

Gaussian perturbations of hard edge random matrix ensembles

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Abstract

We study the eigenvalue correlations of random Hermitian $n \times n$ matrices of the form $S = M + \epsilon H$, where H is a GUE matrix, $\epsilon > 0$, and M is a positive-definite Hermitian random matrix, independent of H , whose eigenvalue density is a polynomial ensemble. We show that there is a soft-to-hard edge transition in the microscopic behaviour of the eigenvalues of S close to 0 if ϵ tends to 0 together with $n \rightarrow +\infty$ at a critical speed, depending on the random matrix M . In a double scaling limit, we obtain a new family of limiting eigenvalue correlation kernels. We apply our general results to the cases where (i) M is a Laguerre/Wishart random matrix, (ii) $M = G^*G$ with G a product of Ginibre matrices, (iii) $M = T^*T$ with T a product of truncations of Haar distributed unitary matrices, and (iv) the eigenvalues of M follow a Muttalib-Borodin biorthogonal ensemble.

1 Introduction

We consider a class of Hermitian random matrices which are perturbed by additive Gaussian noise, and investigate to what extent the microscopic behaviour of the eigenvalues is affected by such a perturbation, in the limit where the size of the matrices tends to infinity. We take the Gaussian noise to be a small multiple of a matrix from the Gaussian Unitary Ensemble (GUE), which consists of Hermitian $n \times n$ matrices H with the probability distribution

$$\frac{1}{Z_n^{\text{GUE}}} e^{-\frac{n}{2} \text{Tr}(H^2)} dH, \quad dH = \prod_{j=1}^n dH_{jj} \prod_{1 \leq i < j \leq n} d\text{Re}H_{ij} d\text{Im}H_{ij}. \quad (1.1)$$

Equivalently, the diagonal entries of H are independent identically distributed (iid) normal random variables $\mathcal{N}(0, 1/n)$ with mean 0 and variance $1/n$ and the upper triangular entries

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are iid complex normal random variables $\mathcal{N}\left(0, \frac{1}{2n}\right) + i\mathcal{N}\left(0, \frac{1}{2n}\right)$. Adding a small multiple of a GUE matrix, ϵH , to another random matrix M can thus be viewed as an entry-wise Gaussian perturbation of M . The above normalization of a GUE matrix is such that the large n limit of the mean distribution λ of eigenvalues exists and is given by the Wigner semi-circle law

$$d\lambda(x) = \frac{1}{2\pi} \sqrt{4 - x^2} dx, \quad x \in [-2, 2]. \quad (1.2)$$

Gaussian perturbations of random matrices are closely related to random matrices with external source. To see this, we note that, given M , the probability distribution of the random matrix $S = M + \epsilon H$ can be written as

$$\frac{1}{Z_n} e^{-\frac{n}{2\epsilon^2} \text{Tr}((S-M)^2)} dS, \quad dS = \prod_{j=1}^n dS_{jj} \prod_{1 \leq i < j \leq n} d\text{Re} S_{ij} d\text{Im} S_{ij}.$$

This is known as the GUE with external source M [12, 40]. In our setting, the external source matrix M is not deterministic but is itself a random matrix.

If M is a unitary invariant random matrix with probability measure

$$\frac{1}{Z_n} e^{-n \text{Tr} V(M)} dM, \quad dM = \prod_{j=1}^n dM_{jj} \prod_{1 \leq i < j \leq n} d\text{Re} M_{ij} d\text{Im} M_{ij},$$

for some potential V , then our model is equivalent to a special case of the Itzykson-Zuber two-matrix model [17, 8], which is defined as a probability measure on pairs of Hermitian matrices (M_1, M_2) , given by

$$\frac{1}{Z_n} e^{-n \text{Tr}(V_1(M_1) + V_2(M_2) - \tau M_1 M_2)} dM_1 dM_2,$$

for certain potentials V_1, V_2 . If we take

$$V_2(x) = x^2/2, \quad V_1(x) = V(x) + \frac{\tau^2}{2} x^2, \quad \tau = 1/\epsilon,$$

then it is straightforward to verify that M_1 and $M_2 - \tau M_1$ are independent, that M_1 has the same distribution as M , and that $\frac{1}{\tau} M_2$ has the same distribution as the sum $S = M + \epsilon H$, see also [16, Section 5].

The eigenvalues of Gaussian perturbations of (deterministic or random) matrices can alternatively be realized as the positions of n non-intersecting Brownian paths with a common endpoint and with (deterministic or random) starting points, see e.g. [24] and the recent work [20].

In what follows, the random Hermitian $n \times n$ matrix M has to be independent of the GUE matrix H and such that the joint probability density function of its eigenvalues is of the form

$$\frac{1}{Z_n} \Delta(x) \det [f_{k-1}(x_j)]_{j,k=1}^n, \quad \Delta(x) = \prod_{1 \leq j < k \leq n} (x_k - x_j), \quad x_1, \dots, x_n \in \mathbb{R}, \quad (1.3)$$

for certain functions $f_0, f_1, \dots, f_{n-1}$, and where Z_n is a normalizing constant. A density function on $\mathbb{R}^n$ of this form is called a polynomial ensemble [26]. For instance, the eigenvalues of unitary invariant random matrix ensembles and of certain products and sums of random matrices follow polynomial ensembles [14, 25]. Polynomial ensembles are special cases of determinantal point processes, their correlation kernel K_n taking the special form

$$K_n(x, y) = \sum_{j=0}^{n-1} p_j(x) q_j(y), \quad (1.4)$$

where p_j is a polynomial of degree j , and q_j is a linear combination of $f_0, \dots, f_{n-1}$, such that the orthogonality conditions

$$\int_{\mathbb{R}} p_j(x) q_k(x) dx = \delta_{jk}, \quad j, k = 0, \dots, n-1, \quad (1.5)$$

are satisfied. Later on, we will focus on polynomial ensembles defined by functions f_j supported on $[0, +\infty)$, but for now, they can be general.

If the joint eigenvalue density of a random matrix M is a polynomial ensemble with correlation kernel K_n , then it was shown in [14] that the eigenvalues of $S = M + \epsilon H$, with H a GUE matrix independent of M and $\epsilon > 0$, also follow a polynomial ensemble, with the transformed eigenvalue correlation kernel

$$K_n^S(x, y) = \frac{n}{2\pi i \epsilon^2} \int_{i\mathbb{R}} \int_{\mathbb{R}} K_n(s, t) e^{\frac{n}{2\epsilon^2}((x-s)^2 - (y-t)^2)} dt ds. \quad (1.6)$$

In addition, if $p_n(x) = \mathbb{E}[\det(xI - M)]$ is the average characteristic polynomial of M , then the average characteristic polynomial of $S = M + \epsilon H$ is given by

$$P_n(x) = \mathbb{E}[\det(xI - S)] = \frac{\sqrt{n}}{\sqrt{2\pi i \epsilon}} \int_{i\mathbb{R}} p_n(s) e^{\frac{n}{2\epsilon^2}(x-s)^2} ds. \quad (1.7)$$

The formulas (1.6) and (1.7) follow from [14, formulas (2.6) and (2.8)] after a simple re-scaling argument, and they will be the starting point of our analysis.

Macroscopic eigenvalue behaviour

The macroscopic large n behaviour of the eigenvalues of $M + \epsilon H$ is well understood thanks to free probability theory: if $M = M_n$ is a sequence of random $n \times n$ matrices whose eigenvalue distributions converge almost surely to a measure μ and if M is independent of the GUE matrix $H = H_n$, then M and H are asymptotically free and we can apply results from free probability theory [31, 36] to describe the limiting eigenvalue distribution of $S = M + \epsilon H$. Writing λ_ϵ for the rescaled semi-circle law,

$$d\lambda_\epsilon(x) = \frac{1}{2\pi\epsilon^2} \sqrt{4\epsilon^2 - x^2} dx, \quad x \in [-2\epsilon, 2\epsilon], \quad (1.8)$$

which is the limiting macroscopic density of the eigenvalues of ϵH , it is well-known that the limiting eigenvalue distribution of S is almost surely given by the free additive convolution $\mu \boxplus \lambda_\epsilon$ of μ and λ_ϵ , see [10] for the definition and properties of the free convolution of a measure μ with λ_ϵ .

Another quantity containing global information about random matrix eigenvalues is the zero counting measure of the average characteristic polynomial. The zeros of the average characteristic polynomial are real and simple (see Lemma 3.1), and the zero counting measure can heuristically be interpreted as a typical eigenvalue configuration. It can therefore be expected that its large n limit coincides with the limiting (mean) eigenvalue distribution in many cases. This is well-known for classical random matrix ensembles and was investigated in a more general framework in [23]. The following result about convergence of the zero counting measure of the average characteristic polynomial of S is not surprising in view of the above-mentioned results from free probability. We will prove it in Section 3 directly using the integral representation (1.7) and without relying on the more sophisticated results from free probability theory.

Theorem 1.1. *Let M be an $n \times n$ Hermitian random matrix such that its eigenvalue density is a polynomial ensemble (1.3), let H be an $n \times n$ GUE matrix independent of M , and let $\epsilon > 0$. Write μ_n for the zero counting measure of the average characteristic polynomial of M , and ν_n for the zero counting measure of the average characteristic polynomial of $S = M + \epsilon H$.*

If, for sufficiently large n , the support of μ_n is contained in some n -independent compact K , and if μ_n converges weakly to a probability measure μ , then ν_n converges weakly to $\mu \boxplus \lambda_\epsilon$, where λ_ϵ is given by (1.8).

Microscopic eigenvalue behaviour

From now on, we consider polynomial ensembles supported on $[0, +\infty)$ or on an interval of the form $[0, b]$. We mean by this that the functions f_j in (1.3) are supported on $[0, +\infty)$ or on $[0, b]$. Ensembles of this kind are said to have a *hard edge* at zero. Classical examples of random matrix ensembles with a hard edge are the Laguerre Unitary Ensemble and the Jacobi Unitary Ensemble. In these ensembles, the microscopic eigenvalue correlations near 0 are described in terms of Bessel functions. As we will see below, other ensembles may lead to other types of microscopic eigenvalue correlations, described in terms of other functions, such as Meijer G-functions or Wright's generalized Bessel functions. A common feature of all hard edge random matrix ensembles which we will study below, is the existence of a scaling limit near the hard edge of the following form:

$$\lim_{n \rightarrow +\infty} \frac{1}{cn^\gamma} K_n \left(\frac{u}{cn^\gamma}, \frac{v}{cn^\gamma} \right) = \mathbb{K}(u, v), \quad v > 0, u \in \mathbb{C}, \quad (1.9)$$

for some values of $c, \gamma > 0$, and for some limiting kernel $\mathbb{K}(u, v)$, which depend on the particular choice of random matrix ensemble.

If we consider a Gaussian perturbation of M of the form $S = M + \epsilon H$, even if $\epsilon > 0$ is small, the matrix S is typically not positive-definite, in other words the hard edge at 0 is removed by the perturbation. It is our aim to understand how scaling limits of the eigenvalue correlation kernel near 0 of the form (1.9) change after the Gaussian perturbation. In particular, we want to see what happens in double scaling limits where the constant ϵ goes to 0 as n goes to infinity, as this is the limit in which the soft edge of the spectrum (which we have for fixed $\epsilon > 0$) turns into a hard edge at the origin, as can be seen in Figures 1, 3, 4, and 5. We now present a general auxiliary result, which we will apply to several concrete examples later on. Given a scaling limit of the form (1.9), it states that the scaling limit is preserved for the eigenvalue correlation kernel of S provided that $\epsilon \rightarrow 0$ sufficiently fast with $n \rightarrow \infty$. If $\epsilon \rightarrow 0$ at a critical speed, the limiting kernel $\mathbb{K}$ is deformed.

Lemma 1.2. *Consider a sequence of $n \times n$ random matrices M such that their eigenvalue densities are polynomial ensembles on $[0, +\infty)$ or on $[0, b]$. We assume there exist constants $\gamma > 1/2$, $c, c_1, c_2, n_0 > 0$ and $\beta \in [0, 1)$ such that the associated correlation kernels K_n satisfy the following conditions:*

1. *there exists a function $\mathbb{K}(u, v)$ such that*

$$\lim_{n \rightarrow +\infty} v^\beta \frac{1}{cn^\gamma} K_n \left(\frac{u}{cn^\gamma}, \frac{v}{cn^\gamma} \right) = v^\beta \mathbb{K}(u, v), \quad (1.10)$$

uniformly for u in any compact subset of $\mathbb{C}$ and v in any compact subset of $[0, +\infty)$. Here $v^\beta \mathbb{K}(u, v)$ and $v^\beta K_n \left(\frac{u}{cn^\gamma}, \frac{v}{cn^\gamma} \right)$ for $v = 0$ have to be understood as the limits as $v \rightarrow 0$ and it is assumed that these limits exist,

2. *for every $(u, v) \in i\mathbb{R} \times [0, +\infty)$ and $n > n_0$,*

$$|K_n(u, v)| \leq c_1 v^{-\beta} n^{\gamma(1-\beta)} e^{c_2 n^\gamma (|u|+|v|)}. \quad (1.11)$$

Let $S = M + \epsilon_n H$ where H is a GUE matrix independent of M , and let K_n^S be the eigenvalue correlation kernel (1.6) for S . Then,

1. *if ϵ_n is such that $\lim_{n \rightarrow +\infty} \epsilon_n n^{\gamma-\frac{1}{2}} = 0$, we have*

$$\lim_{n \rightarrow +\infty} \frac{1}{cn^\gamma} K_n^S \left(\frac{x}{cn^\gamma}, \frac{y}{cn^\gamma} \right) = \mathbb{K}(x, y), \quad x, y > 0, \quad (1.12)$$

2. *if ϵ_n is such that $\lim_{n \rightarrow +\infty} c\epsilon_n n^{\gamma-\frac{1}{2}} = \sigma > 0$, we have*

$$\lim_{n \rightarrow +\infty} \frac{1}{cn^\gamma} K_n^S \left(\frac{x}{cn^\gamma}, \frac{y}{cn^\gamma} \right) = \frac{1}{2\pi i \sigma^2} \int_{i\mathbb{R}} \int_{\mathbb{R}^+} \mathbb{K}(s, t) e^{\frac{1}{2\sigma^2}((x-s)^2 - (y-t)^2)} dt ds, \quad (1.13)$$

uniformly for (x, y) in compact subsets of $\mathbb{C}^2$.

Remark 1. The rate of decay $\mathcal{O}(n^{\frac{1}{2}-\gamma})$ for ϵ_n appears as a critical speed at which the local eigenvalue behaviour changes. When ϵ_n goes to 0 faster than the critical speed, the eigenvalues of the perturbed random matrix $M + \epsilon_n H$ behave locally near 0 as if there were no perturbation. At the critical speed, a new limiting kernel appears at 0, given by (1.13). By a saddle point approximation, it is easy to verify that

$$\lim_{\sigma \rightarrow 0} \frac{1}{2\pi i \sigma^2} \int_{i\mathbb{R}} \int_{\mathbb{R}^+} \mathbb{K}(s, t) e^{\frac{1}{2\sigma^2}((x-s)^2 - (y-t)^2)} dt ds = \mathbb{K}(x, y) \quad (1.14)$$

(the exponential part of the integrand may indeed be seen a sequence converging to the Dirac δ -function), which means that (1.12) and (1.13) are consistent.

Remark 2. Conditions 1 and 2 in the above lemma are designed in such a way that they hold for a large class of random matrix ensembles. In some cases, we can just take $\beta = 0$. However, it may happen that the functions $f_j(x)$ defining the polynomial ensemble (1.3) blow up as $x \rightarrow 0$. This implies that the kernel $K_n(x, y)$ blows up as $y \rightarrow 0$, and thus one cannot expect (1.10) and (1.11) to hold for $\beta = 0$. This is why we allow $\beta \in [0, 1)$.

In the next section, we discuss several concrete examples of random matrix ensembles to which we can apply Lemma 1.2.

2 Examples

2.1 Perturbed Laguerre/Wishart random matrices

We define the generalized Laguerre Unitary Ensemble (LUE) as the set of $n \times n$ positive-definite Hermitian matrices equipped with the probability measure

$$\frac{1}{Z_{n,\alpha,k}^{\text{LUE}}} (\det M)^\alpha e^{-n \text{Tr}(M^k)} dM, \quad dM = \prod_{j=1}^n dM_{jj} \prod_{1 \leq i < j \leq n} d\text{Re} M_{ij} d\text{Im} M_{ij}, \quad \alpha > -1, \quad k \in \mathbb{N}. \quad (2.1)$$

Similarly as for the GUE, the factor n in the exponential ensures the eigenvalues to remain bounded as $n \rightarrow +\infty$ with probability 1. For $\alpha \in \mathbb{N}$ and $k = 1$, a random LUE matrix can be realized as $M = G^* G$, where G is a $(n + \alpha) \times n$ complex Ginibre matrix, which has independent identically distributed complex normal entries $\mathcal{N}(0, \frac{1}{2n}) + i\mathcal{N}(0, \frac{1}{2n})$. In Figure 1, we present numerical samples of the perturbed LUE for different values of ϵ .

The eigenvalues of a random matrix M with probability distribution (2.1) have the joint probability distribution

$$\frac{1}{Z_{n,\alpha,k}^{\text{LUE}}} \Delta(x)^2 \prod_{j=1}^n x_j^\alpha e^{-n x_j^k} dx_j, \quad x_1, \dots, x_n > 0, \quad (2.2)$$

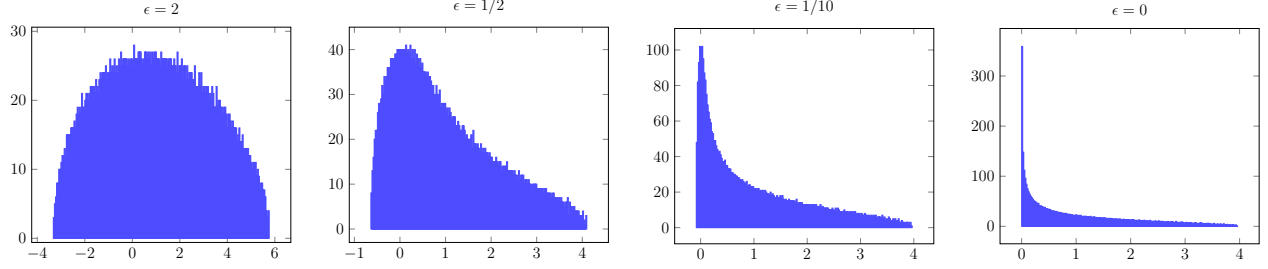


Figure 1: Numerical samples of the eigenvalues of $M + \epsilon H$, with M a 4000×4000 matrix drawn from the LUE (with $\alpha = 0$ and $k = 1$), and H a 4000×4000 matrix drawn from the GUE. The chosen values for ϵ are 2, $1/2$, $1/10$ and 0. The case $\epsilon = 0$ is the unperturbed LUE. The eigenvalues are represented in histograms of 200 intervals.

which is a polynomial ensemble (1.3) with $f_j(x) = x^{j+\alpha}e^{-nx^k}$ on $\mathbb{R}^+$. The limiting eigenvalue distribution μ in this ensemble is almost surely given by a (generalized) Marchenko-Pastur law of the form

$$d\mu(x) = \frac{1}{2\pi} \sqrt{\frac{b-x}{x}} h(x) dx, \quad x \in (0, b), \quad (2.3)$$

for some n -independent $b > 0$ and polynomial h , positive on $[0, b]$. We denote

$$G_\mu(z) = \int \frac{d\mu(x)}{z-x}, \quad z \in \mathbb{C} \setminus [0, b] \quad (2.4)$$

for the Stieltjes transform of μ . The limiting eigenvalue distribution of a Gaussian perturbation $S = M + \epsilon H$ is almost surely the free convolution $\mu \boxplus \lambda_\epsilon$. Its density can be shown to have the form [33]

$$\frac{d(\mu \boxplus \lambda_\epsilon)(x)}{dx} = h_\epsilon(x) \sqrt{(x-a_\epsilon)(b_\epsilon-x)}, \quad x \in [a_\epsilon, b_\epsilon], \quad (2.5)$$

with h_ϵ positive on $[a_\epsilon, b_\epsilon]$. The density vanishes like a square root at both edges for any $\epsilon > 0$, whereas the density of μ blows up at the left edge like an inverse square root.

In [37], large n asymptotics for the eigenvalue correlation kernel $K_n(x, y)$ of M have been obtained using the Deift/Zhou steepest descent method [15] applied to the Riemann-Hilbert problem for generalized Laguerre polynomials. In particular, by [37, Theorem 2.10 (a)], we have

$$\lim_{n \rightarrow +\infty} \frac{1}{cn^2} K_n\left(\frac{x}{cn^2}, \frac{y}{cn^2}\right) = \mathbb{K}_\alpha^{\text{Bessel}}(x, y), \quad (2.6)$$

uniformly for x, y in compact subsets of $(0, +\infty)$, with $c = bh(0)^2$. The limiting kernel $\mathbb{K}_\alpha^{\text{Bessel}}$ is expressed in terms of Bessel functions of the first kind J_α and takes the explicit form

$$\mathbb{K}_\alpha^{\text{Bessel}}(x, y) = x^{-\frac{\alpha}{2}} y^{\frac{\alpha}{2}} \frac{J_\alpha(\sqrt{x})\sqrt{y}J'_\alpha(\sqrt{y}) - J_\alpha(\sqrt{y})\sqrt{x}J'_\alpha(\sqrt{x})}{2(x-y)}. \quad (2.7)$$

The Bessel kernel is usually defined without the factor $x^{-\frac{\alpha}{2}}y^{\frac{\alpha}{2}}$ in front, such that it is symmetric in x and y . The pre-factor is present in our situation because the polynomial ensemble kernel is not symmetric, but it has no effect on the determinants defining the correlation functions associated to the Bessel kernel. Using Lemma 1.2 and results from [37], we will prove the following result.

Theorem 2.1. *Let M be an $n \times n$ random matrix with probability measure (2.1), and let H be an $n \times n$ GUE matrix with probability measure (1.1), independent of M . Write K_n^S for the eigenvalue correlation kernel of $S = M + \epsilon_n H$.*

(i) **(Sub-critical perturbation)** *If $\lim_{n \rightarrow +\infty} \epsilon_n n^{\frac{3}{2}} = 0$, then for $x, y > 0$, we have*

$$\lim_{n \rightarrow +\infty} \frac{1}{cn^2} K_n^S \left(\frac{x}{cn^2}, \frac{y}{cn^2} \right) = \mathbb{K}_\alpha^{\text{Bessel}}(x, y), \quad (2.8)$$

with $c = bh(0)^2$, where h and b are defined by (2.3).

(ii) **(Critical perturbation)** *If $\lim_{n \rightarrow +\infty} c\epsilon_n n^{\frac{3}{2}} = \sigma > 0$, then for $x, y \in \mathbb{C}$, we have*

$$\lim_{n \rightarrow +\infty} \frac{1}{cn^2} K_n^S \left(\frac{x}{cn^2}, \frac{y}{cn^2} \right) = \frac{1}{2\pi i \sigma^2} \int_{i\mathbb{R}} \int_{\mathbb{R}^+} \mathbb{K}_\alpha^{\text{Bessel}}(s, t) e^{\frac{1}{2\sigma^2}((s-x)^2 - (t-y)^2)} dt ds, \quad (2.9)$$

and the limit holds uniformly for (x, y) in compact subsets of $\mathbb{C}^2$.

(iii) **(Super-critical perturbation)** *If $\epsilon_n \rightarrow 0$ in such a way that $\epsilon_n n^{\frac{3}{2}} \rightarrow +\infty$ as $n \rightarrow +\infty$, then for $x, y \in \mathbb{C}$, we have*

$$\lim_{n \rightarrow +\infty} e^{-\varphi_{\epsilon_n, n}(x) + \varphi_{\epsilon_n, n}(y)} \frac{1}{c_{\epsilon_n} n^{\frac{2}{3}}} K_n^S \left(a_{\epsilon_n} - \frac{x}{c_{\epsilon_n} n^{\frac{2}{3}}}, a_{\epsilon_n} - \frac{y}{c_{\epsilon_n} n^{\frac{2}{3}}} \right) = \mathbb{K}^{\text{Ai}}(x, y), \quad (2.10)$$

with a_ϵ as in (2.5), $c_\epsilon = \epsilon^{-2} 6^{1/3} G_\mu''(u_\epsilon)^{-1/3}$, where u_ϵ is the unique negative solution of the equation

$$1 + \epsilon^2 G_\mu'(u) = 0, \quad (2.11)$$

with G_μ the Stieltjes transform defined in (2.4), and with

$$\varphi_{\epsilon, n}(z) = \frac{n^{1/3} z}{c_\epsilon \epsilon^2} (u_\epsilon - a_\epsilon).$$

The limiting kernel $\mathbb{K}^{\text{Ai}}$ is the Airy kernel

$$\mathbb{K}^{\text{Ai}}(u, v) = \frac{\text{Ai}(u)\text{Ai}'(v) - \text{Ai}(v)\text{Ai}'(u)}{u - v}. \quad (2.12)$$

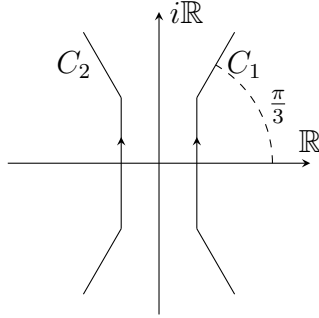


Figure 2: Contours in the definition of the Airy kernel $\mathbb{K}^{\text{Ai}}$

Remark 3. The natural interpretation of these results is as follows: if ϵ_n tends to 0 sufficiently fast, then the perturbation is too weak to have an effect on the large n behaviour of the eigenvalues near 0. In this case we have the same Bessel kernel limit as for the unperturbed LUE, even though 0 is not a hard edge any longer (for any $\epsilon > 0$ and n fixed, there is a non-zero probability of having negative eigenvalues). On the other hand, if ϵ_n tends to zero slowly, one is close to the fixed ϵ case where one has, macroscopically, soft edges, which suggests Airy behaviour. In (2.10), one should note that c_ϵ blows up as $\epsilon \rightarrow 0$: it is of the order $\epsilon^{-8/9}$. The intuition behind this, is that the typical distance between eigenvalues near a_ϵ is of the order $\epsilon^{8/9}n^{-2/3}$. If $\epsilon_n \rightarrow 0$ at the critical speed, the typical distance between eigenvalues is of the order n^{-2} and it is on this scale that the actual transition between the Bessel and the Airy kernel takes place.

Remark 4. One could consider more general LUE type ensembles with eigenvalue density of the form

$$\frac{1}{Z_n} \Delta(x)^2 \prod_{j=1}^n x_j^\alpha e^{-nV(x_j)} dx_j,$$

where the monomial λ^k in (2.1) is replaced by a polynomial $V(\lambda)$. As long as V is such that the limiting eigenvalue density blows up like an inverse square root near 0, it leads no doubt that the results in [37] could be adapted to that case, but we do not investigate this further.

Parts (i) and (ii) of Theorem 2.1 will be direct consequences of Lemma 1.2. We need the results from [37] to show that the conditions of Lemma 1.2 are fulfilled, and also to prove part (iii). Here, instead of (2.12), we will arrive at a different representation of the Airy kernel:

$$\mathbb{K}^{\text{Ai}}(u, v) = \frac{1}{4\pi^2} \int_{C_2} ds \int_{C_1} dt \frac{1}{s-t} \frac{e^{t^3-vt}}{e^{s^3-us}}, \quad (2.13)$$

with C_1 going from $e^{-\frac{i\pi}{3}}\infty$ to $e^{\frac{i\pi}{3}}\infty$ and C_2 its reflexion through the vertical axis, as in Figure 2. Using integration by parts on the integral representation of the Airy function [32, Formula 9.5.4], one can easily check that both kernels (2.12) and (2.13) are indeed the same. It is

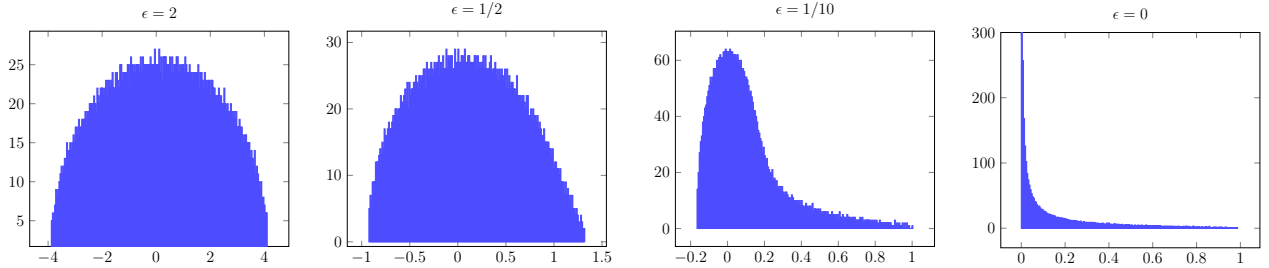


Figure 3: Numerical samples of the eigenvalues of $\frac{3^3}{4^4 4000^3} Y_3^* Y_3 + \epsilon H$, where Y_3 is a product of three 4000×4000 (i.e., $\nu_j = 0$ for $j = 1, 2, 3$) Ginibre matrices, and H is a 4000×4000 matrix drawn from the GUE. Because of the factor in front of Y_3 , the eigenvalues accumulate on $[0, 1]$. The chosen values for ϵ are 2, $1/2$, $1/10$ and 0. The eigenvalues are represented in histograms of 200 intervals. The first column for the $\epsilon = 0$ case has been truncated to 300 eigenvalues, its actual value being around 1500.

worth noting that this type of formula (that other kernels including $\mathbb{K}_\alpha^{\text{Bessel}}$ also possess), is the starting point of the Bertola-Cafasso method [7] for single/multi-time processes.

2.2 Perturbed products of Ginibre matrices

Products of Ginibre matrices have been studied intensively during the last years, see e.g. [2, 3, 4, 19, 26, 27]. The squared singular values of products of $m > 0$ independent complex Ginibre matrices follow also a polynomial ensemble with a hard edge at 0. Let $Y_m = X_m X_{m-1} \dots X_1$, with X_j an $(n + \nu_j) \times (n + \nu_{j-1})$ matrix with complex standard Gaussian iid entries, and with the X_j 's independent. The ν_j 's are assumed to be non-negative integers, and $\nu_0 = 0$. For n fixed, it was proved in [26] that the joint density of the squared singular values of Y_m is a polynomial ensemble. For $m = 1$, we recover the LUE with $k = 1$ and $\alpha = \nu_1$ after rescaling. Numerical samples for perturbed products of Ginibre matrices are presented in Figure 3.

The correlation kernel for the squared singular values of Y_m , or the eigenvalues of $Y_m^* Y_m$, is given by [27, Formula (5.1)]

$$K_n(x, y) = \frac{1}{(2\pi i)^2} \int_{-\frac{1}{2} + i\mathbb{R}} ds \int_{\Sigma_n} dt \prod_{j=0}^m \frac{\Gamma(s + \nu_j + 1) \Gamma(t - n + 1)}{\Gamma(t + \nu_j + 1) \Gamma(s - n + 1)} \frac{x^t y^{-s-1}}{s - t}, \quad (2.14)$$

where Γ denotes the Euler Gamma-function and where Σ_n is a closed contour encircling $0, 1, \dots, n$ in the positive direction in such a way that $\text{Re } t > -\frac{1}{2}$ for $t \in \Sigma_n$. The largest eigenvalue of this ensemble is typically of order n^m [34], and it is therefore more natural for us

to rescale the kernel in the following way

$$\begin{aligned}\widetilde{K}_n(x, y) &:= n^m K_n(n^m x, n^m y) \\ &= \frac{1}{(2\pi i)^2} \int_{-\frac{1}{2} + i\mathbb{R}} ds \int_{\Sigma_n} dt \prod_{j=0}^m \frac{\Gamma(s + \nu_j + 1)}{\Gamma(t + \nu_j + 1)} \frac{\Gamma(t - n + 1)}{\Gamma(s - n + 1)} \frac{x^t y^{-s-1} n^{m(t-s)}}{s - t}.\end{aligned}\quad (2.15)$$

This is the correlation kernel for the eigenvalues of $\frac{1}{n^m} Y_m^* Y_m$. Using this normalization, it has been shown [30, Theorem 3.2] that the zero counting measures of the average characteristic polynomials converge (in the weak-* sense) to the Fuss-Catalan distribution [34]. We may apply Theorem 1.1, and this implies that the counting measures of the average characteristic polynomials of the perturbed random matrix $\frac{1}{n^m} Y_m^* Y_m + \epsilon H$ converges to the free additive convolution of the Fuss-Catalan distribution with the semi-circle law λ_ϵ for any $\epsilon > 0$.

The microscopic behaviour of the eigenvalues near the origin is described by the following scaling limit: we have [27]

$$\lim_{n \rightarrow +\infty} \frac{1}{n^{m+1}} \widetilde{K}_n\left(\frac{x}{n^{m+1}}, \frac{y}{n^{m+1}}\right) = \mathbb{K}_\nu^G(x, y), \quad (2.16)$$

for $x, y > 0$, where

$$\mathbb{K}_\nu^G(x, y) = \frac{1}{(2\pi i)^2} \int_{-\frac{1}{2} + i\mathbb{R}} ds \int_{\Sigma} dt \prod_{j=0}^m \frac{\Gamma(s + \nu_j + 1)}{\Gamma(t + \nu_j + 1)} \frac{\sin \pi s}{\sin \pi t} \frac{x^t y^{-s-1}}{s - t}. \quad (2.17)$$

The contour Σ comes from $+\infty$ in the upper half plane, encircles the positive real axis and goes back to $+\infty$ in the lower half plane, in such a way that $\operatorname{Re} t > -\frac{1}{2}$ for $t \in \Sigma$. The kernel $\mathbb{K}_\nu^G$ can also be expressed in terms of Meijer G -functions. Recently, sine and Airy kernel limits were confirmed rigorously in the bulk and at the right edge [28].

Using Lemma 1.2, we will prove the following result in Section 5.3.

Theorem 2.2. *Let $Y_m = X_m X_{m-1} \dots X_1$, with the X_j 's independent $(n + \nu_j) \times (n + \nu_{j-1})$ complex Ginibre matrices, $\nu_0 = 0$, and let H be an $n \times n$ GUE matrix independent of Y_m . Write K_n^S for the eigenvalue correlation kernel of $S = \frac{1}{n^m} Y_m^* Y_m + \epsilon_n H$.*

(i) **(Sub-critical perturbation)** *If $\lim_{n \rightarrow +\infty} \epsilon_n n^{m+\frac{1}{2}} = 0$, then for $x, y > 0$, we have*

$$\lim_{n \rightarrow +\infty} \frac{1}{n^{m+1}} K_n^S\left(\frac{x}{n^{m+1}}, \frac{y}{n^{m+1}}\right) = \mathbb{K}_\nu^G(x, y). \quad (2.18)$$

(ii) **(Critical perturbation)** *If $\lim_{n \rightarrow +\infty} \epsilon_n n^{m+\frac{1}{2}} = \sigma > 0$, then for $x, y \in \mathbb{C}$, we have*

$$\lim_{n \rightarrow +\infty} \frac{1}{n^{m+1}} K_n^S\left(\frac{x}{n^{m+1}}, \frac{y}{n^{m+1}}\right) = \frac{1}{2\pi i \sigma^2} \int_{i\mathbb{R}} \int_{\mathbb{R}^+} \mathbb{K}_\nu^G(s, t) e^{\frac{1}{2\sigma^2}((s-x)^2 - (t-y)^2)} dt ds. \quad (2.19)$$

Remark 5. In the super-critical regime, one expects Airy behaviour just like in the LUE case. To prove this, one could try to follow the same steps as for the LUE, but this will become considerably harder because the kernel K_n is now expressed in terms of multiple orthogonal polynomials instead of (generalized) Laguerre polynomials. We will come back to this issue later on in Remark 11.

Remark 6. The limiting kernel (2.17) appears also at the hard edge of a matrix from the Cauchy-Laguerre two-matrix model [6, 9]. For $m = 2$, this is the space of pairs of positive-definite Hermitian $n \times n$ matrices with probability measure

$$\frac{1}{Z_n} \frac{\det(M_1)^a \det(M_2)^b e^{-\text{Tr}(V_1(M_1) + V_2(M_2))}}{\det(M_1 + M_2)^n} dM_1 dM_2, \quad (2.20)$$

with $a, b > -1$, $a + b > -1$ and V_1, V_2 two linear scalar potentials. The correlation kernel for the eigenvalues of one of the matrices is given as a double contour integral similar to (2.14) (e.g., for M_1 , we have $\nu_0 = -a$, $\nu_1 = 0$ and $\nu_2 = b$). We expect that Lemma 1.2 can be applied to this case as well.

Remark 7. A different type of (deterministic) perturbation of products of Ginibre matrices has been studied in [21].

2.3 Perturbed products of truncated unitary matrices

Another example is given by the squared singular values of products of $m > 0$ truncated unitary matrices. As for the previous case, we form $Y_m = T_m \dots T_1$, but now T_j is the upper left $(n + \nu_j) \times (n + \nu_{j-1})$ truncation of a unitary matrix of size $\ell_j \times \ell_j$ drawn randomly from the unitary group $\mathcal{U}(\ell_j)$ equipped with the Haar measure, as in [41]. We assume that $\nu_0 = 0$, that $\nu_1, \dots, \nu_m$ are non-negative integers, and that $\ell_j \geq n + \nu_j + 1$ for $j = 1, \dots, m$. See Figure 4 for numerical realizations of this ensemble perturbed by additive Gaussian noise.

If $\ell_1 \geq 2n + \nu_1$, it was shown in [25] that the joint probability density of the squared singular values is a polynomial ensemble, whose kernel is given by

$$K_n(x, y) = \frac{1}{(2\pi i)^2} \int_C ds \int_{\Sigma_n} dt \prod_{j=0}^m \frac{\Gamma(s + 1 + \nu_j) \Gamma(t + 1 + \ell_j - n)}{\Gamma(t + 1 + \nu_j) \Gamma(s + 1 + \ell_j - n)} \frac{x^t y^{-s-1}}{s - t}. \quad (2.21)$$

The contour C leaves at $-\infty$ in the lower half plane, encircles the semi axis $(-\infty, -1)$ and returns to $-\infty$ in the positive half plane, Σ_n being the same contour as in Section 2.2. Moreover, the contours C and Σ_n are not allowed to intersect. In [14], (2.21) was proved under the weaker assumption $\sum_{j=1}^m (\ell_j - n - \nu_j) \geq n$, instead of $\ell_1 \geq 2n + \nu_1$.

This kernel also has a limiting kernel appearing near the hard edge. As n goes to infinity, we also have to let $\ell_1, \dots, \ell_m$ go to infinity. For each ℓ_j , we may choose either to let $\ell_j - n$ go to infinity, or to keep $\ell_j - n$ fixed. For each of these choices, the scaling leads to a different

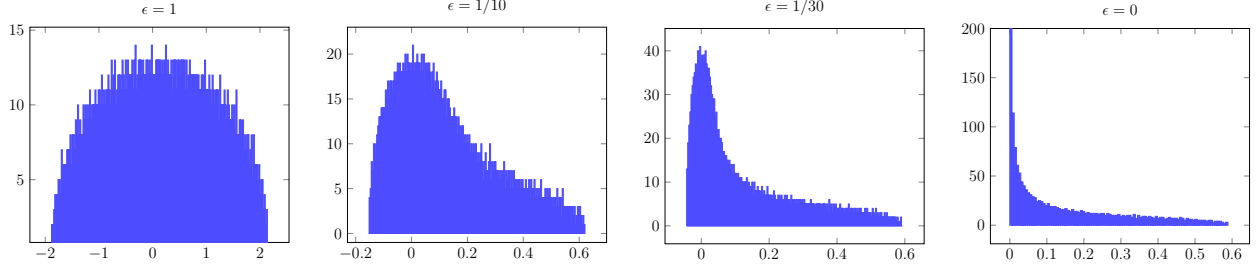


Figure 4: Numerical distribution of the eigenvalues of $Y_3^* Y_3 + \epsilon H$, for H a 2000×2000 GUE matrix and Y_3 is the product of three 2000×2000 truncations of three 4005×4005 Haar matrices. The chosen values for ϵ are 1, $1/10$, $1/30$ and 0. The eigenvalues are represented in a histogram of 200 intervals. The first column for the $\epsilon = 0$ case has been truncated to 200, the actual value being around 550.

limiting kernel. We thus take $J \subseteq \{2, \dots, m\}$ a subset of indices. We then let $\ell_1, \dots, \ell_m$ go to infinity in such a way that

$$\ell_k - n \rightarrow +\infty, \quad \text{if } k \notin J, \quad (2.22)$$

$$\ell_k - n = \mu_k, \quad \text{if } k = j_k \in J. \quad (2.23)$$

Define finally $c_n = n \prod_{j \notin J} (\ell_j - n)$. The kernel (2.21) then has the following scaling limit [25, Theorem 2.8] for $x \in \mathbb{C}, y > 0$,

$$\begin{aligned} \lim_{n \rightarrow +\infty} \frac{1}{c_n} K_n \left(\frac{x}{c_n}, \frac{y}{c_n} \right) &= \mathbb{K}_{\nu, \mu}^T(x, y) \\ &:= \frac{1}{(2\pi i)^2} \int_{-\frac{1}{2} + i\mathbb{R}} ds \int_{\Sigma} dt \prod_{j=0}^m \frac{\Gamma(s+1+\nu_j)}{\Gamma(t+1+\nu_j)} \frac{\sin \pi s}{\sin \pi t} \prod_{k \in J} \frac{\Gamma(t+1+\mu_k)}{\Gamma(s+1+\mu_k)} \frac{x^t y^{-s-1}}{s-t}. \end{aligned} \quad (2.24)$$

The contour Σ is the same as in Section 2.2. Note that if J is empty, the limiting kernel (2.24) reduces to the kernel (2.17). As eigenvalues of a product of truncated unitary matrices, the eigenvalues of M remain bounded as $n \rightarrow +\infty$. It can be verified, in a similar way as we will do in the case of products of Ginibre matrices, that the eigenvalue correlation kernel for M satisfies conditions similar to those of Lemma 1.2, if we replace cn^γ by c_n (see Remark 12). This will allow us to prove the following.

Theorem 2.3. *Let Y_m be a product of truncations of unitary Haar distributed matrices as described above, such that the eigenvalue correlation kernel of $Y_m^* Y_m$ is given by (2.21), and let H be an $n \times n$ GUE matrix independent of Y_m . Write K_n^S for the eigenvalue correlation kernel of $S = Y_m^* Y_m + \epsilon_n H$. Then, we have*

(i) **(Sub-critical perturbation)** *If $\lim_{n \rightarrow +\infty} \epsilon_n c_n n^{-\frac{1}{2}} = 0$, then for $x, y > 0$, we have*

$$\lim_{n \rightarrow +\infty} \frac{1}{c_n} K_n^S \left(\frac{x}{c_n}, \frac{y}{c_n} \right) = \mathbb{K}_{\nu, \mu}^T(x, y). \quad (2.25)$$

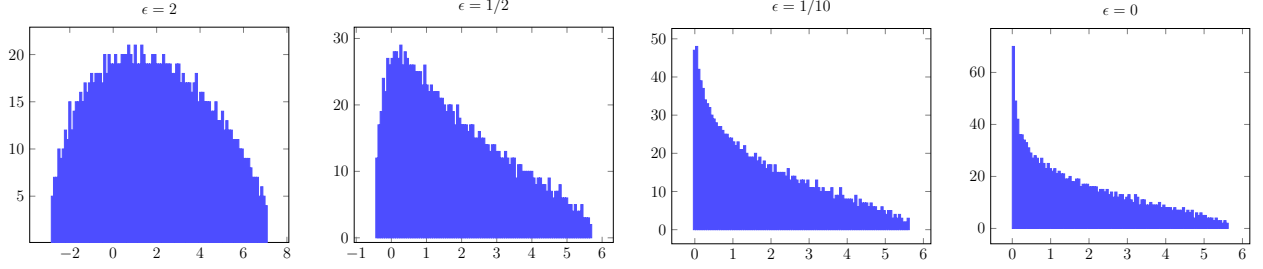


Figure 5: Numerical samples of eigenvalues of $\frac{1}{1500}X^*X + \epsilon H$, for H a GUE matrix and X a 1500×1500 matrix as described below (2.27), with $\theta = 3$ and $\alpha = 1$. Values for ϵ have been taken as 2, $1/2$, $1/10$ and 0. The eigenvalues are represented in histograms with 100 intervals.

(ii) **(Critical perturbation)** If $\lim_{n \rightarrow +\infty} \epsilon_n c_n n^{-\frac{1}{2}} = \sigma > 0$, then for $x, y \in \mathbb{C}$, we have

$$\lim_{n \rightarrow +\infty} \frac{1}{c_n} K_n^S \left(\frac{x}{c_n}, \frac{y}{c_n} \right) = \frac{1}{2\pi i \sigma^2} \int_{i\mathbb{R}} \int_{\mathbb{R}^+} \mathbb{K}_{\nu, \mu}^T(s, t) e^{\frac{1}{2\sigma^2}((s-x)^2 - (t-y)^2)} dt ds. \quad (2.26)$$

Remark 8. If $m = 1$, we have a single truncation of a unitary Haar matrix. Its squared singular values are in the Jacobi Unitary Ensemble, and in this case the kernel $\mathbb{K}_{\nu, \mu}^T$ reduces to the Bessel kernel $\mathbb{K}_{\nu}^{\text{Bessel}}$.

Remark 9. Here again, we expect a super-critical regime leading to the Airy kernel. See Remarks 5 and 11.

2.4 Perturbed Muttalib-Borodin biorthogonal ensembles

The last example consists of random matrices for which the joint probability density of eigenvalues is the Muttalib-Borodin Laguerre ensemble [11, 29]

$$\frac{1}{Z_n} \prod_{j < k} (\lambda_k - \lambda_j) (\lambda_k^\theta - \lambda_j^\theta) \prod_{j=1}^n \lambda_j^\alpha e^{-n\lambda_j}, \quad \theta > 0, \alpha > -1. \quad (2.27)$$

Such densities can be realized as eigenvalue densities of random matrices, see [1, 13, 22]. In [22], the authors constructed a random matrix with this eigenvalue density in the following way, in the case where θ is a positive integer and α a non-negative integer. Define α_j , $j = 1, \dots, n$ by $\alpha_j = \theta(j-1) + \alpha$. Then, consider the matrix X of size $m \times n$, $m \geq n$, whose (j, k) entry is 0 if $j - k > \alpha_k$, and following independent standard complex Gaussian distributions otherwise. In other words, X is a complex Ginibre matrix, but with the entries in a certain region in the lower left corner of the matrix replaced by zeros. The eigenvalues of such matrices are shown for $\theta = 2$ and $\alpha = 1$ in Figure 5. The density of the eigenvalues of $\frac{1}{n}X^*X$ is then given by

(2.27). The eigenvalue correlation kernel of $\frac{1}{n}X^*X$ can be expressed as [22]

$$K_n(x, y) = \frac{1}{(2\pi i)^2} e^{n(x-y)} \int_C ds \int_{C_\alpha} dt \frac{x^{-s-1} y^t}{s-t} n^{-s+t} \frac{\Gamma(s+1)}{\Gamma(t+1)} \prod_{k=1}^n \frac{s-\alpha_k}{t-\alpha_k}, \quad (2.28)$$

with C_α a contour enclosing $\alpha_1, \dots, \alpha_n$ and C starting at $-\infty$ in the lower half plane, enclosing C_α and going back to $-\infty$ in the upper half plane. An alternative expression was given in [39]:

$$K_n(x, y) = \frac{\theta}{(2\pi i)^2} \int_{c+i\mathbb{R}} ds \int_{\Sigma_n} dt \frac{x^{-\theta s-1} y^{\theta t}}{s-t} n^{-\theta s+\theta t} \frac{\Gamma(s+1)\Gamma(\alpha+1+\theta s)\Gamma(t-n+1)}{\Gamma(t+1)\Gamma(\alpha+1+\theta t)\Gamma(s-n+1)}, \quad (2.29)$$

with $c = -\frac{1}{2} + \frac{1}{2} \max\{0, 1 - \frac{\alpha+1}{\theta}\}$, and Σ a closed counter-clockwise contour going around $0, 1, \dots, n-1$ and for which $\operatorname{Re} t > c$.

It admits the hard edge scaling limit [11, 22]

$$\lim_{n \rightarrow +\infty} \frac{1}{n^{1+\frac{1}{\theta}}} K_n \left(\frac{x}{n^{1+\frac{1}{\theta}}}, \frac{y}{n^{1+\frac{1}{\theta}}} \right) = \mathbb{K}_{\alpha, \theta}^{\text{MB}}(x, y), \quad x \in \mathbb{C}, y > 0, \quad (2.30)$$

where the limiting kernel $\mathbb{K}_{\alpha, \theta}^{\text{MB}}$ is given by

$$\mathbb{K}_{\alpha, \theta}^{\text{MB}}(x, y) := \frac{\theta}{(2\pi i)^2} \left(\frac{y}{x} \right)^\alpha \int_{C^\delta} ds \int_{\Sigma} dt \frac{x^{-\theta s-1} y^{\theta t}}{s-t} \frac{\Gamma(\theta s + \alpha + 1)}{\Gamma(\theta t + \alpha + 1)} \frac{\Gamma(s+1)}{\Gamma(t+1)} \frac{\sin \pi s}{\sin \pi t}, \quad (2.31)$$

with Σ starting from $+\infty$ in the upper half plane, enclosing the positive real axis and going back to $+\infty$ in the lower half plane, and C^δ consisting of two rays starting from $-\frac{1}{2}$ and making an angle $0 < \delta < \frac{\pi}{2}$ with the vertical axis, oriented upwards.

Similarly as in the previous cases, we have the following result.

Theorem 2.4. *Let M be a random matrix with eigenvalue density (2.27) and correlation kernel (2.28), and let H be an $n \times n$ GUE matrix independent of X . Write K_n^S for the eigenvalue correlation kernel of $S = M + \epsilon_n H$.*

(i) **(Sub-critical perturbation)** *If $\lim_{n \rightarrow +\infty} \epsilon_n n^{\frac{1}{\theta} + \frac{1}{2}} = 0$, then for $x, y > 0$, we have*

$$\lim_{n \rightarrow +\infty} \frac{1}{n^{\frac{1}{\theta} + 1}} K_n^S \left(\frac{x}{n^{\frac{1}{\theta} + 1}}, \frac{y}{n^{\frac{1}{\theta} + 1}} \right) = \mathbb{K}_{\alpha, \theta}^{\text{MB}}(x, y). \quad (2.32)$$

(ii) **(Critical perturbation)** *If $\lim_{n \rightarrow +\infty} \epsilon_n n^{\frac{1}{\theta} + \frac{1}{2}} = \sigma > 0$, then for $x, y \in \mathbb{C}$, we have*

$$\lim_{n \rightarrow +\infty} \frac{1}{n^{\frac{1}{\theta} + 1}} K_n^S \left(\frac{x}{n^{\frac{1}{\theta} + 1}}, \frac{y}{n^{\frac{1}{\theta} + 1}} \right) = \frac{1}{2\pi i \sigma^2} \int_{i\mathbb{R}} \int_{\mathbb{R}^+} \mathbb{K}_{\alpha, \theta}^{\text{MB}}(s, t) e^{\frac{1}{2\sigma^2}((s-x)^2 - (t-y)^2)} dt ds. \quad (2.33)$$

Remark 10. The hard edge scaling limit (2.30) was derived in [11] with a different expression for the limiting kernel,

$$\mathbb{K}_{\alpha,\theta}^{\text{MB}}(x,y) = \theta y^\alpha \int_0^1 J_{(\alpha+1)/\theta,1/\theta}(xu) J_{\alpha+1,\theta}(yu)^\theta u^\alpha du, \quad (2.34)$$

where $J_{a,b}$ is Wright's generalization of Bessel functions given by

$$J_{a,b}(x) = \sum_{j=0}^{+\infty} \frac{(-x)^j}{j! \Gamma(a + jb)}. \quad (2.35)$$

If $1/\theta \in \mathbb{N}$, this limiting kernel can be expressed in terms of the kernel $\mathbb{K}_\nu^{\text{G}}$ appearing for products of Ginibre matrices, see [26, Section 5].

Yet another hard edge limiting kernel was obtained in [5, 38] in random matrix ensembles with singularities of the form

$$\frac{1}{Z_n} e^{-n \text{Tr}(M + t^k/M^k)} dM, \quad k \in \mathbb{N}, \quad (2.36)$$

on the space of $n \times n$ positive-definite Hermitian matrices. A limiting kernel was obtained which can be expressed in terms of the Painlevé III hierarchy, the corresponding value of γ in (1.9) is $\gamma = 2$. We believe that Lemma 1.2 can also be applied to this ensemble, but a detailed study would lead us too far.

Outline

In Section 3, we prove Theorem 1.1 on the convergence of the zero counting measures of the average characteristic polynomials. In Section 4, we prove the central auxiliary result of this paper, Lemma 1.2. Parts (i) and (ii) of Theorem 2.1 on perturbed LUE matrices are proved in Section 5.1, and part (iii) in Section 5.2. In Section 5.3, the proof of Theorem 2.2 on perturbed Ginibre products is given. The proofs of Theorem 2.3 and Theorem 2.4 on perturbed products of truncated unitary matrices and Muttalib-Borodin ensembles are similar to the Ginibre case, as we explain in Remark 12 without giving details.

3 Proof of Theorem 1.1

In this section we give the proof of Theorem 1.1. We first need a general result about the zeros of the average characteristic polynomial.

Lemma 3.1. *In a polynomial ensemble of the form (1.3), the average characteristic polynomial $p_n(z) = \mathbb{E} \left(\prod_{j=1}^n (z - x_j) \right)$ has n simple real zeros.*

Proof. The average characteristic polynomial has real coefficients and satisfies the orthogonality conditions

$$\int_{\mathbb{R}} p_n(x) f_j(x) dx = 0, \quad j = 0, \dots, n-1.$$

If p_n would have a non-simple or non-real zero z_0 , we can write $p_n(z) = (z - z_0)(z - \bar{z}_0) \left(\sum_{k=0}^{n-2} a_k z^k \right)$ with $a_{n-2} = 1$. By the orthogonality conditions, we have

$$\sum_{k=0}^{n-2} a_k \int_{\mathbb{R}} x^k \left(|x - z_0|^2 f_j(x) \right) dx = 0, \quad j = 0, \dots, n-1.$$

If this homogeneous linear system has a non-zero solution $(a_0, \dots, a_{n-2})$, then the coefficient matrix is of rank $\leq n-2$, so the extended $n \times n$ matrix $\left(\int |x - z_0|^2 x^k f_j(x) dx \right)_{j,k=0,\dots,n-1}$ is at most of rank $n-1$ and has zero determinant.

But on the other hand, by the Andreief identity, we have

$$\det \left(\int_{\mathbb{R}} |x - z_0|^2 x^k f_j(x) dx \right)_{j,k=0,\dots,n-1} = \frac{1}{n!} \int_{\mathbb{R}^n} \Delta(x) \det(f_{j-1}(x_k))_{j,k=1,\dots,n} \prod_{j=1}^n |x_j - z_0|^2 dx_j,$$

which is strictly positive as it is equal to $\frac{Z_n}{n!} \mathbb{E} \left(\prod_{j=1}^n |x_j - z_0|^2 \right)$. This is a contradiction, so p_n has only simple real zeros. $\square$

The proof of Theorem 1.1 now relies on the following two lemmas.

Lemma 3.2. *Under the conditions of Theorem 1.1, the sequence of functions*

$$f_n(z) := \int \log(1 - s/z) d(\mu_n - \mu)(s) \quad (3.1)$$

converges to 0, uniformly for $|z| > r$, where r is such that $r > \max\{|w| : w \in K\}$.

Proof. Point-wise convergence of f_n to 0 follows from the weak-* convergence of μ_n to μ , since $\log(1 - s/z)$ is continuous for $|z| > r$. To prove uniform convergence, we note first that $f_n(z)$ is uniformly bounded: we have

$$|f_n(z)| \leq \int |\log(1 - s/z)| d\mu_n(s) + \int |\log(1 - s/z)| d\mu(s) \leq 2 \max_{|z| \geq r, s \in K} |\log(1 - s/z)|. \quad (3.2)$$

If we define $h_n(z) = f_n(1/z)$, $h_n(z)$ is a uniformly bounded sequence of analytic functions on $|z| < 1/r$, which converges point-wise to 0. By Vitali's theorem, it follows that h_n converges to 0 uniformly for $|z| < 1/r$, and hence $f_n(z)$ converges to 0 uniformly for $|z| > r$. $\square$

For any compactly supported probability measure μ on $\mathbb{R}$, we define

$$g_\mu(z) = \int \log(z - s) d\mu(s), \quad G_\mu(z) = g'_\mu(z) = \int \frac{1}{z - s} d\mu(s), \quad (3.3)$$

where we choose the logarithm corresponding to arguments between $-\pi$ and π .

Lemma 3.3. *Under the conditions of Theorem 1.1, let $P_n(z)$ be the average characteristic polynomial of S . For $|\operatorname{Re} z|$ large enough, we have*

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \log P_n(z) = g_\mu(s_c(z)) + \frac{1}{2\epsilon^2} (z - s_c(z))^2, \quad (3.4)$$

where $s_c(z)$ is the solution in s of $G_\mu(s) + \epsilon^{-2}(s - z) = 0$.

Proof. Writing p_n for the average characteristic polynomial of M and μ_n for its zero counting measure, we have the identity

$$p_n(z) = \prod_{j=1}^n (z - z_j^{(n)}) = e^{ng_{\mu_n}(z)}, \quad (3.5)$$

where $z_j^{(n)}$, $j = 1, \dots, n$ are the zeros of p_n .

From the transformation formula (1.7) for P_n in terms of p_n , we have

$$P_n(z) = \frac{\sqrt{n}}{\sqrt{2\pi i \epsilon}} \int_{i\mathbb{R}} e^{nF_n(s; z)} ds, \quad F_n(s; z) := g_{\mu_n}(s) + \frac{1}{2\epsilon^2} (z - s)^2. \quad (3.6)$$

Since $g'_{\mu_n}(s) = \mathcal{O}(1/s)$ as $s \rightarrow \infty$, uniformly for n large enough, the saddle point equation $F'_n(s; z) = 0$ has, for z sufficiently large, a unique solution $s_c(z)$ such that $s_c(z) \sim z$ as $z \rightarrow \infty$, by the inverse function theorem.

In order to obtain large n asymptotics for $P_n(z)$, we deform the integration contour $i\mathbb{R}$ in (3.6) to the steepest descent path $\gamma = \gamma_n$ passing through $s_c(z)$ and on which the imaginary part of $F_n(s; z)$ is constant. For $|\operatorname{Re} z|$ large, γ makes a small angle with the vertical line through $s_c(z)$, and therefore it remains outside of the compact K . We have

$$\operatorname{Re} F''_n(s; z) = \operatorname{Re} g''_{\mu_n}(s) + \epsilon^{-2} = \epsilon^{-2} + \mathcal{O}(s^{-2}), \quad s \rightarrow \infty,$$

and this implies that $\operatorname{Re} F_n$ achieves its unique local maximum on γ at $s_c(z)$. We can use the saddle point method to approximate the integral in (3.6) in the following way, for z sufficiently large, as $n \rightarrow \infty$.

We may choose an implicit parametrization $\gamma_n(t)$ of the steepest descent path γ_n by imposing

$$F_n(\gamma_n(t); z) - F_n(s_c(z); z) = -t^2, \quad t \in [-\delta, \delta], \quad \operatorname{Im} \dot{\gamma}_n(0) > 0,$$

for some sufficiently small $\delta > 0$, and such that $|\dot{\gamma}_n(t)| = 1$ for $|t| > \delta$. We can then write the integral in (3.6) as

$$\int_{\gamma} e^{nF_n(s; z)} ds = e^{nF(s_c(z); z)} \int_{-\delta}^{\delta} \dot{\gamma}_n(t) e^{-nt^2} dt + \int_{\mathbb{R} \setminus [-\delta, \delta]} \dot{\gamma}_n(t) e^{nF_n(\gamma_n(t); z)} dt. \quad (3.7)$$

Since $\operatorname{Re} F_n(s; z)$ has its unique global maximum on γ_n at $t = 0$ and grows as $t \rightarrow \pm\infty$, the second term is $\mathcal{O}(e^{n[F_n(s_c(z)) - \eta^2]})$ as $n \rightarrow +\infty$. For the first term, note that $\dot{\gamma}_n(0) = i\sqrt{\frac{2}{F_n''(s_c(z); z)}}$. Moreover, there is a constant $C > 0$ independent of n such that

$$|\dot{\gamma}_n(t) - \dot{\gamma}_n(0)| \leq C|t|, \quad t \in [-\delta, \delta].$$

From (3.7), we now get

$$\begin{aligned} \int_{\gamma} e^{nF_n(s; z)} ds &= i\sqrt{\frac{2}{F_n''(s_c(z); z)}} e^{nF(s_c(z); z)} \int_{-\delta}^{\delta} e^{-nt^2} dt \\ &\quad + \mathcal{O}\left(\int_{-\delta}^{\delta} |t| e^{-nt^2} dt\right) + \mathcal{O}\left(e^{n[F_n(s_c(z)) - \eta^2]}\right), \end{aligned} \quad (3.8)$$

as $n \rightarrow \infty$. Evaluating the integrals as $n \rightarrow \infty$ and substituting in (3.6), we finally obtain

$$P_n(z) \sim \frac{1}{\epsilon} F_n''(s_c(z); z)^{-\frac{1}{2}} e^{nF_n(s_c(z); z)}, \quad n \rightarrow +\infty, \quad (3.9)$$

for $|\operatorname{Re} z|$ sufficiently large. It follows that

$$\frac{1}{n} \log P_n(z) = g_{\mu_n}(s_c(z)) + \frac{1}{2\epsilon^2} (z - s_c(z))^2 + o(1), \quad n \rightarrow +\infty.$$

Using the fact that

$$g_{\mu_n(z)} = g_{\mu}(z) + f_n(z), \quad \text{with } f_n(z) = \int \log(1 - s/z) d(\mu_n - \mu)(s) \quad (3.10)$$

and Lemma 3.2, (3.4) now follows easily. □

Proof of Theorem 1.1. Using the general definition of the free convolution, it was noted in [10] that the free convolution of a compactly supported probability measure μ with the semi-circle λ_{ϵ} satisfies the equation

$$G_{\mu \boxplus \lambda_{\epsilon}}(s + \epsilon^2 G_{\mu}(s)) = G_{\mu}(s), \quad (3.11)$$

for s sufficiently large. In other words, if $s_c(z)$ is the solution of $G_{\mu}(s) + \epsilon^{-2}(s - z) = 0$, we have

$$G_{\mu \boxplus \lambda_{\epsilon}}(z) = G_{\mu}(s_c(z)). \quad (3.12)$$

This implies, after a straightforward calculation,

$$\frac{d}{dz} g_{\mu \boxplus \lambda_{\epsilon}}(z) = \frac{d}{dz} \left(g_{\mu}(s_c(z)) + \frac{1}{2\epsilon^2} (z - s_c(z))^2 \right), \quad (3.13)$$

and upon integration, we obtain

$$g_{\mu \boxplus \lambda_\epsilon}(z) = g_\mu(s_c(z)) + \frac{1}{2\epsilon^2}(z - s_c(z))^2, \quad (3.14)$$

since both the left and the right hand side behave like $\log z + \mathcal{O}(z^{-1})$ as $z \rightarrow \infty$.

By Lemma 3.3, we have

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \log P_n(z) = g_{\mu \boxplus \lambda_\epsilon}(z) \quad (3.15)$$

for $|\operatorname{Re} z|$ sufficiently large. This implies that there exists an n -independent compact $\widetilde{K}$ such that P_n has no zeros outside $\widetilde{K}$, for n sufficiently large. By Helly's theorem, $(\nu_n)_n$, and every subsequence of it, has a weak- $*$ converging subsequence. We claim that every such converging subsequence $(\nu_{n_k})_k$ converges to $\mu \boxplus \lambda_\epsilon$. If so, it is easily seen by contraposition that the whole sequence ν_n converges to $\mu \boxplus \lambda_\epsilon$.

To prove the claim, we suppose that a subsequence $(\nu_{n_k})_k$ converges in weak- $*$ sense to some measure $\tilde{\nu}$. We then have

$$\lim_{k \rightarrow +\infty} \frac{1}{n_k} \log P_{n_k}(z) = \lim_{k \rightarrow +\infty} \int \log(z - s) d\nu_{n_k}(s) = g_{\tilde{\nu}}(z), \quad (3.16)$$

for z outside $\widetilde{K}$, and combining this with (3.15), it follows that $g_{\tilde{\nu}} = g_{\mu \boxplus \lambda_\epsilon}$. Since the supports of $\tilde{\nu}$ and $\mu \boxplus \lambda_\epsilon$ are both contained in $\mathbb{R}$, $g_{\tilde{\nu}}$ and $g_{\mu \boxplus \lambda_\epsilon}$ are analytic in $\mathbb{C} \setminus \mathbb{R}$, and by analytic continuation we have in particular that $\operatorname{Re} g_{\tilde{\nu}} = \operatorname{Re} g_{\mu \boxplus \lambda_\epsilon}$ everywhere in $\mathbb{C}$ except possibly on a set of 2-dimensional Lebesgue measure 0. We can then use the unicity theorem [35, Theorem II.2.1] to conclude that $\tilde{\nu} = \mu \boxplus \lambda_\epsilon$. □

4 Proof of Lemma 1.2

We will now give the proof of the central lemma of this paper. We assume that the conditions (1.10) and (1.11) hold.

Replacing x and y by $\frac{x}{cn^\gamma}$ and $\frac{y}{cn^\gamma}$ in (1.6), and substituting $s = \frac{u}{cn^\gamma}$ and $t = \frac{v}{cn^\gamma}$ in the integrals, we obtain the identity

$$\frac{1}{cn^\gamma} K_n^S \left(\frac{x}{cn^\gamma}, \frac{y}{cn^\gamma} \right) = \frac{n^{1-2\gamma}}{2\pi i c^2 \epsilon_n^2} \int_{i\mathbb{R}} \int_{\mathbb{R}^+} \frac{1}{cn^\gamma} K_n \left(\frac{u}{cn^\gamma}, \frac{v}{cn^\gamma} \right) e^{\frac{n^{1-2\gamma}}{2c^2 \epsilon_n^2} ((x-u)^2 - (y-v)^2)} dv du. \quad (4.1)$$

By (1.11), we have

$$\left| \frac{1}{cn^\gamma} K_n \left(\frac{u}{cn^\gamma}, \frac{v}{cn^\gamma} \right) \right| \leq \frac{c_1}{c^{1-\beta}} v^{-\beta} e^{\frac{c_2}{c}(|u|+|v|)},$$

which implies that we can use the dominated convergence theorem in (4.1) if $n \rightarrow +\infty$, $\epsilon_n \rightarrow 0$ in such a way that $\lim_{n \rightarrow +\infty} c\epsilon_n n^{\gamma-\frac{1}{2}} = \sigma > 0$. Using (1.10), we immediately obtain the limit (1.13), point-wise for $x, y \in \mathbb{C}$.

To see that the limit is uniform for (x, y) in compact sets and to treat the case where $\lim_{n \rightarrow +\infty} \epsilon_n n^{\gamma - \frac{1}{2}} = 0$, we need the following technical estimates.

Lemma 4.1. *Let $K_n : \mathbb{C} \times \mathbb{R}^+ \rightarrow \mathbb{C}$ be a sequence of kernels satisfying condition (1.11). Assume that $|x|, |y| \leq r$ and let $\omega > 0$. Then, for n and R sufficiently large, the following estimates hold:*

$$\left| \int_{i\mathbb{R} \setminus [-iR, iR]} \int_0^R \frac{1}{cn^\gamma} K_n \left(\frac{s}{cn^\gamma}, \frac{t}{cn^\gamma} \right) e^{\frac{1}{2\omega^2}((x-s)^2 - (y-t)^2)} dt ds \right| \leq \frac{2c_1}{c^{1-\beta}} \frac{R^{1-\beta}}{1-\beta} \frac{e^{2\frac{Rr}{\omega^2} + \frac{r^2}{\omega^2} + 2\frac{c_2}{c}R}}{R - r - \frac{c_2}{c}\omega^2} \omega^2 e^{-\frac{R^2}{2\omega^2}}; \quad (4.2)$$

$$\left| \int_{-iR}^{iR} \int_R^{+\infty} \frac{1}{cn^\gamma} K_n \left(\frac{s}{cn^\gamma}, \frac{t}{cn^\gamma} \right) e^{\frac{1}{2\omega^2}((x-s)^2 - (y-t)^2)} dt ds \right| \leq \frac{c_1 \sqrt{2\pi}}{c^{1-\beta}} \frac{e^{\frac{r^2}{\omega^2} + 2\frac{Rr}{\omega^2} + 2\frac{c_2}{c}R}}{R - r - \frac{c_2}{c}\omega^2} \omega^2 e^{-\frac{R^2}{2\omega^2}}; \quad (4.3)$$

$$\left| \int_{i\mathbb{R} \setminus [-iR, iR]} \int_R^{+\infty} \frac{1}{cn^\gamma} K_n \left(\frac{s}{cn^\gamma}, \frac{t}{cn^\gamma} \right) e^{\frac{1}{2\omega^2}((x-s)^2 - (y-t)^2)} dt ds \right| \leq \frac{2c_1}{c^{1-\beta}} \frac{e^{2\frac{c_2}{c}R + 2\frac{Rr}{\omega^2} - \frac{R^2}{\omega^2}}}{\left(R - r - \frac{c_2}{c}\omega^2\right)^2} \omega^2. \quad (4.4)$$

Proof. To prove inequality (4.2), we first use condition (1.11) to get

$$\left| \int_{i\mathbb{R} \setminus [-iR, iR]} \int_0^R \frac{1}{cn^\gamma} K_n \left(\frac{s}{cn^\gamma}, \frac{t}{cn^\gamma} \right) e^{\frac{1}{2\omega^2}((x-s)^2 - (y-t)^2)} dt ds \right| \leq \frac{c_1}{c^{1-\beta}} \int_{\mathbb{R} \setminus [-R, R]} e^{\frac{c_2}{c}|s|} \left| e^{\frac{1}{2\omega^2}(x-is)^2} \right| ds \int_0^R t^{-\beta} e^{\frac{c_2}{c}t} e^{-\frac{1}{2\omega^2}(y-t)^2} dt. \quad (4.5)$$

The choices of R and r then allow us to estimate

$$\left| e^{\frac{1}{2\omega^2}(x-is)^2} \right| \leq e^{\frac{r^2}{2\omega^2}} e^{\frac{|s|r}{\omega^2}} e^{-\frac{s^2}{2\omega^2}} \text{ and } e^{-\frac{1}{2\omega^2}(y-t)^2} \leq e^{\frac{r^2}{2\omega^2}} e^{\frac{Rr}{\omega^2}} e^{-\frac{t^2}{2\omega^2}}. \quad (4.6)$$

Using these two things, and the symmetry along the imaginary axis, we have

$$\left| \int_{i\mathbb{R} \setminus [-iR, iR]} \int_0^R \frac{1}{cn^\gamma} K_n \left(\frac{s}{cn^\gamma}, \frac{t}{cn^\gamma} \right) e^{\frac{1}{2\omega^2}((x-s)^2 - (y-t)^2)} dt ds \right| \leq \frac{2c_1}{c^{1-\beta}} \frac{R^{1-\beta}}{1-\beta} e^{\frac{r^2}{\omega^2}} e^{\frac{Rr}{\omega^2}} e^{\frac{c_2}{c}R} \int_R^{+\infty} e^{\frac{c_2}{c}s} e^{\frac{1}{2\omega^2}(2sr-s^2)} ds. \quad (4.7)$$

We may integrate by parts the integral at the right and doing so, we obtain

$$\begin{aligned} \int_R^{+\infty} e^{\frac{c_2}{c}s} e^{\frac{1}{2\omega^2}(2sr-s^2)} ds &= \frac{\omega^2 e^{\frac{c_2}{c}R + \frac{Rr}{\omega^2} - \frac{R^2}{2\omega^2}}}{R - r - \frac{c_2}{c}\omega^2} - \int_R^{+\infty} \frac{\omega^4}{\left(s - r - \frac{c_2}{c}\omega^2\right)^2} e^{\frac{c_2}{c}s + \frac{sr}{\omega^2} - \frac{s^2}{2\omega^2}} ds \\ &\leq \frac{\omega^2 e^{\frac{c_2}{c}R + \frac{Rr}{\omega^2} - \frac{R^2}{2\omega^2}}}{R - r - \frac{c_2}{c}\omega^2}, \end{aligned} \quad (4.8)$$

which yields (4.2).

The proofs for inequalities (4.3) and (4.4) are very similar. For (4.3), the only thing that changes is that the roles of s and t are interchanged. For (4.4), we need to use inequality (4.8) twice. $\square$

We now proceed with the proof of Lemma 1.2. To see that the limit (1.13) is uniform for x, y in compact sets, we assume that $|x|, |y| < r$ and choose $R > r$, as in Lemma 4.1. Define the three ensembles

$$U_1 := [-iR, iR] \times [R, +\infty), U_2 := (i\mathbb{R} \setminus [-iR, iR]) \times [0, R] \text{ and } U_3 := (i\mathbb{R} \setminus [-iR, iR]) \times [R, +\infty),$$

and write $\tilde{\epsilon}_n := c\epsilon_n n^{\gamma-\frac{1}{2}}$. By (4.1), we have

$$\begin{aligned} & \left| \frac{1}{cn^\gamma} K_n^S \left(\frac{x}{cn^\gamma}, \frac{y}{cn^\gamma} \right) - \frac{1}{2\pi i \sigma^2} \int_{i\mathbb{R}} \int_{\mathbb{R}^+} \mathbb{K}(s, t) e^{\frac{1}{2\sigma^2}((x-s)^2 - (y-t)^2)} dt ds \right| \\ & \leq \left| \frac{1}{2\pi i \tilde{\epsilon}_n^2} \int_{-iR}^{iR} \int_0^R \left(v^\beta \frac{1}{cn^\gamma} K_n \left(\frac{s}{cn^\gamma}, \frac{t}{cn^\gamma} \right) - v^\beta \mathbb{K}(s, t) \right) v^{-\beta} e^{\frac{1}{2\tilde{\epsilon}_n^2}((x-s)^2 - (y-t)^2)} dt ds \right| \\ & \quad + \left| \int_{-iR}^{iR} \int_0^R \mathbb{K}(s, t) \left(\frac{1}{2\pi i \tilde{\epsilon}_n^2} e^{\frac{1}{2\tilde{\epsilon}_n^2}((x-s)^2 - (y-t)^2)} - \frac{1}{2\pi i \sigma^2} e^{\frac{1}{2\sigma^2}((x-s)^2 - (y-t)^2)} \right) dt ds \right| \\ & \quad + \frac{1}{2\pi \tilde{\epsilon}_n^2} \sum_{j=1}^3 \left| \frac{1}{cn^\gamma} \iint_{U_j} K_n \left(\frac{s}{cn^\gamma}, \frac{t}{cn^\gamma} \right) e^{\frac{1}{2\tilde{\epsilon}_n^2}((x-s)^2 - (y-t)^2)} ds dt \right| \\ & \quad + \frac{1}{2\pi \sigma^2} \sum_{j=1}^3 \left| \iint_{U_j} \mathbb{K}(s, t) e^{\frac{1}{2\sigma^2}((x-s)^2 - (y-t)^2)} ds dt \right|. \quad (4.9) \end{aligned}$$

In the limit where $n \rightarrow +\infty$ and $\tilde{\epsilon}_n = c\epsilon_n n^{\gamma-1/2} \rightarrow \sigma > 0$, the first term at the right hand side of the above expression tends to 0 uniformly for $|x|, |y| < r$ because of (1.10). The second term at the right tends to 0 as $n \rightarrow \infty$ by the dominated convergence theorem. The remaining terms at the right can be estimated using Lemma 4.1 with $\omega = \tilde{\epsilon}_n$ and become small (uniformly in x and y) as R gets large. Since the left hand side does not depend on R , we can take R as large as we want, and this implies that we have (1.13) uniformly for $|x|$ and $|y|$ smaller than r .

We now deal with the case where $\tilde{\epsilon}_n \rightarrow 0$ as $n \rightarrow +\infty$. By (4.1), we have

$$\begin{aligned} & \frac{1}{cn^\gamma} K_n^S \left(\frac{x}{cn^\gamma}, \frac{y}{cn^\gamma} \right) \\ & = \frac{1}{2\pi i \tilde{\epsilon}_n^2} \int_{[-iR, iR]} \int_{[0, R]} \frac{1}{cn^\gamma} K_n \left(\frac{u}{cn^\gamma}, \frac{v}{cn^\gamma} \right) e^{\frac{1}{2\tilde{\epsilon}_n^2}((x-u)^2 - (y-v)^2)} dv du \\ & \quad + \sum_{j=1}^3 \frac{1}{2\pi i \tilde{\epsilon}_n^2} \iint_{U_j} \frac{1}{cn^\gamma} K_n \left(\frac{u}{cn^\gamma}, \frac{v}{cn^\gamma} \right) e^{\frac{1}{2\tilde{\epsilon}_n^2}((x-u)^2 - (y-v)^2)} dv du, \quad (4.10) \end{aligned}$$

where we have cut the integral in the same 4 parts as before. We may use Lemma 4.1 with $\omega = \tilde{\epsilon}_n$ and we get that the three terms in the sum on the last line tend to 0 as $n \rightarrow +\infty$, provided R is sufficiently large. In the first term, we can deform the integration contour for u to $\square := [-iR, x - iR] \cup [x - iR, x + iR] \cup [x + iR, iR]$. This does not change the integral since K_n is analytic in u . We use the convergence in (1.10), which is uniform for u and v on their respective integration contours, and obtain

$$\begin{aligned} \frac{1}{2\pi i \tilde{\epsilon}_n^2} \int_{[-iR, iR]} \int_{[0, R]} \frac{1}{cn^\gamma} K_n \left(\frac{u}{cn^\gamma}, \frac{v}{cn^\gamma} \right) e^{\frac{1}{2\tilde{\epsilon}_n^2}((x-u)^2 - (y-v)^2)} dv du = \\ \frac{1}{2\pi i \tilde{\epsilon}_n^2} \int_{\square} \int_{[0, R]} \left(\mathbb{K}(u, v) + o(1)v^{-\beta} \right) e^{\frac{1}{2\tilde{\epsilon}_n^2}((x-u)^2 - (y-v)^2)} dv du, \text{ as } n \rightarrow +\infty. \end{aligned} \quad (4.11)$$

Since R is arbitrary, we can take it large enough so that the contribution to the integral on the paths $[-iR, x - iR]$ and $[x + iR, iR]$ is small in n . Then, the contours $[x - iR, x + iR]$ and $[0, R]$ are the steepest descent contours of the u - and v -phase functions, respectively. This allows us to apply the usual saddle point method to the integral, which gives

$$\frac{1}{2\pi i \tilde{\epsilon}_n^2} \int_{[x-iR, x+iR]} \int_{[0, R]} \left(\mathbb{K}(u, v) + o(1)v^{-\beta} \right) e^{\frac{1}{2\tilde{\epsilon}_n^2}((x-u)^2 - (y-v)^2)} dv du = \mathbb{K}(x, y) + o(1), \text{ as } n \rightarrow +\infty, \quad (4.12)$$

for $x, y > 0$ fixed. Note that the error term is not uniform as $y \rightarrow 0$. This completes the proof of Lemma 1.2.

5 Proofs of applications

5.1 Proof of Theorem 2.1: sub-critical and critical cases

For the proof of parts (i) and (ii) of Theorem 2.1, we only need to verify that the conditions of Lemma 1.2, namely (1.10)-(1.11), are satisfied. It should be noted that it is important to have uniformity of (1.10) for u in any compact subset of $\mathbb{C}$ and v in any compact subset of $[0, +\infty)$, and therefore (1.10) is not a direct consequence of (2.6).

Define p_n , $n = 0, 1, \dots$ to be the normalized orthogonal polynomials with respect to the generalized Laguerre weight $w(x) = x^\alpha e^{-nx^k}$. Let Y_n be the matrix-valued function

$$Y_n(z) = \begin{pmatrix} \frac{1}{\kappa_n} p_n(z) & \frac{1}{\kappa_n} C(p_n w)(z) \\ -2\pi i \kappa_{n-1} p_{n-1}(z) & -2\pi i \kappa_{n-1} C(p_{n-1} w)(z) \end{pmatrix}, \quad (5.1)$$

where $\kappa_j > 0$ is the leading coefficient of p_j and Cf is the Cauchy transform

$$Cf(z) = \frac{1}{2\pi i} \int_0^{+\infty} \frac{f(s)}{s - z} ds, \text{ for } z \notin [0, +\infty). \quad (5.2)$$

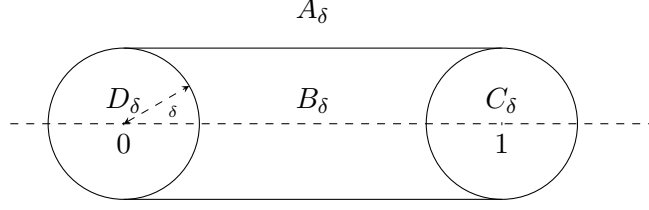


Figure 6: The four regions in which the asymptotics of U_n are expressed differently.

The matrix Y_n is the solution to the usual Fokas-Its-Kitaev Riemann-Hilbert problem for orthogonal polynomials [18]. The eigenvalue correlation kernel K_n for a (generalized) LUE matrix is expressed in terms of Y_n as

$$K_n(x, y) = w(y) \frac{1}{2\pi i(x-y)} \begin{pmatrix} 0 & 1 \end{pmatrix} Y_n^{-1}(y) Y_n(x) \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad (5.3)$$

which is easily verified by the Christoffel-Darboux formula and the fact that $\det Y_n = 1$. Note that the kernel (5.3) differs from its symmetric version defining the same determinantal point process, in which $w(y)$ is replaced by $\sqrt{w(x)w(y)}$. The reason is that we need $K_n(x, y)$ to be polynomial in x in order to apply (1.6).

Define the re-scaled matrix

$$U_n(z) = \left(\frac{2}{kA_k} \right)^{\frac{1}{k}(-n-\frac{\alpha}{2})\sigma_3} Y_n \left(\left(\frac{2}{kA_k} \right)^{\frac{1}{k}} z \right) \left(\frac{2}{kA_k} \right)^{\frac{1}{k}\frac{\alpha}{2}\sigma_3}, \quad A_k = \prod_{j=1}^k \frac{2j-1}{2j}, \quad (5.4)$$

where $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. The rescaled kernel $\widetilde{K}_n$ defined by

$$\widetilde{K}_n(x, y) = \left(\frac{2}{kA_k} \right)^{\frac{\alpha}{k}} y^\alpha e^{-\frac{2n}{kA_k}y^k} \frac{1}{2\pi i(x-y)} \begin{pmatrix} 0 & 1 \end{pmatrix} U_n^{-1}(y) U_n(x) \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad (5.5)$$

corresponds to a re-scaled LUE in which the limiting mean eigenvalue distribution is supported on $[0, 1]$, and given by [37]

$$d\tilde{\mu}(x) = \frac{1}{2\pi} \sqrt{\frac{1-x}{x}} \tilde{h}(x) dx, \quad \tilde{h}(x) = 2 \sum_{j=0}^{k-1} \frac{A_{k-1-j}}{A_k} x^j, \quad x \in (0, 1). \quad (5.6)$$

We have

$$\widetilde{K}_n(x, y) = \left(\frac{2}{kA_k} \right)^{\frac{\alpha+1}{k}} K_n \left(\left(\frac{2}{kA_k} \right)^{\frac{1}{k}} x, \left(\frac{2}{kA_k} \right)^{\frac{1}{k}} y \right). \quad (5.7)$$

We now describe the asymptotics for $U_n(z)$, which were obtained in [37]. Let $\delta > 0$ be sufficiently small and consider the following four regions, as illustrated in Figure 6:

$$D_\delta = \{z \in \mathbb{C} : |z| < \delta\}, \quad C_\delta = \{z \in \mathbb{C} : |z - 1| < \delta\}, \quad (5.8)$$

$$B_\delta = \{z \in \mathbb{C} : 0 < \operatorname{Re} z < 1, |\Im(z)| < \delta, z \notin D_\delta \cup C_\delta\}, \quad A_\delta = \mathbb{C} \setminus (B_\delta \cup C_\delta \cup D_\delta). \quad (5.9)$$

The nature of the asymptotics for $U_n(z)$ is different in each of those regions.

For $z \in A_\delta$, we have

$$U_n(z) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{z^{-\frac{\alpha}{2}}}{2z^{\frac{1}{4}}(z-1)^{\frac{1}{4}}} e^{ng_\mu(z)} \begin{pmatrix} 2^{-\alpha} \varphi(z)^{\frac{1}{2}(\alpha+1)} (1 + \mathcal{O}(1/n)) \\ -ie^{-n\ell} 2^\alpha \varphi(z)^{\frac{1}{2}(\alpha-1)} (1 + \mathcal{O}(1/n)) \end{pmatrix}, \quad (5.10)$$

as $n \rightarrow \infty$, where $\ell = -\frac{2}{k} - 4 \log 2$,

$$g_\mu(z) = \int \log(z-s) d\tilde{\mu}(s) \sim \log z, \quad \text{as } z \rightarrow \infty, \quad (5.11)$$

and

$$\varphi(z) = 2(z - 1/2) + 2z^{1/2}(z - 1)^{1/2}, \quad (5.12)$$

with the square roots defined such that φ is analytic except on $[0, 1]$. It follows that there exists a constant c_1 such that

$$\left\| U_n(z) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\| \leq e^{c_1 n |z|}, \quad z \in A_\delta, \quad (5.13)$$

for n sufficiently large. This is a very rough estimate but it will be enough for our purposes.

For $z \in B_\delta$, we have

$$U_n(z) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{z^{-\frac{\alpha}{2}}}{z^{\frac{1}{4}}(z-1)^{\frac{1}{4}}} e^{n \frac{z^k}{k A_k}} \begin{pmatrix} e^{\frac{1}{2} n \ell} 2^{-\alpha} \cos \left(\eta_1(z) - i n \xi(z) - \frac{\pi}{4} \right) (1 + \mathcal{O}(1/n)) \\ e^{-\frac{1}{2} n \ell} 2^\alpha \cos \left(\eta_2(z) - i n \xi(z) - \frac{\pi}{4} \right) (1 + \mathcal{O}(1/n)) \end{pmatrix}, \quad (5.14)$$

as $n \rightarrow \infty$, for some continuous functions η_1, η_2 independent of n , and with

$$\xi(z) = -\pi i \int_1^z \frac{1}{2\pi i} \frac{(s-1)^{1/2}}{s^{1/2}} \tilde{h}(s) ds, \quad (5.15)$$

with the square roots such that ξ is analytic in $\mathbb{C} \setminus [0, 1]$. It follows that there exists a constant c_2 such that

$$\left\| U_n(z) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\| \leq e^{c_2 n |z|}, \quad z \in B_\delta, \quad (5.16)$$

for n sufficiently large.

For $z \in C_\delta$,

$$U_n(z) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\sqrt{\pi} z^{-\frac{\alpha}{2}}}{z^{1/4}(z-1)^{1/4}} e^{n \frac{z^k}{k A_k}} \times \begin{pmatrix} e^{\frac{1}{2} n \ell} 2^{-\alpha} \left(\cos(\eta_1(z)) f_n(z)^{1/4} \text{Ai}(f_n(z)) - i \sin(\eta_1(z)) f_n(z)^{-1/4} \text{Ai}'(f_n(z)) \right) (1 + \mathcal{O}(1/n)) \\ e^{-\frac{1}{2} n \ell} 2^\alpha \left(-i \cos(\eta_2(z)) f_n(z)^{1/4} \text{Ai}(f_n(z)) - \sin(\eta_2(z)) f_n(z)^{-1/4} \text{Ai}'(f_n(z)) \right) (1 + \mathcal{O}(1/n)) \end{pmatrix},$$

as $n \rightarrow \infty$, with Ai the Airy function, and f_n a conformal map defined in a neighborhood of 1, satisfying $f_n(1) = 0$, $f_n'(1) > 0$ and

$$\frac{2}{3} f_n(z)^{\frac{3}{2}} = n \xi(z). \quad (5.17)$$

Using the asymptotics for the Airy function at large arguments and the fact that $|f_n(z)| \leq C n^{2/3} |z-1|$ for some constant $C > 0$ independent of n , for z sufficiently close to 1, it follows again that there exists a constant c_3 such that

$$\left\| U_n(z) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\| \leq e^{c_3 n |z|}, \quad z \in C_\delta, \quad (5.18)$$

for n sufficiently large.

Finally, we have for $z \in D_\delta$,

$$U_n(z) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{(-1)^n \sqrt{\pi} (-\tilde{f}_n(z))^{1/4} z^{-\frac{\alpha}{2}}}{z^{1/4} (1-z)^{1/4}} e^{n \frac{z^k}{k A_k}} \times \begin{pmatrix} e^{\frac{1}{2} n \ell} 2^{-\alpha} \left(\sin(\zeta_1(z)) J_\alpha(2(-\tilde{f}_n(z))^{1/2}) + \cos(\zeta_1(z)) J'_\alpha(2(-\tilde{f}_n(z))^{1/2}) \right) (1 + \mathcal{O}(1/n)) \\ -i e^{-\frac{1}{2} n \ell} 2^\alpha \left(\sin(\zeta_2(z)) J_\alpha(2(-\tilde{f}_n(z))^{1/2}) + \cos(\zeta_2(z)) J'_\alpha(2(-\tilde{f}_n(z))^{1/2}) \right) (1 + \mathcal{O}(1/n)) \end{pmatrix}, \quad (5.19)$$

as $n \rightarrow \infty$, with $\tilde{f}_n$ a conformal map defined in a neighbourhood of 0 and satisfying

$$2\tilde{f}_n(z)^{\frac{1}{2}} = n \xi(z), \quad \tilde{f}_n(0) = 0, \quad \tilde{f}_n'(0) < 0. \quad (5.20)$$

The functions ζ_1 and ζ_2 are independent of n and such that $\zeta_1(0) = -\zeta_2(0) = \pm \frac{\pi}{2}$. We have $|\tilde{f}_n(z)| \leq C n^2 |z|$ for some positive n -independent constant C , if z is sufficiently close to 0. The asymptotics [32, Formula 10.7.8] of the Bessel function J_α and the fact that $z^{-\alpha} J_\alpha(z)$ is bounded allow us to conclude that there exist constants $c_4, C > 0$ such that

$$\left\| e^{-\frac{n}{2} \ell \sigma_3} U_n(z) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\| \leq C e^{c_4 n |z|}, \quad z \in D_\delta. \quad (5.21)$$

for n sufficiently large.

Proof of Theorem 2.1, parts (i) and (ii). The hard edge scaling limit (2.6) was proved in [37] (using (5.19)) to hold point-wise for $u, v > 0$. From the proof, or from (5.19), it is however readily seen that it holds point-wise for any $u \in \mathbb{C}$. For $\alpha \geq 0$, the Bessel function J_α is bounded near 0, and then it is straightforward to show, again using (5.19), that (1.10) with $\beta = 0$ is uniform for u in any compact in $\mathbb{C}$ and for v in any compact in $[0, +\infty)$. If $-1 < \alpha < 0$, $J_\alpha(z)$ blows up as $z \rightarrow 0$, but we have that $z^{-\alpha}J_\alpha(z)$ is analytic at 0, and this can be used to show that (1.10) with $\beta = -\alpha$ holds uniformly for u in any compact in $\mathbb{C}$ and for v in any compact in $[0, +\infty)$.

To verify condition (1.11) for $(u, v) \in i\mathbb{R} \times \mathbb{R}^+$, we have to consider eight different regions for (u, v) : $A_\delta \times A_\delta$, $A_\delta \times B_\delta$, $A_\delta \times C_\delta$, $A_\delta \times D_\delta$, $\widetilde{D_\delta} \times A_\delta$, $D_\delta \times B_\delta$, $D_\delta \times C_\delta$, and $D_\delta \times D_\delta$. In each of these cases, we can bound the kernel $\widetilde{K}_n$ defined in (5.5) using the estimates (5.13), (5.16), (5.18), and (5.21) corresponding to the different regions. Substituting them in (5.7), we obtain the desired estimate (1.11) in each of the regions.

The results now follow directly from Lemma 1.2. $\square$

5.2 Proof of Theorem 2.1: super-critical case

For part (iii) of Theorem 2.1, we will use saddle point methods and a modified version of the integral representation (1.6) for the correlation kernel K_n^S . We first need some technical results.

Lemma 5.1. *Let μ be the limiting mean eigenvalue distribution given by (2.3), and let λ_ϵ be the rescaled semicircle law (1.8). The free additive convolution $\mu \boxplus \lambda_\epsilon$ is supported on an interval $[a_\epsilon, b_\epsilon]$, and a_ϵ is given by*

$$a_\epsilon = u_\epsilon - \epsilon^2 \int \frac{d\mu(x)}{x - u_\epsilon}, \quad (5.22)$$

where u_ϵ is the unique negative solution of the equation

$$\int \frac{d\mu(x)}{(x - u_\epsilon)^2} = \frac{1}{\epsilon^2}. \quad (5.23)$$

Moreover, for some $\kappa, \widehat{\kappa} > 0$,

$$u_\epsilon = -\kappa\epsilon^{4/3}(1 + o(1)), \quad a_\epsilon = -\widehat{\kappa}\epsilon^{4/3}(1 + o(1)), \quad \epsilon \rightarrow 0. \quad (5.24)$$

Proof. It follows from the general results in [10] on the free additive convolution of a measure with a semi-circle law that the left endpoint a_ϵ is given by (5.22), with u_ϵ solving (5.23), provided that such a solution u_ϵ exists. For general μ and $\epsilon > 0$, it is not always true that (5.23) has a unique solution. However, with μ given by (2.3), the left hand side of (5.23) tends to 0 as $u_\epsilon \rightarrow -\infty$, is increasing, and tends to $+\infty$ as $u_\epsilon \rightarrow 0$. This implies that (5.23) has indeed a unique solution for any $\epsilon > 0$. Small ϵ asymptotics for u_ϵ and a_ϵ can be obtained from (5.23) and (5.22) using residue arguments. This leads to (5.24). $\square$

The following result provides us with suitable paths in the complex plane where the real part of a certain phase function Ψ_ϵ , to be used in the saddle point analysis later on, is monotone.

Lemma 5.2. *Let μ be given by (2.3), let $\epsilon > 0$, and let a_ϵ and u_ϵ be as in Lemma 5.1. Define*

$$\Psi_\epsilon(z) = \frac{1}{2\epsilon^2}(z - a_\epsilon)^2 + \int \log(z - x)d\mu(x), \quad z \in \mathbb{C} \setminus (-\infty, b], \quad (5.25)$$

and define two paths γ_1 and γ_2 by

$$\gamma_1(t) = u_\epsilon + e^{\frac{\pi i}{3}}|u_\epsilon| + t, \quad \gamma_2(t) = u_\epsilon + e^{\frac{2\pi i}{3}}|u_\epsilon| + it, \quad t > 0. \quad (5.26)$$

Then, for $\epsilon > 0$ sufficiently small, the functions

$$t \in \mathbb{R}^+ \mapsto \operatorname{Re}(\Psi_\epsilon(\gamma_1(t))) \text{ and } t \in \mathbb{R}^+ \mapsto \operatorname{Re}(\Psi_\epsilon(\bar{\gamma}_1(t)))$$

are increasing and the functions

$$t \in \mathbb{R}^+ \mapsto \operatorname{Re}(\Psi_\epsilon(\gamma_2(t))) \text{ and } t \in \mathbb{R}^+ \mapsto \operatorname{Re}(\Psi_\epsilon(\bar{\gamma}_2(t)))$$

are decreasing.

Proof. Because of conjugational symmetry, it suffices to check that

$$\frac{d}{dt}\operatorname{Re}(\Psi_\epsilon(\gamma_1(t))) > 0 \text{ and } \frac{d}{dt}\operatorname{Re}(\Psi_\epsilon(\gamma_2(t))) < 0, \quad \text{for } t > 0.$$

On γ_2 , we have after a straightforward computation,

$$\frac{d}{dt}\operatorname{Re}(\Psi_\epsilon(\gamma_2(t))) = \left(t + \frac{\sqrt{3}}{2}|u_\epsilon|\right) \left(\frac{-1}{\epsilon^2} + \int \frac{d\mu(s)}{\left(\frac{3}{2}u_\epsilon - s\right)^2 + \left(t + \frac{\sqrt{3}}{2}|u_\epsilon|\right)^2}\right). \quad (5.27)$$

For $t \geq 0$, we have

$$0 \leq \int \frac{d\mu(s)}{\left(\frac{3}{2}u_\epsilon - s\right)^2 + \left(t + \frac{\sqrt{3}}{2}|u_\epsilon|\right)^2} \leq \int \frac{d\mu(s)}{\left(\frac{3}{2}u_\epsilon - s\right)^2 + \frac{3}{4}u_\epsilon^2}. \quad (5.28)$$

The integral at the right is easily computable using the residue theorem, and by (5.24), it is readily seen that it is of order $\epsilon^{-4/3}$ as $\epsilon \rightarrow 0$. Hence, by (5.27), for ϵ small enough,

$$\frac{d}{dt}\operatorname{Re}(\Psi_\epsilon(\gamma_2(t))) < 0, \quad \text{for all } t \geq 0. \quad (5.29)$$

On γ_1 , we have

$$\frac{d}{dt}\operatorname{Re}(\Psi_\epsilon(\gamma_1(t))) = \frac{1}{\epsilon^2} \left(\frac{u_\epsilon}{2} + t - a_\epsilon\right) + \operatorname{Re} \left(\int \frac{d\mu(x)}{u_\epsilon + e^{\frac{\pi i}{3}}|u_\epsilon| + t - x} \right). \quad (5.30)$$

By Lemma 5.1, we can replace a_ϵ by its expression in terms of u_ϵ given in (5.22), and we get

$$\frac{d}{dt} \operatorname{Re} (\Psi_\epsilon(\gamma_1(t))) = \frac{1}{\epsilon^2} \left(-\frac{u_\epsilon}{2} + t \right) + \int \frac{d\mu(x)}{x - u_\epsilon} + \operatorname{Re} \left(\int \frac{d\mu(x)}{u_\epsilon + e^{\frac{\pi i}{3}} |u_\epsilon| + t - x} \right). \quad (5.31)$$

This is easily seen to be positive if $t \geq b - \frac{u_\epsilon}{2}$. For t smaller, we need to be more careful and exploit the fact that ϵ is small. The Stieltjes transform can be computed explicitly by (5.6) and a residue calculus (or alternatively by observing that μ is an equilibrium measure which satisfies Euler-Lagrange variational conditions): for $z \in \mathbb{C} \setminus [0, b]$,

$$\int \frac{d\mu(x)}{z - x} = \frac{k}{2} z^{k-1} - \frac{h(z)}{2} \frac{\sqrt{z - b}}{\sqrt{z}}, \quad (5.32)$$

with principal branches of the square roots. Evaluating at $z = u_\epsilon$ and at $z = t + u_\epsilon + e^{\frac{\pi i}{3}} |u_\epsilon|$, and substituting in (5.31), we obtain

$$\begin{aligned} \frac{d}{dt} \operatorname{Re} (\Psi_\epsilon(\gamma_1(t))) &= \frac{1}{\epsilon^2} \left(-\frac{u_\epsilon}{2} + t \right) + \frac{h(0)}{2} \sqrt{\frac{b}{|u_\epsilon|}} \\ &\quad - \operatorname{Re} \left(\frac{h(t)}{2} \frac{\sqrt{t + u_\epsilon + e^{\frac{\pi i}{3}} |u_\epsilon| - b}}{\sqrt{t + u_\epsilon + e^{\frac{\pi i}{3}} |u_\epsilon|}} \right) + \mathcal{O}(1), \quad \text{as } \epsilon \rightarrow 0. \end{aligned} \quad (5.33)$$

For $t \gg \epsilon^{4/3}$, the leading order term in this expansion is $\frac{t}{\epsilon^2}$, and it then follows that $\frac{d}{dt} \operatorname{Re} (\Psi_\epsilon(\gamma_1(t))) > 0$ for ϵ sufficiently small.

Writing $t = T|u_\epsilon| \sim \kappa T \epsilon^{4/3}$ as $\epsilon \rightarrow 0$, we have

$$\begin{aligned} \frac{d}{dt} \operatorname{Re} (\Psi_\epsilon(\gamma_1(t))) &= \epsilon^{-2/3} \left(\frac{\kappa}{2} + T\kappa + \frac{h(0)\sqrt{b}}{2\sqrt{\kappa}} \left(1 - \operatorname{Re} \frac{i\sqrt{T - 1 + e^{-\frac{\pi i}{3}}}}{|T - 1 + e^{\frac{\pi i}{3}}|} \right) \right) \\ &\quad + o(\epsilon^{-2/3}), \quad \text{as } \epsilon \rightarrow 0. \end{aligned} \quad (5.34)$$

It is straightforward to check by a trigonometric argument that $\operatorname{Re} \frac{i\sqrt{T - 1 + e^{-\frac{\pi i}{3}}}}{|T - 1 + e^{\frac{\pi i}{3}}|} < 1$ for $T > 0$, and this implies by (5.34) that $\frac{d}{dt} \operatorname{Re} (\Psi_\epsilon(\gamma_1(t))) > 0$ for $t > 0$. $\square$

We can now prove part (iii) of Theorem 2.1. The general strategy of the proof is similar to the one of [28, Theorem 1.2].

Proof of Theorem 2.1, part (iii). We use an alternative expression for K_n^S given by [14, Formula (3.23)]

$$K_n^S(x, y) = \frac{n}{(2\pi i)^2 \epsilon^2} \int_C ds \int_\Sigma d\zeta \frac{1}{s - \zeta} \begin{pmatrix} 0 & 1 \end{pmatrix} Y_n^{-1}(s) Y_n(\zeta) \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{\frac{n}{2\epsilon^2}((x-s)^2 - (y-\zeta)^2)}, \quad (5.35)$$

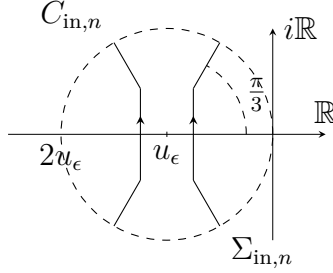


Figure 7: Local contours around the saddle point of the phase function Ψ_ϵ .

where Y_n is defined in (5.1). Here, Σ is a contour leaving from $+\infty$ in the upper half plane, encircling the positive real axis and going back to $+\infty$ in the lower half plane, as in Section 2.2. The second contour C does not intersect with Σ but can otherwise be any path going from $c_1 - i\infty$ to $c_2 + i\infty$. The fact that the two contours do not intersect, explains why the first term of [14, Formula (3.23)] cancels out here. We will choose the contours C and Σ such that they are suitable for a saddle point analysis of the integral in (5.35).

We let Ψ_ϵ , which will serve as a phase function in the saddle point analysis, be defined as in (5.25), and we let $u_\epsilon < 0$ be as in Lemma 5.1. By (5.23) and (5.22), we have that $\Psi'_\epsilon(u_\epsilon) = \Psi''_\epsilon(u_\epsilon) = 0$ and since $u_\epsilon < 0$, $\Psi'''_\epsilon(u_\epsilon) = G''_\mu(u_\epsilon) < 0$. Thus the steepest descent paths for Ψ_ϵ emanating from u_ϵ make angles respectively 0 , $\frac{2\pi}{3}$ and $-\frac{2\pi}{3}$. By (5.24) and (5.32), we have

$$\Psi_\epsilon^{(j)}(u_\epsilon) = -\kappa_j \epsilon^{-\frac{2}{3}(2j-1)}(1 + o(1)), \text{ as } \epsilon \rightarrow 0, \quad (5.36)$$

for $j \geq 3$, where $\kappa_3, \kappa_4, \dots$ are positive constants independent of ϵ .

For the rest of the proof, we set $q_\epsilon = \left| \frac{G''_\mu(u_\epsilon)}{6} \right|$ and we define the two contours $\Sigma_{\text{in},n}$ and $C_{\text{in},n}$ as in Figure 7 by

$$\begin{aligned} \Sigma_{\text{in},n} = & \left\{ u_\epsilon + \frac{1}{4\sqrt{3}n^{\frac{1}{3}}q_\epsilon^{\frac{1}{3}}} + it : t \in \left[-\frac{1}{4n^{\frac{1}{3}}q_\epsilon^{\frac{1}{3}}}, \frac{1}{4n^{\frac{1}{3}}q_\epsilon^{\frac{1}{3}}} \right] \right\} \\ & \cup \left\{ u_\epsilon + te^{i\pi/3} : t \in \left[\frac{1}{2\sqrt{3}n^{\frac{1}{3}}q_\epsilon^{\frac{1}{3}}}, |u_\epsilon| \right] \right\} \cup \left\{ u_\epsilon + te^{-i\pi/3} : t \in \left[\frac{1}{2\sqrt{3}n^{\frac{1}{3}}q_\epsilon^{\frac{1}{3}}}, |u_\epsilon| \right] \right\}, \quad (5.37) \end{aligned}$$

$C_{\text{in},n}$ being the mirror image of $\Sigma_{\text{in},n}$ with respect to the vertical line passing through u_ϵ . Note that the condition $\epsilon n^{\frac{3}{2}} \rightarrow +\infty$ as $n \rightarrow +\infty$ is needed for the path $\Sigma_{\text{in},n}$ not to cross the positive real axis.

We will show that the leading behaviour of K_n^S in (5.35) as $n \rightarrow +\infty$ comes from the integration over these two local paths, and that it converges to the Airy kernel.

Define

$$K_n^{\text{in}}(x, y) = -\frac{n}{(2\pi i)^2 \epsilon^2} \int_{C_{\text{in},n}} ds \int_{\Sigma_{\text{in},n}} d\zeta \frac{1}{s - \zeta} \begin{pmatrix} 0 & 1 \end{pmatrix} Y_n^{-1}(s) Y_n(\zeta) \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{\frac{n}{2\epsilon^2}((x-s)^2 - (y-\zeta)^2)}. \quad (5.38)$$

We first note that the asymptotic expansion (5.10) holds in fact not only in the outer region A_δ , but also when z approaches 0 at a sufficiently slow rate as $n \rightarrow +\infty$. If $n \rightarrow +\infty$ and $z \rightarrow 0$ in such a way that $n^2 z \rightarrow \infty$, (5.10) remains valid, but with a weaker error term $o(1)$ instead of $\mathcal{O}(1/n)$. This follows after a comparison of (5.10) and (5.19), by the asymptotic behaviour of the Bessel function $J_\alpha(z)$ as $z \rightarrow \infty$. Since $\epsilon n^{\frac{3}{2}} \rightarrow +\infty$ as $n \rightarrow +\infty$, this means that we can use (5.10), with weaker error term, for all points of the contour $C_{\text{in},n} \times \Sigma_{\text{in},n}$. We obtain

$$K_n^{\text{in}}(x, y) = -\frac{n}{(2\pi i)^2 \epsilon^2} \int_{C_{\text{in},n}} ds \int_{\Sigma_{\text{in},n}} d\zeta e^{\frac{n}{2\epsilon^2}((x-s)^2 - (y-\zeta)^2)} e^{n(g_\mu(s) - g_\mu(\zeta))} f(s, \zeta) (1 + o(1)),$$

as $n \rightarrow +\infty$. Here f is a certain function independent of n which can be expressed in terms of φ , which satisfies $f(z, z) = 1$ since $\det Y_n = 1$, but whose precise form is not important for us. We define

$$c_\epsilon = \epsilon^{-2} q_\epsilon^{-\frac{1}{3}}, \quad \varphi_{\epsilon,n}(z) = \frac{n^{1/3} z}{c_\epsilon \epsilon^2} (u_\epsilon - a_\epsilon). \quad (5.39)$$

We now let x and y approach the left edge a_ϵ at an appropriate speed and we get

$$\begin{aligned} \frac{1}{c_\epsilon n^{\frac{2}{3}}} K_n^{\text{in}} \left(a_\epsilon - \frac{x}{c_\epsilon n^{\frac{2}{3}}}, a_\epsilon - \frac{y}{c_\epsilon n^{\frac{2}{3}}} \right) &= -\frac{n^{\frac{1}{3}}}{c_\epsilon (2\pi i)^2 \epsilon^2} e^{\varphi_{\epsilon,n}(x) - \varphi_{\epsilon,n}(y)} \\ &\times \int_{C_{\text{in},n}} ds \int_{\Sigma_{\text{in},n}} d\zeta \frac{e^{n\Psi_\epsilon(s) + \frac{\pi n^{\frac{1}{3}}}{\epsilon^2 c_\epsilon}(s - u_\epsilon)}}{e^{n\Psi_\epsilon(\zeta) + \frac{\pi n^{\frac{1}{3}}}{\epsilon^2 c_\epsilon}(\zeta - u_\epsilon)}} \frac{1}{s - \zeta} f(s, \zeta) (1 + o(1)), \quad n \rightarrow +\infty. \end{aligned} \quad (5.40)$$

Using the asymptotics for the derivatives (5.36) and the fact that $f(z, z) = 1$, we may expand the phase functions and f around u_ϵ to get

$$\begin{aligned} \frac{1}{c_\epsilon n^{\frac{2}{3}}} K_n^{\text{in}} \left(a_\epsilon - \frac{x}{c_\epsilon n^{\frac{2}{3}}}, a_\epsilon - \frac{y}{c_\epsilon n^{\frac{2}{3}}} \right) &= -\frac{n^{\frac{1}{3}}}{c_\epsilon (2\pi i)^2 \epsilon^2} e^{\varphi_{\epsilon,n}(x) - \varphi_{\epsilon,n}(y)} \\ &\times \int_{C_{\text{in},n}} ds \int_{\Sigma_{\text{in},n}} d\zeta \frac{e^{-nq_\epsilon(s - u_\epsilon)^3 + n^{\frac{1}{3}} q_\epsilon^{\frac{1}{3}} x(s - u_\epsilon) + r_4^{n,\epsilon}(s)}}{e^{-nq_\epsilon(\zeta - u_\epsilon)^3 + n^{\frac{1}{3}} q_\epsilon^{\frac{1}{3}} y(\zeta - u_\epsilon) + r_4^{n,\epsilon}(\zeta)}} \frac{1}{s - \zeta} (1 + o(1)), \end{aligned} \quad (5.41)$$

as $n \rightarrow +\infty$, since $s - \zeta \rightarrow 0$, where

$$r_4^{n,\epsilon}(z) = \frac{n}{2\epsilon^2} \sum_{j=4}^{+\infty} \frac{\Psi_\epsilon^{(j)}(u_\epsilon)}{j!} (z - u_\epsilon)^j. \quad (5.42)$$

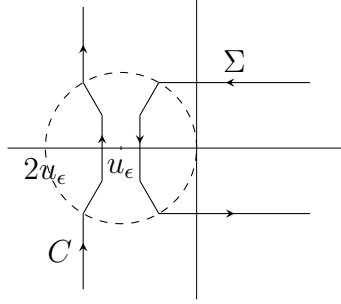


Figure 8: Choices for the contours C and Σ .

We now apply the changes of variables $s \mapsto u$ and $\zeta \mapsto v$ defined implicitly by

$$\begin{aligned} -nq_\epsilon(s - u_\epsilon)^3 + r_4^{n,\epsilon}(s) &= -u^3, \\ -nq_\epsilon(\zeta - u_\epsilon)^3 + r_4^{n,\epsilon}(\zeta) &= -v^3, \end{aligned}$$

such that $u \sim q_\epsilon^{1/3}n^{1/3}(s - u_\epsilon)$ and $v \sim q_\epsilon^{1/3}n^{1/3}(\zeta - u_\epsilon)$ as $n \rightarrow +\infty$. We then get

$$\begin{aligned} e^{-\varphi_{\epsilon,n}(x) + \varphi_{\epsilon,n}(y)} \frac{1}{c_\epsilon n^{\frac{2}{3}}} K_n^{\text{in}} \left(a_\epsilon - \frac{x}{c_\epsilon n^{\frac{2}{3}}}, a_\epsilon - \frac{y}{c_\epsilon n^{\frac{2}{3}}} \right) = \\ - \frac{1}{(2\pi i)^2} \int_{C_n} du \int_{\Sigma_n} dv \frac{e^{-u^3 + xu}}{e^{-v^3 + yv}} \frac{1}{u - v} (1 + o(1)) = \mathbb{K}^{\text{Ai}}(x, y) + o(1), \text{ as } n \rightarrow +\infty, \end{aligned} \quad (5.43)$$

C_n and Γ_n being contours that grow to the contours of the Airy integral formula as $n \rightarrow +\infty$.

Let γ_1 and γ_2 be the contours as in Lemma 5.2. We take the contours C and Σ , depending on n , to be $C = \bar{\gamma}_2 \cup C_{\text{in},n} \cup \gamma_2$ and $\Sigma = \bar{\gamma}_1 \cup \Sigma_{\text{in},n} \cup \gamma_1$ as pictured in Figure 8. We have

$$\left| \int_{C_{\text{in},n}} ds \int_{\gamma_1} d\zeta \frac{f(s, \zeta)}{s - \zeta} \frac{e^{n\Psi_\epsilon(s) + n^{\frac{1}{3}}q_\epsilon^{\frac{1}{3}}x(s - u_\epsilon)}}{e^{n\Psi_\epsilon(\zeta) + n^{\frac{1}{3}}q_\epsilon^{\frac{1}{3}}y(\zeta - u_\epsilon)}} \right| \leq \int_{C_{\text{in},n}} |ds| \int_{\gamma_1} |d\zeta| \left| \frac{f(s, \zeta)}{s - \zeta} \right| \frac{e^{n\text{Re}(\Psi_\epsilon(s)) + n^{\frac{1}{3}}q_\epsilon^{\frac{1}{3}}\text{Re}(x(s - u_\epsilon))}}{e^{n\text{Re}(\Psi_\epsilon(\zeta)) + n^{\frac{1}{3}}q_\epsilon^{\frac{1}{3}}\text{Re}(y(\zeta - u_\epsilon))}}. \quad (5.44)$$

In view of (5.36), we know that for n sufficiently large, the real part of the phase function is increasing on $\Sigma_{\text{in},n} \cap \{\text{Im } z > 0\}$ and decreasing on $C_{\text{in},n} \cap \{\text{Im } z > 0\}$. Moreover, $n^{\frac{1}{3}}q_\epsilon^{\frac{1}{3}}$ becomes small compared to $n\text{Re}(\Psi_\epsilon(s))$. The function $\text{Re}(\Psi_\epsilon(\gamma_1(t)))$ is increasing on $(0, +\infty)$ because of Lemma 5.2, and moreover grows in such a way that $\text{Re}(\Psi_\epsilon(\gamma_1(t))) \sim t^2$ as $t \rightarrow +\infty$, which means that the integral over γ_1 is exponentially small. In a similar way one shows that all parts of the integration contour $(C \times \Sigma) \setminus (C_{\text{in},n} \times \Sigma_{\text{in},n})$ give exponentially small contributions.

The limit (2.10) now follows from (5.43). $\square$

5.3 Proofs of Theorem 2.2, Theorem 2.3, and Theorem 2.4

To prove the results, once more we only need to verify that the conditions of Lemma 1.2 are satisfied.

We start with the Ginibre case, Theorem 2.2. Let K_n be the correlation kernel for the squared singular values of a product of Ginibre matrices, given by (2.14), and let $\widetilde{K}_n$ be given by (2.15).

We will first prove that there exists $c_1 > 0$ independent of n such that

$$\left| \frac{1}{n^{m+1}} \widetilde{K}_n \left(\frac{x}{n^{m+1}}, \frac{y}{n^{m+1}} \right) \right| \leq c_1 y^{-1/2} e^{|x|} \text{ on } i\mathbb{R} \times \mathbb{R}^+. \quad (5.45)$$

This implies condition (1.11) with $\beta = 1/2$. Using the fact that

$$\frac{\Gamma(t - n + 1)}{\Gamma(s - n + 1)} = \frac{\Gamma(n - s)}{\Gamma(n - t)} \frac{\sin \pi s}{\sin \pi t} \quad (5.46)$$

(which follows from the reflection formula of the Γ function), we obtain

$$\frac{1}{n^{m+1}} \widetilde{K}_n \left(\frac{x}{n^{m+1}}, \frac{y}{n^{m+1}} \right) = \frac{1}{(2\pi i)^2} \int_{-\frac{1}{2} + i\mathbb{R}} ds \int_{\Sigma_n} dt \prod_{j=0}^m \frac{\Gamma(s + \nu_j + 1)}{\Gamma(t + \nu_j + 1)} \frac{\Gamma(n - s)}{\Gamma(n - t)} \frac{\sin \pi s}{\sin \pi t} \frac{x^t y^{-s-1}}{s - t} n^{s-t}. \quad (5.47)$$

We can compute the t -integral using the residue theorem. The only poles of the t -integrand are the ones of $1/\sin \pi t$ since $1/\Gamma$ is entire, and the s - and t -contours do not intersect. We obtain

$$\begin{aligned} \frac{1}{n^{m+1}} \widetilde{K}_n \left(\frac{x}{n^{m+1}}, \frac{y}{n^{m+1}} \right) &= \frac{1}{2\pi i} \sum_{k=0}^{n-1} (-1)^k \frac{n^{-k} x^k}{\Gamma(n - k)} \prod_{j=0}^m \frac{1}{\Gamma(k + \nu_j + 1)} \\ &\quad \times \int_{-\frac{1}{2} + i\mathbb{R}} \prod_{j=0}^m \Gamma(s + \nu_j + 1) \Gamma(n - s) (\sin \pi s) y^{-s-1} n^s \frac{1}{s - k} ds. \end{aligned} \quad (5.48)$$

The integral is absolutely convergent since $|\sin \pi (\frac{1}{2} + it)| \sim \frac{e^{\pi|t|}}{2}$ and [32, Formula 5.11.9]

$$\left| \Gamma \left(\frac{1}{2} + \nu_j + it \right) \right| \sim \sqrt{2\pi} |t|^{\nu_j} e^{-\frac{\pi|t|}{2}}, \quad (5.49)$$

as $t \rightarrow \pm\infty$. For s on the integration contour, we have $|y^{-s-1} n^s| = (ny)^{-\frac{1}{2}}$. Moreover, we have [32, Inequality 5.6.6] $|\Gamma(x + iy)| \leq |\Gamma(x)|$ and thus

$$\begin{aligned} \left| \frac{1}{n^{m+1}} \widetilde{K}_n \left(\frac{x}{n^{m+1}}, \frac{y}{n^{m+1}} \right) \right| &\leq \frac{1}{\pi} y^{-\frac{1}{2}} \int_{-\frac{1}{2} + i\mathbb{R}} \prod_{j=0}^m |\Gamma(s + \nu_j + 1)| |\sin \pi s| |ds| \\ &\quad \times \sum_{k=0}^{n-1} \frac{|x|^k}{k!} \frac{n^{-k-\frac{1}{2}} \Gamma(n + 1/2)}{\Gamma(k + 1) \Gamma(n - k)}. \end{aligned} \quad (5.50)$$

It follows from Stirling's Inequality [32, Inequality 5.6.1] that there exists a constant $C > 0$ such that for $n \in \mathbb{N}$, $k = 0, \dots, n-1$,

$$\frac{n^{-k-\frac{1}{2}}\Gamma(n+1/2)}{\Gamma(k+1)\Gamma(n-k)} \leq C. \quad (5.51)$$

In $|x|$, we are left with a truncated exponential series. This implies (5.45).

To prove (1.10) with $\beta = 1/2$, we follow the proof in [27, Section 5.2], where (2.16) was proved uniformly for x, y in compact subsets of $(0, +\infty)$. We need uniformity of (1.10) for u in any compact subset of $\mathbb{C}$ and v in any compact subset of $[0, +\infty)$. Therefore, we fix $R > 0$ and assume that $|u| < R$ and $v \in [0, R]$. Using the asymptotics $\frac{\Gamma(n-s)}{\Gamma(n-t)} = n^{t-s} (1 + \mathcal{O}(n^{-1}))$ as $n \rightarrow \infty$, we obtain from (5.47) that

$$\begin{aligned} & \frac{v^{1/2}}{n^{m+1}} \widetilde{K}_n \left(\frac{u}{n+1}, \frac{v}{n+1} \right) \\ &= \frac{1}{(2\pi i)^2} \int_{-\frac{1}{2}+i\mathbb{R}} ds \int_{\Sigma_n} dt \prod_{j=0}^m \frac{\Gamma(s+\nu_j+1)}{\Gamma(t+\nu_j+1)} \frac{\sin \pi s}{\sin \pi t} \frac{u^t v^{-s-\frac{1}{2}}}{s-t} (1 + \mathcal{O}(n^{-1})), \end{aligned} \quad (5.52)$$

as $n \rightarrow \infty$. Using Stirling's formula and the fact that $|\sin \pi t| \geq |\sinh(\pi \operatorname{Im} t)|$, we can bound $\left| \frac{u^t}{\sin \pi t \prod_{j=0}^m \Gamma(t+\nu_j+1)} \right|$ by a function independent of u which decays rapidly as $t \rightarrow +\infty$. It follows that we can deform the integration contour Σ_n to Σ , provided Σ lies not too close to the real line. For the s -integrand, we can use (5.49) and the fact that $|v^{-s-\frac{1}{2}}| = 1$ to bound it uniformly. By the dominated convergence theorem, we can take $(1 + \mathcal{O}(n^{-1}))$ (with error term independent of u and v) out of the integral, and we obtain (1.10) uniformly in u and v .

Remark 11. Although the kernel for the squared singular values of products of Ginibre matrices cannot be expressed in terms of orthogonal polynomials, it can be expressed in terms of the solution to a Riemann-Hilbert problem for multiple orthogonal polynomials [27, Section 2.2], and it has a Christoffel-Darboux type formula. If large n asymptotics for this Riemann-Hilbert problem were available, one could be optimistic that similar techniques as in Section 5.2 can be used in the super-critical case to prove convergence to the Airy kernel.

Remark 12. The proofs of Theorem 2.3 and Theorem 2.4 are very similar to the proof of Theorem 2.2. For Theorem 2.3, one has to start with the expression (2.21) for the kernel K_n instead of (2.14). Then one shows in a similar way as in the Ginibre case that

$$\lim_{n \rightarrow +\infty} v^{1/2} \frac{1}{c_n} K_n \left(\frac{u}{c_n}, \frac{v}{c_n} \right) = v^{1/2} \mathbb{K}_{\nu, \mu}^T(u, v), \quad (5.53)$$

uniformly for u in any compact subset of $\mathbb{C}$ and v in any compact subset of $[0, +\infty)$, and that there are constants C_1, C_2 such that

$$|K_n(u, v)| \leq C_1 v^{-1/2} c_n^{1/2} e^{C_2 c_n(|u|+|v|)}, \quad (5.54)$$

for every $(u, v) \in i\mathbb{R} \times [0, +\infty)$ and $n > n_0$. The two above conditions are almost the same as the assumptions in Lemma 1.2, except for the fact that c_n now plays the role of cn^γ . From the proof of Lemma 1.2, it is straightforward to check that conditions (5.53) and (5.54) imply (1.12) and (1.13) with cn^γ replaced by c_n . This leads to the proof of Theorem 2.3 without further complications

For Theorem 2.4, one has to use the expression (2.28) or (2.29) instead. The rest of the proof is a straightforward adaptation of the proof of Theorem 2.2, and we do not give the details here.

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