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SHORT COMMUNICATION

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On the applicability of Stokes' hypothesis to low-Mach-number flows

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Abstract Stokes' hypothesis states that the bulk viscosity of a Newtonian fluid can be set to zero. Although not valid for many fluids, it is common practice to invoke this hypothesis in the study of low-Mach-number, variable-density flows. Based on scaling arguments, we provide a necessary condition for neglecting the bulk viscous pressure from the governing equations. More specifically, we show that the Reynolds number defined with respect to the bulk viscosity must be very large. We further show that even when this condition is not satisfied, the bulk viscous pressure does not need to be taken explicitly into account in the computation of the velocity field because it can be combined with the hydrodynamic pressure.

Keywords Bulk viscosity · Bulk viscous pressure · Stokes' hypothesis · Variable-density flows

1 Introduction

One of the fundamental results of non-equilibrium thermodynamics is that for simple and linearly isotropic (Newtonian) fluids, the bulk viscous pressure is proportional to the divergence of the velocity field, with the bulk viscosity of the fluid being the proportionality coefficient [14,16,21]. As the name suggests, the work performed by the bulk viscous pressure is irreversible, i.e., it is dissipated thereby increasing the entropy of the fluid. By contrast, the work performed by the fluid pressure is reversible. For constant-density flows, the velocity field is divergence-free and, therefore, the bulk viscous pressure is identically zero.

According to the kinetic theory of gases, the bulk viscosity of monoatomic gases is zero [2]; hence in this case, the bulk viscous pressure vanishes too. This is explained by the fact that the bulk viscosity is related to the rotational and vibrational modes of the molecules and, as such, it becomes zero for molecules without internal molecular structure. Besides this special case, the bulk viscosity is not zero or even small when compared with the shear viscosity. For many common gases (e.g., N_2, O_2 and light hydrocarbons) the bulk viscosity is of the order of the shear viscosity, while for others (e.g., H_2, water vapor and CO_2) it is significantly higher than the shear viscosity [3,11,15]. For instance, the bulk viscosity of CO_2 is three orders of magnitude higher than its shear viscosity [3]. Similarly, the bulk viscosity of liquids is not small either; see, for example, [3,6].

Still, in the study of fluid flows it is common practice to set the bulk viscosity equal to zero and neglect the bulk viscous pressure. This is the well-known Stokes' hypothesis, and its validity has been the subject of extensive debate; see, for example, [8,11] and references therein. However, given the large values of the bulk viscosity for many fluids, this hypothesis may introduce errors in the prediction of compressible flows of practical relevance, such as those encountered in power plants, propulsion devices and elsewhere. For example, in their recent numerical study, Pan and Johnsen [19] predicted that the bulk viscosity has a non-negligible effect

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on turbulence decay for gases with high bulk-to-shear viscosity ratios. Similarly, for such gases, theoretical [5] and numerical [4] studies have shown that it can influence the structure of supersonic boundary layers.

The present communication is concerned with the role of the bulk viscosity and bulk viscous pressure in low-Mach-number, variable-density flows. These are encountered in technological applications and natural phenomena involving heat transfer or chemical reactions. Therein, the Mach number is much smaller than unity so that compressibility effects are negligible, but density gradients are induced due to spatial variations in the temperature or in the chemical composition of the fluid. Further, the density gradients are sufficiently large so that the Boussinesq approximation is no longer applicable. Then, due to mass conservation, the divergence of the velocity is not zero and, consequently, the bulk viscous pressure cannot *a priori* be neglected. The analysis provided herein is based on the low-Mach-number approximation of the compressible Navier–Stokes–Fourier equations. Essentially, our study examines the effect of the velocity divergence $\nabla\cdot\boldsymbol{u}$ on the dissipative properties of low-Mach-number flows in the context of classical hydrodynamics. The effect of $\nabla\cdot\boldsymbol{u}$ on systems described by extended hydrodynamics [12,13,22] lies beyond the scope of the present study.

2 Scale analysis

For Newtonian fluids, the viscous stress tensor $\boldsymbol{\tau}$ relates linearly to the rate-of-strain tensor $\mathbf{V} = \frac{1}{2}\left(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^{\top}\right)$. More specifically, by virtue of the representation theorem for isotropic tensors [21], $\boldsymbol{\tau}$ is given by the following constitutive relation,

$$\boldsymbol{\tau} \;=\; 2\mu\mathbf{V} + \beta\,(\nabla\cdot\boldsymbol{u})\,\mathbf{I}\,, \tag{1}$$

the velocity divergence being the first invariant (trace) of $\mathbf{V}$. In the above equation, μ stands for the shear viscosity coefficient and β for the second viscosity coefficient, while $\mathbf{I}$ is the identity tensor. By decomposing the rate-of-strain tensor $\mathbf{V}$ into a deviatoric and a diagonal component, $\boldsymbol{\tau}$ can be written as the sum of a deviatoric tensor and a diagonal component that describes bulk viscous stresses,

$$\boldsymbol{\tau} \;=\; 2\mu\left(\mathbf{V} - \frac{1}{3}\,(\nabla\cdot\boldsymbol{u})\,\mathbf{I}\right) + \zeta\,(\nabla\cdot\boldsymbol{u})\,\mathbf{I}\,. \tag{2}$$

In the above equation, ζ is the bulk viscosity and is defined by

$$\zeta \;=\; \beta + \frac{2}{3}\mu\,. \tag{3}$$

In turn, the bulk viscous pressure p^{v} is identified as

$$p^{\mathrm{v}} \;=\; \zeta\,\nabla\cdot\boldsymbol{u}\,, \tag{4}$$

and the deviatoric stress tensor $\boldsymbol{\tau}^{\mathrm{d}}$ is identified as

$$\boldsymbol{\tau}^{\mathrm{d}} \;=\; 2\mu\mathbf{V} - \frac{2}{3}\mu\,(\nabla\cdot\boldsymbol{u})\,\mathbf{I}\,. \tag{5}$$

We remark that both the pressure p and the bulk viscous pressure p^{v} cause isotropic dilatation, whereas $\boldsymbol{\tau}^{\mathrm{d}}$ causes different types of deformation, namely anisotropic dilatation and shear. Further, both μ and ζ take nonnegative values so as to satisfy the 2nd axiom of thermodynamics (entropy inequality).

The low-Mach-number approximation results from the singular perturbation of the compressible Navier–Stokes–Fourier equations at low-Mach numbers, see, e.g., [17,18,20]. According to this procedure, first all flow quantities are made dimensionless with respect to reference values: l_{r}, u_{r}, ρ_{r}, p_{r}, μ_{r}, ζ_{r} etc. The reference values of the state variables, ρ_{r} and p_{r}, correspond to a reference thermodynamic state that is relevant to the flow under study, e.g., the initial state of the fluid. Since the transport coefficients are functions of the state variables, their reference values (μ_{r}, ζ_{r} and so on) are taken at the same reference thermodynamic state.

Upon non-dimensionalization, the parameter $\epsilon = \rho_{\mathrm{r}} u_{\mathrm{r}}^2 p_{\mathrm{r}}^{-1}$, which is proportional to the square of the Mach number of the flow, emerges as a multiplicative factor of certain terms of the governing equations. In other words, ϵ arises naturally as the perturbation parameter of the equations at the low-Mach numbers. Accordingly, all flow quantities and transport coefficients are expanded in asymptotic power-series in terms of ϵ. For example, the expansion of the non-dimensionalized pressure reads,

$$p \;=\; p_0 + \epsilon\, p_1 + \mathcal{O}(\epsilon^2)\,. \tag{6}$$

It is noted that this is not an expansion around a particular equilibrium point or the reference thermodynamic state. Instead, it is an asymptotic expansion for the perturbation of the compressible Navier–Stokes–Fourier equations around the singular zero-Mach-number limit, i.e., when the flow velocity is negligible compared to the speed of sound.

Accordingly, we insert ansatz (6) in the governing equations and collect terms of the same order. Then, by retaining only terms up to $\mathcal{O}(1)$, we recuperate the low-Mach-number approximation of the compressible Navier–Stokes–Fourier equations. In *dimensional* form, the resulting system reads,

$$\nabla p_0 = 0\,, \tag{7}$$

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} + \rho \nabla \cdot \boldsymbol{u} = 0\,, \tag{8}$$

$$\rho \frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} + \nabla p_1 = \nabla \cdot (2\mu \mathbf{V}) - \nabla \left(\frac{2}{3} \mu \nabla \cdot \boldsymbol{u} \right) + \nabla \left(\zeta \nabla \cdot \boldsymbol{u} \right) + \rho \boldsymbol{g}\,, \tag{9}$$

$$\rho c_p \frac{\mathrm{d}T}{\mathrm{d}t} = \frac{\partial p_0}{\partial t} + \nabla \cdot (\kappa \nabla T)\,. \tag{10}$$

For reasons of notational simplicity, in the above equations and henceforth we have dropped the subscript 0 from the leading-order terms of all quantities except for p_0. Also, the symbol $\frac{\mathrm{d}}{\mathrm{d}t}$ stands for the material derivative of a given quantity, $\boldsymbol{g}$ is the gravity vector, while c_p and κ stand for first-order terms of the fluid's isobaric specific heat and conductivity, respectively.

System (7)–(10) is closed with the low-Mach-number approximation of the thermal equation of state, which in general is written as

$$\rho = f(T, p_0)\,. \tag{11}$$

We observe that p_0, which scales with p_r, is uniform in space by virtue of (7). Also, it enters the low-Mach-number approximation of the energy and state equations but not of the momentum equation. In the literature of low-Mach-number flows, p_0 is usually referred to as the "thermodynamic pressure," in the sense that it appears in the low-Mach-number approximation (11) of the thermal equation of state.

By contrast p_1, which scales with $\rho_\mathrm{r} u_\mathrm{r}^2$ and is referred to as the "(hydro)-dynamic pressure," enters the low-Mach-number approximation of the momentum equation but not of the energy equation or the equation of state. Moreover, since it is only the gradient of p_1 that appears in (9), the value of p_1 itself needs to be known only up to a constant, i.e., its exact value is inconsequential. Further, the work of the viscous stresses is negligibly small compared to heat diffusion and, therefore, does not appear in the low-Mach-number approximation of the energy equation either.

It has been argued in [1] that an equivalent condition to Stokes' hypothesis is to assume that the absolute value of $\zeta \nabla \cdot \boldsymbol{u}$ is negligible compared to the fluid pressure p. By virtue of definition (4), this is identical to $|p^\mathrm{v}| \ll p$. It is also remarked in [1] that, no matter how small μ may be, one cannot neglect the components of $\boldsymbol{\tau}^\mathrm{d}$ by comparing their magnitude to p because $\boldsymbol{\tau}^\mathrm{d}$ and p cause different types of deformation. On the other hand, it is meaningful to compare p^v with p because, as mentioned above, they cause the same type of deformation.

In the case of low-Mach-number flows, the condition $|p^\mathrm{v}| \ll p$ is automatically satisfied. Indeed, since p^v scales with $\rho_\mathrm{r} u_\mathrm{r}^2$ and p scales with p_r, their ratio scales with the square of the Mach number.

We note, however, that since the various stresses enter the momentum equation via their gradients, it is more appropriate to compare the magnitude of the gradients of these stresses. Accordingly, in the general case of the compressible Navier–Stokes–Fourier equations, the following condition should be fulfilled if the bulk viscous pressure is to be neglected,

$$|\nabla p^\mathrm{v}| \ll |\nabla p|\,, \qquad \text{(general case)}. \tag{12}$$

For low-Mach-number flows, by virtue of (7) we have that in *dimensional* form, $\nabla p = \nabla p_1$. Then, by inserting this result and expression (4) into (12), we obtain the following condition,

$$|\nabla (\zeta \nabla \cdot \boldsymbol{u})| \ll |\nabla p_1|\,, \qquad \text{(low-Mach-number case)}. \tag{13}$$

This result makes sense because, after all, it is $\nabla(\zeta \nabla \cdot \boldsymbol{u})$ and ∇p_1 that enter the momentum Eq. (9). It is also noted that this is a local criterion and, as such, it should hold everywhere in the flow domain for the bulk viscous pressure to be neglected.

We next observe that: (i) the velocity divergence scales with $u_r l_r^{-1}$, (ii) p_1 scales with $\rho_r u_r^2$, and iii) the gradient operator scales with l_r^{-1}. In view of these scalings, condition (13) yields

$$\frac{\zeta}{\mu_r} \ll Re\,, \tag{14}$$

with Re being the relevant Reynolds number of the flow.

Equivalently, we may define a Reynolds number Re_ζ based on the bulk viscosity,

$$Re_\zeta = \frac{\rho_r l_r u_r}{\zeta_r}\,. \tag{15}$$

Then, criterion (14) translates to

$$Re_\zeta \gg 1\,, \tag{16}$$

with $Re_\zeta = \infty$ corresponding to Stokes' hypothesis.

There are many examples of low-Mach-number flows where this condition is fulfilled. However, there are also cases where this condition is not fulfilled even though p^{v} is orders of magnitude smaller than p. For example, in premixed combustion, the relevant length scale is the flame thickness, which renders the Reynolds number Re_ζ small so that condition (16) may no longer be fulfilled.

Regarding the applicability of condition (16), we remark the following. In general, fluid flow is characterized by the presence of a multitude of length scales. Then, the removal of a term from the governing equations should be based on an appropriate scale analysis. In turn, when applied to p^{v}, this procedure necessarily relies on bounds for the velocity divergence and its gradient. However, such bounds are hard to establish except for very specific cases. For this reason, it is more appropriate to consider (16) as a necessary but not sufficient condition for neglecting the bulk viscous pressure.

3 Combining the hydrodynamic and bulk viscous pressures

In this section, we demonstrate that in the computation of the velocity field of low-Mach-number flows, the bulk viscous pressure does not need to be taken explicitly into account, even in cases when condition (16) is not fulfilled. To this end, we first observe that both p_1 and p^{v} enter the momentum Eq. (9) only via their gradients. Then, under the hypothesis that the components of $\boldsymbol{u}$ are twice differentiable so that (9) makes sense, we can define a modified pressure p' as follows,

$$p' = p_1 - \zeta \nabla \cdot \boldsymbol{u}\,. \tag{17}$$

It is noted that p_1 is a non-dissipative pressure, whereas p^{v} is a viscous dissipative one. We argue however that, for the particular case of low-Mach-number flows, it is legitimate to combine them for the following reasons. First, as mentioned above, they cause the same type of deformation, namely isotropic dilatation. Second, the work of both p_1 and p^{v} is negligible and does not enter the low-Mach-number approximation (10) of the energy equation. Finally, p_1 does not enter the low-Mach-number approximation of either the state equation or the Gibbs relation and, therefore, the effect of p_1 on the thermodynamic state of the fluid is negligible.

Then, by substituting (17) into (9), the momentum balance law takes the following form,

$$\rho \frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} + \nabla p' = \nabla \cdot (2\mu \mathbf{V}) - \nabla \left(\frac{2}{3} \mu \nabla \cdot \boldsymbol{u} \right) + \rho \boldsymbol{g}\,, \tag{18}$$

Accordingly, Equation (18) can replace (9) in the governing system (8)–(10) without modifying either (8) or (10). Indeed, this substitution has been employed by Georgiou & Papalexandris [9,10], albeit without any elaboration.

This approach is applicable provided that p' can be assigned the same boundary conditions as those assigned to p_1. Typically, in numerical computations, p_1 is assigned zero-Neumann data at outflow boundaries and at solid walls. We examine therefore if the same boundary condition can be assigned to $p^{\mathrm{v}} = \zeta \nabla \cdot \boldsymbol{u}$ and, by extension, to p'. Let $\boldsymbol{n}$ stand for the unit normal to a boundary plane. Then, since ζ is a function of the temperature T and thermodynamic pressure p_0, $\zeta = \zeta(T, p_0)$, the normal derivative of p^{v} can be written as

$$\nabla p^{\mathrm{v}} \cdot \boldsymbol{n} = \zeta \nabla(\nabla \cdot \boldsymbol{u}) \cdot \boldsymbol{n} + (\nabla \cdot \boldsymbol{u}) \frac{\partial \zeta}{\partial T} \nabla T \cdot \boldsymbol{n} + (\nabla \cdot \boldsymbol{u}) \frac{\partial \zeta}{\partial p_0} \nabla p_0 \cdot \boldsymbol{n}\,. \tag{19}$$

The last term on the right-hand side of the above equation is zero, by virtue of Eq. (7). Therefore, Eq. (19) reduces to

$$\nabla p^{\mathrm{v}} \cdot \boldsymbol{n} \;=\; \zeta \nabla(\nabla \cdot \boldsymbol{u}) \cdot \boldsymbol{n} + (\nabla \cdot \boldsymbol{u}) \frac{\partial \zeta}{\partial T} \nabla T \cdot \boldsymbol{n}\,. \tag{20}$$

The first term on the right-hand side of (20) involves the second-order derivative of the normal velocity component in the direction of $\boldsymbol{n}$. By setting this term equal to zero, one introduces at most a second-order numerical error at the boundary. However, such an error is introduced anyway by standard second-order discretization schemes. Therefore, setting this term equal to zero is compatible with second-order accurate numerical algorithms.

Regarding the second term on the right-hand side of (20), it vanishes at adiabatically isolated walls and at outflow boundaries because in these cases the normal derivative of the temperature becomes zero. This term also vanishes when a given temperature is specified at a wall of an open domain because, in this case, the velocity divergence vanishes by applying mass continuity (8) at the wall. In fact, enforcing continuity at solid walls is a common technique in numerical methods for the constant-density Navier–Stokes–Fourier equations [7].

In summary, specifying zero-Neumann data for the bulk viscous pressure introduces at most a second-order numerical error at the boundary, thereby ensuring the applicability of the aforementioned approach, i.e., using (18) instead of (9). The proposed combination of the hydrodynamic and bulk viscous pressures allows for the computation of the velocity field in low-Mach number flows when the value of the bulk viscosity is not known with sufficient accuracy.

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