

# Limit state analysis of historical structures using graphic statics related to the Principle of Virtual Works

J.F. Rondeaux & D. Zastavni

*Université catholique de Louvain, Louvain-la-Neuve, Belgium*

**ABSTRACT:** Because of the incomplete knowledge we have of the mechanical characteristics and of the history of loading of historical structures, these are often advantageously analyzed by means of graphic statics, e.g. using thrust lines as possible load paths within masonry arches to demonstrate their stability. In this context, theory of plasticity provides a powerful theoretical framework for the limit state analysis of structures, in particular for those in masonry on condition of several key assumptions: infinite strength in compression but zero in traction; no sliding failure. Starting with the analysis of a simple trussed framework and to basic bended structures, this paper emphasizes that limit analyses implying graphic statics can lead to similar conclusions about the collapse load factor that the ones obtained using the Principle of Virtual Works. Then it states a way to adapt this methodology to ancient masonry structures and presents results obtained for a simple arch analysis.

## 1 INTRODUCTION

Structural analysis of historical structures is a delicate matter because it must deal with complex geometrical data and incomplete knowledge of the actual characteristics of its material. The safety of ancient masonry structures is particularly difficult to assess. However, plastic theory and limit state analysis provides powerful theoretical tools to understand their stability. Initially formulated for steel structures presenting a certain degree of hyperstaticity, plastic theory could advantageously be applied to masonry structures providing certain assumptions were made on the materials. (Heyman 2008)

These assumptions must ensure that the three fundamental structural requirements are achieved:

- The structural material shows a ductile behavior.
- The working deformations are small compared to the overall dimensions of the structure.
- The stability of all structural members should be ensured: a decrease of load may not be observed with increasing deformation.

If so, the three plastic theorems can be used to evaluate the load factor that corresponds to the limit state of the structure before it collapses. This load factor is defined as the ratio between the load leading to the collapse of the structure and the actual load applied to it:

$$\lambda_c = \frac{F_c}{F} \quad (1)$$

The three plastic theorems may be expressed in the following way:

- Static theorem (lower bound/safe theorem): the load factor  $\lambda_s$  calculated on basis of a statically compatible distribution of internal forces and applied loads that respect the yield conditions is lower than or equal to the collapse load factor  $\lambda_c$ ;
- Kinematic theorem (upper bound/unsafe theorem): the load factor  $\lambda_k$  calculated on basis of a kinematically compatible mechanism is greater than or equal to the collapse load factor  $\lambda_c$ ;
- Uniqueness theorem: the collapse factor  $\lambda_c$  is unique.

Synthetically:

$$\lambda_s \leq \lambda_c \leq \lambda_k \quad (2)$$

Despite the brittleness of the singular elements and the discontinuous character of the assembly, masonry taken as a whole can be considered as a ductile material in which failure occurs when the loading induces the formation of a certain number of plastic hinges. (Heyman 1998) This ductile behavior could be evidenced experimentally in the case of masonry arches. (Gilbert 1997)

Applying elastic analysis procedures to masonry is problematic as there is no unique calculable equilibrium state. Contrariwise, using limit analysis

methods to study complex masonry structures, especially when presenting a high degree of hyperstaticity, considerably simplifies the problem. Furthermore, the incomplete knowledge of the actual composition of historical masonry structures and of their history of loading or movements makes the use of limit analyses particularly appropriate for structural assessment.

The very first concern of the structural analysis of such structures is to assess their stability. There is consequently usually no interest to know the actual distribution of efforts inside the structure. In addition the only information provided about these structures is most often exclusively geometrical, obtained from the survey which result is given as 2D drawings, exceptionally 3D. Therefore the use of an exclusively graphical method to assess the stability of such structures should be very appropriate.

In this paper, we first present the graphical principles and geometrical properties on which graphic statics is based. Then we briefly recall the Principle of Virtual Works we use to apply the kinematic theorem of plasticity to a simple trussed framework and to basic structures in bending to determine an upper value for the load factor ( $\lambda_k$ ). Each of these structures is also analyzed from a static point of view by means of the geometrical relationships on which graphic statics is based in order to determine a lower value for the load factor ( $\lambda_s$ ). Finally we discuss the conditions of its applicability to historical masonry structures by means of the analysis of a simple masonry arch and suggest some ideas for further research.

## 2 GRAPHIC STATICS

The main concern of this paper is the application of graphic statics to the limit state analysis of structures. Based on the graphical principle of vectorial addition first expressed for forces by S. Stevin (1548-1620) graphic statics determines the ways a set of forces can respect the equilibrium conditions. The application by J.C. Maxwell (1831-1879) of reciprocal figures properties to graphic statics – also studied around the same time by L. Cremona – provided a potent tool: each segment on the “form diagram” represents the action line of a force which magnitude is expressed in the reciprocal “force diagram” by the length of the corresponding segment. The two following relations between the reciprocal diagrams need to satisfy the reciprocity:

- Each segment in a diagram is related to only one sole segment in the other diagram, parallel to the latter. The parallelism could be replaced by any other lead angle providing that this angle is the same for all the segments.
- All the segments that are connected to a single point in one diagram need to form a closed poly-

gon in its reciprocal. This relationship ensures that the structure represented by the form diagram respects the rotational and translational equilibriums.

It must be underlined that, because of the reciprocity between the two diagrams, each of them can be considered as a form diagram as well as a force diagram. They constitute dual structures that provide graphical insight into each other (Baker 2015). In Maxwell's conjecture it is stated that in order for the reciprocal diagrams to exist, both of them must be perspective projections of reciprocal plane-faced polyhedra. One very interesting result is that it helps assessing mechanisms and states of self-stress in structures. This is done using the relation between the Maxwell rigidity numbers of both diagrams, defined as the difference between the number of mechanisms and states of self-stress of a structure:  $N = m - s$ . (Fig.1)

The rigidity number  $N$  of one of the dual structures and the one  $N^*$  of its reciprocal are related by:

$$N + N^* = -2 \quad (3)$$

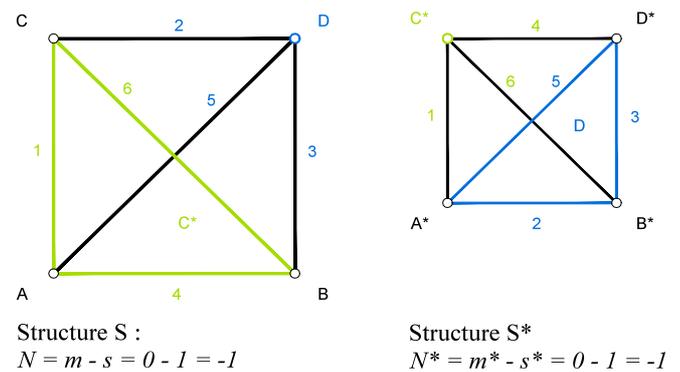


Figure 1. Reciprocal diagrams of dual structures S – S\*

Recent developments in graphic statics, by means of parametrical construction of both diagrams on which nodes geometrical constraints can be applied, offer the opportunity to control the range of possible equilibriums by limiting, for instance, the length of some segments in the force diagram or by constraining the action lines in the form diagram to pass through determined nodes. As a result, every node in either diagram can be constrained to remain inside a graphical region so that it respects the yield conditions and the equilibrium conditions of the structure. This ensures that the conditions of application of the lower bound plastic theorem are satisfied and that graphic statics can be applied to study limit states of structures by determining a lower value for the load factor  $\lambda_s$ .

In the next section we recall the Principle of Virtual Works (PVW) we will apply to some canonical structures in order to find an upper value of the load factor ( $\lambda_k$ ), so that we can compare it to the one determined by the static method ( $\lambda_s$ ).

### 3 PRINCIPLE OF VIRTUAL WORKS

The PVW states that in a structure the work of external forces is equal to the one of the internal stresses. Being  $(y, k, q)$  a set of kinematically compatible deformations that satisfies the displacement boundary conditions of a linear structure in bending; being  $(w, W, M)$  a set of linear or concentrated applied loads and bending moments in equilibrium that satisfies the loading boundary conditions; then the virtual works equation can be written as:

$$\sum_i W_i y_i + \int w y d\theta = \sum_j M_j \theta_j + \int M \kappa d\Omega \quad (4)$$

with  $W_i$  the concentrated loads,  $w$  the linear loads,  $y$  the perpendicular deflections,  $\kappa$  the curvature,  $M_j$  the value of bending moments where the hinge discontinuities  $\theta_j$  are observed and  $M$  the value of bending moment elsewhere.

Adapted to trussed frameworks, this equation can be written as:

$$\sum_n F_n \delta_{n//F} = \sum_b N_b \Delta L_b \quad (5)$$

with  $F_n$  the external load applied to joint  $n$ ,  $N_b$  the internal force acting on bar  $b$ ,  $\delta_{n//F}$  the displacement of node  $n$  in the direction of  $F_n$  and  $\Delta L_b$  the extension of bar  $b$ .

In the next section, PVW is applied to several canonical structures giving an upper value for the load factor  $\lambda_k$  since it allows the application of the kinematic theorem of plasticity.

### 4 GEOMETRICAL LIMIT STATES

In this section we apply both static and kinematic approaches to four canonical structures. The first one is a simple trussed framework presenting a possible self-stress state on which one external horizontal load is applied. Then we study two isostatic beams (simply supported; cantilever) using the same methodology that we finally apply to a hyperstatic beam (propped cantilever). The static approach is based on graphic statics (GS) and the kinematic one on the Principle of Virtual Works (PVW): the corresponding load factors ( $\lambda_s, \lambda_k$ ) are calculated independently. Finally we study the case of a single masonry arch submitted to a horizontal thrust.

#### 4.1 Trussed framework

GS – The two reciprocal diagrams have the particularity to present exactly the same geometry when not submitted to any external loading. (Fig. 1) Therefore they must have the same number of mechanisms and states of self-stress:  $N = m - s = N^* = m^* - s^* = -1$  because of (3). (Baker 2015) It corresponds to a

unique case of self-stress since there is obviously no mechanism possible.

When adding a set of forces ( $F_A, F_B, F_D$ ) in equilibrium to both diagrams, it affects  $N$  and  $N^*$  by decreasing the first of one unity and increasing its reciprocal equally by the addition of a possible mechanism in the dual structure that correspond to an indeterminacy in the construction of the force diagram. (Fig. 2a-b) In this case, the following relation can be deduced graphically from the geometrical properties of similar triangles in its reciprocal force diagram (Fig. 2b):

$$F_D = \frac{N_6 - N_5}{\sqrt{2}} \quad (6)$$

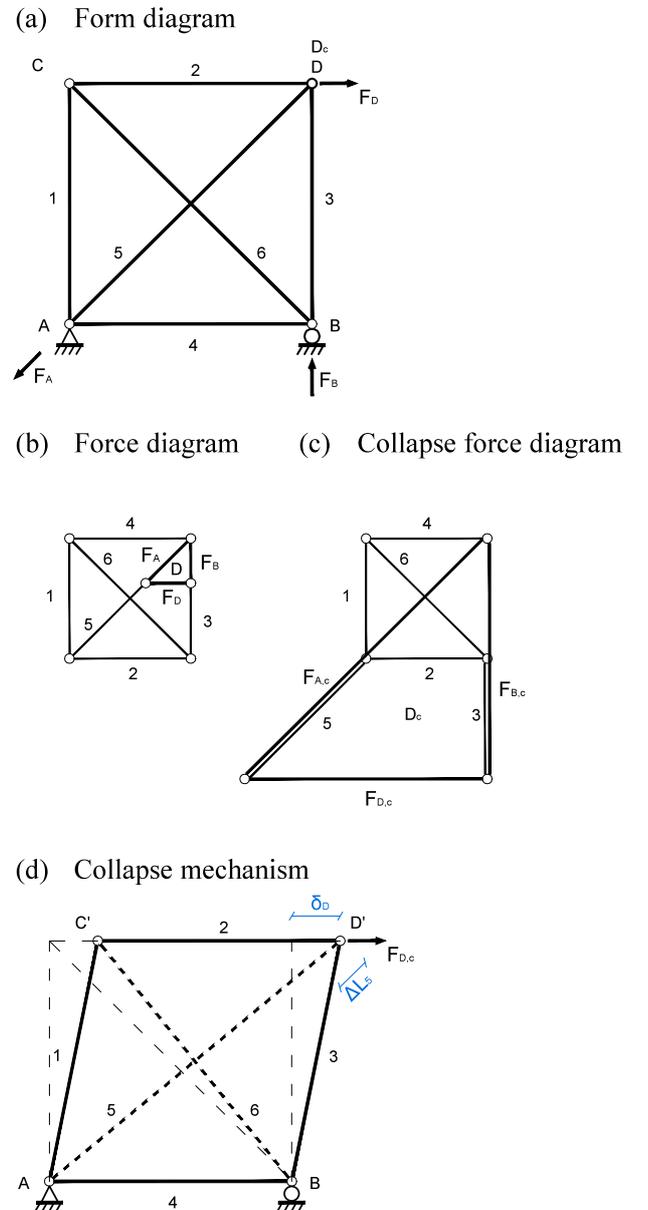


Figure 2. Trussed framework: limit state analysis

The limit state is reached when both bars 5 and 6 are solicited by the ultimate tension  $\pm N_c$  in traction or compression. In this case, the corresponding force diagram is unique for a fixed direction of  $F_D$  (Fig. 2c):

$$F_{D,c} = \frac{N_{6,c}^+ - N_{5,c}^-}{\sqrt{2}} = \frac{N_c + N_c}{\sqrt{2}} = \sqrt{2} \cdot N_c \quad (7)$$

$$\lambda_s = \frac{F_{D,c}}{F_D} = \frac{\sqrt{2} \cdot N_c}{F_D} \quad (8)$$

PVW – Applying the virtual works equation to this hyperstatic simple trussed framework (Fig. 2d) gives the same relation governing the intensity of the forces produced in the two diagonal members in function of the intensity of the applied force  $F_D$ , and consequently the same value for  $\lambda_k$  as for  $\lambda_s$ :

$$F_{D,c} \cdot \delta_D = N_{5,c} \cdot \Delta L_5 + N_{6,c} \cdot \Delta L_6 = 2 \cdot N_c \cdot \frac{\delta_D}{\sqrt{2}} \quad (9)$$

we can easily compare to (8), so that:

$$\lambda_k = \frac{\sqrt{2} \cdot N_c}{F_D} = \lambda_s = \lambda_c \quad (10)$$

We consider now a constant cross section rectangular beam in bending presenting three different end conditions.

#### 4.2 Simply supported beam

GS – The application of the static theorem of plasticity to the isostatic simply supported beam gives a lower value for the load factor. Using the similar triangles relationships between the two reciprocal diagrams (Fig. 3a-b) we can then determinate this value of  $\lambda_s$ :

$$\frac{\mu_1}{L_1} = \frac{F_A}{H}; \frac{\mu_1}{L - L_1} = \frac{F_B}{H}; F_A + F_B = F_1$$

$$F_A = \frac{F_1 \cdot (L - L_1)}{L}; F_B = \frac{F_1 \cdot L_1}{L}$$

$$\lambda_s = \frac{F_{1,c}}{F_1} = \frac{H_c}{H} \cdot \left( \frac{\mu_c}{\mu_1} \right) = \frac{M_c}{F_B \cdot (L - L_1)} = \frac{M_c \cdot L / L_1}{F_1 \cdot (L - L_1)} \quad (11)$$

PVW – The application of the PVW to this same beam gives the classical result for the value of the kinematic load factor  $\lambda_k$  (Fig. 3c):

$$\lambda_k \cdot F_1 \Delta_1 = M_c \cdot (\theta_A + \theta_B) = M_c \cdot \left( \frac{\Delta_1}{L_1} + \frac{\Delta_1}{L - L_1} \right)$$

$$\lambda_k = \frac{M_c \cdot L / L_1}{F_1 \cdot (L - L_1)} \quad (12)$$

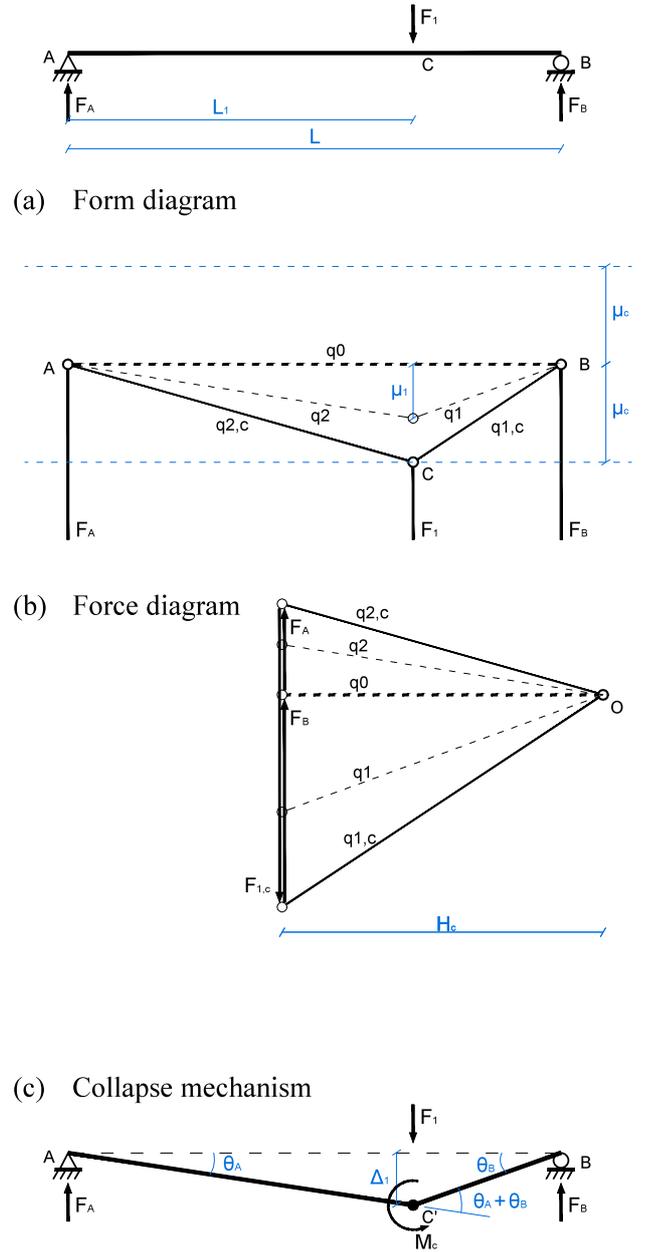


Figure 3. Simply supported beam: limit state analysis

As these two approaches give the same load factor, (11) and (12) ensure that:  $\lambda_k = \lambda_s = \lambda_c$

#### 4.3 Cantilever beam

GS – In order to use the similar triangles relationships in the reciprocal diagrams, we need to find out a way to model the bending moment at the clamped end of the beam in the field of graphic statics. From the equivalence between bending moment and couple of forces, we build up a graphical moment unit composed of two couples of forces in equilibrium. The graphical condition for this to exist is simply that they form similar rectangles in both form and force diagrams. (Fig. 4a-b-c)

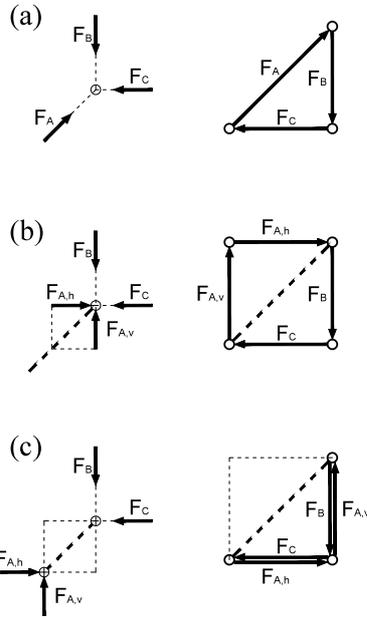


Figure 4. Bending moment in graphic statics

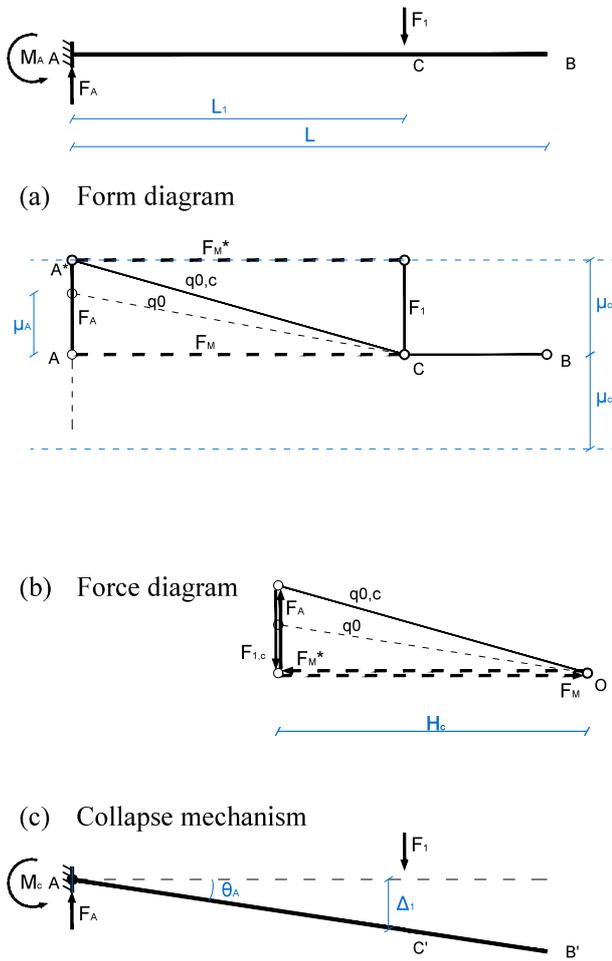


Figure 5. Cantilever beam: limit state analysis

Applying the relationships in similar triangles in the reciprocal diagrams for this isostatic structure (Fig. 5a-b) gives:

$$\lambda_s = \frac{F_{1,c}}{F_1} = \frac{H_c}{H} \cdot \left( \frac{\mu_c}{\mu_A} \right) = \frac{M_c}{F_1 \cdot L_1} \quad (13)$$

PVW – The application of the PVW to the correspondent collapse mechanism (Fig. 5c) gives a similar result for  $\lambda_k$  as the factor found in (13):

$$\lambda_k \cdot F_1 \Delta_1 = \lambda_k \cdot F_1 \cdot \theta_A L_1 = M_c \cdot \theta_A$$

$$\lambda_k = \frac{M_c}{F_1 \cdot L_1} \quad (14)$$

So that (13) and (14) ensure that:  $\lambda_k = \lambda_s = \lambda_c$ .

These results can be extended to multi-forces loadings. For example for the cantilever beam submitted to two forces  $F_1, F_2$  we obtain:

$$\lambda_s = \lambda_k = \frac{M_c}{F_1 L_1 + F_2 L_2} = \lambda_c \quad (15)$$

More interesting is the application of this methodology to hyperstatic structures.

#### 4.4 Propped cantilever beam

GS – In order to construct the reciprocal polygons corresponding to this structure, we apply the principle of superposition of two equilibrium systems, corresponding to the ones analyzed for the two isostatic basic cases, on which we applied alternatively  $F_1$  and a couple of forces  $F_M - F_M^*$  at distance  $\mu_A$ . (Fig. 6a) If we assume the normal force in the beam being zero, then  $F_M - F_M^*$  have the same magnitude (Fig. 6b)  $H$  as the other polar rays ( $q_1, q_2 \dots$ ). As we can see, the resulting combined force polygon presents a mechanism, corresponding to the state of self-stress in the form-diagram, so that the rule exposed for the reticular trusses can also be applied. Relations in similar triangles give:

$$F_1 = F_A + F_B = H \cdot \left( \frac{\mu_A + \mu_1}{L_1} + \frac{\mu_1}{L - L_1} \right)$$

At the limit state when  $F_{1,c}$  is applied,  $\mu_A = \mu_1 = \mu_c$ . Then the value of  $\lambda_s$  is given by:

$$\lambda_s = \frac{F_{1,c}}{F_1} = \frac{F_{A,c} + F_{B,c}}{F_1} = \frac{H}{F_1} \cdot \left( \frac{\mu_c + \mu_c}{L_1} + \frac{\mu_c}{L - L_1} \right) \quad (16)$$

$$= \frac{M_c}{F_1} \cdot \left( \frac{2}{L_1} + \frac{1}{L - L_1} \right)$$

So, with the proper constructions of reciprocal diagrams, we can build up a model able to define in a geometric way the load factor of hyperstatic structures. The graphic statics constructions ensure that the set of efforts in the beam is in equilibrium with the external loads.

PVW – Applying the PVW to this same hyperstatic structure (Fig. 6c) gives:

$$\lambda_k \cdot F_1 \Delta_1 = M_c \theta_A + M_c (\theta_A + \theta_B) = M_c \left( 2 \frac{\Delta_1}{L_1} + \frac{\Delta_1}{L - L_1} \right)$$

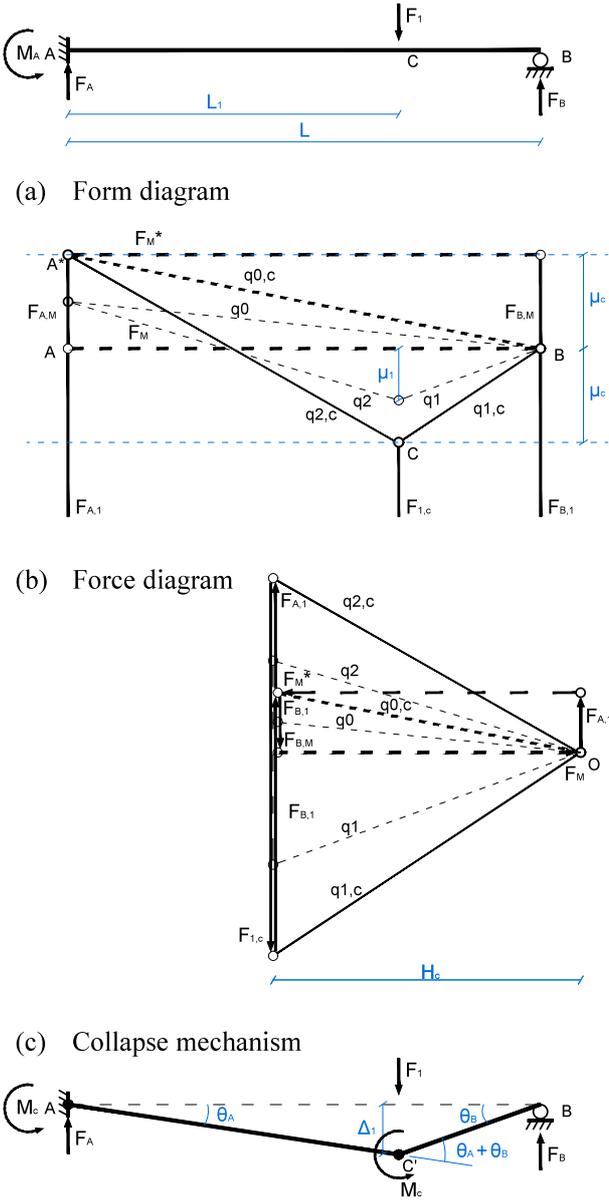


Figure 6. Propped cantilever beam: limit state analysis

$$\lambda_k = \frac{M_c}{F_1} \left( \frac{2}{L_1} + \frac{1}{L-L_1} \right) \quad (17)$$

The result of (17) is once more the same as the one obtained in (16) by the static approach for the corresponding collapse mechanism. Because of (2) we know it gives the value of the actual collapse load factor  $\lambda_c$ .

The graphical interpretation of the allowable positions of the pole  $O$  must take into account that the distance  $H$  on the force diagram linearly affects the value of  $M_c$ . (Fig. 7)

$$M_c = \mu_c \cdot H \cdot (n \cdot k) \quad (18)$$

where  $n$  and  $k$  the scales on which the form- and force diagrams are drawn, respectively.

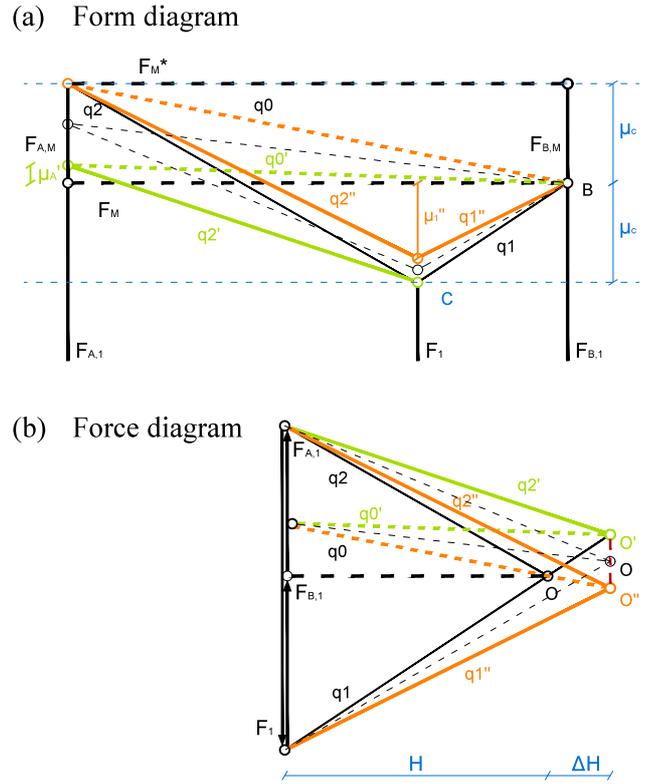


Figure 7. Propped cantilever beam:  $\lambda_s$  graphical interpretation

Therefore, if we want to keep the same envelope of bending moments we used in the form diagram (fixed  $\mu_c$ ), we must also keep the length  $H$  unchanged. It means that the pole  $O$  must move onto a vertical line; this constraint is very simple to implement. The length of the segment on which  $O$  must be placed, gives indirectly the value of the load factor  $\lambda_s$  corresponding to the limit state taken into account since:

$$\lambda_s = \frac{F_c}{F_1} = \frac{(F_1 + \Delta F)}{F_1} \cdot \left( \frac{H}{H} \right) = 1 + \frac{\Delta H}{H} \quad (19)$$

Since the force diagram (Fig. 7b) is drawn at a scale  $k$ , the load factor  $\lambda_s$  as expressed in (19) may be interpreted as the scale factor to apply to the actual force polygon (Fig. 7b - black) to transform it onto a limit state polygon (Fig. 7b - green).

In the next section we will apply a graphical limit state analysis to a single masonry arch submitted to its weight and to a horizontal thrust  $H$ .

#### 4.5 Four voussoirs arch

We construct the successive funicular polygons corresponding to the different limit states of an arch composed by four squared stones (Fig. 8). Each limit polygon on the form diagrams is related to one specific pole ( $O^+$   $O^-$   $O^*$   $O'$   $O''$ ) of the force diagrams.

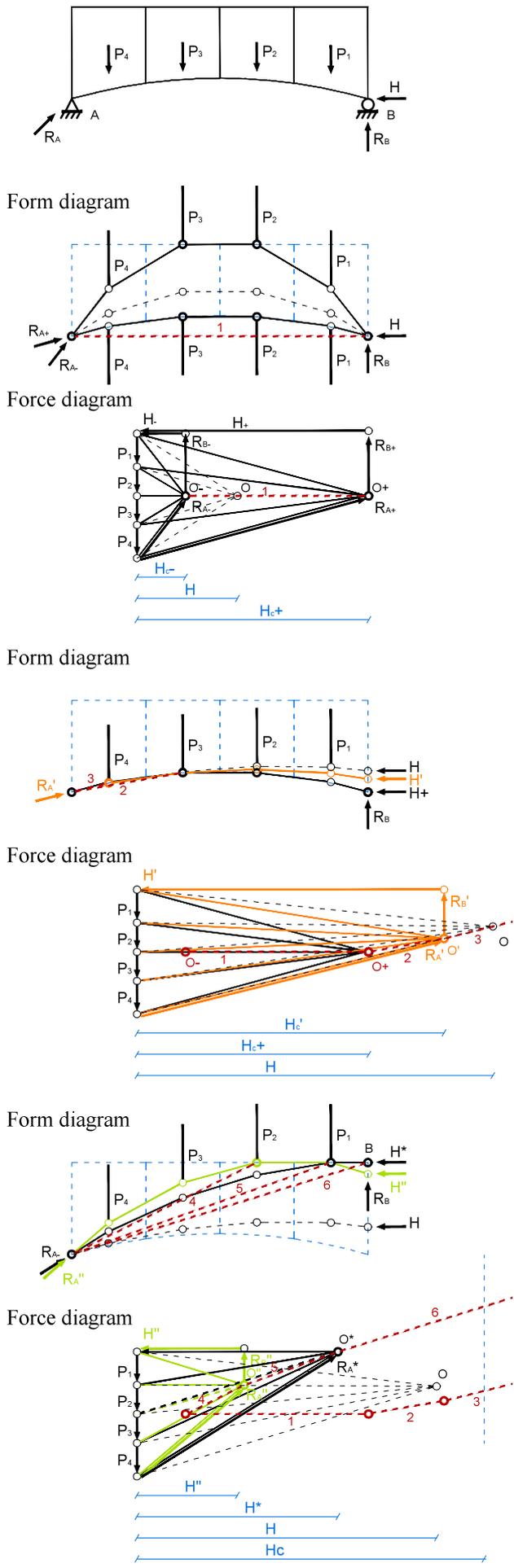


Figure 8. Four *voussoirs* arch: graphical limit state analysis

In order to simplify the construction of the limit form diagrams, we suppose them producing hinges only at the intersection between the action lines of the resulting weight of each of the four *voussoirs* and their outline shape. We also suppose the classical hypotheses about the structural behavior of masonry being applicable: the resistance of masonry is supposed infinite in compression but zero in traction and sliding cannot occur. So a limit state is reached when the funicular polygon forms at least three hinges on the intrados or extrados of the arch (Fig. 8 – bold circles). Each limit polygon corresponds to a specific pole. (Fig. 8 – bold circles) These poles are the nodes of an open polygon in which the pole  $O$  of any other funicular polygon must stay to ensure the stability of the arch. (Fig. 8 – red-dot lines 3-2-1-4-5-6) Bearing in mind that there is no compressive strength limitation, this domain is logically not limited on his right side and forms consequently an open polygon.

In this latter section we have shown that graphic statics is a very powerful tool when dealing with equilibrium of isostatic structures as well as with limit state analysis of hyperstatic structures. More precisely, we have shown how a lower value of the load factor can be calculated for a simple trussed framework and for isostatic and hyperstatic canonical structures in bending by means of graphical considerations on reciprocal diagrams. For each one of these cases, we showed that the load factor  $\lambda_s$  calculated only using geometrical properties of similar triangles is equal to the one  $\lambda_k$  obtained by application of the PVW. After having applied this graphical methodology to a single arch, we give some insights on the application of this method in the case of real and complex historical masonry structures.

## 5 HISTORICAL MASONRY STRUCTURES

Although identical results are given by traditional algebraic formulations in the most frequently used structures (trusses, beams, frames etc.), graphic statics makes the analysis of historical masonry structures simpler when the geometric complexity and unknown material properties disadvantage the algebraic formulation of Virtual Works. Following the three classical assumptions about the behavior of masonry (infinite strength in compression but none in traction; no sliding) this can be seen as an assemblage of stones shaped to pack together into a coherent structural form being maintained by compressive forces only transmitted within the mass of the material, that never reach such a compressive level able to reach the crushing constraint. (Heyman 1995 & 2008)

The idea that tension is not permissible is significant: it means that the action line of the resultant of the compressive forces – line of thrust – must of ne-

cessity lie inside the masonry envelope. Assuming that slip does not occur, we just need the value  $N$  of the normal – compressive – force together with its eccentricity  $e$  from the centerline to formulate the structural assessment. It is usual to work with the bending moment:  $M = N \cdot e$  as second variable instead of using directly this eccentricity  $e$ , because of the better convenience in tracing an  $N$ - $M$  diagram defining the stress state of each peculiar section. When using graphic statics on masonry structures, this transformation is neither opportune nor convenient since we can represent the eccentricity on the form diagram by the distance  $\mu$ .

When the eccentricity  $e$  is such that it corresponds to the limit of the masonry envelope ( $\mu = \mu_c$ ), then the compressive force is forced to pass by this point. This corresponds to the existence of a hinge at that point. This involves the assumption that material possesses an infinite compressive strength: real stone with a finite crushing strength does not permit the line of thrust to pass by a point lying on the border of the masonry envelope (Smars 2000). Taking that into account leads to limit  $e$  to a value:

$$e_{\text{lim}} = \frac{h}{2} - \frac{2N}{3 \cdot b \cdot \sigma_{adm}} \quad (20)$$

that should be used as value for  $\mu_c$ .

The  $M$ - $N$  relations are usually sketched in figures that represent the yield surfaces used in plastic theory, and plastic principles may be applied: a general point ( $N$ ,  $M$ ) lying within the full yield surface represents a safe state for the structure. Applied to masonry and graphic statics by means of the ( $N$ ,  $e$ ) couple, all what the analyst needs to show is that the line of thrust occupies such a position that it lies completely within the masonry. If one such position can be found, it is absolute proof – by the static theorem of plasticity – that the structure is safe.

## 6 CONCLUSION

This paper deals with graphic statics and its application to the limit analysis of a trussed framework, some bended structures as well as to a simple masonry arch. We place the research within the context of plastic theory in order to make use of the three fundamental theorems of plasticity. We applied them to a series of simple structure by means of graphic statics on one side and of the Principle of Virtual Works on the other and compare the results obtained. The way the reciprocal polygons of graphic statics are constructed and constrained ensures that the structure is in equilibrium and that the yield conditions are fulfilled. These are the two conditions for the static theorem of plasticity to be applied in order to determinate a lower bound value of the collapse load factor using graphic statics. For some canonical

isostatic and hyperstatic study cases we show that the graphic statics approach applied to specific collapse modes, gives an equal magnitude for the load factor that the one obtained by the application of the Principle of Virtual Works to the corresponding collapse mechanism. This latter principle is related to the kinematic theorem and provides an upper bound value for the load factor. Consequently the application of the third (uniqueness) theorem ensures that this load factor is the collapse one. If such a collapse mode cannot be identified because of the geometrical complexity of the structure or because of a lack of reliable information about the mechanical properties, as it is often the case for historical buildings, the static approach ensures that any load factor that can be found is lower than or equal to the collapse load factor.

A next step in this research will be to apply this method to structures presenting a higher degree of hyperstaticity as well as a more complex geometry. We will particularly focus in the future on historical masonry structures like retaining walls, three-leaf walls, arches, etc. and take into account a refined model for the mechanical behavior of masonry. New potentialities offered by parametric-oriented CAO software's can also be used to construct and constrain the reciprocal diagrams and provide an efficient program to assess graphically the security of ancient masonry structures or others.

## 7 REFERENCES

- Baker, W. & McRobie, A. & Mitchell, T. & Mazurek A. 2015. Mechanisms and states of self-stress of planar trusses using graphic statics, Part I: Introduction and background. *Proceedings of the International Association for Shell and Spatial Structures Symposium*. Amsterdam.
- Fivet, C. & Zastavni, D. 2015. A fully geometric approach for interactive constraint-based structural equilibrium design. *Computer-Aided Design*. 61:42-57
- Gilbert, M. 1997. Limit analysis applied to masonry arch bridges: state-of-the-art and recent developments. *Proceedings of the 5th International Conference on Arch Bridges*. Madeira.
- Heyman, J. 1995. *The stone skeleton: Structural engineering of masonry architecture*. Cambridge: Cambridge University Press.
- Heyman, J. 2008. *Basic structural theory*. Cambridge: Cambridge University Press.
- O'Dwyer, D. W. 1999. Funicular analysis of masonry vaults. *Computers and Structures* 73.
- Pirard, A. 1950. *La statique graphique – science introductive à l'art de construire*. Liège: Vaillant-Carmanne.
- Ritter, W. 1888. *Anwendungen der Graphischen Statik*. Zürich: Meyer & Zeller.
- Smars, P. 2000. *Etudes sur la stabilité des arcs et voûtes : confrontation des méthodes de l'analyse limite aux voûtes gothiques en Brabant*. PhD thesis. Leuven.
- Timoshenko, S. & Young, D.H. 1945. *Theory of structures*. New-York: McGraw-Hill Book Company.