulations, i.e. setting them to zero, instead of adding ad hoc conditions besides additional auxiliary variables and corresponding constraints used for some linearization where applicable. A simple cleaning-up of the models also allows removing the variables δ^r, providing models for European-like day-ahead market auctions avoiding the use of *any* auxiliary variables or constraints. We now move to the discussion of the content of each Chapter forming the core of the present work.

1.3.1 Chapter 2

The key contributions of the article presented in Chapter 2 is to use the technique just described to prove that MIP formulations avoiding complementarity conditions and *any* auxiliary variables can be proposed to describe European market rules when so-called block orders are considered, which are the only kind of non-convex bids proposed in the Central Western Europe (CWE) region and by Nord Pool. The MILP formulation in the case stepwise linear bid curves are considered is tractable "as is" when given to a high quality MILP solver, and has to some extent attracted the interest of practitioners e.g. modelling markets in power generation companies (private communications). However, the MIQCP formulation arising when piecewise linear bid curves are considered is not tractable "as is" by top solvers today available, and still requires further algorithmic work.

In this respect, another contribution is to show that decomposition techniques known to be efficient [60, 27] can be obtained as a Benders decomposition applied to the new formulations, and that the proofs first developed in a MILP setting where only stepwise bid curves are considered can readily be adapted to the more general setting where piecewise linear bid curves are considered (a MIQCP setting), providing an efficient technique to solve instances in that second setting.

Models and algorithms have been implemented in the algebraic modelling language and software AIMMS and tested using real data kindly provided by EPEX Spot. AIMMS had been chosen because it was, with GAMS, among the only existing algebraic modelling languages allowing to use solver's advanced features such as callbacks to specify so-called lazy constraints or user cuts customizing the branch-and-cut algorithm. Let us observe that the technical computing language Julia together with the modelling layer JuMP, both of which are open source, also allow to use such solver's features, and seem to be the only open source optimization tools allowing this, but were not available at the time of the numerical experiments presented in the article.

1.3.2 Chapter 3

The second contribution emphasizes the bilevel programming view with the approach mentioned above, showing that an *exact* linearization of the so-called ("ad hoc") minimum income conditions (MIC) used in Spain and Portugal can be given in that frame, leading to a MILP model again avoiding the use of *any* auxiliary variables to describe equilibrium for the convex part of the problem and the additional no financial losses conditions stating in particular that start-up costs of dispatched plants must be recovered at the computed market prices (together with a variable cost independent from the marginal cost curves submitted with a given "MIC bid", the relevance of which is questionable). Let us note that a similar exact linearization has been given independently in [32] and in [30, 31] though these contributions introduce many auxiliary variables in the models, cf. the discussion above regarding modelling issues for bilevel programs of the kind considered here. The exact linearization presented in Chapter 3 also leads to a direct and simple economic interpretation regarding the linearization of the "income" used to recover all the costs (start up and variable).

The current approach of the Spanish power exchange OMIE (and hence in the Pan-European market clearing algorithm EUPHEMIA [27]) is to handle the ad hoc non-convex start up costs recovery conditions by relying on a heuristic approach for solving the corresponding problem, first making the condition hold for all committed units by removing those for which the condition is not satisfied (or even for which the probability of satisfaction is too low), then iteratively trying to re-introduce rejected units which are potentially paradoxically rejected, looking at that second stage if welfare is increased with the reintroduction, see [27]. The exact linearization has hence also attracted the interest of practitioners, as the model obtained when only these MIC orders are considered is particularly simple to implement, and shows interesting performances when used "as is" in combination with a few additional considerations. However, the model involving both block orders and MIC orders at the same time remains quite challenging.

This Chapter also examines in the same framework the issue of opportunity costs of so-called paradoxically rejected block orders (PRB). This issue has been first considered empirically in [63] which has studied the number of such PRBs present in the market solutions under various conditions on the size and number of block bids in the input, and has derived from this a likelihood measure of being paradoxically rejected, in relation to these characteristics of the instances.

Regarding this issue and to the best of our knowledge, our contribution in Chapter 3 is the first to provide MIP formulations allowing to compute exactly the minimum total opportunity costs of such PRBs. The formulation is tractable for small to medium instances but becomes much less tractable for large-scale instances of the size encoutered in the whole Central Western Europe (CWE) region (Belgium, France, Germany, Netherlands), though we successfully considered a few of these instances with results presented in [53]. Minimizing the opportunity costs, or also maximizing the traded volume, is compared to the objective of maximizing welfare, and numerical tests are provided using realistic instances corresponding to the Belgian market.

1.3.3 Chapter 4

This Chapter reconsiders the problem of market models with "minimum profit conditions" in uniform price electricity auctions first considered in Chapter 3 when addressing modelling issues of such conditions according to the OMIE-PCR approach in Spain and Portugal (modelling of the so-called "complex orders with minimum income conditions" described above).

Besides the market model used in OMIE-PCR (and hence in EUPHEMIA [27]), several other models have been proposed in the literature, with or without including the start up costs in the welfare objective function.

Our main contribution in that Chapter is to show that appropriately generalizing so-called block orders used in the CWE Region, by adding start up costs in the objective function and using the same modelling and algorithmic ingredients, leads to a market model considering these start up costs recovery conditions in what seems to be a much more relevant way, both from an economic and computational point of view, than current practice in OMIE-PCR or models in the existing literature. The new approach proposed essentially relies on the above mentioned view of the general European pricing approach (roughly equilibrium for market participants and TSOs safe that non-convex bids can be paradoxically rejected) as a variant of IP Pricing where commitment prices are enforced to be positive or null, meaning that one only considers market outcomes where no make-whole payments are needed. Let us emphasize that the European pricing approach viewed in this way is *not* the current approach in OMIE-PCR/EUPHEMIA which instead relies on ad hoc non-convex start up costs recovery conditions, which seems artificial in view of our contribution, as described in more details in the Chapter. Again, by make-whole payments, we mean "uplifts" or "transfer payments" for compensating actual losses at the market prices. This last contribution certainly clarifies the link between IP Pricing and the general European approach, and shows that this last European IP Pricing variant can in a general setting be handled in a highly efficient way, both via a (primal-dual) computationally efficient MIP formulation or via a Benders decomposition with strengthened cuts derived from this same formulation.

Let us mention here another very interesting related result. The revised version of [65] appearing as Chapter 2 in [64] and relying on [54] proposes an analogue of Theorem 4.7 in Chapter 4 in a context which considers general "mixed integer bids", a careful analysis of which shows they encompass the MP bids there proposed (though there is no mention of applications such as the modelling of start up costs and the minimum profit conditions or ramping constraints, etc). As the author indicates, he generalizes the applicability of the cuts of Theorem 6 in [54], similar to those of Theorem 4.7, to these general mixed integer bids (and general convex bids besides) using a completely different technique than the Benders decomposition of Chapter 4.

Again, the contributions in this Chapter have attracted the interest of practitioners, and as highlighted therein, seems to be "the way to go" in terms of bidding product harmonization across Europe, as far as the general European approach relying on uniform prices is considered.

The models and algorithms proposed therein have been implemented in Julia with the package JuMP. Both the source code and the datasets used for the study are freely available online, see [56]. Let us note that Julia and JuMP are both open source tools and the implementation just mentioned has also been the occasion to contribute to JuMP and the CPLEX bindings, by implementing a feature allowing the use of the "local" variant of the "user cut" and "lazy constraint" control callbacks used to customize the branch-and-cut algorithm of the underlying solver. As of now, CPLEX provides the feature but not yet Gurobi. The local variant specifies that the added user cut or lazy constraint only applies at the node where they are added, and the subtree originating from that node.

Finally, let us note that this last Chapter 4, besides its proper contributions, to a great extent subsumes as special cases results presented in the previous Chapters, namely the primal-dual formulations and the Benders decompositions in Chapter 2 (safe the "quadratic cases"), and the exact linearization of the OMIE-PCR minimum income conditions of Chapter 3 which is there reviewed to be further compared to the new approach for modelling minimum profit conditions.

Part II

Three contributions on non-convex uniform price day-ahead electricity auctions

Chapter 2

MIP formulation and algorithms for Central Western Europe market rules

Accepted manuscript version of: Mehdi Madani and Mathieu Van Vyve, Computationally efficient MIP formulation and algorithms for European day-ahead electricity market auctions. European Journal of Operational Research, 242(2):580-593, 2015.

Abstract

We consider the optimization problem implementing current market rules for European day-ahead electricity markets. We propose improved algorithmic approaches for that problem. First, a new MIP formulation is presented which avoids the use of complementarity constraints to express market equilibrium conditions, and also avoids the introduction of auxiliary continuous or binary variables. Instead, we rely on strong duality theory for linear or convex quadratic optimization problems to recover equilibrium constraints. When so-called stepwise bid curves are considered to describe continuous bids, the new formulation allows to take full advantage of state-of-the-art MILP solvers, and in most cases, an optimal solution including market prices can be computed for large-scale instances without any further algorithmic work. Second, the new formulation suggests a Benders-like decomposition procedure. This helps in the case of piecewise linear bid curves that yield quadratic primal and dual objective functions leading to a dense quadratic constraint in the formulation. This procedure essentially strengthens classical Benders cuts locally. Computational experiments using 2011 historical instances for the Central Western Europe region show excellent results. In the linear case, both approaches are very efficient, while for quadratic instances, only the decomposition procedure is appropriate. Finally, when most orders are block orders, and instances are combinatorially very hard, the direct MILP approach is substantially more efficient.

2.1 Introduction

The liberalization of electricity markets in developed countries has led to market design and algorithmic issues addressed now for many years, that still provide with interesting research questions. In Europe, efforts are currently made toward greater integration of electricity markets, with for example the *Price Coupling of Region* (PCR) project supported by the Europex consortium [28]. In the particular case of day-ahead markets and related power exchanges, this integration relies on a common market model whose underlying algorithmic problem is the main subject of this article. This market model has been studied previously, and from different points of view [60, 63, 99, 104]. It is more generally an interesting model for such combinatorial auctions. Our contribution here is the presentation of new algorithmic results relying on a new MIP formulation. Well-known issues in non-convex electricity markets are recalled in Section 2.1.1, European market rules in Section 2.1.2, while Section 2.1.3 details the contribution and structure of the rest of the article.

2.1.1 Day-ahead electricity markets with non-convexities

Day-ahead electricity markets are designed as two-sided auctions in which participants submit orders to buy or sell electric power during some hours of the following day, in some given areas. A market operator collecting these orders is in charge of defining an optimal matching, as well as market prices. Participants agree on a set of rules driving the clearing process, such as rules for bid acceptance and price determination [92]. Order matching and market prices depend in particular on network constraints, and computed prices should ideally support a market equilibrium (for price-taker participants, the market clears for these prices and no excess demand/supply remains, see e.g. [61]). The literature about 'equilibrium among spatially separated markets', thought in a different setting, has been studied in [25, 85, 95]. Samuelson has proposed the term *Cournot-Enke equilibrium* [85] for such kinds of equilibria.

The most complicating feature in day-ahead electricity markets, from both the market design and algorithmic perspectives, is the fact that some orders may be non-convex, in the sense that they yield, in the mathematical formulation of the market clearing problem, objects that don't have the convexity property (e.g. non-convex feasible sets). For example, a participant can submit a block order for which a "fill-or-kill condition" must hold: the order can only be fully accepted or fully rejected. These block orders allow participants to reflect more accurately their production constraints and cost structures. This is mainly due to (i) non-convex production sets (e.g. minimum output levels at which a plant can operate) and (ii) fixed (start-up) costs [63].

A primal program optimizing welfare and defining the optimal selection of bids ensures that the allocation is dispatchable, i.e. respects network security constraints. When there are no non-convexities (e.g. no block orders), it is well-known that optimal dual variables (shadow prices) of this primal program provide with equilibrium prices, as expressed by complementarity constraints relating primal and dual optimal variables. In a mixed integer context, classical strong duality fails, and it is also well-known that a market equilibrium with uniform prices is most of the time mathematically impossible [34, 71, 84, 99, 104]. Uniform prices mean that payments depend only and proportionally on exchanged quantities, via the unique commodity price per time slot and market area, or the price per transmission resource. In particular, this prevents the use of transfer payments for executed bids that would otherwise incur a loss to the bidder. These elements are formally recalled below.

Several interesting approaches have been proposed to clear electricity markets with non-convexities. O'Neill et. al. [71] proposes to solve a primal problem, then to fix integer variables to their optimal value to obtain a convex program whose optimal dual variables are used to form contracts yielding an equilibrium situation. The reference [40] proposes a 'convex hull' approach where transfer payments (uplifts) ensuring a market equilibrium are minimized, while [84] proposes to use uniform prices in such a way that producers recover their costs, and minimizing the duality gap caused by integer constraints. Other propositions are given in [3, 7, 97, 99]. Except [84], all these propositions implement non-uniform pricing schemes.

2.1.2 The current European market model

In Europe, the choice has been made to use uniform prices to avoid discriminating participants. The chosen counterpart is that some *block orders* providing with a gain to the bidder for the computed market prices may be paradoxically rejected, and are not financially compensated. This is for example the solution adopted in coupled markets such as CWE (Central Western Europe market, pooling Belgium, France, Germany, Luxembourg and the Netherlands), which has just been extended to the North Western Europe market (NWE), including Nordic-Baltic countries and Great-Britain. The market clearing optimization problem of these markets is the main topic of this article, see [21] for a full list of requirements. The only requirements of CWE not included in the model we consider below are linked and exclusive block orders, but adding them is straightforward and would only make notations less clear.

The classical way to formulate common European market requirements in a mathematical model is via the addition of dual and complementarity constraints to the primal program defining feasible dispatches. These complementarity constraints form a *subset* of those that would be a consequence of duality theory holding in a well-behaved convex situation (without block orders), see [60, 99]. To handle these formulations, special purpose algorithms have been designed. The two best algorithms so far have been developed independently [21, 60], COSMOS [21] being used in practice in the CWE region since 2009, and on which the algorithm EUPHEMIA [26] used in the NWE region is based. Both are decomposition-based branch-and-bound algorithms solving a main optimization problem and adding cuts to exclude incumbents for which no uniform prices fulfilling auction requirements exist.

On the other hand, so far, *all* mixed integer *linear* programming formulations proposed are using auxiliary variables. For example, [63] uses a formulation with auxiliary binary variables equal to twice the number of hourly (continuous) orders, and is intractable as such when dealing with real large-scale instances. This is similar to the formulation that could be obtained by linearising complementarity constraints, also introducing at least two binary variables per hourly order. A last recent proposition [104] needs a number of auxiliary variables proportional to the number of block orders and submarkets (a given location and time slot), and is also not suitable for large-scale instances, according to numerical results presented.

2.1.3 Contribution and structure of this article

In this article, we provide with a non-trivial reformulation of the European Market Model (EMM) that has several advantages. Precisely, we show how EMM can be modelled as a mixed integer *linear* program *without the introduction of auxiliary variables*, when stepwise bid curves (see definitions below) are considered, beside block orders and network constraints. When piecewise linear bid curves are considered, EMM can be formulated as a mixed integer quadratically constrained program (MIQCP) with one non-linear convex quadratic constraint (with integer variables). In the linear case, the new formulation allows to take full advantage of the power of well-known state-of-the-art MILP solvers, and we are able to solve real large-scale instances without further algorithmic work. In both cases, the new formulation allows the use of a classical Benders decomposition. In particular, we derive in Section 2.4 a Benders-like decomposition procedure with cuts that are stronger than those proposed in [60]. The new cuts are obtained by strengthening classical Benders cuts derived from the new formulation locally (i.e. in branch-and-bound subtrees, using information provided by node solutions). This decomposition algorithm is needed when piecewise linear bid curves are considered, since today solvers are not able to deal with large-scale MIQCP problems of the kind presented below.

The organization of the paper is the following. Section 2.2.1 recalls with the notation of the article well-known results about market equilibrium with uniform prices in the presence of indivisible orders. Section 2.2.2 presents known MPCC formulations [60, 99] of EMM with stepwise bid curves and with general piecewise linear bid curves respectively. Adaptations presented in Section 2.2.2 use the classical Dorn's quadratic programming duality results. In Section 2.3, we present the new MIP formulations in both the linear and the quadratic case. In Section 2.4, we show how to derive a decomposition procedure by the use of a Benders-like argument, again both in the linear and quadratic cases. Finally, Section 2.5 is devoted to computational experiments.

2.2 Non-convex Day-ahead Electricity Markets and the European Market Model

We first fix the notations, describe the market coupling problem [21, 34, 60, 62, 99], and recall why a market equilibrium with uniform prices (see definition below) most of the time doesn't exist in the presence of indivisible orders.

2.2.1 Market equilibrium with uniform prices and non-convexities

Notation and description of the model.

Sets: I is the set of (continuous) hourly orders, J is the set of block orders, and K is the set of network elements (e.g. high voltage power lines or nodes, depending on the network model). The set of bidding locations and time slots are L and T respectively, while N is a set indexing network constraints.

Decision variables: The variables $x_i \in [0,1]$, $i \in I$ and $y_j \in \{0,1\}$, $j \in J$ are decision variables which define the level of execution of a given order. The other variables n_k are used to describe feasible dispatches, according to the network model (see below).

Objective function: The market coupling problem is modelled as a welfare maximisation program. This amounts to maximize the total seller and buyer surplus.

Bid curves and hourly orders

For each time slot $t \in T$ and each location $l \in L$, participants submit a piecewise linear bid curve specified by a finite set of breakpoints $\{(Q_s, P_s)\}_{s \in S}$. These bid curves give the limit (buy or sell) prices, in relation to bid quantities (see Fig. 2.1). Aggregated supply and demand bid curves are then computed, containing all the information needed for the clearing process. Each two consecutive points (Q_s, P_s) and (Q_{s+1}, P_{s+1}) correspond to a *hourly order* i of quantity $Q^i = (Q_{s+1} - Q_s)$. The decision variable x_i determines which fraction of this quantity is executed.

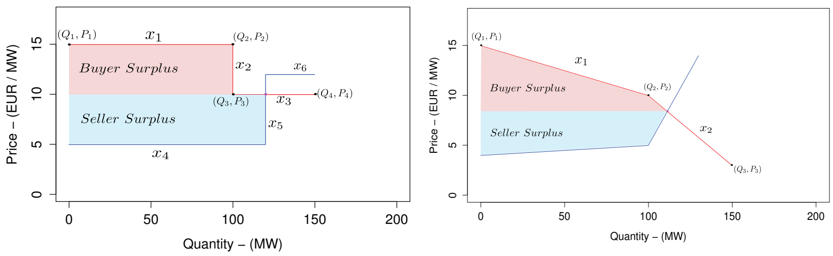

Figure 2.1: stepwise and piecewise linear bid curves

For offer bid curves, $P_s \leq P_{s+1}$ (the curve is non-decreasing), while for demand bid curves, $P_s \geq P_{s+1}$ (the curve is non-increasing). Stepwise bid curves are such that $P_s = P_{s+1}$ if $Q_s \neq Q_{s+1}$ while for piecewise linear bid curves in general, one can have $P_s \neq P_{s+1}$ and $Q_s \neq Q_{s+1}$. A hourly order $i \in I$ always comes from a bid curve corresponding to a given area and a given time slot. However, we will be slightly more general, allowing an order to bid quantities in several areas and time slots. This eases the formal description of the market clearing problem. The parameters associated to a hourly order i are: P^i, $Q^i_{l,t}$, for step orders, and P^i_0, P^i_1, $Q^i_{l,t}$ for interpolated orders which come from general piecewise linear bid curves. For example, according to the left diagram of Fig.1, a decision variable x_1 is associated to an order with bid quantity $Q^1 := (Q_2 - Q_1)$ and a bid price $P^1 := P_1 = P_2$. We deal with piecewise linear bid curves and interpolated orders in Section 2.2.2. Instead of partitioning all orders into the sets of buy orders and sell orders, quantities for buy orders are counted positively, and negatively for sell orders. This is convenient to derive economic interpretations, to state network balance constraints, or the welfare maximizing objective.

Block Orders In practice, a block order $j \in J$ is related to a given area $l \in L$ and specified by a price P^j and quantities Q^j_t for several periods $t \in T$. However, we will again be slightly more general, allowing to consider quantities over multiple areas. The parameters for a block j are P^j and $Q^j_{l,t}$. The *binary* decision variable y_j determines if the order is entirely accepted or entirely rejected. Again, quantities are counted positively for buy orders, and negatively for sell orders.

Linear Network Models

The DC linear network model in [84], or the network models currently used in European day-ahead markets such as the "Available-to-Transfer Capacity" model (ATC) [21, 26, 60] or flow-based models (FB) relying on so-called PTDF matrices [21, 26, 34], are all *linear network representations*. We therefore consider an abstract and very general linear network model to emphasize the fact that the new algorithmic approaches we propose here work with all these usual models. In this setting, the set K contains network elements (inter-connectors or network nodes), variables n_k denote quantities related to each element, and coefficients $e^k_{l,t}$ in (2.4) describe, for a given submarket (l, t), how these elements are related to the net export position of this market. Then, constraints (2.5) describe the most general kind of linear constraints on these network elements. For example, in the case of ATC models, the set K denotes the set of cross-border lines, variables n_k correspond to flows through these lines, and constraints (2.5) would then be capacity constraints on these flows. For flow-based models, they correspond to 'critical network elements' [21].

DA-PRIMAL

$$\max_{x_i, y_j, n_k} \quad \sum_i (\sum_{l,t} Q^i_{l,t} P^i) x_i + \sum_j (\sum_{l,t} Q^j_{l,t} P^j) y_j \tag{2.1}$$

subject to:

$$x_i \leq 1 \qquad \forall i \in I \quad [s_i] \tag{2.2}$$
$$y_j \leq 1 \qquad \forall j \in J \quad [s_j] \tag{2.3}$$
$$\sum_i Q^i_{l,t} x_i + \sum_j Q^j_{l,t} y_j = \sum_k e^k_{l,t} n_k, \ \forall (l,t) \in L \times T \quad [p_{l,t}] \tag{2.4}$$
$$\sum_k a_{m,k} n_k \leq w_m \qquad \forall m \in N \quad [u_m] \tag{2.5}$$
$$x_i, y_j \geq 0, \qquad \forall i \in I, j \in J \tag{2.6}$$
$$y_j \in \mathbb{Z} \qquad \forall j \in J \tag{2.7}$$

Let us consider the *continuous relaxation of* DA-PRIMAL, denoted by **DA-CR**. Its dual is:

DA-CR-DUAL

$$\min_{s_i,s_j,p_{l,t},u_m} \sum_i s_i + \sum_j s_j + \sum_m w_m u_m \tag{2.8}$$

subject to:

$$s_i + \sum_{l,t} Q^i_{l,t} p_{l,t} \geq \sum_{l,t} Q^i_{l,t} P^i \qquad \forall i \in I \quad [x_i] \tag{2.9}$$

$$s_j + \sum_{l,t} Q^j_{l,t} p_{l,t} \geq \sum_{l,t} Q^j_{l,t} P^j \qquad \forall j \in J \ [y_j] \tag{2.10}$$

$$\sum_m a_{m,k} u_m - \sum_{l,t} e^k_{l,t} p_{l,t} = 0 \qquad \forall k \in K \ [n_k] \tag{2.11}$$

$$s_i, s_j, u_m \geq 0 \qquad \forall i \in I, j \in J, m \in N \tag{2.12}$$

And the related complementarity conditions are:

DA-CC

$$s_i(1 - x_i) = 0 \qquad \forall i \in I \tag{2.13}$$

$$s_j(1 - y_j) = 0 \qquad \forall j \in J \tag{2.14}$$

$$x_i(s_i + \sum_{l,t} Q^i_{l,t} p_{l,t} - \sum_{l,t} Q^i_{l,t} P^i) = 0 \qquad \forall i \in I \tag{2.15}$$

$$y_j(s_j + \sum_{l,t} Q^j_{l,t} p_{l,t} - \sum_{l,t} Q^j_{l,t} P^j) = 0 \qquad \forall j \in J \tag{2.16}$$

$$u_m(\sum_k a_{m,k} n_k - w_m) = 0 \qquad \forall m \in N \tag{2.17}$$

Market equilibrium with uniform prices: definitions and classical results

Solving the market coupling problem implies to find prices supporting, ideally, a market equilibrium. *In a convex situation where all orders are continuous orders*, classical shadow prices ($p_{l,t}, u_m$ here) are *uniform equilibrium prices* (see definitions below) for the optimal bid allocation. We review here this equilibrium property implied by dual and complementarity constraints.

Definition 2.1 (Uniform prices)**.** *A price system for the auction process will be called a system of* uniform prices *if all money transfers between market participants depend only and proportionally on a single commodity price $p_{l,t}$ per location $l \in L$ and time slot $t \in T$.*
Definition 2.2 (Bid surplus)**.** *Let $p_{l,t}$ be uniform prices. A* hourly or block order $i \in I \cup J$ *is said to be:*

(i) in-the-money (ITM) if $\sum_{l,t} Q^i_{l,t}(P^i - p_{l,t}) > 0$. This essentially means that for the given market prices, the bidder (producer or consumer) has an economic surplus. For hourly orders, since an order has a precise location and time slot, the sum has only one term $Q^i_{l_0,t_0}(P^i - p_{l_0,t_0})$. So if $Q^i_{l_0,t_0} < 0$ (sell order), then $P^i < p_{l_0,t_0}$ and if $Q^i_{l_0,t_0} > 0$ (buy order), then $P^i > p_{l_0,t_0}$.

(ii) at-the-money if $\sum_{l,t} Q^i_{l,t}(P^i - p_{l,t}) = 0$*. For hourly orders, this means* $Q^i_{l_0,t_0}(P^i - p_{l_0,t_0}) = 0$*, and (assuming* $Q^i_{l_0,t_0} \neq 0$*),* $P^i = p_{l_0,t_0}$*: both bid and market prices are equal.*

(iii) out-of-the-money if it is not ITM nor ATM (i.e. its execution would incur a loss): $\sum_{l,t} Q^i_{l,t}(P^i - p_{l,t}) < 0$.

Definition 2.3 (Network equilibrium and optimality conditions for the TSO problem)**.** *For a given primal solution* (x, y, n) *and prices* p*, there is a* network equilibrium *if there exist network resource prices* $u_m \geq 0, m \in N$ *such that* (2.11) *and* (2.17) *hold. When the abstract network model is specialized to usual linear network models (e.g. ATC or Flow-based [21]), these conditions mean that transmission of electricity has a value only if transmission capacities are scarce. The network resource prices are given by the variables* $u_m \geq 0, m \in N$*. In that situation, for the given market prices, the TSO cannot be more profitable by transporting more or less electricity than in the current situation.*

Let us note that for an ATC network model [21], a price difference $p_{l,t} < p_{k,t}$ can only occur when the line from the market with lower price to the market with higher price is congested. In that case the price difference equals the congestion price.

Definition 2.4 (Market equilibrium with uniform prices)**.** *Let* (x^*, y^*, n^*) *be a feasible point for DA-PRIMAL, i.e. satisfying (2.2)-(2.7), and* p^* *uniform prices. Then* (x^*, y^*, n^*) *and* p^* *form a* market equilibrium with uniform prices *if and only if:*

I. (a) Fully executed orders are ITM or ATM, (b) fractionally executed orders are ATM, (c) rejected orders are ATM or OTM.

II. Network equilibrium conditions, given at Definition 2.3, are satisfied.

The definition essentially means that for the given prices $p^*_{l,t}$, no excess demand or excess supply remains, and no other level of execution could be more profitable to the bidders or the TSO. For example, for a given order i: $\forall x_i \in [0, 1]$, $\sum_{l,t} Q^i_{l,t}(P^i_{l,t} - p^*_{l,t})x_i \leq \sum_{l,t} Q^i_{l,t}(P^i_{l,t} - p^*_{l,t})x^*_i$.

The two following results are then classical. Proofs are given in appendix.

Theorem 2.1. *Let* (x^*, y^*, n^*) *be a feasible point for DA-PRIMAL, i.e. satisfying* $(2.2) - (2.7)$.

(I) A market equilibrium with uniform prices exists if and only if there are dual variables $s^*_i, s^*_j, p^*_{l,t}, u^*_m$ *such that dual and complementarity constraints* $(2.9) - (2.17)$ *are satisfied.*

(II) This is the case if and only if (x^*, y^*, n^*) *is optimal for the continuous relaxation DA-CR,* $s^*_i, s^*_j, p^*_{l,t}, u^*_m$ *is optimal for the dual DA-CR-DUAL, and both objective values are equal.*

Corollary 2.1. *Consider DA-PRIMAL (i.e. the primal program, including integer constraints). There exists a solution* (x, y, n) *and uniform prices* $p^*_{l,t}$ *forming a market equilibrium if and only if the continuous relaxation DA-CR admits an optimal solution* (x^*_i, y^*_j, n^*_k) *with* $y^*_j \in \mathbb{Z}$, $\forall j \in J$.

Theorem 2.1 and Corollary 2.1 show that a market equilibrium with uniform prices exists if and only if there is no duality gap caused by integer constraints, which is not the case for most instances.

2.2.2 The European Market Model: classical MPCC formulation

We describe here the model used everyday in Europe to clear day-ahead markets [21] and recall its classical MPCC formulation [60, 99]. Since a market equilibrium with uniform prices often doesn't exist, the solution adopted in Europe is to compute uniform prices such that hourly orders and the network are both 'at equilibrium', and only *paradoxically rejected block orders* (PRB) are tolerated as deviations from a perfect market equilibrium. These PRB, if executed with the given market prices, would provide with a gain to the bidder, but are rejected by the market operator.

Market clearing price range condition: The following condition is a technical condition (used in Theorems 2.2 and 2.3), *ruling out arbitrarily large market clearing prices, while allowing them to be sufficiently large not to exclude any relevant market clearing solution* (e.g. using Lemma 2.1 in [72] and bounds on all input data assumed to be rational numbers of a priori limited precision):

$$p_{l,t} \in [-\bar{P}, \bar{P}] \qquad \forall (l,t) \in L \times T \tag{2.18}$$

However, let us note that in practice, bid prices are constrained to lie in a range $[-\bar{P}_{bid}, \bar{P}_{bid}]$, and $\bar{P}$ is set to $\bar{P}_{bid}$, which is usually fine, though in some rare cases renders the problem infeasible. See also [69] on related issues about ranges for bid prices and market clearing prices.

Definition 2.5 (European prices)**.** *The main requirements of EMM are: (i) uniform prices, (ii) OTM orders must be rejected (block and hourly orders as well), (iii) ITM* hourly *orders must be accepted, (iv) network equilibrium constraints must be satisfied and (v) computed market prices must lie in a specified interval* $[-\bar{P}, \bar{P}]$.

EMM with stepwise linear bid curves

The classical way to state a maximisation problem formulating European market rules is to write primal, dual and all complementarity constraints excepted those of type (2.14) (see e.g. [99]). According to the interpretation given above, this corresponds to drop for block orders the requirement that they should be accepted if they are ITM. This yields a mathematical program with complementarity conditions (MPCC).

EMM-MPCC:

$$\max_{x,y,n,p,u} \quad \sum_i (\sum_{l,t} Q^i_{l,t} P^i) x_i + \sum_j (\sum_{l,t} Q^j_{l,t} P^j) y_j \tag{2.19}$$

subject to constraints: Primal and dual constraints: (2.2) − (2.7), (2.9) − (2.12), the price range condition (2.18) and the *subset* of complementarity constraints: (2.13), (2.15)− (2.17), but *not subject to* complementarity constraints of type (2.14). □

This formulation involves non-linear constraints, and instances, which are very large in practice, would be hard or even impossible to solve as it is with current MINLP solvers. For this reason, special purpose algorithms have been designed (see above).

EMM with piecewise linear bid curves and quadratic programming duality

The adaptation needed to consider piecewise linear bid curves rely on duality results for convex quadratic programs. We first recall the market equilibrium conditions expressed by dual and complementarity constraints in this different setting. Let us consider the right diagram of Fig.1. A segment of a piecewise linear bid curve now corresponds to a hourly order with a price P_0 at which the order starts to be accepted, a price P_1 at which it is fully accepted, and bid prices for intermediate quantities are obtained by linear interpolation (see e.g. the hourly order associated with the first segment and variable x_1 in the right diagram of Fig.1). For a sell order i, $P_1^i \geq P_0^i$ (because the bid curve is non-decreasing), while for a buy order i, $P_1^i \leq P_0^i$ (because the bid curve is non-increasing). The objective function giving the welfare now depends quadratically on the levels of executions x_i (cf. the area below a bid curve segment limited by an execution level x_i).

DA-PRIMAL-QUAD

$$\max_{x_i,y_j,n_k} \sum_i (\sum_{l,t} Q^i_{l,t} P^i_0 x_i + \sum_{l,t} Q^i_{l,t}(P^i_1 - P^i_0)\frac{x_i^2}{2}) + \sum_j (\sum_{l,t} Q^j_{l,t} P^j_{l,t}) y_j \tag{2.20}$$

subject to (primal constraints remain unchanged): (2.2) − (2.7)

The objective function is trivially concave as factors $Q^i_{l,t}(P^i_1 - P^i_0)$ are non-positive, and the *continuous relaxation of DA-PRIMAL-QUAD*, noted DA-QUAD-CR, is a convex quadratic program. Strong duality still holds in this setting (see e.g. [22, 39, 98]). Compared to the dual DA-CR-DUAL above, the dual objective function has additional quadratic terms:

DA-QUAD-CR-DUAL

$$\min_{s_i,s_j,p_{l,t},u_m,v_i} \sum_i s_i + \sum_j s_j + \sum_m w_m u_m - \sum_i (\sum_{l,t} Q^i_{l,t}(P^i_1 - P^i_0))\frac{v_i^2}{2} \tag{2.21}$$

and feasibility inequalities of type (2.9) in DA-CR-DUAL have an additional linear term:

$$s_i + \sum_{l,t} Q^i_{l,t} p_{l,t} \geq \sum_{l,t} Q^i_{l,t} P^i_0 \ + \sum_{l,t} Q^i_{l,t}(P^i_1 - P^i_0) v_i \qquad \forall i \in I \quad [x_i], \tag{2.22}$$

while other constraints (2.10) $-$ (2.12) remain unchanged.

Lemma 2.1. *If (x, y, n) is an optimal solution of the continuous relaxation DA-QUAD-CR, there exists a dual optimal solution $(s_i, s_j, p_{l,t}, u_m, v_i)$ such that $v_i = x_i \ \ \forall i \in I$.*

Proof. It is a direct application of Dorn's quadratic duality theorem (see e.g. [22], [39] or [98]). □

When stating primal, dual and complementarity constraints, or primal and dual constraints with equality of objective functions, we will thus be allowed to replace v_i with x_i, since such a solution of the dual program exists. This is indeed needed for the economic interpretations.

Complementarity Constraints

Compared to the previous case with stepwise bid curves and complementarity constraints (2.13)$-$(2.17), one has just to replace complementarity constraints of type (2.15) by:

$$x_i(s_i + \sum_{l,t} Q^i_{l,t} p_{l,t} - \sum_{l,t} Q^i_{l,t} P^i_0 - \sum_{l,t} Q^i_{l,t}(P^i_1 - P^i_0)x_i) = 0 \tag{2.23}$$

Lemma 2.1 has been used to replace v_i by x_i. Using this, the definition of ITM, ATM or OTM can be adapted for these interpolated hourly orders, as well as Theorem 2.1 and Corollary 2.1:

Definition 2.6 (Bid surplus for hourly orders, quadratic setting)**.** *Let $p_{l,t}$ be a set of uniform prices and x_i the execution level of the hourly order i. The order is said to be:*

(i) in-the-money (ITM) if $\sum_{l,t} Q^i_{l,t}(P^i_1 - p_{l,t}) > 0$. Recalling that there is in practice only one term in the sum and the sign convention for quantities, this means that $p_{l,t} > P^i_1$ for sell orders and $p_{l,t} < P^i_1$ for buy orders.

(ii) at-the-money (ATM) if $\sum_{l,t} Q^i_{l,t} P^i_0 + \sum_{l,t} Q^i_{l,t}(P^i_1 - P^i_0)x_i = \sum_{l,t} Q^i_{l,t} p_{l,t}$, with x_i the execution level. In this case, again considering orders for one market segment (l_0, t_0): p_{l_0,t_0} is equal to $P^i_0 + (P^i_1 - P^i_0)x_i$, i.e. the market price equals the interpolated bid price given by the piecewise linear bid curve for this level of execution.

(iii) out-of-the-money (OTM) if $\sum_{l,t} Q^i_{l,t} P^i_0 < \sum_{l,t} Q^i_{l,t} p_{l,t}$. This means that $p_{l,t} < P^i_0$ for sell orders and $p_{l,t} > P^i_0$ for buy orders.

The definition 2.4 of a market equilibrium with uniform prices given above is still valid when considering interpolated hourly orders and Definition 2.6. The adaptations of Theorem 2.1 and Corollary 2.1 are straightforward and only concern constraints related to hourly orders: just replace dual and complementarity constraints (2.9) by (2.22) and (2.15) by (2.23), respectively. Lemma A3 in appendix details what needs to be adapted in Lemmas A1 and A2 to prove the analogue of Theorem 2.1 for markets with piecewise linear bid curves.

EMM-QUAD-MPCC:

As in the previous case, a MPCC formulation here denoted EMM-QUAD-MPCC can be given, by just replacing in the formulation EMM-MPCC the welfare objective function by the quadratic one (2.20), as well as dual constraints (2.9) by (2.22) (using Lemma 2.1 to replace v_i by x_i as above), and complementarity constraints (2.15) by (2.23).

2.3 New MIP Formulations

When stepwise bid curves are considered beside block orders, the new formulation consists in an *exact* linearisation of EMM-MPCC, *avoiding the use of any auxiliary variable.* When more generally piecewise linear bid curves are considered as well, EMM-QUAD-MPCC can be reformulated as a MIQCP with one convex quadratic constraint (with integer variables). The advantage of these new formulations is twofold. First, in the MILP case, it allows to solve real large-scale instances without any special purpose algorithm, using state-of-the-art MILP solvers. Second, one can derive in both cases a Benders-like decomposition algorithm, particularly useful to deal with piecewise bid curves where a MIQCP must be solved. The strengthened Benders cuts obtained improve on the exact cuts provided in [60], also using Benders decomposition to solve a formulation similar to the formulation EMM-QUAD-MPCC. These cuts are derived in Section 2.4.

2.3.1 EMM with stepwise bid curves as a MILP

The new formulation involves all primal and dual constraints as well as an equality of objective functions condition (instead of a *subset of* complementarity constraints). To ensure the existence of a solution and to reflect the choice of allowing some ITM block orders to be rejected, dual constraints of type (2.10) are modified, yielding constraints of type (2.31) below, where the M_j are large enough to deactivate the constraint when $y_j = 0$, and chosen in such a way that constraints (2.31) don't reduce the range of prices given by the market rule (2.18). Using the price range conditions above, it is direct to see that $M_j := \sum_{l,t} 2\bar{P}|Q^j_{l,t}|$ is a sufficient choice.

EMM-MILP:

$$\max_{x,y,n,p,u,s} \sum_i (\sum_{l,t} Q^i_{l,t} P^i) x_i + \sum_j (\sum_{l,t} Q^j_{l,t} P^j) y_j \tag{2.24}$$

subject to:

$$p_{l,t} \in [-\bar{P}, \bar{P}] \qquad \forall (l,t) \in L \times T \tag{2.18}$$

$$\sum_i (\sum_{l,t} Q^i_{l,t} P^i) x_i + \sum_j (\sum_{l,t} Q^j_{l,t} P^j) y_j \geq \sum_i s_i + \sum_j s_j + \sum_m w_m u_m \tag{2.25}$$

$$x_i \leq 1 \qquad \forall i \in I \tag{2.26}$$

$$y_j \leq 1 \qquad \forall j \in J \tag{2.27}$$

$$\sum_i Q^i_{l,t} x_i + \sum_j Q^j_{l,t} y_j = \sum_k e^k_{l,t} n_k, \qquad \forall (l,t) \in L \times T \tag{2.28}$$

$$\sum_k a_{m,k} n_k \leq w_m \qquad \forall m \in N \tag{2.29}$$

$$s_i + \sum_{l,t} Q^i_{l,t} p_{l,t} \geq \sum_{l,t} Q^i_{l,t} P^i \qquad \forall i \in I \tag{2.30}$$

$$s_j + \sum_{l,t} Q^j_{l,t} p_{l,t} \geq \sum_{l,t} Q^j_{l,t} P^j - M_j(1 - y_j) \qquad \forall j \in J \tag{2.31}$$

$$\sum_m a_{m,k} u_m - \sum_{l,t} e^k_{l,t} p_{l,t} = 0 \qquad \forall k \in K \tag{2.32}$$

$$x_i, y_j, s_i, s_j, u_m \ \geq 0, \quad y_j \in \mathbb{Z} \qquad \forall i \in I, \forall j \in J, \forall m \in N \tag{2.33}$$

Theorem 2.2. *The formulation EMM-MPCC and the new MILP formulation EMM-MILP are equivalent in the following sense: (i) for each feasible point (x, y, n, p, u, s) of EMM-MPCC, there exists $\tilde{s}$ such that $(x, y, n, p, u, \tilde{s})$ is feasible for EMM-MILP.*

(ii) Conversely, for each feasible point of EMM-MILP (x, y, n, p, u, s), there exists $\tilde{s}$ such that $(x, y, n, p, u, \tilde{s})$ is feasible for EMM-MPCC.

Proof. See appendix. □

2.3.2 EMM with piecewise linear bid curves: new MIQCP formulation

We give here the new formulation analogue to the one presented above, where (2.25) and (2.30) are replaced by their quadratic analogues (2.35) and (2.40) respectively. For the sake of clarity, we rewrite here all constraints in extenso, as they will be used in Section 2.4.

EMM-QUAD-MIQCP:

$$\max \sum_i (\sum_{l,t} Q^i_{l,t} P^i_0 x_i + \sum_{l,t} Q^i_{l,t} (P^i_1 - P^i_0) \frac{x_i^2}{2}) + \sum_j (\sum_{l,t} Q^j_{l,t} P^j_{l,t}) y_j \tag{2.34}$$

subject to:

$$p_{l,t} \in [-\bar{P}, \bar{P}] \qquad \forall (l,t) \in L \times T \qquad (2.18)$$

$$\sum_i (\sum_{l,t} Q^i_{l,t} P^i_0 x_i + \sum_{l,t} Q^i_{l,t}(P^i_1 - P^i_0)\frac{x_i^2}{2}) + \sum_j (\sum_{l,t} Q^j_{l,t} P^j_{l,t}) y_j \geq$$

$$\sum_i s_i + \sum_j s_j + \sum_m w_m u_m - \sum_{l,t} Q^i_{l,t}(P^i_1 - P^i_0)\frac{x_i^2}{2}) \qquad (2.35)$$

$$x_i \leq 1 \qquad \forall i \in I \qquad (2.36)$$

$$y_j \leq 1 \qquad \forall j \in J \qquad (2.37)$$

$$\sum_i Q^i_{l,t} x_i + \sum_j Q^j_{l,t} y_j = \sum_k e^k_{l,t} n_k, \qquad \forall (l,t) \in A \times T \qquad (2.38)$$

$$\sum_k a_{m,k} n_k \leq w_m \qquad \forall m \in N \qquad (2.39)$$

$$s_i + \sum_{l,t} Q^i_{l,t} p_{l,t} \geq \sum_{l,t} Q^i_{l,t} P^i + \sum_{l,t} Q^i_{l,t}(P^i_1 - P^i_0) x_i \qquad \forall i \in I \qquad (2.40)$$

$$s_j + \sum_{l,t} Q^j_{l,t} p_{l,t} \geq \sum_{l,t} Q^j_{l,t} P^j - M_j(1 - y_j) \qquad \forall j \in J \qquad (2.41)$$

$$\sum_m a_{m,k} u_m - \sum_{l,t} e^k_{l,t} p_{l,t} = 0 \qquad \forall k \in K \qquad (2.42)$$

$$x_i, y_j, s_i, s_j, u_m \geq 0 \quad y_j \in \mathbb{Z} \qquad \forall i \in I, \forall j \in J, \forall m \in N \qquad (2.43)$$

Theorem 2.3. *Both EMM-QUAD-MPCC and EMM-QUAD-MIQCP formulations are equivalent in the following sense: (i) for each feasible point (x, y, n, p, u, s) of EMM-QUAD-MPCC, there exists $\tilde{s}$ such that $(x, y, n, p, u, \tilde{s})$ is feasible for EMM-QUAD-MIQCP.*

(ii) Conversely, for each feasible point of EMM-QUAD-MIQCP (x, y, n, p, u, s), there exists $\tilde{s}$ such that $(x, y, n, p, u, \tilde{s})$ is feasible for EMM-QUAD-MPCC.

Proof. See appendix. □

2.4 A Decomposition Method

Here, we derive from our new formulation a Benders-like decomposition algorithm, where cuts are added within the branch and bound tree used to solve the primal program DA-PRIMAL or DA-QUAD-PRIMAL, when no European prices exist for a given node solution incumbent. By a node solution incumbent, we mean a new best primal feasible solution obtained as the optimal solution to the LP relaxation at a given node of the branch and bound tree. It is in this sense similar to the two best algorithms [21, 60] mentioned earlier (the algorithm briefly described in [21] is the proprietary algorithm in charge of solving CWE market instances on which further technical developments for European market integration relies). The cuts we propose are stronger than the cuts proposed in [60]. Quadratic instances of the new formulation cannot be solved with today solvers, and such an algorithm is needed to solve efficiently real life instances. The derivation is in a first stage very close to [5], and in particular relies on the Farkas lemma and the finiteness of the number of vertices of the polytope defining the feasible set of a so-called slave program.

To simplify notations, in all this section, only one area and one time slot are considered, but all of what follows can be carried out with several areas, time slots, and a network model. We are sometimes referring to corresponding previous constraints involving the network structure, but the adaptations needed are minor and direct. *In all this section, we neglect the price range condition (2.18), assuming as explained above that $\bar{P}$ is sufficiently large not to exclude any relevant solution.* Hence, the decomposition as presented here actually solves EMM-MILP minus (2.18). Exposition is made first in the linear case. It is shown hereafter how to handle the quadratic case in a similar way.

2.4.1 The linear case

Consider the primal problem DA-PRIMAL of Section 2.2:

$$\max \quad obj := \sum_i Q^i P^i x_i + \sum_j Q^j P^j y_j,$$

subject to (2.2) − (2.7), with only one market (no network and only one period), i.e. with N empty and $\sum_k e^k_{l,t} n_k := 0$, to simplify notations.

Consider now a branch-and-bound procedure and let (x^*, y^*) be a node solution incumbent. According to constraints (2.25), (2.29) − (2.33) of the new formulation EMM-MILP, a supporting European price exists if and only if there exist s_i, s_j, p_m (p_m denoting the market price) such that:

$$- s_i - Q^i p_m \leq -Q^i P^i \qquad \forall i \in I \quad [u_i] \qquad (2.44)$$

$$- s_j - Q^j p_m \leq -Q^j P^j \; + \; M_j(1 - y^*_j) \qquad \forall j \in J \quad [u_j] \qquad (2.45)$$

$$\sum_i s_i + \sum_j s_j \leq obj^* \qquad [u_\sigma] \qquad (2.46)$$

$$s_i, s_j \geq 0 \qquad (2.47)$$

where obj^* denotes the corresponding optimal value $\sum_i Q^i P^i x^*_i + \sum_j Q^j P^j y^*_j$ of the objective function for this node solution.

According to the Farkas lemma [87], a solution to a linear system $Ax \leq b, x \geq 0$ exists if and only if $\forall y \geq 0,\; yA \geq 0 \Rightarrow yb \geq 0$. The existence of a European price is so equivalent to:

$$\sum_i -Q^i P^i u_i + \sum_j -Q^j P^j u_j + \sum_j M_j(1-y_j^*)u_j + obj^* u_\sigma \geq 0$$

$\forall (u_i, u_j, u_\sigma)$ such that:

$$-u_i + u_\sigma \geq 0 \quad (2.48)$$
$$-u_j + u_\sigma \geq 0 \quad (2.49)$$
$$-\sum_i Q^i u_i - \sum_j Q^j u_j = 0 \quad [p_m] \quad (2.50)$$
$$u_i, u_j, u_\sigma \geq 0, \quad (2.51)$$

The condition being trivially satisfied if $u_\sigma = 0$, we can assume $u_\sigma := 1$ (normalization).

Rearranging terms, a European price exists if and only if :

$$\sum_i Q^i P^i u_i + \sum_j Q^j P^j u_j - \sum_j M_j(1-y_j^*)u_j \leq \sum_i Q^i P^i x_i^* + \sum_j Q^j P^j y_j^*$$

$\forall (u_i, u_j) \in P$ with P defined by the constraints:

$$u_i \leq 1 \quad (2.52)$$
$$u_j \leq 1 \quad (2.53)$$
$$\sum_i Q^i u_i + \sum_j Q^j u_j = 0 \quad (2.54)$$
$$u_i, u_j \geq 0 \quad (2.55)$$

This yields:
Lemma 2.2. *For a given node solution (x_i^*, y_j^*) , a European price exists if and only if:*

$$\max_{(u_i,u_j)\in P} \sum_i Q^i P^i u_i + \sum_j Q^j P^j u_j - \sum_j M_j(1-y_j^*)u_j \leq obj^*. \quad (2.56)$$

Lemma 2.3. *Let (u_i^*, u_j^*) denotes an optimal solution to the optimization problem in the left-hand side of (2.56), stated in Lemma 2.2. Then $y_j^* = 0 \Rightarrow u_j^* = 0$.*

Proof. Because the numbers M_j are very (arbitrarily) large fixed numbers, if $y_j^* = 0$, the objective could not be optimal for any vertex of P with $u_j \neq 0$. Accordingly, this could also be shown by noting that constraints of *the dual* of the left-hand side program are constraints (2.30) – (2.33) with $y_j = y_j^*$ fixed, and that u_j are the shadow prices of constraints (2.31). If $y_j^* = 0$, the corresponding constraint (2.31) is not binding because of the choice of the M_j ($s_j \geq 0$ is binding instead), and $u_j = 0$. □

Note that the numbers M_j are used here only in proofs, and will be avoided in the final procedure described below. The criterion of Lemma 2.2 admits a nice interpretation.

Let us consider the continuous relaxation DA-CR *with the additional constraints that all blocks at 0 in the node solution are held at 0 in this relaxation.* From the two previous lemmas, it follows that the optimal objective value for this modified relaxation cannot be greater than the current node solution value:

Theorem 2.4. *For a node solution (x^*, y^*), consider the polytope $P^{F^*} := P \cap \{(u_i, u_j)|\ u_j = 0\ \ if\ \ y_j^* = 0\}$. Then a European price exists if and only if*

$$\max_{(u_i,u_j)\in P^F} \sum_i Q^i P^i u_i + \sum_j Q^j P^j u_j \leq obj^*, \tag{2.57}$$

where obj^* denotes the optimal value associated with the node solution, in which case equality holds as well. □

Proof. It is a direct corollary of Lemma 2.2 and Lemma 2.3. Also, since (x^*, y^*) is feasible for the left-hand side, if the inequality holds, equality holds as well. □

When no European price exists, Lemma 2.2 provides with a classical Benders cut, where (u_i^*, u_j^*) is optimal for the the left-hand side of (2.56):

Classical Benders cut, linear case

$$\sum_i Q^i P^i u_i^* + \sum_j Q^j P^j u_j^* - \sum_j M_j(1-y_j)u_j^* \leq \sum_i Q^i P^i x_i + \sum_j Q^j P^j y_j. \tag{2.58}$$

Let us note that these cuts are globally valid. Moreover, suppose that (2.58) is violated by (x^*, y^*). As in EMM-MILP (or EMM-MPCC), the welfare $\sum_i Q^i P^i x_i + \sum_j Q^j P^j y_j$ is univocally determined by the selection of accepted and rejected block orders (see DA-FixedBlocks and proof of Theorem 2.2 in appendix or also Corollary 6.1 in [60]), any other solution (x, y) with $y = y^*$ would also violate (2.58), since the right-hand and left-hand sides would be the same. This allows to recover the "no-good" cuts proposed in [60]: $\sum_{j|y_j^*=1} (1 - y_j) + \sum_{j|y_j^*=0} y_j \geq 1$. At this stage, we can already note that there is a finite number of inequalities (2.58) to add, which is bounded by the number of vertices of the bounded polyhedron P. These cuts are not strong as such because of the M_j (a small change in the variables allows to satisfy the new constraint when LP relaxations are considered), but it is possible to strengthen them and the "no-good" cuts locally:

Theorem 2.5 (Strengthened Benders cuts)**.** *For each node solution in the branch-and-bound for which no European price exists, the inequality $\sum_{j|y_j^*=1} (1 - y_j) \geq 1$ is valid in the subtree.*

Proof. Consider any other feasible solution (x, y) in the subtree originating from the current node solution (x_i^*, y_j^*) for which no European price exists, that is for which, according to Lemma 2.2:

$$obj^* < \sum_i Q^i P^i u_i^* + \sum_j Q^j P^j u_j^* - \sum_j M_j (1 - y_j^*) u_j^*,$$

where (u_i^*, u_j^*) is optimal for the left-hand side of (2.56). According to Lemma 2.3, this inequality reduces to:

$$obj^* < \sum_i Q^i P^i u_i^* + \sum_j Q^j P^j u_j^*$$

If $\sum_{j|y_j^*=1} (1 - y_j) = 0$ for the new feasible solution (x, y), using Lemma 2.3, $\sum_j M_j (1 - y_j) u_j^*$ $= 0$ and the Benders cut (2.58) valid for (x, y) reduces to:

$$\sum_i Q^i P^i u_i^* + \sum_j Q^j P^j u_j^* \leq \sum_i Q^i P^i x_i + \sum_j Q^j P^j y_j. \tag{2.59}$$

Using the fact that this other solution (x, y) *is in the subtree originating from* (x^*, y^*),

$$\sum_i Q^i P^i x_i + \sum_j Q^j P^j y_j = obj \leq obj^* < \sum_i Q^i P^i u_i^* + \sum_j Q^j P^j u_j^*,$$

which violates (2.59), and no such new solution can admit a European price. □

2.4.2 The quadratic case

Again, for a node solution (x^*, y^*) in a branch-and-bound solving the primal problem, we apply the Farkas lemma to constraints (2.35) and (2.39) – (2.43) of the new formulation to test the existence of European prices. This yields the equivalent condition (again considering only one area and one time slot to ease the notation):

$$\begin{aligned} \forall (u_i, u_j) \in P, \quad & \sum_i Q^i P_0^i u_i + \sum_i Q^i (P_1^i - P_0^i) x_i^* u_i + \sum_j Q^j P^j u_j - \sum_j M_j (1 - y_j^*) u_j \\ & \leq \sum_i Q^i P_0^i x_i^* + \sum_i Q^i (P_1^i - P_0^i)(x_i^*)^2 + \sum_j Q^j P^j y_j^* \end{aligned} \tag{2.60}$$

where P is the polytope defined by (2.52) – (2.55) above in the linear case.

Note that we can only apply the Farkas lemma to the new formulation because it incorporates 'dual variables' for which $v_i = x_i \ \forall i \in I$: if we consider inequality (2.35) with unknown v_i instead of $v_i = x_i^* \ \ \forall i \in I$ fixed to the given values in the right-hand side (corresponding to the objective function of DUAL-QUAD-CR-DUAL), the inequality is not linear any more in the unknown 'dual variables' and the Farkas lemma doesn't apply.

Mainly two things should be noted concerning this condition. First, it is a *linear* condition which relates two 'quadratic quantities' (with fixed values x^*), which are close to the original quadratic objective function of DA-QUAD. Second, contrary to the condition (2.56) in the linear case, both right and left-hand sides do not correspond exactly to the original objective function of the primal program (here DA-QUAD) or its continuous relaxation with additional terms involving M_j. This last point was used in the preceding arguments to derive the new locally valid strengthened Benders cuts.

Nonetheless, though it is not direct, it is possible to recover the analogue result:

Lemma 2.4. *For a given node solution (x_i^*, y_j^*) , a European price exists if and only if:*

$$\max_{(u_i,u_j)\in P} \sum_i Q^i P^i u_i + \sum_i Q^i (P_1^i - P_0^i) \frac{u_i^2}{2} + \sum_j Q^j P^j u_j - \sum_j M_j (1 - y_j^*) u_j \leq obj^*, \quad (2.61)$$

where obj^* denotes the optimal value of the quadratic objective function associated with the current node solution.

Proof. See appendix. □

Observe however that condition (2.60) asks to solve a linear program and is more efficient as a tester for the existence of European prices than condition (2.61).

We can now adapt to the quadratic case the decomposition algorithm with exactly the same cuts:

Theorem 2.6. *In the quadratic case also, for each node solution in the branch-and-bound for which no European price exists, cuts of the form $\sum_{j|y_j^*=1} (1 - y_j) \geq 1$ are valid in the subtree.*

Proof. The proof is exactly the same as in Theorem 2.5. Just replace condition (2.58) by its counterpart derived from (2.61) (i.e. with quadratic terms). □

Note also that like in the previous linear case, a consequence of Lemma 2.4 is:

Theorem 2.7. *For a node solution (x^*, y^*), consider the polytope $P^{F^*} := P \cap \{(u_i, u_j)| \ u_j = 0 \ \ if \ \ y_j^* = 0\}$. Then a European price exists if and only if*

$$\max_{(u_i,u_j)\in P^{F^*}} \sum_i Q^i P_0^i u_i + \sum_i Q^i (P_1^i - P_0^i) \frac{u_i^2}{2} + \sum_j Q^j P^j u_j \leq obj^*, \quad (2.62)$$

where obj^* denotes the optimal value of the quadratic objective function associated with the node solution, in which case equality holds as well.

2.5 Computational Results

In this section, we mainly address four questions related to the new formulation. First, how state-of-the-art solvers behave on real instances, when the whole model EMM-MILP is provided ? Second, how the Benders-like algorithm behaves in comparison to the first approach ? Third, how efficient is Benders-like algorithm for quadratic instances involving piecewise linear bid curves ? Fourth, how both approaches behave on very combinatorial linear instances ? APX and EPEX kindly provided us with real data from 2011. Statistics computed over the whole year 2011 (i.e. 365 instances) are presented. All instances include full ATC network models as used in actual day-ahead markets and in [60]. In appendix, we present in more details results for 20 representative instances. Computational experiments have been carried out with AIMMS [6] with the solver CPLEX 12.5, on a computer running Windows 7 64 bits, with a four cores CPU i5 @ 3.10 Ghz, and 4 GB of RAM. Even with such a modest platform, results turn out to be very positive. The decomposition procedure has been implemented using lazy constraint callbacks with *locally valid lazy cuts.* Concerning practical requirements for an algorithm, main European power exchanges ask for a time limit of ten minutes, and we have adopted this stopping criterion for all tests below.

For both approaches (the new formulation and the decomposition procedure), we have computed the number of instances solved up to optimality, the (geometric) average time needed to find these optimal solutions, and the (geometric) average of the final absolute MIP gap when only a suboptimal solution is available in time. We also provide with the number of visited nodes for the new MILP approach, and the number of cuts generated in the decomposition approach. Finally all heuristics proposed by CPLEX have been deactivated. This is necessary to obtain an exact algorithm using the decomposition approach, and it turned out to be inefficient when directly using the new formulation. The CPLEX parameter indicating to branch first to the down branch (”branchdir=-1”) have also had a substantial impact on performances of both approaches, the intuition being given by the new locally valid cuts. When a block order is fractionally executed in the continuous relaxation of a given node, the branch where it is fully rejected will be explored first. With this parameter, very good initial feasible solutions are found in a few visited nodes.

2.5.1 Historical instances with stepwise bid curves

Piecewise linear bid curves have been transformed into stepwise bid curves to get MILP instances. To do this, for each two consecutive points of a bid curve such that $Q_i \neq Q_{i+1}$ and $P_i \neq P_{i+1}$, a point (Q^*, P^*) has been inserted in between, with $P^* = P_i$ and $Q^* = Q_{i+1}$.

A particular attention has been devoted to numerical issues. One drawback of the new formulation is the so-called big-M constants involved in the constraints. As it is well-known, this may result in numerically ill-conditioned instances. It appeared that very tight tolerance parameters must be set to obtain correct solutions (e.g. an integer feasibility tolerance of 10^{-9}).

Instances contain orders for 4 areas (Belgium, France, Germany and the Netherlands), and span the whole day (24 hours, excepted twice per year, 23 and 25 hours respectively). There are approximatively 50 000 hourly orders (bid curve segments) and 600 block orders per instance.

	Solved instances	Running time (solved instances, sec)	Final abs. gap (unsolved instances)	Nodes (solved - unsolved) instances	Cuts (solved - unsolved) instances
New MILP formulation	84%	104.42	418.16	43 - 33584	/
Decomposition Procedure	72.78%	6.47	402.05	16 - 1430	8 - 3492

Table 2.1: Historical instances with stepwise bid curves

EMM-MILP allows to solve most of the instances without any algorithmic work and to obtain very good suboptimal solutions when the instance cannot be solved up to optimality. The decomposition procedure is much faster on most instances but most of the time doesn't help to solve hard instances that the MILP approach cannot solve. The fact that the new MILP formulation approach takes in average more time for solved instances is mainly due to the time needed to solve the root node relaxation.

Comparing runs with and without solver's cut generation procedures, it turned out that they were not useful and were indeed slowing down the process in both the decomposition procedure and the full model approaches. In fact, for the full model approach, this may be explained by the presence of big-M's and the fact that most of the cuts generated may be very weak in practice. Concerning the decomposition procedure, in most cases, many good solutions to the primal program are easily found and cuts are not of main interest, the main part of the procedure (from a running time point of view) consisting in rejecting incumbents when no European prices exist. Note also that all CPLEX heuristics have been deactivated.

2.5.2 Historical instances with piecewise linear bid curves

When piecewise linear bid curves are considered, the new formulation EMM-QUAD-MIQCP cannot be solved with today's solvers (e.g. CPLEX or GUROBI), and only the decomposition procedure can be relevantly assessed.

To check for the existence of prices for a given new best node solution, the linear condition (2.60) is used, and the locally valid local cut of Theorem 2.6 is added when no European prices exist.

	Solved instances	Running time (solved instances, sec)	Final abs. gap (unsolved instances)	Nodes (solved - unsolved) instances	Cuts (solved - unsolved) instances
Decomposition Procedure	70.41%	16.70	370.91	11 - 619	7 - 1382

Table 2.2: Historical instances with piecewise linear bid curves

As it can be seen, most of instances are solved up to optimality, and a very small gap remains when only a suboptimal solution is found within ten minutes.

2.5.3 Instances with (almost) only block orders

We have built 50 instances where orders are almost all block orders in the following way. Starting from historical instances, all block orders have been relocated to one area only and are spanning only one hour of the day. Two small continuous orders (one buy order and one sell order) have been added for the sole purpose to have an instance with at least one feasible solution (a matching of orders is possible). The difference between both approaches in this case is remarkable:

	Solved instances	Running time (solved instances, sec)	Final abs. gap (unsolved instances)	Nodes (solved - unsolved) instances	Cuts (solved - unsolved) instances
New MILP formulation	100%	4.17	/	40797 - /	-
Decomposition Procedure	78%	13.82	9303.16	64564 / 937172	1662 / 82497

Table 2.3: Instances with almost only block orders

In this case, the new MILP formulation approach is much more powerful. One possible explanation is the high number of block order selections for which no European prices exist, which are enumerated by the decomposition. On another hand, with the full model approach, the solver may be able to branch more efficiently. The difference of performances between the two approaches was more impressive on a less powerful platform. This difference would therefore certainly be more important for instances with more block orders.

2.6 Conclusions

We have proposed a new formulation for European day-ahead electricity markets that turns out to be (a) tractable and (b) very competitive as long as stepwise preference curves describing hourly orders are considered. More than 80 % of the historical instances of 2011 can be solved up to optimality, and for the other ones, the final gap is very small. We have also compared this approach with a decomposition procedure derived directly from the new formulation, which appeared to solve most instances faster but was not helpful on hard instances that the new formulation approach was not able to solve. Unfortunately, the simple use of the analogue new formulation is no longer successful when piecewise linear preference curves are considered. Today's state-of-the-art MIQCP solvers are not able to deal with large-scale programs with this structure. On the other hand, the Benders-like decomposition approach derived from the new formulation allows managing these cases in an efficient way. Finally, the new MILP formulation performs much better than the decomposition approach on small very combinatorial linear instances, and this could be exploited in auctions with more block orders. Another interesting point is that an approach similar to the new formulation allows considering other objective functions over the set of constraints defining European market rules. In particular, with a similar modelling technique, it would be possible to consider, for example, an objective function minimizing the total opportunity costs of paradoxically rejected block orders. In a article in preparation, we study how this modelling technique can be used from a market design analysis point of view.

Acknowledgment

We greatly thank APX-Endex and Epex for providing us historical data for computational purposes. We also thank very useful comments provided by three anonymous referees. This text presents research results of the P7/36 PAI project COMEX, part of the IPA Belgian Program. This work was also supported by EC-FP7-PEOPLE MINO Marie-Curie Initial Training Network (grant n.316647) and by EC-FP7 COST Action TD1207. The scientific responsibility is assumed by the authors.

2.A Ommited proofs

2.A.1 Proof of Theorem 2.1 and Lemma A3

Lemma A1. *Take a feasible point* (x, y, n) *of DA-PRIMAL, and consider a feasible point of DA-CR-DUAL* $(s_i, s_j, p_{l,t}, u_m)$ *such that complementarity constraints* (2.13) − (2.17) *are satisfied as well. For these prices* $p_{l,t}, u_m$*: (i) fully accepted orders are ITM or ATM, (ii) fractionally accepted orders are ATM and (iii) rejected orders are ATM or OTM. In particular, ITM orders are fully accepted and OTM orders are fully rejected. (iv) Network equilibrium conditions are satisfied.*

Proof. Let us consider an order $i \in I$ or $j \in J$ (hourly or block order, respectively):

(i) if $x_i = 1$ (the order is fully accepted), complementarity constraints of type (2.15) imply $s_i = \sum_{l,t} Q^i_{l,t}(P^i - p_{l,t})$ and since $s_i \geq 0$, the order is in-the-money or at-the-money.

The same for a block order j with $y_j = 1$, using constraints (2.16) instead of (2.15).

(ii) if $0 < x_i < 1$ (the order is partially accepted, only possible for hourly orders $i \in I$), complementarity constraints of type (2.13) imply $s_i = 0$ and those of type (2.15) then imply $\sum_{l,t} Q^i_{l,t} P^i = \sum_{l,t} Q^i_{l,t} p_{l,t}$, i.e. the order is ATM.

(iii) if $x_i = 0$, complementarity constraints (2.13) imply $s_i = 0$ and then dual constraints (2.9) imply $\sum_{l,t} Q^i_{l,t} P^i \leq \sum_{l,t} Q^i_{l,t} p_{l,t}$, i.e. the order is ATM or OTM.

The same for a block order j with $y_j = 0$, using (2.14) instead of (2.13) and (2.10) instead of (2.9).

(iv) By Definition 2.3, network equilibrium conditions are satisfied. □

Lemma A2. *Let* (x, y, n) *be a feasible point of DA-PRIMAL. If* $p_{l,t}, u_m$ *is a price system such that* $(i) - (iv)$ *of Lemma A1 hold, then one can define auxiliary variables* s_i, s_j *such that* (2.9) − (2.17) *hold as well.*

Proof. Assume $p_{l,t}, u_m$ are prices satisfying $(i) - (iv)$ of Lemma A1 and define $s_i, s_j \geq 0$ as follows:

(a) $s_i = \sum_{l,t} Q^i_{l,t}(P^i - p_{l,t}) \geq 0$ if $x_i = 1$ and likewise for s_j if $y_j = 1$,

(b) $s_i = \sum_{l,t} Q^i_{l,t}(P^i - p_{l,t}) = 0$ if $0 < x_i < 1$,

(c) $s_i = 0 \geq \sum_{l,t} Q^i_{l,t}(P^i - p_{l,t})$ if $x_i = 0$ and likewise for s_j if $y_j = 0$.

Conditions $(i)-(iii)$ of Lemma A1 ensure that in cases (a) and (b), $s_i = \sum_{l,t} Q^i_{l,t}(P^i - p_{l,t})$ is non negative (i.e. feasible for dual constraint (2.12)), and that in case (c), $s_i = 0$ is greater or equal to $\sum_{l,t} Q^i_{l,t}(P^i - p_{l,t})$ (conditions (2.9) or (2.10)). Taking into account that (iv) of Lemma A1 ensures by definition that constraints $(2.11), (2.17)$ are satisfied, it is then direct to check that constraints $(2.9) - (2.17)$ are all satisfied. □

Proof of Theorem 2.1.

Proof. (I) Is a direct consequence of Lemma A1 and its converse Lemma A2.

(II) Assume (x_i, y_j, n_k) and $(s_i, s_j, p_{l,t}, u_m)$ are feasible for DA-PRIMAL and DA-CR-DUAL respectively. Then, (x_i, y_j, n_k) is also feasible for the continuous relaxation DA-CR. By duality theory, they satisfy complementarity conditions $(2.13)-(2.17)$ if and only if equality of objective functions $(2.1) = (2.8)$ holds, in which case both are optimal for their respective problem DA-CR and DA-CR-DUAL. □

Proof of Corollary 2.1.

Proof. Theorem 2.1 shows that for a feasible solution (x, y, n) of DA-PRIMAL , a market equilibrium with uniform prices exist if and only if this solution is optimal for the relaxation DA-CR. □

Lemma A3 (Equilibrium for hourly orders in the quadratic case)**.** *Consider a primal feasible point* (x, y, n) *of DA-PRIMAL-QUAD and let* $s_i, p_{l,t}$ *satisfy dual and complementarity constraints* $(2.22), (2.12)$ *and* $(2.13), (2.23)$*. Then (I) (i) Fully executed orders are ITM or ATM, (ii) fractionally executed orders are ATM and (iii) fully rejected orders are OTM or ATM. (II) Conversely, if (i)-(iii) hold, then there exist auxiliary surplus variables* s_i *such that conditions* $(2.22), (2.12), (2.13), (2.23)$ *hold as well.*

Proof. (I)

(i) If $x_i = 1$ (the order is fully accepted), equations of type (2.23) imply

$s_i = \sum_{l,t} Q^i_{l,t} P^i_0 + \sum_{l,t} Q^i_{l,t}(P^i_1 - P^i_0)1 - \sum_{l,t} Q^i_{l,t} p_{l,t} \geq 0$, that is $\sum_{l,t} Q^i_{l,t}(P^i_1 - p_{l,t}) \geq 0$ and the order is ITM or ATM.

(ii) If $0 < x_i < 1$ (the order is partially accepted), equations of type (2.13) imply $s_i = 0$ and equations of type (2.23) then imply $\sum_{l,t} Q^i_{l,t} P^i_0 + \sum_{l,t} Q^i_{l,t}(P^i_1 - P^i_0)x_i = \sum_{l,t} Q^i_{l,t} p_{l,t}$ and the order is ATM.

(iii) If $x_i = 0$, equations (2.13) imply $s_i = 0$ and inequalities (2.22) (with $v_i = x_i = 0$) imply $\sum_{l,t} Q^i_{l,t} P^i_0 \leq \sum_{l,t} Q^i_{l,t} p_{l,t}$, and the order is ATM or OTM.

(II) The converse is shown as in Lemma A2 by defining s_i in an appropriate way: $s_i = \sum_{l,t} Q^i_{l,t} P^i_0 + \sum_{l,t} Q^i_{l,t}(P^i_1 - P^i_0)x_i - \sum_{l,t} Q^i_{l,t} p_{l,t}$ if $0 < x_i \leq 1$, and $s_i = 0$ if $x_i = 0$. □

2.A.2 Proof of Theorem 2 & 3

The proofs rely on strong duality results and we first need to consider a program DA-FixedBlocks, corresponding to DA-PRIMAL with additional constraints (2.66) − (2.67) to fix block order variables y_j to some arbitrary values, corresponding to a partition of J into two subsets J_0 (rejected block orders) and J_1 (accepted block orders). DA-FixedBlocks is an LP and its dual DA-FixedBlocks-DUAL below is well-defined. We also write down related complementarity constraints.

DA-FixedBlocks (primal LP when considering a block bid selection)

$$\max_{x_i, y_j, n_k} \sum_i (\sum_{l,t} Q^i_{l,t} P^i) x_i + \sum_j (\sum_{l,t} Q^j_{l,t} P^j) y_j \tag{2.63}$$

subject to:

$$x_i \leq 1 \qquad \forall i \in I \quad [s_i] \tag{2.64}$$

$$y_j \leq 1 \qquad \forall j \in J \quad [s_j] \tag{2.65}$$

$$y_{j_0} \leq 0 \qquad \forall j_0 \in J_0 \quad [d_{j_0}] \tag{2.66}$$

$$-y_{j_1} \leq -1 \qquad \forall j_1 \in J_1 \quad [d_{j_1}] \tag{2.67}$$

$$\sum_i Q^i_{l,t} x_i + \sum_j Q^j_{l,t} y_j = \sum_k e^k_{l,t} n_k, \qquad \forall (l,t) \in A \times T \quad [p_{l,t}] \tag{2.68}$$

$$\sum_k a_{m,k} n_k \leq w_m \qquad \forall m \in N \quad [u_m] \tag{2.69}$$

$$x_i,\ y_j \geq 0 \tag{2.70}$$

DA-FixedBlocks-DUAL

$$\min \sum_i s_i + \sum_j s_j - \sum_{j_1} d_{j_1} + \sum_m w_m u_m \tag{2.71}$$

subject to:

$$s_i + \sum_{l,t} Q^i_{l,t} p_{l,t} \geq \sum_{l,t} Q^i_{l,t} P^i \qquad \forall i \in I \;\; [x_i] \tag{2.72}$$

$$s_{j_0} + d_{j_0} + \sum_{l,t} Q^{j_0}_{l,t} p_{l,t} \geq \sum_{l,t} Q^{j_0}_{l,t} P^{j_0} \qquad \forall j_0 \in J_0 \;\; [y_{j_0}] \tag{2.73}$$

$$s_{j_1} - d_{j_1} + \sum_{l,t} Q^{j_1}_{l,t} p_{l,t} \geq \sum_{l,t} Q^{j_1}_{l,t} P^{j_1} \qquad \forall j_1 \in J_1 \;\; [y_{j_1}] \tag{2.74}$$

$$\sum_m a_{m,k} u_m - \sum_{l,t} e^k_{l,t} p_{l,t} = 0 \qquad \forall k \in K \; [n_k] \tag{2.75}$$

$$s_i, s_j, d_{j_0}, d_{j_1}, u_m \geq 0 \tag{2.76}$$

and complementarity constraints **DA-FixedBlocks-CC**

$$s_i(1 - x_i) = 0 \qquad \forall i \in I \tag{2.77}$$

$$s_{j_0}(1 - y_{j_0}) = 0 \qquad \forall j_0 \in J_0 \tag{2.78}$$

$$s_{j_1}(1 - y_{j_1}) = 0 \qquad \forall j_1 \in J_1 \tag{2.79}$$

$$y_{j_0} d_{j_0} = 0 \qquad \forall j_0 \in J_0 \tag{2.80}$$

$$(1 - y_{j_1}) d_{j_1} = 0 \qquad \forall j_1 \in J_1 \tag{2.81}$$

$$u_m(\sum_k a_{m,k} n_k - w_m) = 0 \qquad \forall m \in N \tag{2.82}$$

$$x_i(s_i + \sum_{l,t} Q^i_{l,t} p_{l,t} - \sum_{l,t} Q^i_{l,t} P^i) = 0 \qquad \forall i \in I \tag{2.83}$$

$$y_{j_0}(s_{j_0} + d_{j_0} + \sum_{l,t} Q^{j_0}_{l,t} p_{l,t} - \sum_{l,t} Q^{j_0}_{l,t} P^{j_0}) = 0 \qquad \forall j_0 \in J_0 \tag{2.84}$$

$$y_{j_1}(s_{j_1} - d_{j_1} + \sum_{l,t} Q^{j_1}_{l,t} p_{l,t} - \sum_{l,t} Q^{j_1}_{l,t} P^{j_1}) = 0 \qquad \forall j_1 \in J_1 \tag{2.85}$$

Proof of Theorem 2.2.

Proof. (i) Let $(x_i, y_j, n_k, p_{l,t}, u_m, s_i, s_j)$ be a feasible point of the MPCC formulation.

Let us define $J_0 := \{j | y_j = 0\}$, $J_1 := \{j | y_j = 1\}$, $d_{j_1} := 0 \; \forall j_1 \in J_1$ and $d_{j_0} := M_{j_0}$ $\forall j_0 \in J_0$.

For $j_0 \in J_0$, since $d_{j_0} := M_{j_0} := \sum_{l,t} 2\bar{P}|Q^{j_0}_{l,t}|$, we can define new $\tilde{s}_{j_0} = 0$ such that dual constraints of type (2.73) and complementarity constraints of type (2.78) above are satisfied. The new point $(x_i, y_j, n_k, p_{l,t}, u_m, s_i, \tilde{s}_j, d_{j_0}, d_{j_1})$ satisfies constraints (2.64)−(2.70), (2.72)−(2.85), that is all primal, dual and complementarity constraints corresponding to the primal and dual optimization problems where block order variables are fixed to the values given by the initial point considered. Therefore, by strong duality for LP, for this selection J_0, J_1, (x, y, n) is optimal for DA-FixedBlocks and $(p_{l,t}, u_m, s_i, \tilde{s}_j, d_{j_0}, d_{j_1})$ is optimal for

DA-FixedBlocks-DUAL. Moreover, (2.63) = (2.71) and since $d_{j_1} = 0 \; \forall j_1 \in J_1$ in the objective (2.71), it follows that (2.25) holds. Due to constraints $(2.73) - (2.74)$ and the given values of d_{j_0}, d_{j_1}, it is direct to check that the projection $(x_i, y_j, n_k, p_{l,t}, u_m, s_i, \tilde{s}_j)$ satisfies constraints (2.31). This shows that the projection $(x_i, y_j, n_k, p_{l,t}, u_m, s_i, \tilde{s}_j)$ satisfies $(2.25) - (2.33)$, so is a feasible point of EMM-MILP.

(ii) Now let $(x_i, y_j, n_k, p_{l,t}, u_m, s_i, s_j)$ be a feasible point of EMM-MILP.

Let us define $J_0, J_1, d_{j_0}, d_{j_1}$ as above at (i). The point $(x_i, y_j, n_k, p_{l,t}, u_m, s_i, s_j, d_{j_0}, d_{j_1})$ satisfies all primal and dual conditions $(2.64) - (2.70)$, $(2.72) - (2.76)$ of the optimization problems DA-FixedBlocks and DA-FixedBlocks-DUAL above, as well as the condition of equality of objective functions (2.63) = (2.71). By duality theory (implying related complementarity constraints), it satisfies constraints $(2.77) - (2.85)$. We can now define new $\tilde{s}_{j_0} := s_{j_0} + d_{j_0}$ to satisfy constraints (2.10) for $j_0 \in J_0 \subseteq J$. Constraints (2.78) (the same as (2.14)) may not be satisfied any more but the projection of the new point thus obtained, $(x_i, y_j, n_k, p_{l,t}, u_m, s_i, \tilde{s}_j)$ is a feasible point of the EMM-MPCC formulation, as it satisfies primal conditions $(2.2) - (2.7)$, dual conditions $(2.9) - (2.12)$ and the required complementarity constraints $(2.13), (2.15) - (2.17)$. □

Proof of Theorem 2.3

Proof. The proof is almost identical to the proof of Theorem 2.2. It is just needed to adapt primal and dual problems DA-B-FixedBlocks and DA-FixedBlocks-DUAL to the quadratic setting, i.e. considering respective objective functions $(2.20), (2.21)$, and the adapted dual and complementarity constraints $(2.22), (2.23)$. Replace then in the proof constraint (2.25) by (2.35), dual constraints of type (2.9) and the same (2.72) by (2.22), and complementarity constraints of type (2.15) and the same (2.83) by (2.23), taking into account Lemma 2.1 according to which we can consider optimal dual variables $v_i = x_i \;\; \forall i \in I$. □

2.A.3 Proof of Lemma 2.4

Proof. (i) If (European) equilibrium prices exist, condition (2.60) holds, and necessarily:

$\forall (u_i, u_j) \in P$,

$$\sum_i Q^i P_0^i u_i + \sum_j Q^j P^j u_j - \sum_j M_j (1 - y_j^*) u_j$$

$$\leq \quad \sum_i Q^i P_0^i x_i^* + \sum_i Q^i (P_1^i - P_0^i)[(x_i^*)^2 - x_i^* u_i] + \sum_j Q^j P^j y_j^*$$

$$\leq \quad \sum_i Q^i P_0^i x_i^* + \sum_i Q^i (P_1^i - P_0^i)[\frac{(x_i^*)^2}{2} - \frac{u_i^2}{2}] + \sum_j Q^j P^j y_j^*,$$

where the first inequality is condition (2.60) rearranged, and where for the last inequality, we use the fact that if c_{ij} are coefficients of a negative semi-definite matrix, then:

$$\sum_{ij} c_{ij}x_i(x_j - u_j) \leq \frac{1}{2}(\sum_{ij} c_{ij}x_ix_j - \sum_{ij} c_{ij}u_iu_j).$$

Rearranging, we now get the necessary condition (2.61):

$$\max_{(u_i,u_j)\in P} \sum_i Q^iP_0^iu_i + \sum_i Q^i(P_1^i - P_0^i)\frac{(u_i)^2}{2} + \sum_j Q^jP^ju_j - \sum_j M_j(1-y_j^*)u_j \leq obj^*,$$

where obj^* is the value of the quadratic objective function of the model for the current node solution.

(ii) Let us prove that this condition is also sufficient and let obj^* correspond to the optimal value associated to a node solution (x_i^*, y_j^*). Using the EMM-QUAD-MIQCP formulation, we show that if (2.61) holds, the left-hand side QP provides with European equilibrium prices for the current node solution. This QP in (2.61) is the continuous relaxation DA-QUAD-CR with an additional term $-\sum_j M(1-y_j^*)u_j$ in the objective function (taking into account the minor adaptations to consider a network representation if needed).

The node solution $(x_i^*, y_j^*) \in P$, so is feasible for this QP in (2.61) and is therefore optimal for it (terms with the M_j cancel if $u_j = y_j^*$, so the expression is exactly the same on both sides).

By Lemma 2.1, for this QP in (2.61), there exist dual optimal variable values $(s_i^*, s_j^*, p_{l,t}^*, v_i^*)$ such that $v_i^* = x_i^*$. Mutatis mutandis to take a network model into account, constraints of the dual of this left-hand side QP are exactly constraints $(2.40) - (2.43)$ with (x, y) fixed to (x^*, y^*), which are therefore satisfied by these optimal dual variable values. Using strong duality for quadratic programs [22], we now show that constraint (2.35) (equality of objective functions) is satisfied as well:

$$\begin{aligned}
&\sum_i s_i^* + \sum_j s_j^* - \sum_i Q^i(P_1^i - P_0^i)\frac{(x_i^*)^2}{2} \\
&\qquad = \sum_i Q^iP_0^ix_i^* + \sum_i Q^i(P_1^i - P_0^i)\frac{(x_i^*)^2}{2} + \sum_j Q^jP^jy_j^* - \sum_j M_j(1-y_j^*)y_j^* \\
&\qquad \leq obj^* = \sum_i Q^iP_0^ix_i^* + \sum_i Q^i(P_1^i - P_0^i)\frac{(x_i^*)^2}{2} + \sum_j Q^jP^jy_j^*.
\end{aligned}$$

Rearranging this inequality shows that constraint (2.35) is satisfied. Hence, for our node solution (x_i^*, y_j^*), we can define $(s_i^*, s_j^*, p_{l,t}^*)$ such that all constraints $(2.35) - (2.43)$ are satisfied, and a European equilibrium price exists for the solution (x_i^*, y_j^*) (or European prices when several areas or time slots are considered). One just needs to consider the optimal dual solution of the QP in (2.61) for which $v_i = x_i^*$. □

2.B Tables

			New MILP formulation			Decomposition approach			
Instance	# Hourly orders	# Block orders	Run. Time	Final Gap	Nodes	Run. Time	Final Gap	Nodes	Cuts
1	51231	766	600.37	495.89	27478	600.276	1463.35	842	3442
2	46700	477	64.912			2.106			0
3	50148	731	277.042		18331	600.401	106.75	2204	3594
4	49999	566	64.819		12	2.527		17	2
5	52073	683	57.097			2.168			0
6	49304	513	47.283		84	6.537		46	27
7	47924	658	79.577		473	99.7		458	706
8	48645	604	51.028		3	2.371			0
9	45141	571	36.348		2	1.685		2	0
10	46472	655	136.891		5292	600.651	179.12	993	3625
11	47199	686	54.335		77	6.708		85	29
12	52369	692	69.156		2	2.73		2	0
13	54147	640	93.773		9	3.369		21	3
14	55361	618	85.692		7	3.635		6	5
15	55774	550	92.368		57	6.567		75	16
16	53789	591	59.857		9	3.885		7	7
17	59384	685	117.781		91	3.37		15	4
18	60169	699	600.339	252.83	27679	600.588	268.65	1408	3042
19	57992	578	71.32		10	122.27		133	570
20	51687	703	600.308	235.22	39225	600.604	517.47	1046	3173

Table 2.4: Linear Instances

Instance	# Hourly orders	# Block orders	Run. Time	Final Gap	Nodes	Cuts
1	51231	766	600.21	1160.50	565	1343
2	46700	477	5.27			0
3	50148	731	600.51	145.21	961	1382
4	49999	566	8.47		13	2
5	52073	683	8.30		1	0
6	49304	513	10.95		32	7
7	47924	658	13.01		59	16
8	48645	604	5.82			0
9	45141	571	4.31		1	0
10	46472	655	601.10	256.69	357	1478
11	47199	686	14.12		63	15
12	52369	692	7.66			0
13	54147	640	20.03		34	21
14	55361	618	600.48	202.48	388	1291
15	55774	550	180.29		247	366
16	53789	591	42.67		40	58
17	59384	685	44.43		82	66
18	60169	699	600.56	204.42	753	1138
19	57992	578	15.23		7	7
20	51687	703	600.42	1740.25	457	1280

Table 2.5: Quadratic Instances (decomposition approach only)

			New MILP Formulation			Decomposition approach			
Instance	# Hourly orders	# Block orders	Run. Time	Final Gap	Nodes	Run. Time	Final Gap	Nodes	Cuts
1	2	526	7.18		75561	600.14	28497.24	493464	132118
2	2	508	12.18		168467	540.00		1777391	121336
3	2	612	1.34		15348	2.32		8784	367
4	2	594	9.95		114400	15.41		81721	2756
5	2	671	4.74		53026	4.88		18312	847
6	2	766	8.80		90938	129.04		1156506	17312
7	2	714	1.82		17111	10.25		70038	1517
8	2	497	1.16		16210	459.08		1090631	106874
9	2	460	0.53		6216	0.56		4219	84
10	2	579	0.31		2474	1.01		2437	199
11	2	668	0.16		725	0.19		473	15
12	2	684	0.70		6733	2.45		29995	310
13	2	650	1.84		19433	7.58		71328	988
14	2	682	1.48		13224	2.43		10835	374
15	2	487	14.68		192265	600.01	6099.59	794340	142957
16	2	477	1.09		15481	302.75		699328	69114
17	2	597	0.16		792	0.47		5716	20
18	2	740	3.12		28904	28.44		105697	4312
19	2	794	5.91		57537	113.37		366836	14008
20	2	823	1.01		9677	600.03	209922.61	155204	63899

Table 2.6: Instances with (almost) only block orders

Chapter 3

A MIP framework for non-convex uniform price day-ahead electricity auctions

Accepted manuscript version of: Mehdi Madani and Mathieu Van Vyve. A MIP framework for non-convex uniform price day-ahead electricity auctions. EURO Journal on Computational Optimization, pages 1-22, 2016 (in press).

Abstract

It is well-known that a market equilibrium with uniform prices often does not exist in non-convex day-ahead electricity auctions. We consider the case of the non-convex, uniform-price Pan-European day-ahead electricity market "PCR" (Price Coupling of Regions), with non-convexities arising from so-called complex and block orders. Extending previous results, we propose a new primal-dual framework for these auctions, which has applications in both economic analysis and algorithm design. The contribution here is threefold. First, from the algorithmic point of view, we give a non-trivial exact (i.e. not approximate) linearization of a non-convex 'minimum income condition' that must hold for complex orders arising from the Spanish market, avoiding the introduction of any auxiliary variables, and allowing us to solve market clearing instances involving most of the bidding products proposed in PCR using off-the-shelf MIP solvers. Second, from the economic analysis point of view, we give the first MILP formulations of optimization problems such as the maximization of the traded volume, or the minimization of opportunity costs of paradoxically rejected block bids. We first show on a toy example that these two objectives are distinct from maximizing welfare. Third, we provide numerical experiments on realistic large-scale instances. They illustrate the efficiency of the approach, as well as the economic trade-offs that may occur in practice.

3.1 Introduction

3.1.1 Equilibrium in non-convex day-ahead electricity auctions

An extensive literature now exists on non-convex day-ahead electricity markets or electricity pools, dealing in particular with market equilibrium issues in the presence of indivisibilities, see e.g. [40, 53, 54, 71, 70, 84, 99, 104] and references therein. Research on the topic has been fostered by the ongoing liberalization and integration of electricity markets around the world during the past two decades. Due to the peculiar nature of electric power systems, non-convexities of production sets cannot be neglected, and bids introducing non-convexities in the mathematical formulation of the market clearing problem have been proposed for many years by power exchanges or electricity pools, allowing participants to reflect more accurately their operational constraints and cost structure.

It is now well known that due to these non-convexities, a market equilibrium with uniform prices may fail to exist (a single price per market area and time slot, no transfer payments, no losses incurred, and no excess demand nor excess supply for the given uniform market prices). To deal with this issue, almost all ideas proposed revolve around getting back, or getting close, to a convex situation where strong duality holds and shadow prices exist. For example, a now classic proposition in [71] is to fix integer variables to optimal values for a welfare maximizing primal program whose constraints describe physically feasible dispatches of electricity, and compute multi-part equilibrium prices using dual variables of these fixing constraints. The same authors, in an unpublished working paper, have later adapted this proposition to the context of European power auctions, proposing to allow and compensate so-called paradoxically accepted block bids, thus deviating from a pure uniform price system. We briefly review their proposition at the end of Section 3.3. A recent proposition for electricity pools in [84] is to use a 'primal-dual' formulation (i.e. involving both executed quantities as primal variables, and market clearing prices as dual variables), where 'getting close' is materialized by minimising the duality gap introduced by integer constraints, and where additional constraints are added to ensure that producers are recovering their costs. The goal is to use uniform prices, while minimizing the inevitable deviation from market equilibrium, and providing adequate incentives to producers.

These last conditions, which have been used in Spain for many years, are usually called 'minimum income conditions' (MIC). The natural way to model them is through imposing a lower bound on the revenue expressed as the product between executed quantities and market prices, yielding non-convex quadratic constraints. However, [83] propose an exact linearization of the revenue related to a set of bids of a strategic bidder participating in a convex market, relying on KKT conditions explicitly added to model the lower-level market clearing problem of a bilevel program, and linearized by introducting auxiliary binary variables.

Regarding the proposition in [84], a market equilibrium exists only if the optimal duality gap is null, which is rarely the case with real instances, and the proposition chooses *not* to enforce network equilibrium conditions (corresponding to optimality conditions of transmission system operators), nor that demand bids are not loosing money. The same remarks apply to [37], where the only non-convexities arise from the minimum income conditions for producers (indivisibilities such as minimum power output or 'fill-or-kill conditions' are not considered).

Several other auction designs have been previously considered, which often propose implementing a non-uniform pricing scheme, see e.g. [84] for a review.

The fact that equilibrium is not enforced for the convex part of the market clearing problem (i.e. for convex bids and the network model), is a key auction design difference compared with the choices made by power exchanges in Europe. In this article, we deal mainly with this last market model, which is further described in the next section, and illustrated in the toy example given in Section 3.2.1.

3.1.2 The PCR market

We consider the Pan-European day-head electricity market being developed under the Price Coupling of Regions project (PCR), as publicly described in [27]. Essentially, it is a near-equilibrium auction mechanism using uniform prices, and where the sole deviation from a perfect market equilibrium is the allowance of so-called "paradoxically rejected *non-convex* bids", generating opportunity costs, because these bids, which are 'in-the-money', would be profitable for the computed clearing prices (see e.g. [27] for more information on these market rules). On the other side, all convex bids as well as TSOs must be 'at equilibrium' for the computed market clearing prices. This integrated market is coupling the CWE region (France, Belgium, Germany, the Netherlands, Luxembourg) with NordPool (Norway, Sweden, Denmark, Finland, Baltic countries), as well as Italy and OMIE (Spain, Portugal).

From the algorithmic point of view, when considering specifically the CWE region, we have previously shown that the market clearing problem can be restated as a MILP, without introducing any auxiliary variables to linearise the needed complementarity conditions modelling the near-equilibrium, see [54]. We also proposed a Benders-like decomposition procedure with locally strenghtened Benders cuts. Leaving aside a peculiar kind of bids from the Italian market (so-called PUN bids), introducing complex bids with a MIC condition yields a non-convex MINLP. The production-quality algorithm in use, EUPHEMIA ([27]), an extension of COSMOS previously used to clear the CWE market, is a sophisticated branch-and-cut algorithm handling all market requirements. However, due to the introduction of MIC bids, the algorithm is a heuristic, though COSMOS on which it relies is an exact branch-and-cut.

3.1.3 Contribution and structure of this article

We provide here a new primal-dual framework for PCR-like auctions, which is mainly a continuation of ideas presented in [53, 54]. The objective is to present a unified approach to algorithmic and economic modelling issues concerning these European auctions, with useful computational applications. The approach essentially consists in using strong duality adequately to enforce complementarity conditions modelling equilibrium for the convex part of the market clearing problem, as required by European power exchanges.

This approach is similar to the bilevel approach suggested in [104], though we do not need to introduce any auxiliary variables to linearize quadratic terms when considering the equality of objective functions modelling optimality of a second level problem modelling equilibrium for the convex part (see below, in particular Sections 3.2.4 and 3.3.3). Moreover, we could choose to consider some additional continuous variables with clear economic interpretations as bounds on opportunity costs or losses of (non-convex) block bids. This approach is used in [54] to derive a powerful Benders decomposition, where it is also shown how to consider the case of piecewise linear bid curves yielding a QP setting, using strong duality for convex quadratic programs. The MIP framework proposed here is presented in Section 3.2. It includes the following extensions.

First, so-called complex bids used in Spain and Portugal are added to the model, and we give an exact (i.e. not approximate) linearisation of a non-linear non-convex 'minimum income condition' (MIC) that must hold for these bids ([27]), which model revenue adequacy for producers. These conditions are provided for many years by the Spanish power exchange OMIE ([59]), and are also considered (in a different auction design setting) in [37, 84]. This enables us to give a MILP formulation of the PCR market clearing problem which avoids complementarity constraints and the use of any auxiliary variables, while taking into account these MIC conditions. This is developed in Section 3.3.

Second, we show in Section 3.2 how to consider in the main MILP model, together with main decision variables such as prices and bid execution levels, additional variables which correspond to upper bounds on opportunity costs of block bids, and upper bounds on losses. Let us emphasize that current European market rules forbid paradoxically accepted block bids (executed bids incurring losses), and stating these particular conditions amount to requiring that some of the added variables must be null. This is used in Section 3.3.4 to develop economic analysis applications. For example, the framework can be used in particular to provide the first (and reasonably tractable) MILP formulations of optimization problems such as the minimization of incurred opportunity costs, or the maximization of the traded volume. Let us note that in a convex context, no opportunity costs are incurred and any market clearing solution is welfare maximizing, so maximizing the traded volume only amounts to choosing peculiar tie-breaking rules in case of indeterminacy. Yet it is shown on the toy example in the introductory Section 3.2.1 that in a non-convex context, these two objectives are both distinct from maximizing welfare. To our knowledge, if opportunity costs of rejected block bids have been considered empirically in the past (e.g in [62, 63]), this point is new and could provide useful information to day-ahead auctions stakeholders. (We have presented partial results about opportunity costs in a simplified setting at the EEM 14 conference, see [53].)

Finally, numerical experiments using realistic large-scale instances are presented in Section 3.4. They show that our proposition allows solving up to optimality market clearing instances with MIC bids, which correspond to the Spanish market design. This is the first time that real-life instances of this type of problems are solved to optimality. This straightforward approach does not behave as well for instances including both MIC and block bids. However, a simple heuristic approach already yields provably high-quality solutions. Regarding the economic analysis applications, results presented illustrate the trade-offs that may occur for realistic large-scale instances, for example between optimizing welfare and optimizing the traded volume. Again this is the first time that optimal solutions for such problems can be computed apart from toy examples of small sizes.

3.2 A new primal-dual framework

Below, Section 3.2.1 describes the context and issues for day-ahead markets with indivisibilities such as in the CWE region (with block bids). Then, Section 3.2.2 introduces the welfare maximization problem without equilibrium restrictions, i.e. neglecting adequate market clearing price existence issues. Section 3.2.3 derives a related dual program parametrized by the integer decisions, and several important economic interpretations relating dual variables, uniform prices and deviations from a perfect market equilibrium (losses and opportunity costs of executed/rejected non-convex bids). Finally, Section 3.2.4 presents the basis of the new primal-dual framework proposed.

3.2.1 Uniform prices and price-based decisions in the CWE region: a toy example

We use here a toy example [53] illustrating two key points. First, a market equilibrium may not exist in the presence of indivisible orders. Second, under European market rules where paradoxically rejected non-convex bids are allowed, a welfare maximizing solution is not necessarily a traded volume maximizing solution nor it is necessarily an opportunity costs minimizing solution. The toy example consists in a market clearing instance involving two demand continuous bids (e.g. two steps of a stepwise demand bid curve), and two sell block bids. Parameters are summarized in Table 3.1 :

Bids	Power (MW)	Limit price (EUR/MW)
A: Buy bid 1	11	50
B: Buy bid 2	14	10
C: Sell block bid 1	10	5
D: Sell block bid 2	20	10

Table 3.1: Toy market clearing instance

First, obviously, it is not possible to execute both sell block bids, as they offer a total amount of power of 30 MW, while the total demand is at most 25 MW. As they are indivisible, if there is a trade, either (i) bid C is fully executed or (ii) bid D is fully executed. Second, at equilibrium, by definition, for the given market prices, no bidder should prefer another level of execution of its bid. In particular, in-the-money (ITM) bids must be fully executed, out-of-the-money (OTM) bids must be fully rejected, and fractionally executed bids must be right at-the-money (ATM).

So in the first case (i), A is partially accepted and sets the market clearing price to 50 EUR/MW, if any equilibrium with uniform prices exists. But in that case, block bid D is rejected while ITM: an opportunity cost of $20(50-10) = 800$ is incurred. This situation is accepted under the near-equilibrium European market rules described above. A direct computation shows that the welfare is then equal to 10(50-50) + 10(50-5) = 450, while the traded volume is 10. Similar computations in the case (ii) yield the market outcome summarized in Table 3.2.

	Price	Traded Volume	Welfare	Opportunity costs
Matching C	50	10	450	800
Matching D	10	20	440	50

Table 3.2: Market outcome

In this toy example, case (i) maximizes welfare but generates (much) more opportunity costs and half the traded volume.

3.2.2 Unrestricted welfare optimization

We formulate here the classical welfare optimization problem with an abstract and very general power transmission network representation that is still linear. It covers e.g. DC network flow models or the so-called ATC and Flow-based models used in PCR (see [27]). The usual network equilibrium conditions involving locational market prices apply, see [54].

Hourly bids originate from continuous bid curves, and can be fractionally accepted. They are hence modelled below with continuous variables $x_i, x_{hc} \in [0,1]$ for each step of a given step-wise bid curve, describing which fraction of the corresponding bid quantity P_i (resp. P_{hc}) is accepted.

In order to better represent their operational constraints or cost structure, participants are also allowed to submit indivisible bids, called 'block bids' in the PCR vocabulary, which usually span multiple time periods. They are used for example to model minimum power output. They are modelled below with binary variables y_j.

Binary variables u_c are introduced to model the conditional acceptance of a set of hourly bids $hc \in H_c$, controlled via constraints (3.4). The conditional acceptance relative to a minimum income condition is dealt with in Section 3.3. These bids are used to allow participants to express e.g. their start-up costs which should be covered if they are dispatched. However, the conditional acceptance only depends on the adequacy of the revenue, and in this respect they are distinct from block bids.

Constraint (3.6) is the balance equation at location l at time t, the right-hand side corresponding to the net export position expressed as a linear combination of abstract network elements n_k. Constraint (3.7) is the capacity constraint of network resource m, constraining the use of the elements n_k.

Finally, let us note that we model binary requirements as integrality, see conditions (3.9), plus bound constraints (3.3),(3.5),(3.8). Indeed, dual variables of bound constraints (3.2)-(3.3) and (3.5) have a nice economic interpretation as 'surplus variables', and some technical developments presented below, such as Lemma 3.3, rely on these dual variables.

Notation

Notation used throughout the text is provided here for quick reference. The interpretation of any other symbol is given within the text itself.

Sets and indices:

i	Index for hourly bids, in set I
j	Index for block bids, in set J
c	Index for MIC bids, in set C
hc	Index for hourly bids associated to the MIC bid c, in set H_c
l	Index for locations, $l(i)$ (resp. $l(hc)$) denotes the location of bid i (resp. hc)
t	Index for time slots, $t(i)$ (resp. $t(hc)$) denotes the time slot of bid i, (resp. hc)
$I_{lt} \subseteq I$	Subset of hourly bids associated to location l and time slot t
$HC_{lt} \subseteq HC$	Subset of MIC hourly suborders, associated to location l and time slot t
$J_l \subseteq J$	Subset of block bids associated to location l

Parameters:

P_i, P_{hc}	Power amount of hourly bid i (resp. hc), $P < 0$ for sell bids, and $P > 0$ for demand bids
P_j^t	Power amount of block bid j at time t, same sign convention
λ^i, λ^{hc}	Limit bid price of hourly bid i, hc
λ^j	Limit bid price of block bid j
$a_{m,k}$	Abstract linear network representation parameters
$e_{l,t}^k$	Parameters used to describe net export positions using variables n_k
w_m	Capacity of the abstract network resource m

Primal decision variables:

$x_i \in [0, 1]$	fraction of power P_i which is executed
$x_{hc} \in [0, 1]$	fraction of power P_{hc} (related to the MIC bid c) which is executed
$y_j \in \{0, 1\}$	binary variable which determines if the quantities P_j^t are fully accepted or rejected
$u_c \in \{0, 1\}$	binary variable controling the execution or rejection of the MIC bid c (i.e. of the values of x_{hc})
n_k	variables used for the abstract linear network representation, related to net export positions

Dual decision variables:

π_{lt}	uniform price (locational marginal price) for power in location l and time slot t
$v_m \geq 0$	dual variable pricing the network constraint m,
$s_i \geq 0$	dual variable interpretable as the surplus associated to the execution of bid $i \in I$
$s_j \geq 0$	dual variable interpretable as the surplus associated to the execution of bid $j \in J$
$s_{hc} \geq 0$	dual variable interpretable as the (potential) surplus associated to the execution of bid hc
$s_c \geq 0$	dual variable interpretable as the surplus associated to the execution of the MIC bid c

$$\max_{x,y,u,n} \sum_i (\lambda^i P_i) x_i \quad + \quad \sum_{c,h \in H_c} (\lambda^{hc} P_{hc}) x_{hc} \quad + \quad \sum_{j,t} (\lambda^j P_j^t) y_j \tag{3.1}$$

subject to:

$$x_i \leq 1 \qquad \forall i \in I \ [s_i] \tag{3.2}$$

$$y_j \leq 1 \qquad \forall j \in J \ [s_j] \tag{3.3}$$

$$x_{hc} \leq u_c \qquad \forall h \in H_c, c \in C \ [s_{hc}] \tag{3.4}$$

$$u_c \leq 1 \qquad \forall c \in C [s_c] \tag{3.5}$$

$$\sum_{i \in I_{lt}} P_i x_i + \sum_{j \in J_l} P_j^t y_j + \sum_{hc \in HC_{lt}} P_{hc} x_{hc} = \sum_k e_{l,t}^k n_k, \qquad \forall (l,t) \quad [\pi_{l,t}] \tag{3.6}$$

$$\sum_k a_{m,k} n_k \leq w_m \qquad \forall m \in N \quad [v_m] \tag{3.7}$$

$$x, y, u \geq 0, \tag{3.8}$$

$$y, u \in \mathbb{Z} \tag{3.9}$$

3.2.3 Duality, uniform prices and opportunity costs

Let us now consider partitions $J = J_r \cup J_a$, $C = C_r \cup C_a$, and the following constraints, fixing all integer variables to some arbitrarily given values:

$$-y_{j_a} \leq -1 \qquad \forall j_a \in J_a \subseteq J \quad [d_{j_a}^a] \tag{3.10}$$

$$y_{j_r} \leq 0 \qquad \forall j_r \in J_r \subseteq J \quad [d_{j_r}^r] \tag{3.11}$$

$$-u_{c_a} \leq -1 \qquad \forall c_a \in C_a \subseteq C \quad [du_{c_a}^a] \tag{3.12}$$

$$u_{c_r} \leq 0 \qquad \forall c_r \in C_r \subseteq C \quad [du_{c_r}^r] \tag{3.13}$$

Dropping integer constraints (3.9) not needed any more, this yields an LP whose dual is:

$$\min \sum_i s_i \quad + \quad \sum_j s_j \quad + \quad \sum_c s_c \quad + \quad \sum_m w_m v_m \quad - \quad \sum_{j_a \in J_a} d^a_{j_a} \quad - \quad \sum_{c_a \in C_a} du^a_{c_a} \tag{3.14}$$

subject to:

$$s_i + P_i \pi_{l(i),t(i)} \geq P_i \lambda^i, \qquad \forall i \quad [x_i] \tag{3.15}$$

$$s_{hc} + P_{hc} \pi_{l(hc),t(hc)} \geq P_{hc} \lambda^{hc}, \qquad \forall h \in H_c, c \; [x_{hc}] \tag{3.16}$$

$$s_{j_r} + d^r_{j_r} + \sum_t P^t_{j_r} \pi_{l(j_r),t} \geq \sum_t P^t_{j_r} \lambda^{j_r}, \qquad \forall j_r \in J_r [y_{j_r}] \tag{3.17}$$

$$s_{j_a} - d^a_{j_a} + \sum_t P^t_{j_a} \pi_{l(j_a),t} \geq \sum_t P^t_{j_a} \lambda^{j_a}, \qquad \forall j_a \in J_a \; [y_{j_a}] \tag{3.18}$$

$$s_{c_r} + du^r_{c_r} \geq \sum_{h \in H_{c_r}} s_{hc_r}, \qquad \forall c_r \in C_r \; [u_{c_r}] \tag{3.19}$$

$$s_{c_a} - du^a_{c_a} \geq \sum_{h \in H_{c_a}} s_{hc_a}, \qquad \forall c_a \in C_a \; [u_{c_a}] \tag{3.20}$$

$$\sum_m a_{m,k} v_m - \sum_{l,t} e^k_{l,t} \pi_{l,t} = 0 \qquad \forall k \in K \; [n_k] \tag{3.21}$$

$$s_i, s_j, s_c, s_{hc}, d^r_{j_r}, d^a_{j_a}, du^r_{c_r}, du^a_{c_a}, v_m \geq 0 \tag{3.22}$$

We now write down the complementarity constraints corresponding to these primal and dual programs parametrized by the integer decisions. Economic interpretations are stated afterwards:

$$
\begin{aligned}
&s_i(1-x_i)=0 && \forall i\in I && (3.23)\\
&s_j(1-y_j)=0 && \forall j\in J && (3.24)\\
&s_{hc}(u_c-x_{hc})=0 && \forall h,c && (3.25)\\
&s_c(1-u_c)=0 && \forall c\in C && (3.26)\\
&v_m(\sum_k a_{m,k}n_k-w_m)=0 && \forall m\in N && (3.27)\\
&(1-y_{j_a})d^a_{j_a}=0 && \forall j_a\in J_a && (3.28)\\
&y_{j_r}d^r_{j_r}=0 && \forall j_r\in J_r && (3.29)\\
&(1-u_{c_a})du^a_{c_a}=0 && \forall c_a\in C_a && (3.30)\\
&u_{c_r}du^r_{c_r}=0 && \forall c_r\in C_r && (3.31)\\
&x_i(s_i+P_i\pi_{l(i),t(i)}-P_i\lambda^i)=0 && \forall i\in I && (3.32)\\
&x_{hc}(s_{hc}+P_{hc}\pi_{l(hc),t(hc)}-P_{hc}\lambda^{hc})=0 && \forall h,c && (3.33)\\
&y_{j_r}(s_{j_r}+d^r_{j_r}+\sum_t P^t_{j_r}(\pi_{l(j_r),t}-\lambda^{j_r}))=0 && \forall j_r\in J_r && (3.34)\\
&y_{j_a}(s_{j_a}-d^a_{j_a}+\sum_t P^t_{j_a}(\pi_{l(j_a),t}-\lambda^{j_a}))=0 && \forall j_a\in J_a && (3.35)\\
&u_{c_r}(s_{c_r}+du^r_{c_r}-\sum_{h\in H_{c_r}} s_{hc_r})=0 && \forall c_r\in C_r && (3.36)\\
&u_{c_a}(s_{c_a}-du^a_{c_a}-\sum_{h\in H_{c_a}} s_{hc_a})=0 && \forall c_a\in C_a && (3.37)
\end{aligned}
$$

Lemma 3.1 (Economic interpretation of d^a, d^r [53])**.** *Take a pair of points (x, y, u, n) and $(s, \pi_{l,t}, d^a, d^r, du^a, du^r)$ respectively satisfying primal conditions (3.2)-(3.13) and dual conditions (3.15)-(3.22), such that complementarity constraints (3.23)-(3.37) are satisfied. For the uniform prices $\pi_{l,t}$: (i) $d^a_{j_a}$ is an upper bound on the actual loss (if any) $-\min[0, \sum_t P^t_{j_a}(\lambda^{j_a} - \pi_{l(j_a),t})]$ of the executed block order j_a, (ii) $d^r_{j_r}$ is an upper bound on the opportunity cost $\max[0, \sum_t P^t_{j_r}(\lambda^{j_r} - \pi_{l(j_r),t})]$ of the rejected order j_r.*

Proof. (i) Conditions (3.35) show that for an accepted block $y_{j_a} = 1$, we have $s_{j_a} - d^a_{j_a} = \sum_t P^t_{j_a}(\lambda^{j_a} - \pi_{l(j_a),t})$, the right-hand side corresponding to the gain (if positive) or loss (if negative) of the bid. As $s_{j_a} \geq 0$, the loss (i.e. the negative part $[\sum_t P^t_{j_a}(\lambda^{j_a} - \pi_{l(j_a),t})]^- \geq 0$) is bounded by $d^a_{j_a}$.

(ii) For a rejected block bid, $y_{j_r} = 0$, and conditions (3.24) imply $s_{j_r} = 0$, which used in dual conditions (3.17) directly yields the result, as $d^r_{j_r} \geq 0$. □

The following lemma proposes analogous interpretations for the case of MIC orders. Intuitively, neglecting for now the so-called MIC condition, the shadow cost of forcing a MIC order to be rejected (given by du^r) is at least equal to the sum of all maximum missed surpluses generated by its hourly suborders at the given market prices, while there is no 'shadow cost' at forcing it to be accepted (its suborders are then cleared using standard rules for hourly bids):

Lemma 3.2 (Interpretation of du^a, du^r)**.** *(i) $du^r_{c_r}$ is an upper bound on the sum of maximum missed individual hourly surpluses* $\sum_{h \in H_{c_r}} (\max[0, P_{hc_r}(\lambda^{hc_r} - \pi_{l(hc_r),t(hc_r)})])$ *of the rejected MIC order c_r. (ii) We can always assume $du^a_{c_a} = 0, \forall c_a \in C_a$.*

Proof. (i) Conditions of type (3.26) show that $s_{c_r} = 0$, while (3.16) and (3.22) give $s_{hc_r} \geq \max[0, P_{hc_r}(\lambda^{hc_r} - \pi_{l(hc_r),t(hc_r)})]$. Using these two facts in (3.19) provides the result.

(ii) As $\forall h \in H_c, s_{hc} \geq 0$, using (3.20), it follows that $\forall c_a \in C_a, s_{c_a} - du^a_{c_a} \geq 0$. Then, we can pose $\tilde{du^a_{c_a}} := 0$ and make a change of variable $\tilde{s_{c_a}} := s_{c_a} - du^a_{c_a}$ in (3.2)-(3.37) (systems of conditions equivalent in the usual sense). □

3.2.4 The new primal-dual framework

This new 'primal-dual approach' makes use of an equality of objective functions (3.39) to enforce all the economically meaningful complementarity conditions (3.23)-(3.37), where the additional variables $d^a_{j_a}, d^r_{j_r}, du^a_{c_a}, du^r_{c_r}$ represent deviations from a perfect market equilibrium affecting non-convex bids, cf. Lemma 3.1 and Lemma 3.2 above. Leaving aside for the time being the question of MIC bid selections, which is dealt with in Section 3.3, the problem is that we do not know a priori, for a given criterion, what is the best block bid selection $J = J_r \cup J_a$. However, the feasible set UMFS described below allows determining the optimal block bid selection, whatever the desired objective function is, and the pair of optimal points for the corresponding primal and dual programs stated above, where welfare is maximized with fixed combinatorial decisions, enforcing equilibrium for the convex part of the market clearing problem, and in particular spatial equilibrium. As the selection J_a, J_r (resp. C_a, C_r) is not known in advance, the feasible set is described using 'deviation variables' $d^a_j, d^r_j, du^a_c, du^r_c$ for all $j \in J, c \in C$, and constraints (3.52),(3.54) for example ensure that $d^a_{j_r} = 0$ for a given rejected block j_r, and conversely that $d^r_{j_a} = 0$ for an accepted block j_a. Therefore, constraints (3.50) below enforce constraints (3.17)-(3.18) in all cases. This is formalised in Theorem 3.1 and helps considering many interesting issues (welfare or traded volume maximization, minimization of opportunity costs, etc), in a computationally efficient way. This is also a key step towards the main extension presented in the next section, proposing an exact linearisation to deal with MIC bids using a MILP formulation.

In theory, admissible market clearing prices may lie outside the price range allowed for bids, see [69]. For modelling purposes, we need to include the following technical constraint limiting the market price range

$$\pi_{l,t} \in [-\bar{\pi}, \bar{\pi}] \qquad \forall l \in L, t \in T. \tag{3.38}$$

$\bar{\pi}$ can be choosen large enough to avoid excluding any relevant market clearing solution (see [54]). Note that in practice, power exchanges actually do impose that the computed prices $\pi_{l,t}$ stay within a given range in order to limit market power and price volatility.

Uniform Market Clearing Feasible Set (UMFS):

$$\sum_i (\lambda^i P_i) x_i + \sum_{c,h \in H_c} (\lambda^{hc} P_{hc}) x_{hc} + \sum_{j,t} (\lambda^j P_j^t) y_j \geq \sum_i s_i + \sum_j s_j + \sum_c s_c - \sum_{j \in J} d_j^a + \sum_m w_m v_m \quad (3.39)$$

$$x_i \leq 1 \qquad \forall i \in I[s_i] \quad (3.40)$$

$$y_j \leq 1 \qquad \forall j \in J\ [s_j] \quad (3.41)$$

$$x_{hc} \leq u_c \qquad \forall h \in H_c, c \in C\ [s_{hc}] \quad (3.42)$$

$$u_c \leq 1 \qquad \forall c \in C[s_c] \quad (3.43)$$

$$\sum_{i \in I_{lt}} P_i x_i + \sum_{j \in J_l} P_j^t y_j + \sum_{hc \in HC_{lt}} P_{hc} x_{hc} = \sum_k e_{l,t}^k n_k, \qquad \forall (l,t)\ [\pi_{l,t}] \quad (3.44)$$

$$\sum_k a_{m,k} n_k \leq w_m \qquad \forall m \in N\ \ [v_m] \quad (3.45)$$

$$x, y, u \geq 0, \quad (3.46)$$

$$y, u \in \mathbb{Z} \quad (3.47)$$

$$s_i + P_i \pi_{l(i),t(i)} \geq P_i \lambda^i, \qquad \forall i \quad [x_i] \quad (3.48)$$

$$s_{hc} + P_{hc} \pi_{l(hc),t(hc)} \geq P_{hc} \lambda^{hc}, \qquad \forall h \in H_c, c\ [x_{hc}] \quad (3.49)$$

$$s_j + d_j^r - d_j^a + \sum_t P_j^t \pi_{l(j),t} \geq \sum_t P_j^t \lambda^j, \qquad \forall j \in J\ [y_j] \quad (3.50)$$

$$(s_c + du_c^r) \geq \sum_{h \in H_c} s_{hc} \qquad \forall c \in C[u_c] \quad (3.51)$$

$$d_j^r \leq M_j (1 - y_j) \qquad \forall j \in J \quad (3.52)$$

$$du_c^r \leq M_c (1 - u_c) \qquad \forall c \in C \quad (3.53)$$

$$d_j^a \leq M_j\ y_j \qquad \forall j \in J \quad (3.54)$$

$$du_c^a = 0 \qquad \forall c \in C \quad (3.55)$$

$$\sum_m a_{m,k} v_m - \sum_{l,t} e_{l,t}^k \pi_{l,t} = 0 \qquad \forall k \in K[n_k] \quad (3.56)$$

$$s_i, s_j, s_c, s_{hc}, d^a, d^r, du^a, du^r, v_m \geq 0 \quad (3.57)$$

Constants M_j are choosen large enough in Constraints (3.52), (3.54) so that Constraints (3.50) are not restraining the range $[-\bar{\pi}, \bar{\pi}]$ of possible values for $\pi_{l,t}$ (or the possibility to paradoxically reject and accept block bids). They must indeed correspond to the maximum opportunity cost in conditions (3.52), or loss in conditions (3.54), that could be incurred to a block bid, for clearing prices in the allowed range. For this purpose, assuming that both the bid and market clearing prices satisfy (3.38), it is sufficient to set $M_j := K \sum_t |P_j^t|$ with $K = 2\bar{\pi}$. This value of K could be improved. For example, in constraints (3.52), one could set $K := (\bar{\pi} - \lambda^j)$ for a sell block bid, and $K := (\lambda^j - (-\bar{\pi}))$ for a buy block bid. Values of the M_c are determined similarly with respect to (3.53) and condition (3.51): they must be such that the value of s_c can be null whatever the values of the s_{hc} are. Economically speaking, the surplus s_c of a rejected MIC bid will be null even if the *potential* surpluses s_{hc} of its suborders are not.

Also, we have made use of Lemma 3.2 to set, without loss of generality, all the variables $du_c^a := 0$ in UMFS. This is clarified in the proof of Theorem 3.1.

Theorem 3.1. *(I) Let $(x, y, u, n, \pi, v, s, d^a, d^r, du^a, du^r)$ be any feasible point of UMFS satisfying the price range condition (3.38), and let us define $J_r = \{j|y_j = 0\}, J_a = \{j|y_j = 1\}, C_r = \{c|u_c = 0\}, C_a = \{c|u_c = 1\}$.*

Then the projection $(x, y, u, n, \pi, v, s, d^a_{j_a \in J_a}, d^r_{j_r \in J_r}, du^a_{c_a \in C_a}, du^r_{c_r \in C_r})$ satisfies all conditions in (3.2)-(3.37).

(II) Conversely, any point

$MCS = (x, y, u, n, \pi, v, s, d^a_{j_a \in J_a}, d^r_{j_r \in J_r}, du^a_{c_a \in C_a}, du^r_{c_r \in C_r})$ feasible for constraints (3.2)-(3.37) related to a given arbitrary block order selection $J = J_r \cup J_a$ and MIC selection $C = C_r \cup C_a$ which respects the price range condition (3.38) can be 'lifted' to obtain a feasible point $\tilde{MCS} = (x, y, u, n, \pi, v, \tilde{s}, \tilde{d^a}, \tilde{d^r}, \tilde{du^a}, \tilde{du^r})$ of UMFS.

Proof. See appendix. □

3.3 Including MIC bids

3.3.1 Complex orders with a minimum income condition

A MIC order is basically a set of hourly orders with the classical clearing rules but with the additional condition that a given 'minimum income condition' must be satisfied. Otherwise, all hourly bids associated to the given MIC bid are rejected, even if some of them are ITM. The minimum income condition of the MIC order c ensures that some fixed cost F_c together with a variable cost $V_c \times P_c$ are recovered, where P_c is the total executed quantity related to the order c, and V_c a given variable cost.

With the notation described in Section 3.2.2, the minimum income condition for a MIC bid c has the form:

$$(u_c = 1) \Longrightarrow \sum_{h \in H_c} (-P_{hc} x_{hc}) \pi_{l(hc),t(hc)} \geq F_c + \sum_{h \in H_c} (-P_{hc} x_{hc}) V_c, \tag{3.58}$$

where H_c denotes the set of hourly orders associated with the MIC order c. The left-hand side represents the total income related to order c, given the market prices $\pi_{l,t}$ and executed amount of power $\sum_{h \in H_c} -P_{hc} x_{hc}$, while the right-hand side corresponds to the fixed and variable costs of production. At first sight, this condition is non-linear and non-convex, because of the terms $x_{hc}\pi_{l(hc),t(hc)}$ in the left-hand side.

In previous works, MIC constraints are either approximated, see [37, 84], or the full model is decomposed and solved heuristically, as in the approach described in EUPHEMIA ([27]). More specifically only the primal part (3.1)–(3.9) is considered in a master problem, and prices are computed only when integer solutions are found. Let us however recall that [37, 84] on the one hand and [27] on the other hand are considering distinct market models. We show in the next section how MIC conditions can be linearized without approximation in the common European market model considered by [27]. To the best of our knowledge, this is the first exact linearisation proposed for this type of conditions.

3.3.2 Exact linearization of the MIC conditions

The following lemma is the key reason for which it is possible to express the a priori non-linear non-convex MIC condition (3.58) as a linear constraint. As we have in UMFS both the surplus variables s_c and the contributions to welfare $(P_{hc}x_{hc})\lambda^{hc}$, we can use them to express the income in a linear way:

Lemma 3.3. *Consider any feasible point of UMFS. Then, the following holds:*

$$\forall c \in C, \sum_{h \in H_c} (P_{hc}x_{hc})\pi_{l(hc),t(hc)} = \sum_{h \in H_c} (P_{hc}\lambda^{hc})x_{hc} - s_c \tag{3.59}$$

Proof. We first define C_r and C_a as in Theorem 3.1. For $c_r \in C_r$, the identity is trivially satisfied, because if a MIC bid is rejected, all related hourly bids are rejected: $\forall h \in H_{c_r}, x_{hc_r} = 0$, and on the other side, $s_{c_r} = 0$ because of complementarity constraints (3.26).

Let us now consider an accepted MIC bid $c_a \in C_a$. We first show that the following identity holds:

$$(P_{hc_a}x_{hc_a})\pi_{l(hc_a),t(hc_a)} = (P_{hc_a}\lambda^{hc_a})x_{hc_a} - s_{hc_a} \tag{3.60}$$

Consider for x_{hc_a} the following two possibilities, noting that $u_{c_a} = 1$:

(a) if $x_{hc_a} = 0$, the identity (3.60) is trivially satisfied, as $s_{hc_a} = 0$ according to complementarity constraints (3.25).

(b) if $0 < x_{hc_a}$, (3.33) gives $s_{hc_a} = P_{hc_a}\lambda^{hc_a} - P_{hc_a}\pi_{l(hc_a),t(hc_a)}$, so multiplying the equation by x_{hc_a} and using (3.25) guaranteeing $s_{hc_a}x_{hc_a} = s_{hc_a}u_{c_a} = s_{hc_a}$, we get identity (3.60).

Summing up (3.60) over $h \in H_{c_a}$ yields:

$$\sum_{h \in H_{c_a}} (P_{hc_a}x_{hc_a})\pi_{l(hc_a),t(hc_a)} = \sum_{h \in H_{c_a}} (P_{hc_a}\lambda^{hc_a})x_{hc_a} - \sum_{hc_a} s_{hc_a}$$

Finally, using complementarity constraints (3.37), we get:

$$\sum_{h \in H_{c_a}} (P_{hc_a} x_{hc_a}) \pi_{l(hc_a),t(hc_a)} = \sum_{h \in H_{c_a}} (P_{hc_a} \lambda^{hc_a}) x_{hc_a} - (s_{c_a} - du^a_{c_a}),$$

where $du^a_{c_a} := 0$ by Lemma 3.2 in the definition of UMFS, providing the required identity (3.59). □

Using Lemma 3.3, the MIC condition (3.58) can be stated in a linear way as follows:

$$s_c - \sum_{h \in H_c} (P_{hc} \lambda^{hc}) x_{hc} \geq F_c + \sum_{h \in H_c} (-P_{hc} x_{hc}) V_c - \overline{M_c}(1 - u_c) \tag{3.61}$$

where $\overline{M_c}$ is a fixed number large enough to deactivate the constraint when $u_c = 0$. As $u_c = 0$ implies $s_c = 0$ and $x_{hc} = 0$, we set $\overline{M_c} := F_c$.

3.3.3 Welfare maximization with MIC bids, without any auxiliary variables

We propose here a formulation of the welfare maximization problem including MIC bids, avoiding any auxiliary variables, by eliminating the variables d^a, d^r, du^r from the formulation UMFS.

Let us consider UMFS with the additional MIC conditions (3.61) for $c \in C$. We can make a first simplification of the model by replacing both kinds of conditions (3.51), (3.53) by the conditions (3.74) below. Also, under PCR market rules, as no bid can be paradoxically executed, according to Lemma 3.1, we must set $d^a_j = 0, \forall j \in J$. With these last conditions added, in the same way, we can clean up the mathematical formulation by replacing (3.50) and (3.52) by conditions (3.73) below, as well as removing constraints (3.54)-(3.55) (constraints (3.55) are not needed any more as we removed all occurrences of du^a_c in consequence). This yields an equivalent MILP formulation without any auxiliary variables, and in particular no more binary variables than the number of block and MIC bids:

PCR-FS

$$\sum_i (\lambda^i P_i) x_i + \sum_{c,h \in H_c} (\lambda^{hc} P_{hc}) x_{hc} + \sum_{j,t} (\lambda^j P_j^t) y_j$$
$$\geq \sum_i s_i + \sum_j s_j + \sum_c s_c + \sum_m w_m v_m \tag{3.62}$$
$$x_i \leq 1 \qquad \forall i \in I \ [s_i] \tag{3.63}$$
$$y_j \leq 1 \qquad \forall j \in J \ [s_j] \tag{3.64}$$
$$x_{hc} \leq u_c \qquad \forall h \in H_c, c \in C \ [s_{hc}] \tag{3.65}$$
$$u_c \leq 1 \qquad \forall c \in C [s_c] \tag{3.66}$$
$$\sum_{i \in I_{lt}} P_i x_i + \sum_{j \in J_l} P_j^t y_j + \sum_{hc \in HC_{lt}} P_{hc} x_{hc}$$
$$= \sum_k e_{l,t}^k n_k, \qquad \forall (l,t) \quad [\pi_{l,t}] \tag{3.67}$$
$$\sum_k a_{m,k} n_k \leq w_m \qquad \forall m \in N \quad [v_m] \tag{3.68}$$
$$x, y, u \geq 0, \tag{3.69}$$
$$y, u \in \mathbb{Z} \tag{3.70}$$
$$s_i + P_i \pi_{l(i),t(i)} \geq P_i \lambda^i, \qquad \forall i \qquad [x_i] \tag{3.71}$$
$$s_{hc} + P_{hc} \pi_{l(hc),t(hc)} \geq P_{hc} \lambda^{hc}, \qquad \forall h \in H_c, c \ [x_{hc}] \tag{3.72}$$
$$s_j + M_j(1 - y_j) + \sum_t P_j^t \pi_{l(j),t} \geq \sum_t P_j^t \lambda^j, \qquad \forall j \in J \ [y_j] \tag{3.73}$$
$$s_c + M_c(1 - u_c) \geq \sum_{h \in H_c} s_{hc} \qquad \forall c \in C [u_c] \tag{3.74}$$
$$\sum_m a_{m,k} v_m - \sum_{l,t} e_{l,t}^k \pi_{l,t} = 0 \qquad \forall k \in K [n_k] \tag{3.75}$$
$$s_c - \sum_{h \in H_c} (P_{hc} \lambda^{hc}) x_{hc} \geq$$
$$F_c + \sum_{h \in H_c} (-P_{hc} x_{hc}) V_c - \overline{M_c}(1 - u_c) \qquad \forall c \in C \tag{3.76}$$
$$s_i, s_j, s_c, s_{hc}, d^r, du^r, v_m \geq 0 \tag{3.77}$$

The welfare optimization problem is then stated as follows:

$$\max_{PCR-FS} \quad \sum_i (\lambda^i P_i) x_i + \sum_{c,h \in H_c} (\lambda^{hc} P_{hc}) x_{hc} + \sum_{j,t} (\lambda^j P_j^t) y_j \tag{3.78}$$

Let us note that a solution will always exist, provided that bid curves can be matched and network constraints be satisfied. This follows from the fact that in the worst case, all non-convex bids could be rejected, since the paradoxical rejection of a non-convex bid is allowed in the market model. In particular, minimum income conditions (3.58) or (3.76) are trivially satisfied for rejected MIC bids, i.e. when $u_c = 0$.

3.3.4 Considering other objective functions for economic analysis purposes

Maximizing the traded volume

The following program aims at maximizing the traded volume under the same market rules:

$$\max_{PCR-FS} \quad \sum_{i|P_i>0} P_i x_i \quad + \quad \sum_{c,h\in H_c|P_{hc}>0} P_{hc}x_{hc} \quad + \quad \sum_{(j,t)|P_j^t>0} P_j^t y_j \tag{3.79}$$

An alternative formulation of the objective function is the following:

$$\max_{PCR-FS} \quad \frac{1}{2}(\sum_i |P_i| x_i \quad + \quad \sum_{c,h\in H_c} |P_{hc}| x_{hc} \quad + \quad \sum_{(j,t)} |P_j^t| y_j) \tag{3.80}$$

Minimizing opportunity costs of PRB

It suffices to consider the following objective function over UMFS and the additional constraints $d_j^a = 0, \forall j \in J$:

$$\min \sum_j d_j^r \tag{3.81}$$

Let us note that constraints like (3.52) also allow to control which block bids could be paradoxically rejected on an individual basis, or could be used to forbid the paradoxical rejection of bids which are too deeply in-the-money, by specifying a threshold via the values of M_j.

Finally, let us note that a result previously proposed in [70] about an alternative market model could be recovered almost directly using the framework proposed here. This result basically states that (a) there is always enough welfare to allow and compensate paradoxically accepted block bids (PAB), (b) allowing PAB generates globally more welfare. The fact that this approach generates more welfare is because allowing PAB corresponds to discarding constraints $d_j^a = 0$, providing a relaxation of the European market rules. Moreover, the condition of equality of objective functions (3.39) directly shows that welfare, which is positive under very mild assumptions, can be decomposed as the sum of individual bid surpluses minus the compensations d_j^a that should be paid to block bids losing money to make them whole. However, this corresponds to a non-uniform pricing scheme that we don't consider further in the present article.

3.4 Numerical Experiments

We provide here a proof-of-concept of the approach, presenting numerical experiments, using realistic *large-scale instances*, on: (a) welfare maximization with MIC bids, (b) traded volume maximisation under CWE rules (i.e. without MIC bids), and (c) opportunity costs minimization also under CWE market rules. The models have been implemented in C++ using IBM ILOG Concert Technology interfaced with R for input-output management as well as post-processing analysis, and solved using CPLEX 12.5.1 using 4 threads on a platform with 2x Xeon X5650 (6 cores @ 2.66 GHz), 16 GB of RAM, running Fedora Linux 20. One potential advantage of the new primal-dual approach is the possibility to benefit directly from parallel computing routines of state-of-the-art solvers like CPLEX.

3.4.1 Welfare maximization with MIC bids

We first consider solving market instances with MIC bids only. The instances involve hourly bids from four areas, one of these areas also containing MIC bids. Solving instances up to optimality is tractable, see Table 3.3, though some instances are challenging from a numerical stability point of view, due to the introduction of so-called big M numbers in the formulations. A particular attention should be paid to the tolerance parameters of the solver, for example the integer feasibility tolerance parameter. The branching direction has been set to -1 (priority to the 'down branch'), to guarantee finding a good feasible solution quickly. The intuition is that minimum income conditions do not apply to rejected orders. Moreover, eliminating MIC bids tend to increase the prices and make the minimum income conditions of other MIC orders satisfied. Heuristics have therefore been deactivated. Cuts have also been deactivated, as we observed they were not helpful and slowed down the algorithm.

Let us emphasize that this new formulation provides dual LP bounds, and can be solved *exactly* by state-of-the-art MIP solvers. To our knowledge, this is the first tractable exact approach proposed. (Another, apparently non-tractable approach would be to proceed by decomposing the problem and adding e.g. no-good cuts rejecting the current MIC bids selection when no admissible prices exist, with a very slow convergence rate.)

#inst	Run. Time (s.)	Nodes	Abs. Gap	Rel. Gap	#Hourly bids	# MIC bids
1	70	58			47107	70
2	274	896			49299	74
3	432	1111			48119	71
4	144	104			52434	72
5	37	19			41623	74
6	901	3238	1299754.97	0.04%	45371	69
7	22	23			36819	73
8	624	1055			53516	69
9	255	504			62770	76
10	216	418			45731	74

Table 3.3: Welfare optimization with MIC bids

Adding block bids makes the problem much more difficult to solve. The solver parameter values used are the same as above. It turns out that many block bids are fractionally accepted in continuous relaxations of the branch-and-cut tree. As a consequence, the first feasible solutions found are of poor quality, with a few block and MIC bids accepted. Instead, as an easy-to-implement heuristic approach, we first solve an instance with all block bids fixed to zero, and determine an admissible MIC bids selection (ideally optimal for this subproblem). Second, we fix this MIC bids selection, and introduce block bids, to determine a potentially very good solution to the initial problem. It should be noted that a solution for this second stage always exists, as in the worst case, all block bids could be rejected. Third, the obtained solution is used as a MIP start for the initial model with both block and MIC bids. With this approach, the number of block bids accepted is of the same order as when no MIC bid is present besides these block bids, and the relative gap is improved, compared to the basic approach of providing the solver with the formulation 'as is'. However, the absolute MIP gap remains substantial. Results are presented in Table 3.4, with a running time limit of 900 seconds for each of the first two stages, and 1200 seconds for the last stage.

#inst	Nodes	Abs. Gap	Rel. Gap	#Hourly bids	# MIC bids	# Blocks bids
1	13214	1015887.92	0.04%	47107	70	502
2	7913	4129620.69	0.14%	49299	74	589
3	11375	2748987.16	0.12%	48119	71	516
4	2873	3009748.14	0.10%	52434	72	591
5	12213	1425671.83	0.05%	41623	74	588
6	6443	5999741.05	0.19%	45371	69	567
7	22250	337651.70	0.01%	36819	73	550
8	6925	4747440.57	0.19%	53516	69	691
9	3658	2937928.67	0.08%	62770	76	604
10	3194	3100317.15	0.12%	45731	74	537

Table 3.4: Welfare optimization with MIC and block bids

3.4.2 Traded volume maximization

To optimize the traded volume, welfare maximization itself turns out to be a useful *heuristic*. At a first stage, we solve this welfare maximization problem, and for the given optimal block bid selection, maximize the traded volume (dealing with a possible indeterminacy of the traded volume for that welfare maximizing solution). At a second stage, we use this solution as a MIP start to solve the initial problem. This helps in practice, at least providing a useful upper bound, even in some cases proving optimality of the welfare maximizing solution for the traded volume maximization problem. We also observed that provided the reasonably good solution obtained from maximizing welfare, well-known heuristics such as 'solution polishing' in CPLEX could quickly provide better solutions in terms of traded volume. Therefore, this heuristic is first applied when starting the second stage solving the initial traded volume maximization problem itself, provided the solution obtained at the first stage by maximizing the welfare. Let us note however that maximizing the traded volume, or minimizing opportunity costs, is more difficult than maximizing welfare, though we are able to solve instances of reasonable size, and not toy examples only. For illustration purposes, instances corresponding to the Belgian market have been used. Table 3.5 summarizes the trade-off between both kinds of objectives for ten such instances.

#	Welf. max sol. Max Trad. Vol.	Maximizing traded volume Max Trad. Vol.	 Best bound	Δ Vol.	Δ Welf.	# Hourly bids	# Blocks bids
1	24589.84	24589.84	24589.84	0.00	0.00	1939	54
2	25794.53	25928.19	26654.32	133.66	5672.35	1711	67
3	23633.48	23696.99	23696.99	63.51	171.40	1706	54
4	35137.32	35285.32	35285.32	148.00	4292.13	1893	56
5	21296.94	21433.08	21433.08	136.15	2460.55	1713	39
6	23361.72	23871.27	23871.27	509.55	56518.94	1700	46
7	23542.38	23679.64	23679.64	137.26	1877.94	1749	35
8	35974.15	36270.11	36270.11	295.96	21403.03	1533	58
9	24988.63	24988.63	24988.63	0.00	0.00	1787	33
10	35307.91	35507.32	37434.39	199.41	16082.15	1418	62

Table 3.5: Comparison of the maximum traded volume in both cases

For example, instances # 2 or # 3 show concrete examples where it is possible to obtain more traded volume than when just optimizing welfare (as in the toy example presented above), the better solution for # 3 even being proven optimal. Instance # 1 shows an example where the welfare maximizing solution is proven optimal for the traded volume maximization problem. Sometimes the traded volume can be significantly larger (2% or more), as in instance #6.

3.4.3 Minimizing opportunity costs

We proceed as above, (a) first solving the welfare maximizing solution, (b) looking for the minimum opportunity costs possible for this solution, and (c) use this solution as a start solution for the proper opportunity costs minimization problem. Let us note that prices and opportunity costs obtained from stage (b) can substantially differ from the prices computed in practice, as these prices are determined in a different way from what is specified by tie breaking rules in case of price/volume indeterminacy. Let us recall that welfare is uniquely determined by the block bid selections and hence not affected by stage (b), see [54]. We also refer to [53] for a table showing results for a few real CWE instances from 2011.

Results are given in Table 3.6. Optimal solutions are found in the majority of the instances (9 out of 10). Again, for example, instances # 1 or # 2 show that solutions to both problems do not coincide in general (as in the toy example of Section 3.2.1). Opportunity cost can sometimes be reduced by 75% or more, for example for instance #3. In the case of instances # 5 and # 9, the welfare maximizing solution is proven optimal for the opportunity costs minimization problem.

#	W-MAXSOL Min OC	Minimizing Opp. Costs Min OC	Best bound	Δ OC.	Δ Welf.	# Hourly	# Blocks
1	13096.96	5624.37	5624.37	7472.58	2501.76	1939	54
2	6559.97	2124.96	2124.96	4435.01	963.19	1711	67
3	3913.16	978.61	978.61	2934.55	171.40	1706	54
4	483.71	348.00	348.00	135.71	138.65	1893	56
5	1715.30	1715.30	1715.30	0.00	0.00	1713	39
6	49366.33	46405.44	46405.44	2960.90	1577.61	1700	46
7	8771.51	8771.51	8771.51	0.00	0.00	1749	35
8	17249.96	7399.43	7399.43	9850.53	236.38	1533	58
9	256.81	256.81	256.81	0.00	0.00	1787	33
10	64777.46	61579.08	3198.25	3198.37	1591.57	1418	62

Table 3.6: Comparison of opportunity costs in both cases

3.5 Conclusions

The new primal-dual approach proposed here allows deriving powerful algorithmic tools, and dealing with economic issues of interest for day-ahead auction participants or organizers. We have been able to give a MILP formulation of the market clearing problem in the presence of MIC bids, avoiding the introduction of any auxiliary variables, relying on an exact linearisation of the minimum income condition. To the best of our knowledge, it is the first tractable exact approach proposed to deal with such kind of bids, and numerical experiments show good results, though the approach is still challenging when both block and MIC bids are considered together. From the economic analysis point of view, the approach allowed us to examine the trade-off occurring in practice between different objectives such as welfare maximization, traded volume maximization, and minimization of opportunity costs of paradoxically rejected block bids. It also seems these are the first tractable formulations proposed to examine these economic issues. The trade-offs for the examined instances were rather small, though they could be more important in absolute terms if the number and size of non-convex bids are allowed to increase. We also plan to release a Julia package implementing the models and algorithms, to foster exchanges and provide adaptable tools to the academic community working on related research topics.

Acknowledgements: We greatly thank APX, BELPEX, EPEX SPOT, OMIE and N-Side for providing us with data used to generate realistic instances. We also thank organizers of the 11th International (IEEE) Conference on European Energy Market (Krakow, May 2014), as well as organizers of the COST Workshop on Mathematical Models and Methods for Energy Optimization (Budapest, Sept. 2014), for allowing us to present partial results developed here. This text presents research results of the P7/36 PAI project COMEX, part of the IPA Belgian Program. The work was also supported by EC-FP7 COST Action TD1207. The scientific responsibility is assumed by the authors.

3.A Omitted proofs in main text

Let us first consider the equality of primal and dual objective functions of Section 3.2.2:
Observation 3.1. *By strong duality for linear programs, for a pair of primal and dual feasible points corresponding to a block bid selection and a MIC bid selection, i.e. satisfying respectively (3.2)-(3.13) and (3.15)-(3.22), the complementarity constraints (3.23)-(3.37) hold if and only if we have the equality* (3.1) = (3.14).

3.A.1 Proof of Theorem 3.1

Proof. We emphasize again, and use below, the fact that according to Lemma 3.2, we can assume without loss of generality $du^a_{c_a} = 0, \forall c_a \in C_a$ in (3.14)-(3.22).

(I) Let $MCS = (x, y, n, u, \pi, v, s, d^a, d^r, du^a, du^r)$ be a feasible point of UMFS and let us define $J_r := \{j \in J | y_j = 0\}$, $J_a := \{j \in J | y_j = 1\}$ and likewise for C_r, C_a with respect to the values of the variables u_c. Consider the projection
$\tilde{MCS} = (x, y, n, u, \pi, v, s, d^a_{j_a \in J_a}, d^r_{j_r \in J_r}, du^a_{c_a \in C_a}, du^r_{c_r \in C_r})$. Constraints (3.50)-(3.55) ensure that $\tilde{MCS}$ satisfies constraints (3.17)-(3.20): constraints (3.52)-(3.55) are 'dispatching' constraints (3.50)-(3.51) to constraints (3.17)-(3.20). Therefore $\tilde{MCS}$ satisfies primal conditions (3.2)-(3.13) and dual conditions (3.15)-(3.22). Condition (3.54) ensures that $d^a_j = 0$ for $j \in J_r$, and with (3.55), it shows that condition (3.39) implies the equality (3.1) = (3.14). By Observation 3.1, we can then replace this equality by the needed complementarity conditions (3.23)-(3.37).

(II) Conversely let $\tilde{MCS} = (x, y, n, u, \pi, v, s, d^a_{j_a \in J_a}, d^r_{j_r \in J_r}, du^a_{c_a \in C_a}, du^r_{c_r \in C_r})$ be a point satisfying primal conditions (3.2)-(3.13), dual conditions (3.15)-(3.22), and complementarity conditions (3.23)-(3.37), associated to a given block and MIC bid selection $J = J_a \cup J_r, C = C_a \cup C_r$. Observation 3.1 ensures that this point also satisfies the equality (3.1) = (3.14). Let us set additional values $d^r_j = 0$, for $j \in J_a$, also $d^a_j = 0$ for $j \in J_r$, and similarly $du^a_c = 0$ for $c \in C_r$, $du^r_c = 0$ for $c \in C_a$, giving a point $MCS = (x, y, n, u, \pi, v, s, d^a, d^r, du^a, du^r)$. The new point satisfies condition (3.39), since only terms $d^a_j = 0, j \in J_r, du^a_c = 0$ are added to the equality (3.1) = (3.14). It remains to verify that all the remaining constraints defining UMFS are satisfied as well. All these additional values trivially satisfy constraints (3.52)-(3.55). Therefore, it is needed to show that conditions (3.50)-(3.55) are also satisfied *for all* $j \in J, c \in C$. Due to (3.24), in condition (3.17), $s_{j_r} = 0$ and we can set $d^r_{j_r} := P_{j_r} \lambda^{j_r} - P_{j_r} \pi$ without altering the satisfaction of any condition. Due to the price range condition and the choice of the parameters M_j, these $d^r_j, j \in J_r$ satisfy conditions (3.52) which therefore hold for all $j \in J$. In condition (3.18), $s_{j_a}, d^a_{j_a}$ can be redefined without modifying the values of $(s_{j_a} - d^a_{j_a})$ and hence without altering satisfaction of any other constraint. Due to the large values of the parameters M_j, this again can be done so as to satisfy conditions (3.54) for $j \in J_a$, hence for all $j \in J$. Then, (3.17)-(3.18), and the 'dispatcher conditions' (3.52)-(3.54) imply (3.50). Finally, concerning the analogue constraints related to the MIC bids, and first using Lemma 3.1 to set $du^a_{c_a} = 0$ for all $c_a \in Ca$, it is straightforward to show in a similar way that (3.19)-(3.20) together with the M_c and the additional null values for part of the du^r (resp. du^a) given above allow satisfying (3.51), (3.53). □

Chapter 4

Revisiting minimum profit conditions in uniform price day-ahead electricity auctions

CORE Discussion Paper 2016/43 updated version of: Mehdi Madani and Mathieu Van Vyve, Revisiting minimum profit conditions in uniform price day-ahead electricity auctions (submitted).

[Core Discussion papers 2016: http://www.uclouvain.be/en-633757.html]

Abstract

We examine the problem of clearing day-ahead electricity market auctions where each bidder, whether a producer or consumer, can specify a minimum profit or maximum payment condition constraining the acceptance of a set of bid curves spanning multiple time periods in locations connected through a transmission network with linear constraints. Such types of conditions are for example considered in the Spanish and Portuguese day-ahead markets. This helps describing the recovery of start-up costs of a power plant, or analogously for a large consumer, utility reduced by a constant term. A new market model is proposed with a corresponding MILP formulation for uniform locational price day-ahead auctions, handling bids with a minimum profit or maximum payment condition in a uniform and computationally-efficient way. An exact decomposition procedure with sparse strengthened Benders cuts derived from the MILP formulation is also proposed. The MILP formulation and the decomposition procedure are similar to computationally-efficient approaches previously proposed to handle so-called block bids according to European market rules, though the clearing conditions could appear different at first sight. Both solving approaches are also valid to deal with both kinds of bids simultaneously, as block bids with a minimum acceptance ratio, generalizing fully indivisible block bids, are but a special case of the MP bids introduced here. We argue in favour of the MP bids by comparing them to previous models for minimum profit conditions proposed in the academic literature, and to the model for minimum income conditions used by the Spanish power exchange OMIE.

4.1 Introduction

4.1.1 Minimum profit conditions and Near-Equilibrium in non-convex day-ahead electricity auctions

Day-ahead electricity markets are organized markets where electricity is traded for the 24 hours of the next day. They can take the form of single or two sided auctions (pool with mandatory participation to match forecast demand or auctions confronting elastic offer and demand). The prices set in day-ahead markets are used as reference prices for many electricity derivatives, and such markets are taking more importance with the ongoing liberalization and coupling of electricity markets around the world in general, and in Europe in particular.

Clearing these auctions amounts to finding - ideally- a partial equilibrium using submitted bids describing demand and offer profiles, depending on the utility, production costs and operational constraints of market participants. A market operator, typically power exchanges in Europe, is in charge of computing a market clearing solution.

It is well-known that for a well-behaved convex welfare optimization problem where strong duality holds, duality theory provides equilibrium prices. However, to describe their operational constraints or cost structure, participants can specify for example a minimum output level of production (indivisibilities), or that the revenue generated by the traded power at the market clearing prices should cover some start-up costs if the plant is started. Similar bids could be specified for the demand side. This leads to the study of partial market equilibrium with uniform prices where indivisibilities and fixed costs must be taken into account, deviating from a well-behaved convex configuration studied in classical microeconomic textbooks, e.g. in [61]. The need for bidding products introducing non-convexities is due in particular to the peculiar nature of electricity and the non-convexities of production sets of the power plants.

When considering a market clearing problem with non-convexities such as indivisibilities (so-called block bids in the Pan-European PCR market [27]), or start-up cost recovery conditions (so-called complex bids with a minimum income condition also called MIC bids in PCR), most of the time no market equilibrium exists, see e.g. the toy example in Section 4.2.1 for an instance involving MIC bids, and in [55] for an instance involving block bids.

Let us also mention that in coupled day-ahead electricity markets, representation of the network is a particularly important matter. Besides the potential issues due to the simplifications or approximations made to represent a whole network, it is of main importance for participants to understand clearly the reason for price differences occurring between different locations. Economically speaking, locational prices should ideally form a spatial equilibrium, as historically studied in [25, 85], which could equivalently be interpreted as requiring optimality conditions for TSOs, relating locational price differences to the scarcity and marginal prices of transmission resources.

Near-equilibrium under minimum profit conditions in uniform price day-ahead electricity auctions is the main topic of the present contribution, and is also considered in references [27, 37, 36, 35, 84], which are discussed in Section 4.3.1 below.

4.1.2 Contribution and structure of this article

The main contribution of the present paper is to show how to handle minimum profit (or maximum payment) conditions in a new way which turns out to generalize both block orders with a minimum acceptance ratio used in France, Germany or Belgium, and, mutatis mutandis, complex orders with a minimum income condition used in Spain and Portugal. The new approach consists in new bids, which we call MP bids (for minimum profit or maximum payment), and the corresponding mathematical programming formulation is a MILP modelling all the corresponding market clearing conditions without *any* auxiliary variables, similar to an efficient MIP formulation previously proposed for block orders [54]. An efficient Benders decomposition with sparse strenghtened cuts similar to the one proposed in [54] is also derived. These MP bids hence seem an appropriate tool to foster market design and bidding products convergence among the different regions which form the coupled European day-ahead electricity markets of the Pan-European PCR project.

We start by providing in Section 4.2.1 a toy example illustrating the key points dealt with in the reminder of the article. It illustrates the issues arising when considering minimum profit conditions, and alternatives to take them into account in the computation of market clearing solutions. We describe in Section 4.2.2 the notation used and a basic 'unrestricted' welfare maximization problem where such minimum profit conditions are first not enforced, also recalling the nice equilibrium properties which would hold in a convex market clearing setting.

Section 4.3 is devoted to modelling minimum profit conditions or more generally MP conditions, as with the approach proposed, the statement of a maximum payment condition for demand-side orders is formally identical. After reviewing previous contributions considering minimum profit conditions, we derive economic interpretations for optimal dual variables of a welfare maximization program where an *arbitrary* MP bids combination has been specified. We then develop the core result, showing how to consider MP bids in a computationally-efficient way, relying on previous results to provide a MILP formulation without complementarity constraints nor any auxiliary variable to model these MP conditions. Section 4.4 shows how to adapt all results when ramping constraints of power plants are considered.

Section 4.5 derives from the MILP formulation provided in Section 4.3 a Benders decomposition procedure with locally strengthened Benders cuts. These cuts are valid in subtrees of a branch-and-bound solving a primal welfare maximization program, rooted at nodes where an incumbent should be rejected because no uniform prices exist such that MP conditions are all satisfied. They complement the classical Benders cuts which we show to correspond indeed to 'no-good cuts' basically rejecting the current MP bids combination, and which are globally valid.

Numerical experiments are presented in Section 4.6. Implementations have been made in Julia/JuMP [51] and are provided together with sample datasets in an online Git repository [56]. They show the efficiency and merit of the new approach, in particular compared to the current practice in OMIE-PCR.

4.2 Near-equilibrium and minimum profit conditions

4.2.1 Position of the problem: a toy example

In the following toy example whose data is provided in Table 4.1 and depicted on Figure 4.1, a bid curve (in blue) represents some elastic demand. To satisfy this demand, there are two offer bids from two plants, each having different start-up costs (100 EUR and 200 EUR respectively), but the same marginal cost of 10 EUR / MW. Both plants bid their marginal cost curve and their start-up cost to the auctioneer.

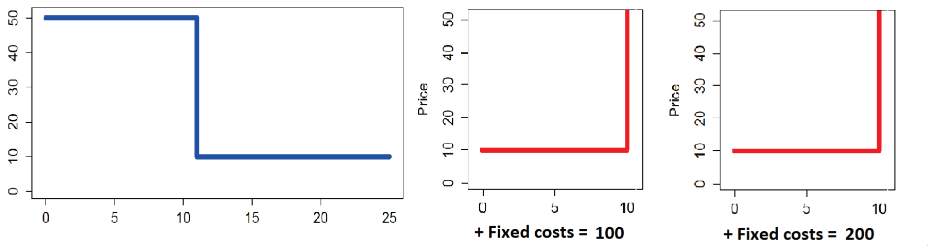

Figure 4.1: Marginal cost/utility curves (see Table 4.1 for related start-up costs)

Neglecting first the minimum income conditions stating that all costs should be recovered for online plants (i.e. both start-up and marginal costs), we can clear the market auction by matching the aggregated marginal costs (resp. utility) bid curves, as done in the left part of Figure 4.2. In that case, the determined market clearing price would be 10 EUR /MW, and obviously, both power plants won't recover their costs for that market clearing price.

However, if we allow the potentially paradoxical rejection of bids involving start-up costs, which is also tolerated in all previous propositions considering minimum profit conditions exposed in [27, 35, 37, 36, 84], then a 'satisfactory solution' could be obtained by either rejecting bid B or bid C. In that case, matching marginal cost/utility curves as in the right Figure 4.2, we see that the market clearing price will rise to 50 EUR / MW and that, whatever the chosen offer B or C, the corresponding plant will recover all its costs. Similar examples could be given for demand bids with a maximum payment condition.

These observations help understanding why it is not possible to get a market equilibrium such that all MP conditions are satisfied. It may be required to expel some bids from the market clearing solution that would be profitable for the market clearing prices obtained in that situation. On the other hand, including such 'paradoxically rejected bids' would modify prices such that the MP condition of some bid would not be satisfied any more.

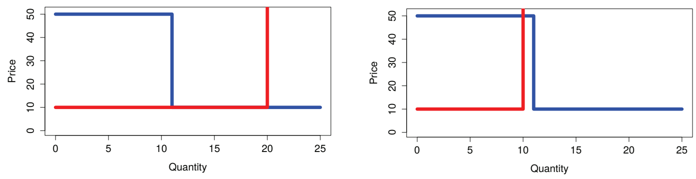

Figure 4.2: Matching MP bids

Bids	Power (MW)	Limit price (EUR/MW)	Start-up costs
D1: Demand bid 1	11	50	-
D2: Demand bid 2	14	10	-
MP1: Offer MP bid 1	10	10	100
MP2: Offer MP bid 2	10	10	200

Table 4.1: Toy market clearing instance

	Market Price	Revenue		Costs		Profits	
		MP1	MP2	MP1	MP2	MP1	MP2
Matching MP1 & MP2	10	100	100	200	300	-100	-200
Matching MP1	50	500	0	200	0	300	0
Matching MP2	50	0	500	0	300	0	200

Table 4.2: Market outcomes

The second point is that, even if in both matchings the costs are recovered for the chosen plant, both matchings are not equivalent from a welfare point of view if we include fixed costs in the computation of the welfare.

Under current OMIE-PCR market rules, both matching possibilities are not distinguished because fixed costs are not included in the welfare maximizing objective function which only considers marginal costs (resp. utility) of selected plants (resp. consumers). In such a case, welfare is considered to be 400 whatever the chosen matching. Let us note that in the same way, in [37], the fixed costs that should be recovered are not included in the welfare objective.

If we pay attention to fixed costs when computing welfare, matching MP1 yields a welfare of 300 while matching MP2 yields a welfare of 200. Such a choice in terms of inclusion of fixed costs in the welfare objective function is similar to what is done in [84].

4.2.2 Unrestriced welfare optimization

Notation used throughout the text is provided here for quick reference. The interpretation of any other symbol is given within the text itself.

Notation and Abbreviations

Abbreviations:

MP bids	Stands for bids with either a minimum profit or a maximum payment condition
MIC bids	Stands for complex orders with a minimum income condition used in OMIE-PCR
ITM	Stands for 'in-the-money'
ATM	Stands for 'at-the-money'
OTM	Stands for 'out-of-the-money'

Sets and indices:

i	Index for hourly bids, in set I
c	Index for MP bids, in set C
hc	Index for hourly bids associated to the MIC bid c, in set H_c
l	Index for locations, $l(i)$ (resp. $l(hc)$) denotes the location of bid i (resp. hc)
t	Index for time slots, $t(i)$ (resp. $t(hc)$) denotes the time slot of bid i, (resp. hc)
$I_{lt} \subseteq I$	Subset of hourly bids associated to location l and time slot t
$HC_{lt} \subseteq HC$	Subset of MP hourly suborders, associated to location l and time slot t

Parameters:

Q_i, Q_{hc}	Power amount of hourly bid i (resp. hc), $Q < 0$ for sell bids, and $Q > 0$ for demand bids
$r_{hc} \in [0, 1]$	minimum ratio parameter used to express minimum output levels
P^i, P^{hc}	Limit bid price of hourly bid i, hc
$a_{m,k}$	Abstract linear network representation parameters
w_m	Capacity of the network resource m
F_c	Start-up or fixed cost associated to bid c

Primal decision variables:

$x_i \in [0, 1]$	fraction of power Q_i which is executed
$x_{hc} \in [0, 1]$	fraction of power Q_{hc} (related to the MIC bid c) which is executed
$u_c \in \{0, 1\}$	binary variable conditioning the execution or rejection of the MP bid c (i.e. of the values of x_{hc})
n_k	variables used for the abstract linear network representation, related to net export positions

Dual decision variables:

π_{lt}	locational uniform price of electricity at location l and time slot t
$v_m \geq 0$	dual variable pricing the network constraint m,
$s_i \geq 0$	dual variable interpretable as the surplus associated to the execution of bid $i \in I$
$s_{hc}^{max} \geq 0$	dual variable related to the (potential) surplus associated to the execution of bid hc
$s_{hc}^{min} \geq 0$	dual variable related to the (potential) surplus associated to the execution of bid hc
$s_c \geq 0$	dual variable interpretable as the surplus associated to the execution of the MP bid c

A classical hourly order corresponds to a step of a stepwise offer or demand bid curve relating accepted power quantities to prices. For each such step, the variable $x_i \in [0, 1]$ denotes which fraction of this step will be accepted in the market clearing solution. In the same way, variables x_{hc} denote these accepted fractions for bid curves associated to a bid with a minimum profit condition or maximum payment condition (MP bids).

Concerning these MP bids, binary variables u_c are introduced to model the conditional acceptance of a set of hourly bids $hc \in H_c$, controlled via constraints (4.3), while constraints (4.4) enforce minimum acceptance ratios where applicable. They are used for example to model minimum power outputs of power plants. The conditional acceptances will be expressed as price-based decisions (as called in [104, 32]) using the primal-dual formulation developed in Section 4.3.2, involving both quantity and price variables. Parameters F_c correspond to fixed/start-up costs incurred if the MP bid is accepted. Let us also note that a block bid spanning multiple time periods as described in [27, 60, 54] could be described as an MP bid c by using a suitable choice of associated bid curves and minimum acceptance ratios, and setting the corresponding fixed cost parameter F_c to 0 in (4.1). It turns out that in such a case, minimum profit or maximum payment conditions as described below will exactly correspond to the European market clearing conditions for block orders described in [27, 60, 54], essentially stating that no loss should be incurred to any accepted block bid, but allowing some block bids to be paradoxically rejected.

Constraint (4.6) is the balance equation at location l at time t, where the right-hand side is the net export position expressed as a linear combination of abstract network elements. Constraint (4.7) is the capacity constraint of the abstract network resource m. This abstract linear network representation covers e.g. DC network flow models or the so-called ATC and Flow-based models used in PCR (see [27]). The usual network equilibrium conditions involving locational market prices apply, as they will be enforced by dual and complementarity conditions (4.14), (4.20), see [54].

The objective function aims at maximizing welfare. For the sake of conciseness, we do not consider ramping constraints of power plants in the main parts of the text, though they can straightforwardly be included in all the developments carried out, as shown in Section 4.4.

UWELFARE:

$$\max_{x,y,u,n} \sum_i (P^i Q_i) x_i \quad + \quad \sum_{c,h \in H_c} (P^{hc} Q_{hc}) x_{hc} \quad - \quad \sum_c F_c u_c \tag{4.1}$$

subject to:

$$x_i \leq 1 \qquad \forall i \in I\ [s_i] \tag{4.2}$$
$$x_{hc} \leq u_c \qquad \forall h \in H_c, c \in C\ [s_{hc}^{max}] \tag{4.3}$$
$$x_{hc} \geq r_{hc} u_c \qquad \forall h \in H_c, c \in C\ [s_{hc}^{min}] \tag{4.4}$$
$$u_c \leq 1 \qquad \forall c \in C[s_c] \tag{4.5}$$
$$\sum_{i \in I_{lt}} Q_i x_i + \sum_{hc \in HC_{lt}} Q_{hc} x_{hc} = \sum_k e_{l,t}^k n_k, \qquad \forall (l,t) \quad [\pi_{l,t}] \tag{4.6}$$
$$\sum_k a_{m,k} n_k \leq w_m \qquad \forall m \in N \quad [v_m] \tag{4.7}$$
$$x_i, u_c \geq 0, (x_{hc}\ free) \tag{4.8}$$
$$u \in \mathbb{Z} \tag{4.9}$$

4.2.3 Dual and complementarity conditions of the continuous relaxation

We denote by UWELFARE-CR-DUAL the dual of the continuous relaxation of the welfare maximization program stated above.

UWELFARE-CR-DUAL:

$$\min \sum_i s_i \quad + \quad \sum_c s_c \quad + \quad \sum_m w_m v_m \tag{4.10}$$

subject to:

$$s_i + Q_i \pi_{l(i),t(i)} \geq Q_i P^i, \qquad \forall i \qquad [x_i] \tag{4.11}$$
$$(s_{hc}^{max} - s_{hc}^{min}) + Q_{hc} \pi_{l(hc),t(hc)} = Q_{hc} P^{hc}, \qquad \forall h \in H_c, c\ [x_{hc}] \tag{4.12}$$
$$s_c \geq \sum_{h \in H_c} (s_{hc}^{max} - r_{hc} s_{hc}^{min}) - F_c, \qquad \forall c \in C\ [u_c] \tag{4.13}$$
$$\sum_m a_{m,k} v_m - \sum_{l,t} e_{l,t}^k \pi_{l,t} = 0 \qquad \forall k \in K\ [n_k] \tag{4.14}$$
$$s_i, s_c, s_{hc}, v_m \geq 0 \tag{4.15}$$

Complementarity conditions:

$$s_i(1 - x_i) = 0 \qquad \forall i \in I \qquad (4.16)$$
$$s_{hc}^{max}(u_c - x_{hc}) = 0 \qquad \forall h, c \qquad (4.17)$$
$$s_{hc}^{min}(x_{hc} - r_{hc}u_c) = 0 \qquad \forall h, c \qquad (4.18)$$
$$s_c(1 - u_c) = 0 \qquad \forall c \in C \qquad (4.19)$$
$$v_m(\sum_k a_{m,k}n_k - w_m) = 0 \qquad \forall m \in N \qquad (4.20)$$
$$x_i(s_i + Q_i\pi_{l(i),t(i)} - Q_iP^i) = 0 \qquad \forall i \in I \qquad (4.21)$$
$$u_c(s_c - \sum_{h\in H_c}(s_{hc}^{max} - r_{hc}s_{hc}^{min}) + F_c) = 0 \qquad \forall c \in C \qquad (4.22)$$

As it is well-known, these dual and complementarity conditions, which are optimality conditions for the continuous relaxation of (4.1)-(4.9) denoted UWELFARE-CR, exactly describe the nice equilibrium properties we would like to have for a market clearing solution. This could be easily seen from the economic interpretations given in Lemmas 4.1,4.2, 4.4 and Theorem 4.1 below.

Hence, equilibrium and integrality conditions for u cannot be both satisfied unless the continuous relaxation UWELFARE-CR admits a solution which is integral in u. In the particular case where there is no fixed cost ($\forall c \in C, F_c = 0$), no minimum acceptance ratios ($r_{hc} = 0$ for all $hc \in H_c, c \in C$), and there is no condition restraining the conditional acceptances modelled by the binary variables u_c via constraints (4.3), it is always optimal to set all $u_c := 1$ and the problem amounts to solving a classical convex market clearing problem where equilibrium can be found which optimizes welfare.

Also, even setting $F_c := 0$ in (4.1), adding MP conditions to the constraints (4.2)-(4.9), (4.11)-(4.22) to deal with them as in OMIE-PCR (cf. the toy example above with the remark about distinguishable cases, and also Section 4.3.3) would in most cases render the problem infeasible. Hence, equilibrium restrictions must be relaxed, and this can be done in different ways, which is the topic of the next section.

4.3 Modelling Near-equilibrium with MP Conditions

Section 4.3.1 reviews previous propositions to handle minimum profit conditions, including the current practice in OMIE-PCR, while Section 4.3.2 proposes a new approach which seems to be both more appropriate economically speaking, and computationally more efficient. Section 4.3.3 makes further technical comparisons between the current OMIE-PCR practice and the new proposition, and recalls an exact linearisation for minimum income conditions used by OMIE proposed in a previous contribution. Ramping conditions are not explicitly considered here, but Section 4.4 shows how all results could be derived when these are included as well in the models.

4.3.1 Modelling minimum profit conditions: literature review

As stated above, when one considers MP conditions or indivisibilities, it is needed to relax market equilibrium conditions to get feasible solutions. A first idea to relax these equilibrium conditions is to relax the complementarity conditions (4.16)-(4.22) while making them satisfied as closely as possible. With the present context and notation, the proposition in [37] is essentially to minimize the slacks, i.e. the deviations from 0, of the left-hand sides in (4.16)-(4.22), while adding ad-hoc non-convex quadratic constraints guaranteeing non-negative profits for producers, which are then approximated with linear constraints. The idea is generalized in [35] which also considers the possibility of relaxing integrality conditions and to minimize a weighted sum of deviations from complementarity, of deviations from integrality (which could be required to be null), and of uplift variables included in the statement of the minimum profit conditions, corresponding to side payments to ensure revenue adequacy for producers. Leaving aside relaxation of integrality conditions and uplifts, to minimize deviations from complementarity, for each left-hand side expression $g_l \geq 0$, slack variables ϵ_l are added together with constraints $\epsilon_l \geq g_l$, and the sum of the ϵ_l is minimized. Let us note that in the models considered, the fixed costs involved in the minimum profit conditions are not part of the welfare maximizing function in [37], while they are included in the welfare in [35].

The model and idea suggested in [35] is considered further in [84], where there is no uplift variable in the statement of minimum profit conditions, therefore requiring revenue adequacy from the uniform market prices only, and where it is observed that minimizing the slacks amounts to minimizing the duality gap given with our notation by (4.10) minus (4.1), subject to primal and dual constraints (4.2)-(4.15). The contribution [84] observes that this is a significant improvement over the formulation proposed in [35].

In all these propositions, the choice is made to use uniform prices, to ensure minimum profit conditions for producers, and to minimize the deviations from a market equilibrium by minimizing the sum of slacks of all complementarity conditions. In such a case, there is no control on which deviations from market equilibrium are allowed, and in particular, network equilibrium conditions which correspond to optimality conditions of TSOs are often not satisfied.

In the Pan-European PCR market, the choice has been made to ensure network equilibrium conditions as well as equilibrium conditions for all 'classical convex bids' corresponding to steps of classical bid curves. The only allowed deviations from a market equilibrium are that some 'non-convex bids' involving minimum power output constraints or minimum profit (resp. maximum payment) conditions could be paradoxically rejected as in the toy example given above in Section 4.2.1. Let us note that such a 'paradoxical rejection' is also allowed in all other propositions.

Concerning complex bids with a minimum income condition used in OMIE-PCR [27, 28], minimum profit conditions are of the form:

$$(u_c = 1) \Longrightarrow \sum_{h \in H_c} (-Q_{hc} x_{hc}) \pi_{l(hc),t(hc)} \geq \widetilde{F}_c + \sum_{h \in H_c} (-Q_{hc} x_{hc}) V_c, \tag{4.23}$$

where for the given market prices $\pi_{l,t}$, classical bid curves and the network are 'at equilibrium', describing in particular the fact that ITM hourly bids are fully executed, OTM hourly bids are fully rejected, and ATM hourly bids could be executed or rejected. In the condition, $\widetilde{F}_c$ corresponds to a start-up cost, and V_c to a variable cost of production, while $\sum_{h \in H_c} (-Q_{hc} x_{hc}) \pi_{l(hc),t(hc)}$ denotes the revenue generated at the given market prices.

We have shown in a previous article [55], in which other related economic aspects are considered, how to give an exact linearization of this kind of constraints in the whole European market model which can then be formulated as a MILP without *any* auxiliary variables, relying on strong duality for linear programs to enforce equilibrium for the network, classical hourly bids, and hourly bids related to accepted MIC bids. This is reviewed (and extended to include minimum power output level conditions) below in Section 4.3.3. Let us also note here that an exact linearisation similar to the one proposed in [55] has been independently proposed in [32]. Though the derivation therein is technically different and e.g. needs to introduce many auxiliary continuous variables and constraints for a McCormick convexification of bilinear binary-continuous terms, a parallel could be made between ideas of the two approaches, which is beyond the scope of the present contribution.

The following Table comparatively summarizes some core characteristics of the previous propositions to model minimum profit conditions and the present one presented below:

Proposition	Start-up costs in the Welfare	Variable costs in the Min. Profit. Cond.	*Strict* spatial price equilibrium
Garcia-Bertrand et al. [37]	No	marginal costs	No
Garcia-Bertrand et al. [36]	No	marginal costs	No
Gabriel et al. [35]	Yes	marginal costs	No
Ruiz et al. [84]	Yes	marginal costs	No
OMIE-PCR [27]	No	Ad-hoc var. costs	Yes
Present contribution	Yes	marginal costs	Yes

Table 4.3: Comparison of propositions

4.3.2 A new proposition for modelling MP conditions

We use a slightly modified version of a MIP framework introduced in [55], to enforce equilibrium for the convex bids and the network, and which is computationally efficient in particular because it avoids explicitly adding complementarity conditions modelling equilibrium for this convex part, and also any auxiliary variables. It is used to present two distinct models for minimum profit conditions in this setting: one used in practice for many years by OMIE now coupled to PCR, and the new one involving the 'MP bids' introduced in the present contribution.

Duality, uniform prices and deviations from equilibrium

Let us consider the primal welfare maximization problem UWELFARE stated in Section 4.2.2. Let us now consider a partition $C = C_r \cup C_a$, and the following constraints, fixing all integer variables to some arbitrarily given values (unit-commitment-like decisions):

$$-u_{c_a} \leq -1 \qquad \forall c_a \in C_a \subseteq C \quad [du^a_{c_a}] \tag{4.24}$$
$$u_{c_r} \leq 0 \qquad \forall c_r \in C_r \subseteq C \quad [du^r_{c_r}] \tag{4.25}$$

Dropping integer constraints (4.9) not needed any more, this yields an LP whose dual is:

$$\min \sum_i s_i \quad + \quad \sum_c s_c \quad + \quad \sum_m w_m v_m \quad - \quad \sum_{c_a \in C_a} du^a_{c_a} \tag{4.26}$$

subject to:

$$s_i + Q_i \pi_{l(i),t(i)} \geq Q_i P^i, \qquad \forall i \quad [x_i] \tag{4.27}$$
$$(s^{max}_{hc} - s^{min}_{hc}) + Q_{hc} \pi_{l(hc),t(hc)} = Q_{hc} P^{hc}, \qquad \forall h \in H_c, c \; [x_{hc}] \tag{4.28}$$
$$s_{c_r} + du^r_{c_r} \geq \sum_{h \in H_c} (s^{max}_{hc} - r_{hc} s^{min}_{hc}) - F_c, \qquad \forall c_r \in C_r \; [u_{c_r}] \tag{4.29}$$
$$s_{c_a} - du^a_{c_a} \geq \sum_{h \in H_c} (s^{max}_{hc} - r_{hc} s^{min}_{hc}) - F_c, \qquad \forall c_a \in C_a \; [u_{c_a}] \tag{4.30}$$
$$\sum_m a_{m,k} v_m - \sum_{l,t} e^k_{l,t} \pi_{l,t} = 0 \qquad \forall k \in K \; [n_k] \tag{4.31}$$
$$s_i, s_c, s_{hc}, du^r_{c_r}, du^a_{c_a}, v_m \geq 0 \tag{4.32}$$

We now write down the complementarity constraints corresponding to these primal and dual programs parametrized by the integer decisions. Economic interpretations are stated afterwards:

$$
\begin{array}{llr}
s_i(1-x_i)=0 & \forall i \in I & (4.33)\\
s^{max}_{hc}(u_c - x_{hc}) = 0 & \forall c, h \in H_c & (4.34)\\
s^{min}_{hc}(x_{hc} - r_{hc}u_c) = 0 & \forall c, h \in H_c & (4.35)\\
s_c(1-u_c)=0 & \forall c \in C & (4.36)\\
v_m(\sum_k a_{m,k}n_k - w_m) = 0 & \forall m \in N & (4.37)\\
(1-u_{c_a})du^a_{c_a} = 0 & \forall c_1 \in C_1 & (4.38)\\
u_{c_r}du^r_{c_r} = 0 & \forall c_r \in C_r & (4.39)\\
x_i(s_i + Q_i\pi_{l(i),t(i)} - Q_iP^i) = 0 & \forall i \in I & (4.40)\\
u_{c_r}(s_{c_r} + du^r_{c_r} - \sum_{h \in H_{c_r}} (s^{max}_{hc_r} - r_{hc_r}s^{min}_{hc_r}) + F_{c_r}) = 0 & \forall c_r \in C_r & (4.41)\\
u_{c_a}(s_{c_a} - du^a_{c_a} - \sum_{h \in H_{c_a}} (s^{max}_{hc_a} - r_{hc_a}s^{min}_{hc_a}) + F_{c_a}) = 0 & \forall c_a \in C_a & (4.42)
\end{array}
$$

In what follows, we consider uniform prices, that is all payments depend only and proportionally on a single price $\pi_{l,t}$ for each location l and time period t.

In the following Lemmas, it is important to keep in mind the sign convention adopted, according to which a bid quantity $Q > 0$ for a buy bid, and $Q < 0$ for a sell bid, cf. the description of notation above.

Lemma 4.1 (Interpretation of s_i and equilibrium for hourly bids)**.** *Let us consider a solution to (4.2)-(4.9), (4.24)-(4.25), (4.27)-(4.42). Variables s_i correspond to surplus variables, i.e.:*

$$
s_i = (Q_iP^i - Q_i\pi_{l(i),t(i)})x_i \tag{4.43}
$$

Moreover, the following equilibrium conditions hold, meaning that for the given market prices $\pi_{l,t}$, no other level of execution x^*_i could be preferred to x_i:

- *An hourly bid i which is fully executed, i.e. for which $x_i = 1$, is ITM or ATM, and the surplus is given by $s_i = (Q_iP^i - Q_i\pi_{l(i),t(i)})x_i = Q_iP^i - Q_i\pi_{l(i),t(i)} \geq 0$,*
- *An hourly bid i which is fractionally executed is ATM, i.e. $(Q_iP^i - Q_i\pi_{l(i),t(i)}) = 0 = s_i$*
- *Fully rejected bids i, i.e. for which $x_i = 0$, are OTM or ATM, and then $s_i = 0$, which also corresponds to the surplus: $s_i = 0 = (Q_iP^i - Q_i\pi_{l(i),t(i)})x_i = (Q_iP^i - Q_i\pi_{l(i),t(i)})^+$,*

  Hence, ITM hourly bids are fully accepted, OTM hourly bids are fully rejected, and ATM hourly bids i can be either accepted or rejected, fully or fractionally.

Proof. If $x_i = 1$, conditions (4.40) ensure that $s_i = Q_iP^i - Q_i\pi_{l(i),t(i)} \geq 0$ (since $s_i \geq 0$), and the bid is ITM or ATM. Multiplying the obtained equality by $x_i = 1$, we get identity (4.43).

If $0 < x_i < 1$, $s_i = 0 = s_i x_i$ according to (4.33), and (4.40) then gives $s_i = Q_i P^i - Q_i \pi_{l(i),t(i)} = 0$: the bid is ATM. Multiplying these equalities by x_i, we get identity (4.43).

If $x_i = 0$, $s_i = 0$ according to (4.33), which used in dual conditions (4.27) gives $Q_i P^i - Q_i \pi_{l(i),t(i)} \leq 0$: the bid is OTM or ATM. As $s_i = x_i = 0$, identity (4.43) is trivially satisfied. □

Lemma 4.2 (Interpretation of $s^{max}_{hc}, s^{min}_{hc}$)**.** *Provided that $u_c = 1$:*

$$(s^{max}_{hc} - r_{hc} s^{min}_{hc}) = (Q_{hc} P^{hc} - Q_{hc} \pi_{l(hc),t(hc)}) x_{hc} \tag{4.44}$$

while if $u_c = 0$, then the left-hand side is disconnected from the right-hand side which is 0. Economically speaking, this means that for rejected MP bids, the left-hand side only corresponds to a potential surplus.

Proof. Multiplying (4.28) by x_{hc} yields $s^{max}_{hc} x_{hc} - s^{min}_{hc} x_{hc} = (Q_{hc} P^{hc} - Q_{hc} \pi_{l(hc),t(hc)}) x_{hc}$. Using complementarity conditions (4.34)-(4.35) where $u_c = 1$, according to which $s^{max}_{hc} x_{hc} = s^{max}_{hc}$ and $s^{min}_{hc} x_{hc} = s^{min}_{hc} r_{hc}$, we get the required identity (4.44). □

For rejected MP bids, the sole deviation from an equilibrium affecting the corresponding hourly bids is that some of them could be rejected paradoxically, since at equilibrium, they should or could be rejected if they are out-of-the-money or at-the-money. The situation for accepted MP bids is more interesting. Essentially, the situation is very similar to the case of classical hourly bids described by Lemma 4.1, excepted that here, some 'MP hourly bids' could be incurring a loss due to the minimum acceptance ratio, and several configurations should be distinguished:

Lemma 4.3 (Equilibrium and deviations for MP hourly bids of accepted MP bids)**.** *Let us consider hourly bids associated to an accepted MP bid c, i.e. such that $u_c = 1$. If:*

- $0 \leq r_{hc} < x_{hc} < u_c = 1$, *then* $s^{max}_{hc} = s^{min}_{hc} = 0$, *and the bid hc is at-the-money:*

  $(s^{max}_{hc} - r_{hc} s^{min}_{hc}) = 0 = (Q_{hc} P^{hc} - Q_{hc} \pi_{l(hc),t(hc)}) = (Q_{hc} P^{hc} - Q_{hc} \pi_{l(hc),t(hc)}) x_{hc}$
- $0 \leq r_{hc} = x_{hc} < u_c = 1$, *then* $s^{max}_{hc} = 0$ *and* $(s^{max}_{hc} - r_{hc} s^{min}_{hc}) = (-r_{hc} s^{min}_{hc}) = (Q_{hc} P^{hc} - Q_{hc} \pi_{l(hc),t(hc)}) x_{hc} \leq 0$. *Noting that* $s^{min}_{hc} \geq 0$ *and* $x_{hc} \geq r_{hc} \geq 0$, *the bid is ATM or OTM, and for* $r_{hc} > 0$, *a loss could be incurred in that case.*
- $0 \leq r_{hc} < x_{hc} = u_c = 1$, *then* $s^{min}_{hc} = 0$ *and* $(s^{max}_{hc} - r_{hc} s^{min}_{hc}) = s^{max}_{hc} = (Q_{hc} P^{hc} - Q_{hc} \pi_{l(hc),t(hc)}) x_{hc} \geq 0$*: the bid is ITM or ATM.*
- *In the special case where* $r_{hc} = 1 = x_{hc} = u_c$, *nothing could be inferred on* $s^{max}_{hc}, s^{min}_{hc}$, *and the bid could be ITM, ATM or OTM, depending on the sign of* $(s^{max}_{hc} - r_{hc} s^{min}_{hc})$.

Proof. This follows a direct discussion of the equality (4.44) of Lemma 4.2, using complementarity conditions (4.34)-(4.35), with $u_c = 1$. □

The following Lemma is key to derive Theorem 4.1 and then Corollary 4.1. These are the main ingredients to derive a MILP formulation avoiding any auxiliary variables of the new model for minimum profit conditions.

Lemma 4.4 (Interpretation of du^a, du^r). *(i) $\forall c_a \in C_a, du^a_{c_a}$, is an upper bound on the loss of order c_a, given by*

$$[\sum_{h\in H_{c_a}} (s^{max}_{hc_a} - r_{hc_a} s^{min}_{hc_a}) - F_{c_a}]^- = [\sum_{h\in H_{c_a}} (Q_{hc_a} P^{hc_a} - Q_{hc_a} \pi_{l(hc_a),t(hc_a)}) x_{hc_a} - F_{c_a}]^-,$$

where $[a]^-$ denotes the negative part of a, i.e. $-\min[0, a]$.

(ii) $du^r_{c_r}$ is an upper bound on the sum of the maximum missed individual hourly surpluses (some of which could be negative) minus the fixed cost F_{c_r} of the rejected MP bid c_r, that is:

$$du^r_{c_r} \geq \sum_{h\in H_{c_r}} (s^{max}_{hc_r} - r_{hc_r} s^{min}_{hc_r}) - F_{c_r} \geq \sum_{h\in H_{c_r}} (Q_{hc_r} P^{hc_r} - Q_{hc_r} \pi_{l(hc_r),t(hc_r)}) - F_{c_r}.$$

Proof. (i) Since $u_{c_a} = 1$, and using conditions (4.42), we have:

$s_{c_a} - du^a_{c_a} = \sum_{h\in H_{c_a}} (s^{max}_{hc_a} - r_{hc_a} s^{min}_{hc_a}) - F_{c_a}$. Since, $s_{c_a}, du^a_{c_a} \geq 0$, the observation follows (cf. also Lemma 4.2 for the identity used to replace $(s^{max}_{hc_a} - r_{hc_a} s^{min}_{hc_a})$).

(ii) Conditions of type (4.36) show that $s_{c_r} = 0$, which used in (4.29) provide the first inequality. Then, as $r_{hc_r} \in [0, 1]$ and $s^{min}_{hc_r} \geq 0$, one has $(s^{max}_{hc_r} - r_{hc_r} s^{min}_{hc_r}) \geq (s^{max}_{hc_r} - s^{min}_{hc_r}) = Q_{hc} P^{hc} - Q_{hc} \pi_{l(hc),t(hc)}$ where this last equality is given by (4.28). The result immediately follows. □

Theorem 4.1 (MP conditions and shadow costs of acceptance du^a). *Let us consider a given partition $C_a \cup C_r$ and a solution to (4.2)-(4.9), (4.24)-(4.25), (4.27)-(4.42):*

- *For an accepted sell bid $c_a \in C_a$, i.e. for which $\forall hc_a \in H_{c_a}$, $Q_{hc_a} < 0$:*

  $$(-\sum_{h\in H_{c_a}} Q_{hc} \pi_{l(hc),t(hc)} x_{hc}) \geq (-\sum_{h\in H_{c_a}} Q_{hc} P^{hc} x_{hc}) + F_{c_a} \iff du^a_{c_a} = 0,$$

  where the left-hand side of the equivalence expresses that the revenue from trade is greater or equal to the sum of marginal costs plus the fixed cost F_c, which is a minimum profit condition.

- *For an accepted buy bid $c_a \in C_a$, i.e. for which $\forall hc_a \in H_{c_a}$, $Q_{hc_a} > 0$:*

  $$(\sum_{h\in H_{c_a}} Q_{hc} \pi_{l(hc),t(hc)} x_{hc}) \leq (\sum_{h\in H_{c_a}} Q_{hc} P^{hc} x_{hc}) - F_{c_a} \iff du^a_{c_a} = 0,$$

  where the left-hand side of the equivalence expresses that the total payments are lesser or equal to the total utility reduced by the constant term F_c, which is a maximum payment condition.

Proof. It is a direct consequence of Lemma 4.4. If $du^a_{c_a} = 0$, then

$\sum_{h\in H_{c_a}} (Q_{hc_a} P^{hc_a} - Q_{hc_a} \pi_{l(hc_a),t(hc_a)}) x_{hc_a} - F_{c_a} \geq 0$, which rearranged provides the result (the converse holding as well: if this last inequality holds, the $du^a_{c_a}$ can be set to 0 without altering the satisfaction of the other constraints). □

Corollary 4.1. *MP conditions could be expressed by requiring that shadow costs of acceptance could be set to zero, i.e.:*

$$\forall c_a \in C_a,\ du^a_{c_a} = 0 \tag{4.45}$$

Naturally, not all MP bid selections C_a, C_r are such that these conditions hold for all accepted MP bids $c_a \in C_a$, cf. e.g. the toy example presented in Section 4.2.1. Moreover, admissible selections C_a, C_r for which all shadow costs of acceptance could be set to zero are not known in advance. However, following [55], we can provide a MILP formulation without any auxiliary variables, exactly describing those admissible partitions C_a, C_r, together with a corresponding solution to (4.2)-(4.9), (4.24)-(4.25), (4.27)-(4.42). This is developed in the next subsection.

A MILP without auxiliary variables modelling MP conditions

To state Theorem 4.2 about the formulation UMFS, we need to include the following technical constraint limiting the market price range

$$\pi_{l,t} \in [-\bar{\pi}, \bar{\pi}] \qquad \forall l \in L, t \in T. \tag{4.46}$$

$\bar{\pi}$ can be chosen large enough to avoid excluding any relevant market clearing solution (see for example the discussion of the analogue condition (18) in [54]). Under this assumption, the parameters M_c below are chosen large enough not to arbitrarily constraint the range of values of the variables d^a, d^r. As these values respectively correspond to upper bounds on actual losses and upper bounds on opportunity costs, the M_c can be straighforwardly computed from the bid data provided by the market participants and the market price range condition (4.46). Note that in practice, power exchanges actually do impose that the computed prices $\pi_{l,t}$ stay within a given range in order to limit market power and price volatility, see e.g. [27].

Uniform Market Clearing Feasible Set (UMFS):

$$\sum_i (P^i Q_i) x_i + \sum_{c,h\in H_c} (P^{hc} Q_{hc}) x_{hc} - \sum_c F_c u_c \geq \sum_i s_i + \sum_c s_c - \sum_{c\in C} du^a_c + \sum_m w_m v_m \tag{4.47}$$

$$
\begin{aligned}
& x_i \leq 1 && \forall i \in I\ [s_i] && (4.48)\\
& x_{hc} \leq u_c && \forall h \in H_c, c \in C\ [s_{hc}^{max}] && (4.49)\\
& x_{hc} \geq r_{hc} u_c && \forall h \in H_c, c \in C\ [s_{hc}^{min}] && (4.50)\\
& u_c \leq 1 && \forall c \in C\ [s_c] && (4.51)\\
& \sum_{i \in I_{lt}} Q_i x_i + \sum_{hc \in HC_{lt}} Q_{hc} x_{hc} \\
& \qquad\qquad = \sum_k e_{l,t}^k n_k, && \forall (l,t)\ [\pi_{l,t}] && (4.52)\\
& \sum_k a_{m,k} n_k \leq w_m && \forall m \in N\ \ [v_m] && (4.53)\\
& x, u \geq 0, && && (4.54)\\
& u \in \mathbb{Z} && && (4.55)\\
& s_i + Q_i \pi_{l(i),t(i)} \geq Q_i P^i, && \forall i \in I \qquad [x_i] && (4.56)\\
& (s_{hc}^{max} - s_{hc}^{min}) + Q_{hc} \pi_{l(hc),t(hc)} = Q_{hc} P^{hc}, && \forall h \in H_c, c\ [x_{hc}] && (4.57)\\
& s_c + du_c^r - du_c^a \geq \sum_{h \in H_c} (s_{hc}^{max} - r_{hc} s_{hc}^{min}) - F_c, && \forall c \in C\ [u_c] && (4.58)\\
& du_c^r \leq M_c (1 - u_c) && \forall c \in C && (4.59)\\
& du_c^a \leq M_c u_c && \forall c \in C && (4.60)\\
& \sum_m a_{m,k} v_m - \sum_{l,t} e_{l,t}^k \pi_{l,t} = 0 && \forall k \in K [n_k] && (4.61)\\
& s_i, s_c, s_{hc}^{max}, s_{hc}^{min}, du^a, du^r, v_m \geq 0 && && (4.62)
\end{aligned}
$$

Theorem 4.2. *(I) Let $(x, u, n, \pi, v, s, du^a, du^r)$ be any feasible point of UMFS satisfying the price range condition (4.46), and let us define $C_r = \{c | u_c = 0\}, C_a = \{c | u_c = 1\}$.*

Then the projection $(x, u, n, \pi, v, s, du^a_{c_a \in C_a}, du^r_{c_r \in C_r})$ satisfies all conditions in (4.2)-(4.9), (4.24)-(4.42).

(II) Conversely, any point

$MCS = (x, u, n, \pi, v, s, du^a_{c_a \in C_a}, du^r_{c_r \in C_r})$ feasible for constraints (4.2)-(4.9), (4.24)-(4.42) related to a given arbitrary MIC selection $C = C_r \cup C_a$ which respects the price range condition (4.46) can be 'lifted' to obtain a feasible point $\tilde{MCS} = (x, u, n, \pi, v, \tilde{s}, \tilde{du^a}, \tilde{du^r})$ of UMFS.

Sketch of the proof. This is a straightforward adaptation of Theorem 1 in [55]. Essentially: (I) any feasible point of UMFS defines a corresponding partition $C_a \cup C_r$ of accepted and rejected MP bids, and conditions (4.58)-(4.60) ensure that (4.29)-(4.30) are satisfied whatever the corresponding partition is. It is then direct to check that conditions in (4.2)-(4.9), (4.24)-(4.42) are all satisfied, since, provided (4.27)-(4.32), (4.47) can then equivalently be replaced by the complementarity conditions (4.33)-(4.42) as optimality conditions for the program (4.1) subject to (4.2)-(4.9), (4.24)-(4.25). (Let us note that as du^a, du^r are upper bounds on losses or missed surpluses, see Lemma 4.4, the involved big-

Ms in (4.59)-(4.60) are appropriately defined using the technical condition (4.46) bounding the range of possible market prices.)

(II) Conversely, for any partition $C_a \cup C_r$ and a solution to (4.2)-(4.9), (4.24)-(4.42) such that the condition (4.46) is satisfied, we only need to define the additional values $du_c^a = 0$ for $c \in C_r$ and $du_c^r = 0$ for $c \in C_a$. Since the big-Ms have been suitably defined using (4.46), and using (4.29)-(4.30), it is straightforward to check that (4.58)-(4.60) will be satisfied for all $c \in C$, and hence all conditions (4.47)-(4.62) defining UMFS are satisfied (again relying on the equivalence of (4.33)-(4.42) and (4.47) as optimality conditions for (4.1) subject to (4.2)-(4.9), (4.24)-(4.25) provided that (4.2)-(4.9),(4.24)-(4.25) and the dual conditions (4.27)-(4.32) are satisfied). □

As we want to enforce MP conditions, we need to add to UMFS the following conditions:

$$\forall c \in C,\ du_c^a = 0 \tag{4.63}$$

Since we set all the du_c^a to 0, constraints (4.60) are not needed any more, and constraints (4.58)-(4.59) reduce to (4.76) below. We hence get the following MILP formulation which we denote 'MarketClearing-MPC', enforcing all MP conditions, and which doesn't make use of any auxiliary variable.

MarketClearing-MPC

$$\max \ \sum_i (P^i Q_i) x_i + \sum_{c,h \in H_c} (P^{hc} Q_{hc}) x_{hc} - \sum_c F_c u_c \tag{4.64}$$

subject to:

$$\sum_i (P^i Q_i) x_i + \sum_{c,h \in H_c} (P^{hc} Q_{hc}) x_{hc} - \sum_c F_c u_c$$

$$\geq \sum_i s_i + \sum_c s_c + \sum_m w_m v_m \qquad [\sigma] \tag{4.65}$$

$$x_i \leq 1 \qquad \forall i \in I \ [s_i] \tag{4.66}$$

$$x_{hc} \leq u_c \qquad \forall h \in H_c, c \in C \ [s_{hc}^{max}] \tag{4.67}$$

$$x_{hc} \geq r_{hc} u_c \qquad \forall h \in H_c, c \in C \ [s_{hc}^{min}] \tag{4.68}$$

$$u_c \leq 1 \qquad \forall c \in C [s_c] \tag{4.69}$$

$$\sum_{i \in I_{lt}} Q_i x_i + \sum_{hc \in HC_{lt}} Q_{hc} x_{hc}$$

$$= \sum_k e_{l,t}^k n_k, \qquad \forall (l,t) \quad [\pi_{l,t}] \tag{4.70}$$

$$\sum_k a_{m,k} n_k \leq w_m \qquad \forall m \in N \quad [v_m] \tag{4.71}$$

$$x, u \geq 0, \tag{4.72}$$

$$u \in \mathbb{Z} \tag{4.73}$$

$$s_i + Q_i \pi_{l(i),t(i)} \geq Q_i P^i, \qquad \forall i \in I \qquad [x_i] \tag{4.74}$$

$$(s_{hc}^{max} - s_{hc}^{min}) + Q_{hc} \pi_{l(hc),t(hc)} = Q_{hc} P^{hc}, \qquad \forall h \in H_c, c \ [x_{hc}] \tag{4.75}$$

$$s_c \geq \sum_{h \in H_c} (s_{hc}^{max} - r_{hc} s_{hc}^{min}) - F_c - M_c(1 - u_c) \qquad \forall c \in C[u_c] \tag{4.76}$$

$$\sum_m a_{m,k} v_m - \sum_{l,t} e_{l,t}^k \pi_{l,t} = 0 \qquad \forall k \in K[n_k] \tag{4.77}$$

$$s_i, s_c, s_{hc}, v_m \geq 0 \tag{4.78}$$

4.3.3 Comparison to 'Minimum income conditions' used by OMIE-PCR

The way minimum profit conditions are handled in OMIE-PCR, described in Section 4.3.1, presents two substantial differences compared to the MP bids introduced above. First, start-up costs are not included in the welfare maximizing objective function, and second there is the presence of a variable cost V_c which could have no relation to the marginal cost curves described by the hourly bids $hc, c \in H_c$. In [55], we have shown how such 'minimum income conditions' could be linearized exactly without any auxiliary variables, in the frame of the PCR market rules. We adapt here this result to take into account minimum acceptance ratios (corresponding e.g. to minimum output levels) modelled by conditions (4.4), which were not considered in [55]. This helps considering more formally the differences between MP bids and classical bids with a minimum income condition currently in use in OMIE-PCR.

Let us denote by $\widetilde{F}_c$ the actual start-up cost attached to some bid c provided by a producer. As in OMIE-PCR, start-up costs $\widetilde{F}_c$ are not considered in the welfare objective function, it is first needed to set all parameters $F_c = 0$ in MarketClearing-MPC, but then, nothing ensures that these start-up costs are recovered for executed bids. It is therefore needed to explicitly include a condition equivalent to (4.23), and this can be done in a linear way without any auxiliary variables and any approximation, using the following Lemma:

Lemma 4.5 (Adaptation of Lemma 3 in [55])**.** *Consider any feasible point of MarketClearing-MPC in the case where all parameters F_c are set to 0. Then, the following holds:*

$$\forall c \in C, \sum_{h \in H_c} (-Q_{hc} x_{hc}) \pi_{l(hc),t(hc)} = s_c - \sum_{h \in H_c} (Q_{hc} P^{hc}) x_{hc} \tag{4.79}$$

Proof. The identity is trivially satisfied if $u_c = 0$, thanks to conditions (4.3) and (4.36) which are enforced for any feasible point of MarketClearing-MPC.

For $u_c = 1$, summing up (4.44) in Lemma 4.2 over $hc \in H_c$, we get:

$$\sum_{hc \in H_c} (s_{hc}^{max} - r_{hc} s_{hc}^{min}) = \sum_{hc \in H_c} (Q_{hc} P^{hc} - Q_{hc} \pi_{l(hc),t(hc)}) x_{hc} \tag{4.80}$$

Then, noting that MarketClearing-MPC enforces (4.42) with $du^a = 0$, and that we have set all $F_c = 0$ not to consider start-up costs in the welfare objective, we can replace the left-hand side of (4.80) by s_c to get the required identity. □

Let us note that the economic interpretation of the algebraic identity provided by (4.79) is straightforward: the total income in the left-hand side can be decomposed as the total marginal costs plus the total surplus s_c collecting individual surpluses of all the individual bid curves associated to the MIC order.

Using Lemma 4.5, the MIC condition (4.23) can then be stated in a linear way as follows:

$$s_c - \sum_{h \in H_c} (Q_{hc} P^{hc}) x_{hc} \geq \widetilde{F}_c + \sum_{h \in H_c} (-Q_{hc} x_{hc}) V_c - \overline{M_c}(1 - u_c) \tag{4.81}$$

where $\overline{M_c}$ is a fixed number large enough to deactivate the constraint when $u_c = 0$. As $u_c = 0$ implies $s_c = 0$ and $x_{hc} = 0$, we set $\overline{M_c} := \widetilde{F}_c$.

Let us emphasise that once this is done and that we have a linear description of the feasible set handling minimum income conditions as done in OMIE-PCR, many objective functions could be considered, including objective functions involving startup and variable costs in the measure of welfare instead of the marginal costs described by the bid curves associated to a given MIC order.

From a modelling point of view there are therefore two main differences between the MP bids proposed here and the OMIE-PCR MIC orders. The first one is that we need to explicitly state constraints (4.81), apart from the single constraint (4.65) that essentially enforce all complementary conditions simultaneously. This is because in the OMIE-PCR model, the fixed and variable costs of the MIC orders are not part of the objective function to be maximised. This is linked to the second difference that in the OMIE-PCR model, there are two different variable costs for MIC orders: one that appears in the objective function to be maximised P^{hc}, and another one V_c that appears in the MIC condition (4.81). It is questionable whether these two costs actually correspond to real costs of a power plant. This makes the task of regulators in charge of monitoring market behaviour of participants more difficult. Indeed it is not clear any more what is the normal or justifiable market behaviour, and what constitutes gaming or a possible exercise of market power.

4.4 Handling ramping constraints

Ramping constraints are also called 'load-gradient' conditions in the PCR vocabulary, see [27]. Let us suppose one wants to include in the primal program UWELFARE (4.1)-(4.9) ramping constraints for each MP bid representing the technical conditions for operating the corresponding power plant. Our goal is to show how to adapt all results of the present contribution regarding minimum profit (resp. maximum payment) conditions in this setting. Ramping constraints to add are of the form:

$$\sum_{hc \in H_c | t(hc)=t+1} (-Q^{hc}) x_{hc} - \sum_{hc \in H_c | t(hc)=t} (-Q^{hc}) x_{hc} \leq RU_c \ u_c \qquad \forall t \in \{1, ..., T-1\}, \forall c \in C \qquad [g_{c,t}^{up}] \tag{4.82}$$

$$\sum_{hc \in H_c | t(hc)=t} (-Q^{hc}) x_{hc} - \sum_{hc \in H_c | t(hc)=t+1} (-Q^{hc}) x_{hc} \leq RD_c \ u_c \qquad \forall t \in \{1, ..., T-1\}, \forall c \in C \qquad [g_{c,t}^{down}] \tag{4.83}$$

The occurrences of u_c might seem unnecessary and optional as the conditions would be trivially satisfied for $u_c = 0$. However, these occurrences are technically required to derive the appropriate dual program and adapt straightforwardly all previous results. They also make the continuous relaxation of the resulting Integer Program stronger. The corresponding complementarity conditions that will be enforced as all other complementarity conditions in Theorem 4.2 are:

$$g_{c,t}^{up}(RU_c \ u_c - \sum_{hc \in H_c | t(hc)=t} Q^{hc} x_{hc} + \sum_{hc \in H_c | t(hc)=t+1} Q^{hc} x_{hc}) = 0 \qquad \forall t \in \{1, ..., T-1\}, \forall c \in C \tag{4.84}$$

$$g_{c,t}^{down}(RD_c \ u_c - \sum_{hc \in H_c | t(hc)=t+1} Q^{hc} x_{hc} + \sum_{hc \in H_c | t(hc)=t} Q^{hc} x_{hc}) = 0 \quad \forall t \in \{1, ..., T-1\}, \forall c \in C \tag{4.85}$$

Such constraints do not exist for $t = 0$ or $t = T$, but the following convention is useful for writing what follows while avoiding distinguishing different cases: $g^{up}_{c,0} = g^{down}_{c,0} = g^{up}_{c,T} = g^{down}_{c,T} = 0$.

The dual constraints (4.28), (4.29) and (4.30) should then respectively be replaced by:

$$(s^{max}_{hc} - s^{min}_{hc}) + (Q^{hc} g^{down}_{c,t(hc)-1} - Q^{hc} g^{up}_{c,t(hc)-1}) + (Q^{hc} g^{up}_{c,t(hc)} - Q^{hc} g^{down}_{c,t(hc)}) + Q_{hc}\pi_{l(hc),t(hc)} = Q_{hc}P^{hc}, \quad \forall h \in H_c, \forall c \in C \; [x_{hc}] \quad (4.86)$$

$$s_{c_r} + du^r_{c_r} \geq \sum_{h \in H_{c_r}} (s^{max}_{hc} - r_{hc} s^{min}_{hc}) + \sum_t (RU_{c_r} g^{up}_{c_r,t(hc)} + RD_{c_r} g^{down}_{c_r,t(hc)}) - F_{c_r}, \quad \forall c_r \in C_r \; [u_{c_r}] \quad (4.87)$$

$$s_{c_a} - du^a_{c_a} \geq \sum_{h \in H_{c_a}} (s^{max}_{hc} - r_{hc} s^{min}_{hc}) + \sum_t (RU_{c_a} g^{up}_{c_a,t(hc)} + RD_{c_a} g^{down}_{c_a,t(hc)}) - F_{c_a}, \quad \forall c_a \in C_a \; [u_{c_a}] \quad (4.88)$$

with the corresponding consequence in the formulation of UMFS (used in Theorem 4.2) of replacing (4.58) by

$$s_c + du^r_c - du^a_c \geq \sum_{h \in H_c} (s^{max}_{hc} - r_{hc} s^{min}_{hc}) + \sum_t (RU_c g^{up}_{c,t(hc)} + RD_c g^{down}_{c,t(hc)}) - F_c, \quad \forall c \in C \; [u_c] \quad (4.89)$$

It is shown below that this is all we need to handle ramping constraints. Adaptation of Lemma 4.4 and Theorem 4.1 are then straightforward, as it suffices to replace in the proofs the occurrences of $\sum_{h \in H_c} (s^{max}_{hc} - r_{hc} s^{min}_{hc})$ by its analogue provided by the left-hand side of (4.92) below, and the corresponding adaptations needed e.g. in MarketClearing-MPC immediately follows.

These assertions rest on the follwing adaptation of Lemma 4.2:

Lemma 4.6 (Adaptation of Lemma 4.2 to handle ramping constraints)**.** *Provided that* $u_c = 1$*, :*

1.

$$(s^{max}_{hc} - r_{hc} s^{min}_{hc}) + (Q^{hc} g^{down}_{c,t(hc)-1} - Q^{hc} g^{up}_{c,t(hc)-1}) x_{hc} + (Q^{hc} g^{up}_{c,t(hc)} - Q^{hc} g^{down}_{c,t(hc)}) x_{hc} = (Q_{hc}P^{hc} - Q_{hc}\pi_{l(hc),t(hc)}) x_{hc} \quad (4.90)$$

2.

$$\sum_{hc \in H_c} (Q^{hc} g^{down}_{c,t(hc)-1} - Q^{hc} g^{up}_{c,t(hc)-1}) x_{hc} + \sum_{hc \in H_c} (Q^{hc} g^{up}_{c,t(hc)} - Q^{hc} g^{down}_{c,t(hc)}) x_{hc} = \sum_t (RU_c g^{up}_{c,t(hc)} + RD_c g^{down}_{c,t(hc)}) \quad (4.91)$$

3.

$$\sum_{h \in H_c} (s^{max}_{hc} - r_{hc} s^{min}_{hc}) + \sum_t (RU_c g^{up}_{c,t(hc)} + RD_c g^{down}_{c,t(hc)}) = \sum_{h \in H_c} [Q_{hc} P^{hc} - Q_{hc} \pi_{l(hc),t(hc)}] x_{hc} \quad (4.92)$$

Proof. 1. is obtained by multiplying equation (4.86) by the corresponding dual variable x_{hc} and by using as in Lemma 4.2 the complementarity conditions (4.34)-(4.35) with $u_c = 1$, according to which $s^{max}_{hc} x_{hc} = s^{max}_{hc}$ and $s^{min}_{hc} x_{hc} = s^{min}_{hc} r_{hc}$.

2. Summing equations (4.84) and (4.85) then summing up over t and rearranging the terms provides the result, noting that it is assumed that $u_c = 1$.

3. is a direct consequence of 1. and 2., obtained by summing up (4.90) over $hc \in H_c$ and using the identity provided by (4.91)

□

4.5 A decomposition procedure with Strengthened Benders cuts

The contribution in this Section is essentially to show how the Benders decomposition procedure with strengthened cuts described in [54] for fully indivisible block bids applies to the present context of newly introduced bids with a minimum profit/maximum payment condition (MP bids), providing an efficient method for large-scale instances where both block and MP bids are present, as such decomposition approaches (see also [60, 27]) are known to be efficient to handle block bids. The present extension includes as a special case instances involving block bids with a minimum acceptance ratio as described in [27].

This Benders decomposition procedure solves MarketClearing-MPC , working with (an implicit decription of) the projection G of the MarketClearing-MPC feasible set described by (4.65)-(4.78) on the space of primal decision variables (x_i, x_{hc}, u_c, n_k). In particular, we start with a relaxation of G, denoted G_0 and described by constraints (4.66)-(4.73), and then add Benders cuts to G_0 which are valid inequalities for G derived from a so-called worker program until a feasible - hence optimal - solution is found. The worker program generates cuts to cut off incumbents for which no prices exist such that all MP conditions could be enforced, see Theorems 4.3 and 4.4. It is shown that these Benders cuts correspond indeed to 'no-good cuts' rejecting the current MP bids combination, see Theorem 4.5. We show how these cuts could be strengthened, providing stronger and sparser cuts which are valid for G when they are computed to cut off solutions which are optimal for the master program (potentially with cuts added at previous iterations where applicable), cf. Theorem 4.6. Instead of adding these cuts iteratively after solving the augmented master program each time up to optimality, it could be preferable to generate them *within* a branch-and-cut algorithm solving this master program (hence also MarketClearing-MPC, as MP conditions will be checked and enforced when needed). In that case, the strengthened cuts are locally valid, i.e. in branch-and-bound subtrees originating from incumbents rejected by the worker program during the branch-and-cut algorithm solving the master program, see Theorem 4.7. Adding cuts after solving master programs up to optimality is similar to the original approach described in the seminal paper [5], while adding cuts inside a branch-and-cut, which is often more efficient, is sometimes called the "modern version" of a Benders decomposition. Let us note that the classical Benders cuts of Theorem 4.4 or their "no-good" equivalent of Theorem 4.5 are always globally valid, as opposed to their strengthened version of Theorem 4.7.

Let us also mention a very interesting result. The revised version of [65] appearing as Chapter 2 in [64] and relying on [54] proposes an analogue of Theorem 4.7 in a context which considers general "mixed integer bids", a careful analysis of which shows they encompass the MP bids proposed here (though there is no mention of applications such as the modeling of start-up costs and the minimum profit conditions or ramping constraints, etc). As noted therein, the author generalizes the applicability of the cuts of Theorem 6 in [54], similar to those of Theorem 4.7 below, to these general mixed integer bids (and general convex bids besides) using a completely different technique than the present Benders decomposition which relies on other considerations and the primal-dual formulations presented above (shadow costs of acceptance in Theorem 4.1, etc).

Let us consider a master branch-and-bound solving (4.64) subject to the initial constraints (4.66)-(4.73), and let $(x_i^*, x_{hc}^*, u_c^*, n_k^*)$ be an incumbent satisfying (4.66)-(4.73) of MarketClearing-MPC.

A direct application of the Farkas Lemma to the remaining linear conditions (4.65), (4.74)-(4.78), which is detailed in appendix, yields:

Theorem 4.3 (Worker program of the decomposition)**.** *Let $(x_i^*, x_{hc}^*, u_c^*, n_k^*)$ be an incumbent satisfying the "primal conditions" (4.66)-(4.73), then there exists (π, s, v) such that all MP conditions modelled by the other conditions (4.65), (4.74)-(4.78) in MarketClearing-MPC are satisfied if and only if:*

$$\max_{(x,u,n)\in P} \sum_i (P^i Q_i) x_i + \sum_{c,h\in H_c} (P^{hc} Q_{hc}) x_{hc} - \sum_c F_c u_c - M_c(1-u_c^*)u_c \leq \Big(\sum_i (P^i Q_i) x_i^* + \sum_{c,h\in H_c} (P^{hc} Q_{hc}) x_{hc}^* - \sum_c F_c u_c^*\Big), \quad (4.93)$$

where P is the polyhedron defined by the linear conditions (4.66)-(4.72), that is the linear relaxation of (4.66)-(4.73). This condition is also equivalent to

$$\max_{(x,u,n)\in P|u_c=0 \ if \ u^*_c=0} \quad \sum_i (P^i Q_i)x_i + \sum_{c,h\in H_c} (P^{hc}Q_{hc})x_{hc} - \sum_c F_c u_c$$
$$\leq (\sum_i (P^i Q_i)x^*_i + \sum_{c,h\in H_c} (P^{hc}Q_{hc})x^*_{hc} - \sum_c F_c u^*_c), \quad (4.94)$$

where no "big M's" are involved.

Proof. See appendix. □

A direct consequence of Theorem 4.3 is:

Theorem 4.4 (Classical Benders cuts). *Suppose $(x^*_i, x^*_{hc}, u^*_c, n^*_k)$ doesn't belong to G, i.e. there are no prices such that MP conditions could all be satisfied, i.e. for which the test of Theorem 4.3 fails.*

Then, the following Benders cut is a valid inequality for G and cuts off the current incumbent $(x^*_i, x^*_{hc}, u^*_c, n^*_k)$:

$$\sum_i (P^i Q_i)x^{\#}_i + \sum_{c,h\in H_c} (P^{hc}Q_{hc})x^{\#}_{hc} - \sum_c F_c u^{\#}_c - M_c(1-u_c)u^{\#}_c$$
$$\leq (\sum_i (P^i Q_i)x_i + \sum_{c,h\in H_c} (P^{hc}Q_{hc})x_{hc} - \sum_c F_c u_c), \quad (4.95)$$

where $(x^{\#}_i, x^{\#}_{hc}, u^{\#}_c, n^{\#}_k)$ is an optimal solution to the left-hand side worker program in (4.93) (resp. (4.94)).

Lemma 4.7. *In the feasible set of MarketClearing-MPC, welfare is univocally determined by an MP bids combination, i.e., by given arbitrarily values for the variables u_c.*

Proof. Let us consider a feasible point of MarketClearing-MPC and the corresponding MP bids combination $C_a \cup C_r$. As detailed in Theorem 4.2 and its proof, this point is then feasible for (4.2)-(4.9), (4.24)-(4.25), (4.27)-(4.32) and (4.33)-(4.42), which are optimality conditions for the welfare maximization program (4.1)-(4.9), (4.24)-(4.25) where only the integer values of the variables u_c have been fixed. □

Observation 4.1. *An optimal solution of the left-hand side of (4.93) is always such that $u^{\#}_c = 0$ if $u^*_c = 0$, because of the penalty coefficients M_c, or alternatively because $u^{\#}_c$ corresponds to the optimal dual variable of (4.76) which is not tight when $u^*_c = 0$.*

Theorem 4.5 (No-good / Combinatorial Benders cuts). *Suppose $(x^*_i, x^*_{hc}, u^*_c, n^*_k)$ doesn't belong to G, i.e. there are no prices such that MP conditions could all be satisfied, i.e. for which the test of Theorem 4.3 fails.*

Then, the following 'no-good cut' is a valid inequality for G and cuts off the current incumbent:

$$\sum_{c|u_c^*=1} (1-u_c) + \sum_{c|u_c^*=0} u_c \geq 1, \tag{4.96}$$

basically excluding the current MP bids combination.

Proof. This is a direct consequence of Theorem 4.4. Suppose we need to cut off $(x_i^*, x_{hc}^*, u_c^*, n_k^*)$ by adding (4.95). For any other solution (x_i, x_{hc}, u_c, n_k) such that $u_c = u_c^*$ for all $c \in C$, the left-hand side value of (4.95) will trivially be the same as with u^*. The right-hand side will also be the same as with u^* according to Lemma 4.7, because welfare is univocally determined by the values of the u_c. Hence any such solution will also violate (4.95) and it is therefore needed to change the value of at least one of the u_c, providing the result. □

Theorem 4.6 (Globally valid strengthened Benders cuts)**.** *Let $(x_i^*, x_{hc}^*, u_c^*, n_k^*)$ be an optimal solution for the master program (4.64) subject to (4.66)-(4.73), potentially with additional valid inequalities. If the test of Theorem 4.3 fails, the following sparse cut is a valid inequality for G:*

$$\sum_{c|u_c^*=1} (1-u_c) \geq 1, \tag{4.97}$$

meaning that at least one of the currently accepted MP bids should be excluded in any valid market clearing solution satisfying MP conditions.

Proof. This is also a consequence of Theorem 4.4. First, observe that (4.97) trivially implies (4.96) and hence cuts off $(x_i^*, x_{hc}^*, u_c^*, n_k^*)$, according to Theorem 4.5. It remains to show that it is also a valid inequality for G.

Let $(x_i^*, x_{hc}^*, u_c^*, n_k^*)$ be the optimal solution considered that violates (4.95), i.e., such that:

$$\sum_i (P^i Q_i) x_i^\# + \sum_{c,h \in H_c} (P^{hc} Q_{hc}) x_{hc}^\# - \sum_c F_c u_c^\# - M_c (1-u_c^*) u_c^\# \\ > (\sum_i (P^i Q_i) x_i^* + \sum_{c,h \in H_c} (P^{hc} Q_{hc}) x_{hc}^* - \sum_c F_c u_c^*), \tag{4.98}$$

which using Observation 4.1 reduces to:

$$(\sum_i (P^i Q_i) x_i^* + \sum_{c,h \in H_c} (P^{hc} Q_{hc}) x_{hc}^* - \sum_c F_c u_c^*) \\ < \sum_i (P^i Q_i) x_i^\# + \sum_{c,h \in H_c} (P^{hc} Q_{hc}) x_{hc}^\# - \sum_c F_c u_c^\# \tag{4.99}$$

Suppose (x_i, x_{hc}, u_c, n_k) is feasible for (4.66)-(4.73) (with the potential added valid inequalities obtained at previous iterations). Because of the optimality of $(x_i^*, x_{hc}^*, u_c^*, n_k^*)$,

$$(\sum_i (P^i Q_i) x_i + \sum_{c,h \in H_c} (P^{hc} Q_{hc}) x_{hc} - \sum_c F_c u_c)$$
$$\leq (\sum_i (P^i Q_i) x_i^* + \sum_{c,h \in H_c} (P^{hc} Q_{hc}) x_{hc}^* - \sum_c F_c u_c^*)$$
$$< \sum_i (P^i Q_i) x_i^{\#} + \sum_{c,h \in H_c} (P^{hc} Q_{hc}) x_{hc}^{\#} - \sum_c F_c u_c^{\#} \quad (4.100)$$

Now suppose (x_i, x_{hc}, u_c, n_k) does not satisfy (4.97), i.e., that $\sum_{c|u_c^*=1} (1 - u_c) = 0$. Then, combined with Observation 4.1 exactly as to reduce (4.98) to (4.99), the valid cut (4.95) that this other solution must satisfy to potentially be in G reduces to:

$$\sum_i (P^i Q_i) x_i^{\#} + \sum_{c,h \in H_c} (P^{hc} Q_{hc}) x_{hc}^{\#} - \sum_c F_c u_c^{\#}$$
$$\leq (\sum_i (P^i Q_i) x_i + \sum_{c,h \in H_c} (P^{hc} Q_{hc}) x_{hc} - \sum_c F_c u_c), \quad (4.101)$$

which contradicts (4.100). Hence, (4.97) must hold for any other (x_i, x_{hc}, u_c, n_k) that is in G. □

Now, suppose we want to use the sparse cuts of Theorem 4.6 *within* the branch-and-bound tree solving the master program, instead of adding them after solving up-to-optimality the master program (together with the cuts obtained at previous iterations where applicable). Then these cuts are valid *locally*, i.e. in the subtrees originating from the incumbents to cut off, as their validity depends on the local optimality of this incumbent to cut off:

Theorem 4.7 (Locally valid strengthened Benders cuts)**.** *Let again $(x_i^*, x_{hc}^*, u_c^*, n_k^*)$ be an incumbent obtained via an LP relaxation at a given node of the branch-and-cut solving the master program (4.64) subject to (4.66)-(4.73). If the test of Theorem 4.3 fails, the following sparse cut is locally valid, i.e. is valid in the subtree of the branch-and-bound originating from the current node providing the incumbent $(x_i^*, x_{hc}^*, u_c^*, n_k^*)$:*

$$\sum_{c|u_c^*=1} (1 - u_c) \geq 1, \quad (4.102)$$

meaning that at least one of the currently accepted MP bids should be excluded in any solution found deeper in the subtree.

Proof. This is also a consequence of Theorem 4.4 and the proof is a slight variant of the proof of Theorem 4.6. Since in the present case $(x_i^*, x_{hc}^*, u_c^*, n_k^*)$ is just an incumbent and no longer globally optimal for the master program, to reproduce the argument providing (4.100), we use the *local* optimality of the incumbent $(x_i^*, x_{hc}^*, u_c^*, n_k^*)$ obtained via an LP relaxation, and the fact that the other solutions considered (x_i, x_{hc}, u_c, n_k) lie in the subtree originating from the current node providing $(x_i^*, x_{hc}^*, u_c^*, n_k^*)$. □

4.6 Numerical Experiments

Implementation of the models and algorithms proposed above have been made in Julia using JuMP[51], an open source package providing an algebraic modelling language embedded within Julia, CPLEX 12.6.2 as the underlying MIP solver, and ran on a laptop with an i5 5300U CPU with 4 cores @2.3 Ghz and 8GB of RAM. The source code and sample data sets used to compute the tables presented below are available online, see [56] . Let us note that thanks to Julia/JuMP, it is easy to switch from one solver to another, provided that all the required features are available. *Raw implementations* of the primal-dual formulation MarketClearing-MPC, and the classic and modern Benders decompositions all fit within 250 lines of code including input-output data management (see the file 'dam.jl' provided online), while some solution checking tools provided in an auxiliary file span about 180 lines of code.

Our main purpose here is to compare the new approach proposed to the market rules used until now by the power exchange OMIE (part of PCR). We thus have considered realistic datasets corresponding to the case of Spain and Portugal. Notable differences compared to real data for example available at [68] is that the marginal costs of the first steps of each bid curve associated to a given MIC order have been replaced by the variable cost of that MIC order whenever they were below the variable cost, and as a consequence, a minimum acceptance ratio of 0.6 has been set for the first step of each of these bid curves. The rationale for such modifications is the following: marginal costs for the first steps of the bid curves are sometimes very low (even almost null) certainly to ensure a reasonable level of acceptance of the corresponding offered quantities for operational reasons, and increasing them would decrease too much the accepted quantities at some hours, which is counterbalanced by setting an appropriate acceptance ratio at each hour in case the MP order is part of the market outcome solution. Let us recall that an MP order can only be accepted if the losses incurred at some hours (due to the minimum acceptance ratios forcing paradoxical acceptances and which are 'measured' by the dual variables s_{hc}^{min}) are sufficiently compensated by the profits made at some other hours of the day. All the costs have then been uniformly scaled to obtain interesting instances where e.g. the MP conditions are not all verified if only the primal program (4.1)-(4.9) is solved. As network aspects are not central here, a simple two nodes network corresponding to coupling Spain and Portugal is considered.

As both market models, though different, pursue the same goal of modelling start-up costs and marginal costs recovery conditions while representing in some ways indivisibilities of production (with minimum acceptance ratios or using very low marginal costs for the first amounts of power produced in some original datasets), Tables 4.4 & 4.5 propose a comparison from a computational point of view, which shows the benefits of the new approach. A key issue with the current practice is the absence of the fixed costs in the objective function and the occurrence of an 'ad-hoc' variable cost in the minimum income conditions which is not related to the marginal costs used in the objective function. The objective function in the continuous relaxations somehow 'goes in a direction' which may not be the most appropriate with respect to the enforcement of the minimum income conditions. On the other side, the new approach seems more natural as it enforces minimum profit conditions by requiring that the 'shadow costs of acceptance' du^a must all be null, see Corollary 4.1.

Inst.	Welfare	Abs. gap	Solver's cuts	Nodes	Runtime	# MP Bids	# Curve Steps
1	151218658.27	0.00	24	388	72.63	92	14494
2	115365156.34	0.00	15	181	38.08	90	14309
3	112999837.94	1644425.79	21	4085	600.17	91	14329
4	107060355.83	0.00	16	0	7.63	89	14370
5	100118316.52	0.00	15	347	96.06	89	15091
6	97572068.18	0.00	18	1116	143.65	86	14677
7	87937471.32	1091700.74	27	4958	600.11	87	14979
8	89866979.23	0.00	87	1707	296.41	87	16044
9	86060320.81	0.00	97	361	57.27	81	15177
10	90800596.61	3755055.95	59	2430	600.02	90	16475

Table 4.4: Instances with 'MIC Orders' as in OMIE-PCR

Inst.	Welfare	Abs. gap	Solver's cuts	Nodes	Runtime	# MP Bids	# Curve Steps
1	151487156.16	0.00	11	9	17.36	92	14494
2	115475592.36	0.00	11	0	16.38	90	14309
3	114220400.20	0.00	24	0	17.23	91	14329
4	107219935.90	0.00	35	7	17.48	89	14370
5	100743738.16	0.00	14	0	14.74	89	15091
6	98359291.45	0.00	10	0	15.67	86	14677
7	89251699.16	0.00	84	3	22.92	87	14979
8	90797407.15	0.00	27	0	21.58	87	16044
9	86403721.22	0.00	35	7	25.04	81	15177
10	94034444.59	0.00	20	0	19.58	90	16475

Table 4.5: Instances with MP bids - MarketClearing-MPC formulation

Table 4.6 is to be compared with Table 4.5 e.g. in terms of runtimes and visited nodes, as it solves exactly the same market model. Heuristics of the solver have been here deactivated as primal feasible solutions found need to be obtained as optimal solutions of the LP relaxation at the given node for the local cuts of Theorem 4.7 to be valid (cf. its statement above). As it can be seen, the Benders decomposition is faster by an order of magnitude for the instances at hand.

4.7 Conclusions

A new approach to minimum profit or maximum payment conditions has been proposed in the form of a bidding product called 'MP bid', which turns out to generalize both block orders with a minimum acceptance ratio used in France, Germany or Belgium, and, mutatis mutandis, complex orders with a minimum income condition used in Spain and Portugal. The corresponding market clearing conditions such as minimum profit or maximum payment conditions can be expressed with a 'primal-dual' MILP model involving both primal decision variables such as unit commitment or power output variables, and

Inst.	Welfare	Lazy cuts	Solver's cuts	Nodes	Runtime	# MP Bids	# Curve Steps
1	151487156.16	2	0	5	2.66	92	14494
2	115475592.36	1	18	5	1.38	90	14309
3	114220400.20	1	28	3	1.81	91	14329
4	107219935.90	2	14	11	1.78	89	14370
5	100743738.16	1	12	3	1.36	89	15091
6	98359291.45	1	3	3	1.36	86	14677
7	89251699.16	1	29	8	1.54	87	14979
8	90797407.15	1	11	3	1.66	87	16044
9	86403721.22	2	1	13	2.24	81	15177
10	94034444.59	1	40	4	1.54	90	16475

Table 4.6: Instances with MP bids - Benders decomposition of Theorem 4.7

dual decision variables such as market prices or economic surpluses of market participants, while avoiding the introduction of *any* auxiliary variables, whether continuous or binary. Moreover, it can be used to derive a Benders decomposition with strengthened cuts of a kind which is known to be efficient to handle block bids. These MP bids hence seem an appropriate tool to foster market design and bidding products convergence among the different regions which form the coupled European day-ahead electricity markets of the Pan-European PCR project. Also, compared to the MIC orders currently in use at OMIE-PCR, they have the following additional advantages. Firstly, they lead to optimisation models that can be solved much more quickly. Secondly, the proposed instruments seem to be more aligned with the operating constraints and cost structure of the power plants that they are supposed to represent in the market. Finally, they are more natural (from an economic point of view) and simpler (from a modelling point of view), leading to a market model easier to understand for participants and monitor for regulators. All the models and algorithms have been implemented in Julia/JuMP and are available online together with sample datasets to foster research and exchange on the topic. The models and algorithms can also be used to clear instances involving block bids only (small extensions could also be added to handle linked and exclusive block orders as described in [27] if desired). European day-ahead electricity markets will certainly be subject to a major evolution in the coming years, as many challenges are still to be faced, which calls for further research within the academic and industrial communities. The present contribution is a proposal made in that frame.

Acknowledgements: We greatly thank OMIE and N-Side for providing us with data used to generate realistic instances. This text presents research results of the P7/36 PAI project COMEX, part of the Interuniversity Attraction Poles (IAP) Programme of the Belgian Science Policy Office.This work was also supported by EC-FP7-PEOPLE MINO Marie-Curie Initial Training Network (grant number 316647) and by EC-FP7 COST Action TD1207. The scientific responsibility is assumed by the authors.

4.A Omitted proofs in main text

4.A.1 Proof of Theorem 4.3

Reminder of the Farkas Lemma [87], which is used in the proof afterward:

$\exists x : Ax <= b, x \geq 0$ if and only if $\forall y : y \geq 0, yA \geq 0 \Rightarrow yb \geq 0$

Proof. Applying the Farkas lemma, given an incumbent $(x_i^*, x_{hc}^*, u_c^*, n_k^*)$, a solution $(s_i, s_{hc}^{max}, s_{hc}^{min}, s_c, \pi_{l,t}, v_m)$ to the remaining linear conditions (4.65), (4.74)-(4.78) exist if and only if:

$$\sum_i (P^i Q_i) x_i + \sum_{c,h \in H_c} (P^{hc} Q_{hc}) x_{hc} - \sum_c F_c u_c - M_c(1 - u_c^*) u_c \leq \sigma\Big(\sum_i (P^i Q_i) x_i^* + \sum_{c,h \in H_c} (P^{hc} Q_{hc}) x_{hc}^* - \sum_c F_c u_c^*\Big) \quad (4.103)$$

$\forall(\sigma, x_i, x_{hc}, u_c, n_k)$ such that:

$$x_i \leq \sigma \qquad \forall i \in I \ [s_i] \quad (4.104)$$

$$x_{hc} \leq u_c \qquad \forall h \in H_c, c \in C \ [s_{hc}^{max}] \quad (4.105)$$

$$x_{hc} \geq r_{hc} u_c \qquad \forall h \in H_c, c \in C \ [s_{hc}^{min}] \quad (4.106)$$

$$u_c \leq \sigma \qquad \forall c \in C [s_c] \quad (4.107)$$

$$\sum_{i \in I_{lt}} Q_i x_i + \sum_{hc \in HC_{lt}} Q_{hc} x_{hc} = \sum_k e_{l,t}^k n_k, \qquad \forall (l,t) \ \ [\pi_{l,t}] \quad (4.108)$$

$$\sum_k a_{m,k} n_k \leq w_m \qquad \forall m \in N \ \ [v_m] \quad (4.109)$$

$$x_i, x_{hc}, u_c, \sigma \geq 0 \quad (4.110)$$

Since the condition described by (4.103)-(4.110) is trivially satisfied when $\sigma = 0$ (technically assuming that network conditions (4.108)-(4.109) could be satisfied when $x_i = x_{hc} = 0$), we can normalize, i.e. set $\sigma := 1$ and the condition becomes

$$\max_{\forall (x_i, x_{hc}, u_c, n_k) \in P} \sum_i (P^i Q_i) x_i + \sum_{c,h \in H_c} (P^{hc} Q_{hc}) x_{hc} - \sum_c F_c u_c - M_c(1 - u_c^*) u_c \leq \sum_i (P^i Q_i) x_i^* + \sum_{c,h \in H_c} (P^{hc} Q_{hc}) x_{hc}^* - \sum_c F_c u_c^*, \quad (4.111)$$

where P is the polyhedron defined by the linear conditions (4.66)-(4.72), that is the linear relaxation of (4.66)-(4.73). This provides the first result (4.93).

Now, observe that an optimal solution of the left-hand side of (4.93) or (4.111) is always such that $u_c^{\#} = 0$ if $u_c^{*} = 0$, because of the penalty coefficients M_c, or alternatively because $u_c^{\#} = 0$ corresponds to the optimal dual variable of (4.76) which is not tight when $u_c^{*} = 0$. This proves (4.94). □

Appendix A

Convex Quadratic Programming Duality

Let us note that the duality results presented here could be derived from the general Langragean duality theory: both classical Lagrangean duality results in convex programming and the derivation of Dorn's duality results for convex quadratic programs are for example exposed in [39] (relying also on a general fact about attained bounds by quadratic programs with bounded objective function values), while [89] derives Dorn's main results from KKT conditions and considers extensions to the non-convex quadratic programming case. Let us note that a quadratic dual has also been proposed by Jack Bonnell Dennis [19], and that an extensive technical review of duality results in non-linear programming including a unified presentation of these duality results can be found in [91]. For our needs, we focus here on presenting Dorn's results as first derived by Dorn, i.e. via duality for linear programs, while adopting a more modern approach, in particular distinguishing between the weak and strong duality parts of the general theorem and underlining the role of algebraic arguments in these two parts. Compared to the historical presentation, the equivalence between complementarity conditions and equality of objective functions for particular pairs of feasible primal and dual solutions is also detailed. Additional general basic facts on quadratic programming proposed by Wolf [102] and used at some points are also proposed as they are also of interest and not always mentioned in classical introductory textbooks in non-linear programming.

A.1 Basic facts

Let us consider the following convex quadratic program:

$$(QP) \qquad \max_x \frac{1}{2}x^TQx + c^Tx \tag{A.1}$$

s.t.

$$Ax \leq b \tag{A.2}$$

$$x \geq 0 \tag{A.3}$$

where Q is a negative semi-definite matrix, the objective function to maximize hence being concave.

Observation A.1. $Qx = Qy \Rightarrow x^TQy = y^TQx = x^TQx = y^TQy$

Lemma A.1. *[102] If Q is negative semi-definite (the same holds if Q is positive semi-definite),*

$$x^TQx = 0 \Rightarrow Qx = 0$$

Proof. As Q is negative semi-definite, $\forall t \in \mathbb{R}, y \in \mathbb{R}^n$, $(y + tx)^TQ(y + tx) \leq 0$,

which expanded, and since $x^TCx = 0$, gives:

$t^2x^TCx + y^TCy + 2y^TCx = y^TCy + 2ty^TCx \leq 0$ for all t and y.

Since the inequality should hold for all t, $y^TCx = 0$. Since $y^TCx = 0$ for all y, $Cx = 0$. □

Lemma A.2. *[102, Lemma 2]*

If QP has an optimal solution x^*, any other feasible solution $\tilde{x}$ is optimal if and only if it satisfies $Qx^* = Q\tilde{x}$ and $cx^* = c\tilde{x}$.

Proof. Let x^* and $\tilde{x}$ be two optimal solutions to (QP). As the objective function, here denoted f, is concave, $\forall\lambda \in [0, 1], f(\lambda x^* + (1-\lambda)\tilde{x}) \geq \lambda f(x^*) + (1-\lambda)f(\tilde{x}) = f(x^*) = f(\tilde{x})$ which is the optimal value, hence $f(\lambda x^* + (1-\lambda)\tilde{x}) = f(x^*)$ and f is constant over any segment joining x^* and $\tilde{x}$, the whole segment still lying in the feasible set.

Hence, setting $w := (\tilde{x} - x^*)$, $\forall\lambda \in [0, 1], f(x^* + \lambda w) = f(x^*)$, that is:

$$c(x^* + \lambda w) + \frac{1}{2}(x^* + \lambda w)^TQ(x^* + \lambda w) = cx^* + \frac{1}{2}(x^*)^TQx^* \tag{A.4}$$

which rearranged gives $\lambda cw + \lambda^2\frac{1}{2}w^TQw + \lambda w^TQx^* = 0$ for all $\lambda \in [0, 1]$, and:

$$\forall\lambda \in]0, 1], \qquad (cw + w^TQx^*) + \lambda\frac{1}{2}w^TQw = 0 \tag{A.5}$$

This last equation trivially implies $w^TQw = 0$ and by Lemma A.1, $Qw = 0$, hence:

$$Qx^* = Q\tilde{x}.$$

As $Qw = 0$, $w^TQx^* = 0$, and (A.5) reduces to $cw = 0$, hence:

$$cx^* = c\tilde{x}$$

Finally, for the converse assertion, it is direct to check that any other feasible $\tilde{x}$ satisfying these two conditions will provide the same objective value as any optimal x^*, and will therefore also be optimal. □

A third (trivial) Lemma is of interest as it is used in the brief proof of the Weak Duality Theorem below:

Lemma A.3. *If the real numbers c_{ij} are coefficients of a negative semi-definite matrix Q, then, for every x, u:*

$$\sum_{ij} c_{ij} x_i (x_j - u_j) \leq \frac{1}{2}(\sum_{ij} c_{ij} x_i x_j - \sum_{ij} c_{ij} u_i u_j).$$

Proof. For any u, x, as Q is negative semi-definite:

$$(x-u)Q(x-u) \leq 0, \text{ i.e. } \sum_{i,j} c_{ij}(x_i - u_i)(x_j - u_j) \leq 0$$

$$\Leftrightarrow \sum_{i,j} c_{ij} x_i x_j + \sum_{i,j} c_{ij} u_i u_j - 2\sum_{i,j} c_{ij} x_i u_j \leq 0$$

$$\Leftrightarrow -\sum_{i,j} c_{ij} x_i u_j \leq -\frac{1}{2}(\sum_{i,j} c_{ij} x_i x_j + \sum_{i,j} c_{ij} u_i u_j)$$

And adding $\sum_{i,j} c_{ij} x_i x_j$ on both sides:

$$\Leftrightarrow \sum_{i,j} c_{ij} x_i x_j - \sum_{i,j} c_{ij} x_i u_j \leq \frac{1}{2}(\sum_{i,j} c_{ij} x_i x_j - \sum_{i,j} c_{ij} u_i u_j)$$

□

Finally, the following Lemma is key to derive the strong duality theorems for convex quadratic programs from strong duality for linear programs:
Lemma A.4. *Let x^* be an optimal solution to (QP). Then it is also optimal for the following program where the objective has been replaced by an analogue linear form where x^* appears:*

$$\max_x c^T x + (x^*)^T Q x \tag{A.6}$$

subject to (A.2)-(A.3)

N.B. This result could also be easily derived via KKT optimality conditions as follows. If x^* is optimal for (QP), there exist optimal multipliers λ^* such that (x^*, λ^*) satisfies the KKT conditions of the problem which are necessary and sufficient (the problem is convex and classical constraint qualifications are satisfied). One then directly verifies that it would hence be a solution to the KKT conditions of the program A.6, proving that x^* would therefore be optimal for it as well. Here, in order to rely only on linear programming and basic algebra, the proof provided in [23] is presented.

Proof. Suppose on the contrary that there exists a solution $\overline{x}$ such that :

$$c^T\overline{x} + (x^*)^T Q\overline{x} > c^T x^* + (x^*)^T Q x^*, \tag{A.7}$$

contradicting the optimality of x^* for A.6. We show that we can then contradict the optimality of x^* for (QP) using $\overline{x}$ and basic algebraic arguments, hence proving optimality of x^* for A.6.

Let us consider a convex combination of $x^*, \overline{x}$: $\tilde{x} = x^* + k(\overline{x} - x^*)$, for some $0 < k \leq 1$. Any such convex combination remains feasible. It is now shown that appropriately choosing k would provide a point $\tilde{x}$ contradicting the optimality of x^*:

$$\frac{1}{2}\tilde{x}^T Q\tilde{x} + c^T\tilde{x} = (\frac{1}{2}(x^*)^T Q x^* + c^T x^*) + \frac{1}{2}k(\overline{x} - x^*)^T Q k(\overline{x} - x^*) + c^T k(\overline{x} - x^*) + x^* Q k(\overline{x} - x^*)$$

It remains to choose k to enforce $\frac{1}{2}k(\overline{x} - x^*)^T Q k(\overline{x} - x^*) + c^T k(\overline{x} - x^*) + (x^*)^T Q k(\overline{x} - x^*) > 0$, or rearranging terms, to have:

$$k\left[\frac{k}{2}(\overline{x} - x^*)^T Q(\overline{x} - x^*) + [c^T + (x^*)^T Q](\overline{x} - x^*)\right] > 0 \tag{A.8}$$

Rewritting (A.7) shows that $[c^T + (x^*)^T Q](\overline{x} - x^*) > 0$, and hence that:

$$k^* = -\frac{[c^T + (x^*)^T Q](\overline{x} - x^*)}{\frac{1}{2}(\overline{x} - x^*)^T Q(\overline{x} - x^*)} > 0. \tag{A.9}$$

Note that we can assume here that the denominator is strictly negative, as otherwise, Lemma A.4 is easily proven to hold: according to Lemma A.1, $Q\overline{x} = Qx^*$, and using Observation A.1 in that case shows that (A.7) reduces to $c^T\overline{x} > c^T x^*$, since we have then $(x^*)^T Q x^* = \overline{x}^T Q x^*$. Hence, $\overline{x}$ would trivially provides a feasible point contradicting the optimality of x^* for (QP).

Now, let us observe that any k such that $0 < k < k^*$ ensures that both factors of the left-hand side of (A.8) are strictly positive, making the condition holds.

To enforce (A.8) with a $0 < k \leq 1$, it hence suffices to choose k such that:

$$0 < k < min\left[-\frac{[c^T + (x^*)^T Q](\overline{x} - x^*)}{\frac{1}{2}(\overline{x} - x^*)^T Q(\overline{x} - x^*)}; 1\right] \tag{A.10}$$

This shows that contradicting the optimality of x^* for A.6 implies contradicting the optimality of x^* for (QP) and completes the proof. $\square$

A.2 Dorn's quadratic programming dual

The following program (DQP) is called the Dorn's dual of (QP), and standard duality theory assertions hold for this pair of programs, see Theorems A.1, A.2 and A.3.

$$(DQP) \qquad \min_{u,v} b^T u - \frac{1}{2} v^T Q v \tag{A.11}$$

s.t.

$$A^T u - Qv \geq c \tag{A.12}$$
$$u \geq 0 \tag{A.13}$$

It directly follows from Observation A.1 that if (u, v) is an optimal solution to (DQP), then $(u, \tilde{v})$ is also optimal for all $\tilde{v}$ such that $Q\tilde{v} = Qv$. Indeed, all optimal solutions verify this condition, as shown by Lemma A.2.

Theorem A.1 (Weak duality)**.** *For every pair x and (u, v) of respectively primal and dual feasible points, i.e. satisfying (A.2)-(A.3) and (A.12)-(A.13), the following holds:*

$$\frac{1}{2} x^T Q x + cx \leq b^T u - \frac{1}{2} v^T Q v \tag{A.14}$$

Proof. The proof is almost as short and straightforward as for linear programming. We start by multiplying (A.2) by the dual variable u and (A.12) by its dual variable x, which provides:

$$u^T A x \leq u^T b \tag{A.15}$$
$$c^T x \leq x^T A^T u - x^T Q v \tag{A.16}$$

Using the first of these two inequalities for substitution in the second one gives:

$$c^T x \leq u^T b - x^T Q v \tag{A.17}$$

Now observe that, according to Lemma A.3, the term $(-x^T Q v)$ is bounded above by $(-\frac{1}{2} x^T Q x - \frac{1}{2} v^T Q v)$, hence:

$$cx \leq b^T u - \frac{1}{2} v^T Q v - \frac{1}{2} x^T Q x \tag{A.18}$$

which rearranged provides the result. □

Theorem A.2 (Strong duality for convex QP, Theorem 'Dual' in [23])**.** *(I) If x^* is an optimal solution for (QP), then there exists an optimal solution (u^*, v^*) for (DQP) such that $v^* = x^*$ and such that the optimal objective values are equal:*

$$\frac{1}{2}(x^*)^T Qx^* + cx^* = b^T u^* - \frac{1}{2}(x^*)^T Qx^*.$$

(II) Conversely, suppose that (u^*, v^*) is optimal for (DQP), then there exists an optimal solution x^* for (QP) such that $Qx^* = Qv^*$ and such that the optimal objective values are equal.

Proof. See Section A.3 for a proof relying on linear programming strong duality and Lemma A.4. □

Though the dual variable v somehow corresponds to the primal variable x, one should be careful in observing that, given an optimal dual solution (u^*, v^*), v^* could be neither optimal nor feasible for the primal, as shown by the toy example below: what is guaranteed by the strong duality theorem is the existence of an optimal primal x^* such that $Qx^* = Qv^*$. Let us observe however that if Q is non-singular, then both (QP) and (DQP) admit unique optimal solutions x^* and (u^*, v^*) such that $x^* = v^*$. If Q is singular, it may happen, as in the toy example, that (QP) still admits an optimal solution which is unique, but not the dual.

Example A.1. *Let $Q = \begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix}$, $c = (0, 1), A = Id$ and $b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, (QP) and (DQP) are:*

$$\max_{x_1,x_2} -x_1^2 + x_2 \text{ s.t.} \qquad\qquad \min_{u_1,u_2,v_1,v_2} u_1 + u_2 + v_1^2$$

$$\begin{aligned} x_1 &\leq 1 & [u_1] \qquad\qquad u_1 + 2v_1 &\geq 0 & [x_1] \\ x_2 &\leq 1 & [u_2] \qquad\qquad u_2 &\geq 1 & [x_2] \\ x_1, x_2 &\geq 0 & \qquad\qquad u_1, u_2 &\geq 0 & \end{aligned}$$

It is staightforward to check that $(0, 1)$ is optimal for (QP) and $(0, 1, 0, 0)$ is optimal for (DQP), as would indeed be any point of the form $(u_1, u_2, v_1, v_2) = (0, 1, 0, v_2)$. However, such an optimal point for (DQP) could provide (v_1, v_2) which is not optimal (e.g. $(0, 0)$), or even not feasible (e.g. $(0, 2)$) for (QP).

Theorem A.3. *Let x^* and (u^*, v^*) be primal and dual feasible respectively. Then the following conditions are equivalent:*

1. *x^* and (u^*, v^*) are optimal for their respective programs*
2. *$\frac{1}{2}(x^*)^T Qx^* + cx^* = b^T u^* - \frac{1}{2}(v^*)^T Qv^*$ [equality of objective function values]*
3. *$Qx^* = Qv^*$ and complementarity constraints hold:*

$$\forall i, x_i^*(A^T u^* - Qv^* - c)_i = 0 \tag{A.19}$$
$$\forall j, u_j^*(Ax^* - b)_j = 0 \tag{A.20}$$

or equivalently, as both $(u^*)^T(Ax^* - b)$ and $(-x^*)^T(A^T u^* - Qv^* - c)$ are non-positive, and in matrix form:

$$(u^*)^T(Ax^* - b) - (x^*)^T(A^Tu^* - Qv^* - c) = 0 \tag{A.21}$$

Proof. (1) ⇒ (2) is a direct consequence of the strong duality theorem, while (2) ⇒ (1) is a direct consequence of the weak duality theorem.

For (2) ⇒ (3), as the points x^* and (u^*, v^*) are then optimal solutions for their respective programs, by Lemma A.2 and the strong duality Theorem A.2, we know that $Qx^* = Qv^*$, which trivially implies $(x^*)^TQx^* = (x^*)^TQv^* = (v^*)^TQx^* = (v^*)^TQv^*$ (see Observation above), and we thus have:

$$\begin{aligned}&((u^*)^TAx^* - (u^*)^Tb) - ((x^*)^TA^Tu^* - (x^*)^TQv^* - c^Tx^*)\\&= c^Tx^* - (u^*)^Tb + (x^*)^TQv^* =^{\star\star} \frac{1}{2}(x^*)^TQx^* + c^Tx^* - b^Tu^* + \frac{1}{2}(v^*)^TQv^* =^{\star\star\star} 0\end{aligned} \tag{A.22}$$

where $=^{\star\star\star}$ is simply (2) rearranged, and the sequence of equalities (A.22) show that the condition $(A.21)$, i.e. (3), holds if (2) holds.

For (3) ⇒ (2) the same equalities (A.22) are used safe that the equality $=^{\star\star}$ here holds because $Qx^* = Qv^*$ is part of the stated condition (3). □

A.3 Proof of Dorn's strong duality theorem

As mentioned above, Dorn's strong duality theorem could be derived from general Langrangean duality results as presented in [39] or from KKT optimality conditions as described e.g. in [89]. We instead follow here the original approach of Dorn, deriving the result as a consequence of strong duality for linear programs plus a few additional algebraic arguments.

To slightly simplify the proof of Part (II) of the Theorem as presented in [23], we shall rely on [33] proving by recurrence that any quadratic function achieves its maximum (resp. minimum) on any closed polyhedral convex set on which it is bounded from above (resp. below). Details are provided below when proving (II). Another option in our context would simply be to make the extra assumption that the feasible set is a polytope (i.e. bounded and hence compact), as this assumption holds in all the applications considered in the present thesis.

Proof. (I) Let x^* be an optimal solution for (QP). To prove the result, we show that there exists a feasible (u^*, x^*) such that the dual program attains there the primal lower bound given by the weak duality theorem (Theorem A.1). Such a (u^*, x^*) must hence satisfy:

$$A^Tu^* - Qx^* \geq c \tag{A.23}$$
$$u^* \geq 0 \tag{A.24}$$

$$\frac{1}{2}(x^*)^T Qx^* + cx^* \geq b^T u^* - \frac{1}{2}(x^*)^T Qx^* \tag{A.25}$$

Note that (A.25) rearranged is simply:

$$b^T u^* \leq cx^* + (x^*)^T Qx^* \tag{A.26}$$

To show that such a u^* exists, we consider (DQP) where v is fixed to x^*, and denoted $(RDQP)$. As (QP) admits an optimal solution,$(RDQP)$ also admits an optimal solution u^*: otherwise, relying on linear programming duality, one can contradict Lemma A.4 where the linear programming dual of $(RDQP)$ appears. By linear programming strong duality, optimality of u^* is characterised by the existence of y such that:

$$Ay \leq b \tag{A.27}$$
$$A^T u^* - Qx^* \geq c \tag{A.28}$$
$$y \geq 0, u^* \geq 0 \tag{A.29}$$
$$b^T u^* \leq (c + Qx^*)^T y \tag{A.30}$$

These last optimality conditions indeed imply (A.26), completing the proof. This follows from the fact that $(c+Qx^*)^T y = c^T y + y^T Qx^* \leq cx^* + (x^*)^T Qx^*$ for any y primal feasible, i.e. satisfying $Ax \leq b, x \geq 0$, this last inequality being a consequence of Lemma A.4.

(II) As mentioned above in the introduction, a possibility is to proceed as in [23], essentially applying Part (I) of the Theorem to the dual modulo a few algebraic reformulations. We present here a slightly shorter alternative.

First, note that if (QP) has an optimal solution, it must be such that (a) $Qx^* = Qv^*$ and (b) the optimal objective values are equal. Otherwise, using (I), one could construct an optimal solution $(\tilde{u}, x^*)$ to (DQP) (a) such that $Qx^* \neq Qv^*$, contradicting Lemma A.2, or (b) such that the optimal dual objective value would be different from the current one, also a contradiction. The problem is hence reduced to proving that (QP) has indeed an optimal solution.

The result then directly follows from the fact that any quadratic function achieves its maximum (resp. minimum) on any closed polyhedral convex set on which it is bounded from above (resp. below) [33]: if (DQP) has an optimal solution (u^*, v^*), by weak duality, (QP) is bounded from above and the maximum of the objective is achieved for a x^*.

(As mentioned above, the use of the result in [33] could be avoided if one makes the extra assumption that the feasible set is compact, which always holds in the applications considered in the present text.) □

Appendix B

Spatial price equilibrium

In 1952, Paul Samuelson published a seminal paper [85] showing how a spatial price equilibrium that he calls a "Cournot-Enke equilibrium" - of which the precise definition is recalled below - could be determined by solving a social welfare optimization problem. It seems that this paper is the first proposing the equivalence between market equilibrium and welfare optimization, and moreover in the general context of spatially separated markets, rather than in the simpler particular case where there is only one given market at hand. Regarding this, Paul Samuelson points out in his paper that "the first explicit statement that competitive market price is determined by the intersection of supply and demand functions seems to have been given by A. A. Cournot in 1838 in connection, curiously enough, with the more complicated problem of price relations between two spatially separate market".

The work of Cournot evoked [17, Chapter 10] is entitled "De la communication des marchés", where several interesting observations are made regarding markets coupled with what would be called today a transportation model. In particular, it is observed therein that coupling two markets can result both in a decrease of production of a good, as well as a decrease of the total value of production at the new market prices, though it always results in an increase of social welfare. The topic is standard and the article of Samuelson pleasant to read, yet it does not contain any explicit mathematical development, certainly as the target audience at the time were economists not always technically acquainted with the newly born field of linear programming.

Our main purpose here is to present the key underlying ideas in the context of the abstract network model used in the three contributions presented in Part II, and which could then be specialized e.g. (a) to a transportation model as considered by Samuelson (called Available-to-Transfer Capacity model in the PCR jargon), or (b) to a DC model where Kirchhoff's laws are linear or their non-linear AC versions have been linearised. As in the classical case, the spatial price equilibrium properties still include optimality conditions for 'arbitrageurs' between local markets, here Transmission System Operators.

The seminal paper [85] focuses on linear programming, but also mentions that the results presented still hold for more general monotonically increasing (resp. decreasing) offer (resp. demand) curves. Regarding this possibility, Takayama [93] considers piecewise linear curves and use KKT conditions to derive the appropriate economic interpretations.

The presentation here also considers piecewise linear curves and makes explicit use of Dorn's dual presented in Appendix A. It is all we need in our context where (offer or demand) bid curves submitted to power exchanges are either stepwise (e.g. in Belgium and the Netherlands) or piecewise linear (e.g. in France or Germany).

In a few words, for a spatial price equilibrium to hold given some production, consumption and trading decisions such that the market clears, prices must be determined for each location where the commodity is traded, such that among price-taker market participants - among which an arbitrageur potentially buying in one market to resell in another - no one could be better off with another decision. (By price-taker, we mean here market participants without market power taking market prices as given to take their decisions, and *not* participant indifferent to the given market prices.) In other words, for the given prices, the current state of affairs corresponds to optimal decisions of participants, and the market clears. Paul Samuelson's result is that, under some classical assumptions regarding the welfare optimization problem, locational equilibrium prices can be determined as optimal dual variables to the constraints which, for each market, relate net import-export positions to the flows of transported commodities between these markets.

Here, one should be cautious when defining the notion of market equilibrium and include optimality conditions of the 'arbitrageurs'/Transmission System Operators, to avoid confusion as examples presented in [103] may suggest.

B.1 Spatial Price Equilibrium with an abstract linear network model

The context is the following: a commodity is traded in different locations connected with a capacitated network - such as high-voltage transmission lines in the case of electricity - which can be described by linear inequalities. Let us note that in the case of AC power flows, linear transmission models are only approximations of the real power flow models which are non-linear and much more difficult to deal with.

Each market participant has preferences regarding the limit price at which she is willing to buy or sell the commodity, and these limit prices are related to the quantities traded. Limit offer prices correspond, in an ideal so-called competitive market, to the marginal costs of the producers which we assume increase with the quantity that is produced. Limit demand prices correspond to the consumption utility, and are assumed to be decreasing with the quantity consumed.

Let us consider the following welfare maximizing program, where $x_i \in [0,1]$ is the decision variable determining which fraction of demand $Q_i > 0$ or offer $Q_i < 0$ is accepted in the market clearing solution. The *marginal* utility/cost of bid i is described by the line segment joining $(P_0^i, 0)$ to (P_1^i, Q^i) for demand bids with $Q^i > 0$ and $P_0^i > P_1^i$, or $(P_0^i, 0)$ to $(P_1^i, -Q^i)$ for offer bids with $Q^i < 0$ and $P_0^i < P_1^i$. Conditions (B.3) are balance constraints relating the net export position of a given market (on the left-hand side) to a certain usage of network resources described by a linear form in the right-hand side. Conditions (B.4) describe in a linear way the scarcity of the network resources n_k. These resources have a marginal cost given by the parameters c_k The welfare maximizing objective is then given by:

$$\max_{x_i} \sum_i P_0^i Q^i x_i + \frac{1}{2} \sum_i (P_1^i - P_0^i) Q^i x_i^2 - \sum_k c_k n_k \tag{B.1}$$

subject to

$$\begin{aligned}
& x_i \leq 1 && \forall i \in I \quad [s_i] && \text{(B.2)}\\
& \sum_{i \in I_{lt}} Q^i x_i = \sum_k e_{l,t}^k n_k, \ \forall (l,t) \in L \times T && [\pi_{l,t}] && \text{(B.3)}\\
& \sum_k a_{m,k} n_k \leq w_m && \forall m \in N \quad [u_m] && \text{(B.4)}\\
& x_i \geq 0, && \forall i \in I && \text{(B.5)}
\end{aligned}$$

The (Dorn's) dual program is written with $v_i := x_i$ as we know there is always an optimal dual solution (u, v) with $v = x$ provided that the primal has an optimal solution x:

$$\min_{s_i, \pi_{l,t}, u_m, v_i} \sum_i s_i + \sum_m w_m u_m - \sum_i Q^i (P_1^i - P_0^i) \frac{x_i^2}{2} \tag{B.6}$$

s.t.

$$\begin{aligned}
& s_i + Q^i \pi_{l(i),t(i)} \geq Q^i P_0^i \ + Q^i (P_1^i - P_0^i) x_i && \forall i \in I \ \ [x_i], && \text{(B.7)}\\
& \sum_m a_{m,k} u_m - \sum_{l,t} e_{l,t}^k \pi_{l,t} = -c_k && \forall k \in K \ [n_k] && \text{(B.8)}\\
& s_i, u_m \geq 0 && \forall i \in I, m \in N && \text{(B.9)}
\end{aligned}$$

And the related complementarity conditions:

$$\begin{aligned}
& s_i (1 - x_i) = 0 && \forall i \in I && \text{(B.10)}\\
& x_i (s_i + Q^i p_{l(i),t(i)} - Q^i P_0^i - Q^i (P_1^i - P_0^i) x_i) = 0 && \forall i \in I && \text{(B.11)}\\
& u_m (\sum_k a_{m,k} n_k - w_m) = 0 && \forall m \in N && \text{(B.12)}
\end{aligned}$$

We now briefly review the fact that the primal, dual and complementarity conditions just written, which are optimality conditions for the welfare optimization problem (B.1)-(B.5), include in particular optimality conditions for *price-taker* profit maximizing market participants including Transportation/Transmission System Operators acting as 'arbitrageurs' between local markets. This shows that the market prices $\pi_{l,t}$ support a spatial price equilibrium.

Let us note that fixing π in (B.6)-(B.9) renders the problem separable by market participant. Accordingly, a Langrangian dualization of the balance constraints (B.3) with dual multipliers π provides a dual aiming at finding prices π minimizing the maximum sum of economic surpluses attainable with this balance constraints relaxed (payments dependent on the prices π now appear in the objective), a problem easily seen to be separable by market participant as well (including arbitrageurs between local markets). It is then straightforward to see that the separated maximization subproblems thus obtained for each (type of) market participant are those presented in the next sections. The results following in the next brief sections can hence readily be obtained by using the strong duality property holding for the primal welfare maximization problem (B.1)-(B.5) and the Langrangian dual obtained by dualizing the balance constraints (B.3) (strong duality generalizing classical duality for linear programs, see [39]).

B.1.1 Price-taker Market Participants

For the given market prices $\pi_{l,t}$, the following problem is solved by each profit maximizing price-taker market participant or bid i.

$$\max_{x_i} P_0^i Q^i x_i + \frac{1}{2}(P_1^i - P_0^i) Q^i x_i^2 - \pi_{l(i),t(i)} Q^i x_i \tag{B.13}$$

subject to:

$$x_i \leq 1 \qquad [s_i] \tag{B.14}$$

$$x_i \geq 0 \tag{B.15}$$

Optimality conditions for this problem, namely dual and complementarity conditions (see Appendix A), are exactly given by (B.7),(B.9) and (B.10)-(B.11), where again the dual and complementarity conditions are written with '$v := x$', cf. the remark above just before writing the previous dual (B.6)-(B.9).

Note that the corresponding economic interpretation of these dual and complementarity conditions are described in the first contribution presented in Chapter 2, see Definition 6 and the paragraph which follows. Essentially, the level of acceptance x_i of the bid is then completely appropriate regarding the market price and the expressed preferences. In case x_i is fractional, the market price is given by the appropriate interpolation between the prices P_0 and P_1.

B.1.2 The Transmission System Operator

The problem of the Transportation/Transmission System Operator (TSO) is to optimize import/export decisions given the locational market prices π_{lt}, assuming "infinite market depth" when buying or selling at each given location. As the import/export positions are given by $\sum_k e^k_{lt} n_k$, which is negative when the local market (l,t) sells/exports, i.e. sells to the TSO, and positive when the local market imports, i.e. buys from the TSO, the corresponding optimization problem is:

$$\max_{n_k} \sum_{l,t} [\sum_k e^k_{lt} n_k] \pi_{lt} \tag{B.16}$$

subject to:

$$\sum_k a_{m,k} n_k \leq w_m \qquad \forall m \in N \quad [u_m] \tag{B.17}$$

Again, optimality conditions are included in the optimality conditions for the welfare optimization program described above, being given by the dual conditions (B.8)-(B.9) and complementarity conditions (B.12).

It should be noted that, for example, Propositions 3 and 4 in [13] adapted from [43] are a special case of this fact and the notion of spatial price equilibrium.

B.1.3 Solution to the transmission/transportation problem

Beyond the notion of spatial price equilibrium and as initially observed by Samuelson, one may also check straightforwardly that any optimal solution to the welfare maximization problem (B.1)-(B.5) should provide optimal values n_k for the following transportation problem (here in our slightly more general context), where import/export quantities are known and fixed, and which aims at minimizing the corresponding transportation costs:

$$\max_{n_k} \sum_k (-c_k) n_k \qquad \equiv \min_{n_k} \sum_k c_k n_k \tag{B.18}$$

subject to:

$$\sum_k e^k_{l,t} n_k = \sum_{i \in I_{lt}} Q^i x_i, \ \forall (l,t) \in L \times T \quad [p_{l,t}] \tag{B.19}$$

$$\sum_k a_{m,k} n_k \leq w_m \qquad \forall m \in N \quad [u_m] \tag{B.20}$$

$$\tag{B.21}$$

Bibliography

[1] P. J. Agrell and P. Bogetoft. International benchmarking of electricity transmission system operators. In *11th International Conference on the European Energy Market (EEM14)*, pages 1–5, May 2014. doi: 10.1109/EEM.2014.6861311.

[2] René Aid. *Electricity Derivatives*. SpringerBriefs in Quantitative Finance. Springer International Publishing, 2015. ISBN 9783319083957.

[3] Veronica Araoz and Kurt Jörnsten. Semi-lagrangean approach for price discovery in markets with non-convexities. *European Journal of Operational Research*, 214(2):411 – 417, 2011. ISSN 0377-2217. doi: http://dx.doi.org/10.1016/j.ejor.2011.05.009. URL `http://www.sciencedirect.com/science/article/pii/S0377221711004140`.

[4] Raik Becker, Caroline Deilen, Tim Felling, Björn Felten, Lenja Niesen, Paul Osinski, and Christoph Weber. Study on the impact of price zones in different configurations in europe, creg note (f)160519-cdc-1530. Technical report, CREG and University of Duisburg-Essen, 2016. URL `http://www.creg.info/pdf/Studies/F1530EN.pdf`.

[5] J.F. Benders. Partitioning procedures for solving mixed-variable programming problems. *Numer. Math.*, 4:238–252, 1962.

[6] J. Bisschop and R. Entriken. *AIMMS The Modeling System*. Paragon Decision Technology, 1993.

[7] Mette Bjørndal and Kurt Jörnsten. Equilibrium prices supported by dual price functions in markets with non-convexities. *European Journal of Operational Research*, 190(3):768 – 789, 2008. ISSN 0377-2217. doi: http://dx.doi.org/10.1016/j.ejor.2007.06.050. URL `http://www.sciencedirect.com/science/article/pii/S0377221707006340`.

[8] Carl Blumstein, Lee S. Friedman, and Richard Green. The history of electricity restructuring in California. *Journal of Industry, Competition and Trade*, 2(1):9–38, 2002. ISSN 1573-7012. doi: 10.1023/A:1020822720063. URL `http://dx.doi.org/10.1023/A:1020822720063`.

[9] Roger E. Bohn. *Spot Pricing of Public Utility Services*. PhD thesis, MIT, 1982.

[10] François Boisseleau. *The role of power exchanges for the creation of a single European electricity market: market design and market regulation*. PhD thesis, Delft University, University of Paris IX Dauphine, 2004.

[11] M. Boiteux. Sur la gestion des monopoles publics astreints a l'equilibre budgetaire. *Econometrica*, 24(1):22–40, 1956. ISSN 00129682, 14680262. URL http://www.jstor.org/stable/1905256.

[12] Marcel Boiteux and Olivier Coutard. Libres propos sur l'économie des réseaux. une interview de marcel boiteux. *Flux*, 15(36):46–55, 1999. ISSN 1154-2721. doi: 10.3406/flux.1999.1276. URL http://www.persee.fr/doc/flux_1154-2721_1999_num_15_36_1276.

[13] Jacqueline Boucher and Yves Smeers. Alternative models of restructured electricity systems, part 1: No market power. *Operations Research*, 49(6):821–838, 2001. ISSN 0030364X, 15265463. URL http://www.jstor.org/stable/3088597.

[14] C. Breuer and A. Moser. Optimized bidding area delimitations and their impact on electricity markets and congestion management. In *11th International Conference on the European Energy Market (EEM14)*, pages 1–5, May 2014. doi: 10.1109/EEM.2014.6861218.

[15] Mary B. Cain, Richard P. O'Neill, and Anya Castillo. History of optimal power flow and formulations. Technical report, FERC, 2012. URL http://www.ferc.gov/industries/electric/indus-act/market-planning/opf-papers/acopf-1-history-formulation-testing.pdf.

[16] The European Commission. COMMISSION REGULATION (EU) 2015/1222 of 24 july 2015 establishing a guideline on capacity allocation and congestion management, 2015. URL http://eur-lex.europa.eu/legal-content/EN/TXT/?uri=CELEX:32015R1222.

[17] Augustin Cournot. *Recherches sur les principes mathématiques de la théorie des richesses*. L. Hachette, 1838.

[18] Kenneth Van den Bergh, Jonas Boury, and Erik Delarue. The flow-based market coupling in central western europe: Concepts and definitions. *The Electricity Journal*, 29(1):24 – 29, 2016. ISSN 1040-6190. doi: http://dx.doi.org/10.1016/j.tej.2015.12.004. URL http://www.sciencedirect.com/science/article/pii/S104061901530004X.

[19] Jack Bonnel Dennis. *Mathematical Programming and Electrical Networks*. PhD thesis, MIT, 1959.

[20] Valeria Di Cosmo and Muireann Á. Lynch. Competition and the single electricity market: Which lessons for Ireland? *Utilities Policy*, 41:40 – 47, 2016. ISSN 0957-1787. doi: http://dx.doi.org/10.1016/j.jup.2016.05.002. URL http://www.sciencedirect.com/science/article/pii/S0957178716301163.

[21] Rouquia Djabali, Joel Hoeksema, and Yves Langer. Cosmos public description v1.1. 2011. URL http://www.apxendex.com/wp-content/uploads/2012/09/COSMOS_public_description.pdf.

[22] W.S. Dorn. *Duality in quadratic programming*. Courant Institute of Mathematical Sciences, New York University, 1958.

[23] W.S. Dorn. Duality in quadratic programming. *Quarterly of Applied Mathematics*, 18(2):155–162, 1960. ISSN 0033569X, 15524485. URL http://www.jstor.org/stable/43636320.

[24] Jonas Egerer, Jens Weibezahn, and Hauke Hermann. Two price zones for the German electricity market — Market implications and distributional effects. *Energy Economics*, 59:365 – 381, 2016. ISSN 0140-9883. doi: http://dx.doi.org/10.1016/j.eneco.2016.08.002. URL http://www.sciencedirect.com/science/article/pii/S014098831630202X.

[25] Stephen Enke. Equilibrium among spatially separated markets: Solution by electric analogue. *Econometrica*, 19(1):40–47, 1951. ISSN 00129682. URL http://www.jstor.org/stable/1907907.

[26] Euphemia. Euphemia public description v0.6. 2013. URL http://www.belpex.be/wp-content/uploads/Euphemia-public-description-Nov-2013.pdf.

[27] Euphemia. Euphemia public description v1.4. 2016. URL https://www.epexspot.com/document/34460/Euphemia%3A%20Public%20documentation%20-%20January%202016.

[28] Europex. Online publications, 2012. URL http://www.europex.org.

[29] James E. Falk. Lagrange multipliers and nonconvex programs. *SIAM Journal on Control*, 7(4):534–545, 1969. doi: 10.1137/0307039. URL http://dx.doi.org/10.1137/0307039.

[30] R. Fernandez-Blanco, J. M. Arroyo, and N. Alguacil-Conde. On the solution of revenue- and network-constrained day-ahead market clearing under marginal pricing;part i: An exact bilevel programming approach. *IEEE Transactions on Power Systems*, PP(99):1–1, 2016. ISSN 0885-8950. doi: 10.1109/TPWRS.2016.2551046.

[31] R. Fernandez-Blanco, J. M. Arroyo, and N. Alguacil-Conde. On the solution of revenue- and network-constrained day-ahead market clearing under marginal pricing;part ii: Case studies. *IEEE Transactions on Power Systems*, PP(99):1–1, 2016. ISSN 0885-8950. doi: 10.1109/TPWRS.2016.2551078.

[32] Ricardo Fernández-Blanco, José M. Arroyo, and Natalia Alguacil. Bilevel programming for price-based electricity auctions: a revenue-constrained case. *EURO Journal on Computational Optimization*, 3(3):163–195, 2015. ISSN 2192-4414. doi: 10.1007/s13675-015-0037-8. URL http://dx.doi.org/10.1007/s13675-015-0037-8.

[33] Marguerite Frank and Philip Wolfe. An algorithm for quadratic programming. *Naval Research Logistics Quarterly*, 3(1-2):95–110, 1956. ISSN 1931-9193. doi: 10.1002/nav.3800030109. URL http://dx.doi.org/10.1002/nav.3800030109.

[34] Steven A. Gabriel, Antonio J. Conejo, J. David Fuller, Benjamin F. Hobbs, and Carlos Ruiz. *Complementarity Modeling in Energy Markets*. Springer Publishing Company, Incorporated, 2012. ISBN 1441961224, 9781441961228.

[35] Steven A. Gabriel, Antonio J. Conejo, Carlos Ruiz, and Sauleh Siddiqui. Solving discretely constrained, mixed linear complementarity problems with applications in energy. *Computers and Operations Research*, 40(5):1339 – 1350, 2013. ISSN 0305-0548. doi: http://dx.doi.org/10.1016/j.cor.2012.10.017. URL http://www.sciencedirect.com/science/article/pii/S0305054812002298.

[36] Raquel García-Bertrand, AntonioJ. Conejo, and StevenA. Gabriel. Multi-period near-equilibrium in a pool-based electricity market including on/off decisions. *Networks and Spatial Economics*, 5(4):371–393, 2005. ISSN 1566-113X. doi: 10.1007/s11067-005-6209-y. URL http://dx.doi.org/10.1007/s11067-005-6209-y.

[37] Raquel Garcia-Bertrand, Antonio J. Conejo, and Steven Gabriel. Electricity market near-equilibrium under locational marginal pricing and minimum profit conditions. *European Journal of Operational Research*, 174(1):457 – 479, 2006. ISSN 0377-2217. doi: http://dx.doi.org/10.1016/j.ejor.2005.03.037. URL http://www.sciencedirect.com/science/article/pii/S0377221705003814.

[38] A. M. Geoffrion. *Lagrangean relaxation for integer programming*, pages 82–114. Springer Berlin Heidelberg, Berlin, Heidelberg, 1974. ISBN 978-3-642-00740-8. doi: 10.1007/BFb0120690. URL http://dx.doi.org/10.1007/BFb0120690.

[39] A.M. Geoffrion. Duality in nonlinear programming: a simplified applications-oriented development. *SIAM review*, 13(1):1–37, 1971.

[40] P.R. Gribik, W.W. Hogan, and S.L. Pope. Market-clearing electricity prices and energy uplift. Technical report, John F. Kennedy School of Government, Harvard University, 2007. URL https://www.hks.harvard.edu/fs/whogan/Gribik_Hogan_Pope_Price_Uplift_123107.pdf.

[41] L. Hancher, A. de Hauteclocque, and M. Sadowska. *Capacity Mechanisms in the EU Energy Market: Law, Policy, and Economics*. OUP Oxford, 2015. ISBN 9780191066184. URL https://books.google.be/books?id=8GG8CgAAQBAJ.

[42] J. R. Hicks. Annual survey of economic theory: The theory of monopoly. *Econometrica*, 3(1):1–20, 1935. ISSN 00129682, 14680262. URL http://www.jstor.org/stable/1907343.

[43] William W. Hogan. Contract networks for electric power transmission. *Journal of Regulatory Economics*, 4(3):211–242, 1992. ISSN 1573-0468. doi: 10.1007/BF00133621. URL http://dx.doi.org/10.1007/BF00133621.

[44] Bowen Hua and Ross Baldick. A convex primal formulation for convex hull pricing. Technical report, 2016. URL https://arxiv.org/pdf/1605.05002.pdf.

[45] IEA. *Electricity Market Reform - An IEA Handbook*. OECD Publishing, 1999. doi: http://dx.doi.org/10.1787/9789264180987-en. URL /content/book/9789264180987-en.

[46] IEA. *Competition in Electricity Markets*. OECD Publishing, 2001. doi: http://dx.doi.org/10.1787/9789264192768-en. URL /content/book/9789264192768-en.

[47] Paul L Joskow. Incentive regulation and its application to electricity networks. *Review of Network Economics*, 7(4), 2008.

[48] Paul L. Joskow. *Incentive Regulation in Theory and Practice: Electric Transmission and Distribution Networks*, chapter 5. University of Chicago Press, 2014.

[49] D.S. Kirschen and G. Strbac. *Fundamentals of Power System Economics*. Wiley, 2004.

[50] George Liberopoulos and Panagiotis Andrianesis. Critical review of pricing schemes in markets with non-convex costs. *Operations Research*, 64(1):17–31, 2016. doi: 10.1287/opre.2015.1451. URL http://dx.doi.org/10.1287/opre.2015.1451.

[51] Miles Lubin and Iain Dunning. Computing in operations research using julia. *INFORMS Journal on Computing*, 27(2):238–248, 2015. doi: 10.1287/ijoc.2014.0623. URL http://dx.doi.org/10.1287/ijoc.2014.0623.

[52] M. Madani, M. Van Vyve, A. Marien, M. Maenhoudt, P. Luickx, and A. Tirez. Non-convexities in European day-ahead electricity markets: Belgium as a case study. In *2016 13th International Conference on the European Energy Market (EEM)*, pages 1–5, June 2016. doi: 10.1109/EEM.2016.7521190.

[53] Mehdi Madani and Mathieu Van Vyve. Minimizing opportunity costs of paradoxically rejected block orders in European day-ahead electricity markets. In *European Energy Market (EEM), 2014 11th International Conference on the*, pages 1–5, May 2014. doi: 10.1109/EEM.2014.6861237.

[54] Mehdi Madani and Mathieu Van Vyve. Computationally efficient MIP formulation and algorithms for European day-ahead electricity market auctions. *European Journal of Operational Research*, 242(2):580 – 593, 2015. ISSN 0377-2217. doi: http://dx.doi.org/10.1016/j.ejor.2014.09.060. URL http://www.sciencedirect.com/science/article/pii/S0377221714007991.

[55] Mehdi Madani and Mathieu Van Vyve. A MIP framework for non-convex uniform price day-ahead electricity auctions. *EURO Journal on Computational Optimization*, pages 1–22, 2015. ISSN 2192-4406. doi: 10.1007/s13675-015-0047-6. URL http://dx.doi.org/10.1007/s13675-015-0047-6.

[56] Mehdi Madani and Mathieu Van Vyve. Source code and sample datasets for 'revisiting minimum profit conditions in uniform price day-ahead electricity auctions', 2016. URL http://github.com/madanim/revisiting_mp_conditions/.

[57] Marcelino Madrigal Martinez. *Optimization models and techniques for implementation and pricing of electricity markets*. PhD thesis, University of Waterloo, 2000.

[58] P. Mäntysaari. *EU Electricity Trade Law: The Legal Tools of Electricity Producers in the Internal Electricity Market*. Springer International Publishing, 2015. ISBN 9783319165127.

[59] Nestor Sanchez Maria. Day-ahead electricity market, proposals to adapt complex conditions in OMEL. Master's thesis, ESCUELA TÉCNICA SUPERIOR DE INGENIERÍA (ICAI), UNIVERSIDAD PONTIFICIA COMILLAS, 2010.

[60] Alexander Martin, Johannes C. Müller, and Sebastian Pokutta. Strict linear prices in non-convex European day-ahead electricity markets. *Optimization Methods and Software*, 0(0):1–33, 0. doi: 10.1080/10556788.2013.823544. URL http://www.tandfonline.com/doi/abs/10.1080/10556788.2013.823544.

[61] Andreu Mas-Colell, Michael D. Whinston, and Jerry R. Green. *Microeconomic Theory*. Oxford University Press, 1995.

[62] Leonardo Meeus. *Power Exchange Auction Trading Platform Design*. PhD thesis, Katholieke Universiteit Leuven, July 2006.

[63] Leonardo Meeus, Karolien Verhaegen, and Ronnie Belmans. Block order restrictions in combinatorial electric energy auctions. *European Journal of Operational Research*, 196(3):1202 – 1206, 2009. ISSN 0377-2217. doi: http://dx.doi.org/10.1016/j.ejor.2008.04.031. URL http://www.sciencedirect.com/science/article/pii/S0377221708003901.

[64] Johannes Müller. *Auctions in Exchange Trading Systems: Modeling Techniques and Algorithms*. PhD thesis, University of Erlangen-Nürnberg, 2014. URL https://opus4.kobv.de/opus4-fau/files/5339/mueller2014+-+Auctions+in+Exchange+Trading+Systems.pdf.

[65] Johannes C Müller. Competitive equilibrium relaxations in general auctions. *arXiv preprint arXiv:1307.2852*, 2013. URL https://arxiv.org/pdf/1307.2852v2.

[66] David M. Newbery. Problems of liberalising the electricity industry. *European Economic Review*, 46(4–5):919 – 927, 2002. ISSN 0014-2921. doi: http://dx.doi.org/10.1016/S0014-2921(01)00225-2. URL http://www.sciencedirect.com/science/article/pii/S0014292101002252.

[67] Fernando S. Oliveira, Carlos Ruiz, and Antonio J. Conejo. Contract design and supply chain coordination in the electricity industry. *European Journal of Operational Research*, 227(3):527 – 537, 2013. ISSN 0377-2217. doi: http://dx.doi.org/10.1016/j.ejor.2013.01.003. URL http://www.sciencedirect.com/science/article/pii/S037722171300012X.

[68] OMIE. Publicly available day-ahead and intraday market data, 2016. URL http://www.omie.es/aplicaciones/datosftp/datosftp.jsp.

[69] Richard O'Neill, David Mead, and Partha Malvadkar. On market clearing prices higher than the highest bid and other almost paranormal phenomena. *The Electricity Journal*, 18(2):19 – 27, 2005. ISSN 1040-6190. doi: http://dx.doi.org/10.1016/j.tej.2005.02.004. URL http://www.sciencedirect.com/science/article/pii/S1040619005000114.

[70] Richard P O'Neill, Paul M Sotkiewicz, and Michael H Rothkopf. Equilibrium prices in power exchanges with non-convex bids. Technical report, Working Paper, revised July, 2007. URL http://bear.warrington.ufl.edu/centers/purc/docs/papers/0601_ONeill_Equilibrium_Prices_in.pdf.

[71] R.P. O'Neill, P.M. Sotkiewicz, B.F. Hobbs, M.H. Rothkopf, and W.R. Stewart. Efficient market-clearing prices in markets with nonconvexities. *European Journal of Operational Research*, 164(1):269–285, 2005.

[72] C.H. Papadimitriou and K. Steiglitz. *Combinatorial Optimization: Algorithms and Complexity*. Dover Books on Computer Science Series. Dover Publications, 1998. ISBN 9780486402581.

[73] Anthony Papavasiliou. *Coupling renewable energy supply with deferrable demand*. PhD thesis, University of California, Berkeley, 2011.

[74] Anthony Papavasiliou and Shmuel S Oren. Multiarea stochastic unit commitment for high wind penetration in a transmission constrained network. *Operations Research*, 61(3):578–592, 2013.

[75] Anthony Papavasiliou and Shmuel S Oren. Large-scale integration of deferrable demand and renewable energy sources. *IEEE Transactions on Power Systems*, 29 (1):489–499, 2014.

[76] G. Plancke, C. De Jonghe, and R. Belmans. The implications of two German price zones in a European-wide context. In *2016 13th International Conference on the European Energy Market (EEM)*, pages 1–5, June 2016. doi: 10.1109/EEM.2016.7521290.

[77] F. P. Ramsey. A contribution to the theory of taxation. *The Economic Journal*, 37(145):47–61, 1927. ISSN 00130133, 14680297. URL `http://www.jstor.org/stable/2222721`.

[78] B. Ring and W.W. Hogan. On minimum-uplift pricing for electricity markets. Technical report, John F. Kennedy School of Government, Harvard University, 2003. URL `https://www.hks.harvard.edu/fs/whogan/minuplift_031903.pdf`.

[79] B. J. Ring. *Dispatch based pricing in decentralised power systems*. PhD thesis, University Of Canterbury, 1995.

[80] Fred C. Schweppe Roger E. Bohn, Michael C. Caramanis. Optimal pricing in electrical networks over space and time. *The RAND Journal of Economics*, 15(3):360–376, 1984. ISSN 07416261. URL `http://www.jstor.org/stable/2555444`.

[81] M.M. Roggenkamp and F. Boisseleau. *The Regulation of Power Exchanges in Europe*. Energy & law. Intersentia, 2005. ISBN 9789050953177.

[82] J. Rosellón and T. Kristiansen. *Financial Transmission Rights: Analysis, Experiences and Prospects*. Lecture Notes in Energy. Springer London, 2013. ISBN 9781447147879.

[83] C. Ruiz and A.J. Conejo. Pool strategy of a producer with endogenous formation of locational marginal prices. *Power Systems, IEEE Transactions on*, 24(4):1855–1866, Nov 2009. ISSN 0885-8950. doi: 10.1109/TPWRS.2009.2030378.

[84] C. Ruiz, A.J. Conejo, and S.A. Gabriel. Pricing non-convexities in an electricity pool. *Power Systems, IEEE Transactions on*, 27(3):1334–1342, 2012. ISSN 0885-8950. doi: 10.1109/TPWRS.2012.2184562.

[85] Paul A. Samuelson. Spatial price equilibrium and linear programming. *The American Economic Review*, 42(3):pp. 283–303, 1952. URL `http://www.jstor.org/stable/1810381`.

[86] D. A. Schiro, T. Zheng, F. Zhao, and E. Litvinov. Convex hull pricing in electricity markets: Formulation, analysis, and implementation challenges. *IEEE Transactions on Power Systems*, 31(5):4068–4075, Sept 2016. ISSN 0885-8950. doi: 10.1109/TPWRS.2015.2486380.

[87] Alexander Schrijver. *Theory of Linear and Integer Programming*. John Wiley & Sons, Chichester, 1986.

[88] G.B. Sheblé. *Computational Auction Mechanisms for Restructured Power Industry Operation*. Power Electronics and Power Systems. Springer US, 2012. ISBN 9781461551577.

[89] Hanif D. Sherali. Dorn's duality for quadratic programs revisited: The nonconvex case. *European Journal of Operational Research*, 65(3):417 – 424, 1993. ISSN 0377-2217. doi: http://dx.doi.org/10.1016/0377-2217(93)90121-3. URL `http://www.sciencedirect.com/science/article/pii/0377221793901213`.

[90] Ramteen Sioshansi. Pricing in centrally committed electricity markets. *Utilities Policy*, 31:143 – 145, 2014. ISSN 0957-1787. doi: http://dx.doi.org/10.1016/j.jup.2014.09.009. URL `http://www.sciencedirect.com/science/article/pii/S0957178714000708`.

[91] J. Stoer and C. Witzgall. *Convexity and optimization in finite dimensions*. Number vol. 1 in Grundlehren der mathematischen Wissenschaften; Bd. 163. Springer-Verlag, 1970.

[92] S. Stoft. *Power System Economics: Designing Markets for Electricity*. IEEE Press. IEEE Press, 2002. ISBN 9780471150404.

[93] G. G. Judge T. Takayama. Spatial equilibrium and quadratic programming. *Journal of Farm Economics*, 46(1):67–93, 1964. ISSN 10711031. URL `http://www.jstor.org/stable/1236473`.

[94] Milad Tahanan, Wim van Ackooij, Antonio Frangioni, and Fabrizio Lacalandra. Large-scale unit commitment under uncertainty. *4OR*, 13(2):115–171, 2015.

[95] T. Takayama and G.G. Judge. *Spatial and temporal price and allocation models*. Contributions to economic analysis. North-Holland Pub. Co., 1971. ISBN 9780720431735.

[96] Fehmi Tanrisever, Kursad Derinkuyu, and Geert Jongen. Organization and functioning of liberalized electricity markets: An overview of the Dutch market. *Renewable and Sustainable Energy Reviews*, 51:1363 – 1374, 2015. ISSN 1364-0321. doi:

http://dx.doi.org/10.1016/j.rser.2015.07.019. URL http://www.sciencedirect.com/science/article/pii/S1364032115006668.

[97] Eugeniusz Toczyłowski and Izabela Zoltowska. A new pricing scheme for a multi-period pool-based electricity auction. *European Journal of Operational Research*, 197(3):1051 – 1062, 2009. ISSN 0377-2217. doi: http://dx.doi.org/10.1016/j.ejor.2007.12.048. URL http://www.sciencedirect.com/science/article/pii/S0377221708002762.

[98] S. Vajda. *Mathematical Programming*. Dover Books on Computer Science Series. Dover Publications, 2009. ISBN 9780486472133.

[99] M. Van Vyve. Linear prices for non-convex electricity markets: models and algorithms. Technical report, Université catholique de Louvain, Center for Operations Research and Econometrics (CORE), 2011.

[100] C. Wang, T. Peng, P. B. Luh, P. Gribik, and L. Zhang. The subgradient simplex cutting plane method for extended locational marginal prices. *IEEE Transactions on Power Systems*, 28(3):2758–2767, Aug 2013. ISSN 0885-8950. doi: 10.1109/TPWRS.2013.2243173.

[101] Robert Wilson. Architecture of power markets. *Econometrica*, 70(4):1299–1340, 2002. ISSN 00129682, 14680262. URL http://www.jstor.org/stable/3082000.

[102] Philip Wolfe. The simplex method for quadratic programming. *Econometrica*, 27(3):382–398, 1959. ISSN 00129682, 14680262. URL http://www.jstor.org/stable/1909468.

[103] Felix Wu, Pravin Varaiya, Pablo Spiller, and Shmuel Oren. Folk theorems on transmission access: Proofs and counterexamples. *Journal of Regulatory Economics*, 10(1):5–23, 1996. ISSN 1573-0468. doi: 10.1007/BF00133356. URL http://dx.doi.org/10.1007/BF00133356.

[104] E.J. Zak, S. Ammari, and K.W. Cheung. Modeling price-based decisions in advanced electricity markets. In *European Energy Market (EEM), 2012 9th International Conference on the*, pages 1–6, May 2012. doi: 10.1109/EEM.2012.6254813.

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Revisiting European day-ahead electricity market auctions: MIP models and algorithms

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In Europe, orders are submitted to power exchanges integrated under the Price Coupling of Region project, to sell or buy substantially large amounts of electricity for the next day. The orders involved render the underlying microeconomic optimization problem “non-convex”, departing from more classical assumptions in microeconomic theory. Uniform prices are computed, in the sense that every market participant in a given location and hour of the day will pay or receive the same electricity price and no other side payments are considered. This is done at the expense of having some bids "paradoxically rejected" at the computed market prices, as some bids may propose a price which is "good enough" but are yet rejected. It is also at the expense of welfare optimality, as most of the time, no welfare optimal solution can be supported by uniform prices such that no financial losses are incurred. The present thesis proposes mixed integer programming models and algorithms for such non-convex uniform price auctions. In particular, a new bidding product is proposed which generalizes both block orders used in the Central Western Europe Region or Northern countries, and, mutatis mutandis, complex orders with a minimum income condition used in Spain and Portugal.

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