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Output externalities on total factor productivity

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# DISCUSSION PAPER

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#### Output externalities on total factor productivity

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#### Abstract

The impact that output has on future total factor productivity —i.e. the dynamic complementarities shown to be empirically relevant in Cooper and Johri (1997)— is not internalized by competitive agents. As a result, the allocation that a planner would choose cannot be reached as a competitive equilibrium outcome (neither for infinitely-lived agents nor for overlapping generations): the market return to savings and wage rate are too low. The planner's allocation can nonetheless be implemented by a fiscal policy subsidizing as needed the returns to savings and the wage rate. The exact policy differs depending on whether just past investment or total output influences productivity: in the first case only capital returns need to be subsidized, while in the second case labor income needs to be subsidized too. The policy is balanced period-by-period by means of a lump-sum tax.

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### 1. INTRODUCTION

It is well known that output is not just about pumping capital and labor into the production process. Aspects other than just quantities matter. Total factor productivity (TFP) is precisely thought to capture how labor and capital contribute to the output of an economy in ways other than by their amounts. In practice, TFP variations are measured as the residual increase in output not accounted for by increases in capital and labor, and hence the concept offers no explanation about its determinants. A broad range of candidates have been put forward to explain changes in total factor productivity, including changes in technology, innovations, human capital accumulation, learning by doing, organizational and management improvements, among others.<sup>1</sup> Such a panoply shows that we are still far from really understanding what drives changes in total factor productivity. But it obviously matters (for policy choices) to know what exactly drives more forcefully improvements in TFP, in particular whether it is just investment or the entire output.

That R+D investments have a huge potential in raising productivity is well understood, but the impact on productivity of all other expenditures comprised in the output —including expenditures that would normally be classified as consumption is far from being negligible too, as established in Cooper and Johri (1997) in the context of a stochastic growth model of infinitely-lived agents. In effect, among the drivers of total factor productivity can be found the way work is organized, the specific skills developed on the jobs as well as, more generally, human capital accumulation. Although the latter, usually measured by years of schooling, may well be classified as an expenditure distinct from consumption, its effectiveness as well as that of organizational skills and the working of society's institutions and legal framework— is no doubt greatly enhanced by expenditures that definitely fall into the category of mere consumption, from reading habits to food consumption at early stages (through the enhancement of cognitive skills).<sup>2</sup> More generally, a

<sup>&</sup>lt;sup>1</sup>Candidly enough, Easterly and Levine (2001) report that "we do not have empirical evidence [...] that confidently assesses the relative importance of each of these conceptions of TFP in explaining economic growth. Economists need to provide much more shape and substance to the amorphous term TFP". Even more to the point, the title in Prescott (1998) says it all.

<sup>&</sup>lt;sup>2</sup>To better realize to what extent merely consumed output may impact TFP just think of the ultimate consumption good: food. Empirical evidence shows that malnutrition from gestation to 24 months affects brain development to the point of translating into a 7% lower performance at 8 years old in mathematics, 19% lower in reading, 13% lower in writing, which compounds into a 20% lower future earnings —as reported in Save the Children Foundation (2013) (www.savethechildren.org.uk), from Young Lives (www.younglives.org.uk), a longitudinal study of 12,000 children across four countries throughout their lives. If this reflects, as it might be expected,

higher economic activity conveyed by a higher output reflects also an increase in the number, length, and density of supply chains and organizational links within and between firms that can only facilitate even higher increases in output in the future for the same increase in the inputs. All in all, it is only natural to expect to see a positive impact of past realizations of overall output per capita —i.e. including consumption per capita — on the total factor productivity of an economy, if only because more productive economies (i.e. with a longer history of higher per capita outputs) can afford a better trained and educated work force, a more secure and better enforced legal framework for property rights and contracts, a better organization of complex production processes, and a denser network of working links within and between firms that all allow for a more efficient use of capital and labor. The intuition therefore extends that of the dynamic complementarities quantitatively assessed in Cooper and Johri (1997) and supposed to represent the learning spillovers introduced in Durlauf (1991).

While Cooper and Johri (1997) have shown these dynamic complementarities to play an empirically relevant role in the propagation of the business cycle, I focus in this paper on finding policies allowing to exploit them in order to improve upon the laissez-fare outcome. In effect, economic agents acting in a competitive environment do not internalize in their decisions the additional positive impact that their saving, labor supply, and consumption decisions have on future output, via current output, through the improvement of future total factor productivity. As a result, competitive equilibria lead to allocations that differ from those that a planner internalizing such impact would choose. I show this to be the case for both infinitely-lived agents economies and overlapping generations economies, and I provide fiscal policies allowing to support the planner's choice as a market outcome.

More precisely, I show in Section 2 that, because of the impact of past output per capita on current total factor productivity, the competitive equilibrium of an infinitely-lived agent economy differs from the allocation that a planner internalizing the effect would choose for that economy (Proposition 3). In particular, the market systematically remunerates the agents too little for their savings and labor: the wage rate and the return to capital are, at every period, lower than those that the planner would choose instead. As a consequence, capital and labor supply are too low.

I show next, nonetheless, how a policy of subsidies and lump-sum taxes decentralizes the planner's choice (Proposition 5), allowing for fully exploiting the dynamic

an impact on labor productivity, its size is far from immaterial.

complementarities. The policy requires subsidizing linearly both capital and labor incomes at a *common* rate depending positively on the sensitivity of total factor productivity to past output. The policy is funded by means of a lump-sum tax. The exact decentralization requires from the policy-maker to choose the rate depending on the *future marginal productivity of capital* though, and hence to have perfect foresight.<sup>3</sup> Notwithstanding, for the widely used case of logarithmic utilities, a neoclassical production, and a total factor productivity that is a power function of past output, Proposition 6 establishes that when the policy is made to depend on lagged, observed variables instead, then *convergence* towards the planner's choice obtains.

In order to make sure that the results are not special to the infinitely-lived agents setup, I show next in Section 3 analogous results for an overlapping generations economy with heterogeneous agents and population growth. Specifically, the steady state that a utilitarian planner would choose (maximizing the sum of utilities of a distribution of identical generations) cannot be a competitive equilibrium outcome when past outputs have an impact on total factor productivity (Proposition 9). As in the infinitely-lived agents case, the market systematically remunerates the agents too little for their labor: the return to capital and the wage rate are, at every period, lower than the one the planner would choose instead. As in the previous setup, a policy of subsidies and lump-sum taxes allows, nonetheless, to implement the planner's choice as a competitive outcome (Propositions 10 and 11). Interestingly enough, form the analysis it follows that for an egalitarian planner's steady state the population growth rate needs to exceed the return to capital.

Finally, Section 4 addresses the issue of whether the entire past output is actually needed to capture the externalities on total factor productivity, instead of just past investment, as underlined by the literature on learning-by-doing starting with Arrow (1962), Levhari (1966), and Sheshinski (1967) —the driving idea being that past investments in physical capital increase the productivity of labor through a process of gradually driving up the skills needed to operate increasingly complex machinery or, more generally, working processes— or even the spillovers from R+D investments by other firms as in Romer (1986) —similarly, with the externality coming from the knowledge spillovers across firms. In effect, one might legitimately conjecture that, from the positive correlation between investment and output, it might eventually make no difference to pick either as the channel of the externality in order to capture the effect. Section 5 clarifies, nonetheless, that whether the externality is exerted through just investment or total output makes a difference indeed. Specifically, I show that implementing the planner's choice requires two

<sup>&</sup>lt;sup>3</sup>Although the agents are routinely and less scrupulously assumed to have perfect foresight anyway.

qualitatively different policies, depending on whether just past investment or the entire past output affects total factor productivity: subsidizing the returns to capital suffices in the case investment is the only source of externality, while subsidizing the returns to labor too is necessary in the case the entire output has an impact on future total factor productivity.

# 2. Output externalities on TFP with infinitely-lived agents

## 2.1 Competitive equilibria and planner's allocations.

Consider an infinitely-lived representative agent choosing the consumption  $c_t$ , labor supply  $l_t$ , and capital savings  $k_{t+1}$ , for all  $t = 0, 1, \ldots$ , that maximize his discounted sum of utilities subject to a sequence of budget constraints, i.e.

$$\max_{c_t, l_t, k_{t+1} \ge 0} \sum_{t=0}^{\infty} \beta^t \left[ u(c_t) - v(l_t) \right]$$

$$c_t + k_{t+1} \le w_t l_t + r_t k_t$$
(1)

given an initial capital endowment  $k_0$  and sequences  $w_t, r_t$  of factor prices. Under standard assumptions guaranteeing the concavity of the objective and the interiority of the solution, the agent's choice is necessarily characterized by the first-order conditions

$$\begin{pmatrix} \beta^{t} u'(c_{t}) \\ -\beta^{t} v'(l_{t}) \\ 0 \end{pmatrix} = \lambda_{t} \begin{pmatrix} 1 \\ -w_{t} \\ 1 \end{pmatrix} + \lambda_{t+1} \begin{pmatrix} 0 \\ 0 \\ -r_{t+1} \end{pmatrix}$$
(2)

for some multipliers  $\lambda_t > 0$ , and the budget constraints binding

$$c_t + k_{t+1} = w_t l_t + r_t k_t (3)$$

for all  $t \ge 0$ .

Output is produced out of capital (i.e. saved output)<sup>4</sup> and labor through a constant returns to scale neoclassical production function whose total factor productivity depends on the previous period output, so that the output at t is given by

$$y_t = A(y_{t-1})F(k_t, l_t)$$
 (4)

<sup>&</sup>lt;sup>4</sup>For the sake of notational simplicity, and without loss of generality, capital is supposed to depreciate entirely in one period.

I will assume a positive correlation between output and total factor productivity, i.e. A' > 0, and that A'' < 0, which guarantees the convexity of the planner's problem below.

Under the assumption of perfect competition in the capital and labor markets, factors are remunerated by their marginal productivities and, therefore, the factor prices are

$$w_{t} = A(y_{t-1})F_{L}(k_{t}, l_{t})$$
  

$$r_{t} = A(y_{t-1})F_{K}(k_{t}, l_{t})$$
(5)

Conditions (2) through (5) characterize necessarily and sufficiently the competitive equilibria of this economy, as summarized in the next proposition —whose proof is straightforward from the conditions above.<sup>5</sup>

**Proposition 1.** A competitive equilibrium allocation of the infinitely-lived agents economy with output-driven TFP, given an initial capital endowment  $k_0$  and an initial total factor productivity  $A(y_{-1})$ , is characterized by sequences of consumption  $\tilde{c}_t$ , labor supply  $\tilde{l}_t$ , capital savings  $\tilde{k}_{t+1}$ , and output  $\tilde{y}_t$ , satisfying

$$\frac{v'(l_t)}{u'(c_t)} = A(y_{t-1})F_L(k_t, l_t)$$

$$\frac{1}{\beta}\frac{u'(c_t)}{u'(c_{t+1})} = A(y_t)F_K(k_{t+1}, l_{t+1})$$

$$c_t + k_{t+1} = A(y_{t-1})F(k_t, l_t)$$

$$y_t = A(y_{t-1})F(k_t, l_t)$$
(6)

each period  $t \ge 0$ , given  $k_0, y_{-1}$ .

Note that, under competitive conditions, the agent only takes into account his budget constraints in (1) and not the impact of past output on productivity in (4) and hence on the factor prices that remunerate his labor and savings. A planner would internalize this impact instead, and therefore would solve the problem

$$\max_{\substack{c_t, l_t, k_{t+1}, y_t \ge 0 \\ c_t + k_{t+1} \le y_t \\ y_t \le A(y_{t-1})F(k_t, l_t)}} \sum_{t=0}^{\infty} \beta^t \left[ u(c_t) - v(l_t) \right]$$
(7)

<sup>&</sup>lt;sup>5</sup>The binding budget constraints in (3) are, because of the linear homogeneity of the neoclassical production function, equivalent to the feasibility of the allocation.

given  $k_0, y_{-1}$ , where the planner solves as well for the sequence of outputs  $y_t$ , internalizing this way its effect on total factor productivity.<sup>6</sup> The planner's choice, given an initial condition  $k_0, y_{-1}$ , is characterized by the conditions stated in the next proposition, whose proof is provided in Appendix.

**Proposition 2.** The planner's allocation for the infinitely-lived agents economy with output-driven TFP, given an initial capital endowment  $k_0$  and an initial total factor productivity  $A(y_{-1})$ , is characterized by sequences of consumption  $\bar{c}_t$ , labor supply  $\bar{l}_t$ , capital savings  $\bar{k}_{t+1}$ , and output  $\bar{y}_t$ , satisfying

$$\frac{v'(l_t)}{u'(c_t)} = A(y_{t-1})F_L(k_t, l_t) \left[ 1 + \frac{A'(y_t)F(k_{t+1}, l_{t+1})}{A(y_t)F_K(k_{t+1}, l_{t+1})} \right]$$

$$\frac{1}{\beta} \frac{u'(c_t)}{u'(c_{t+1})} = A(y_t)F_K(k_{t+1}, l_{t+1}) \left[ 1 + \frac{A'(y_{t+1})F(k_{t+2}, l_{t+2})}{A(y_{t+1})F_K(k_{t+2}, l_{t+2})} \right]$$

$$c_t + k_{t+1} = A(y_{t-1})F(k_t, l_t)$$

$$y_t = A(y_{t-1})F(k_t, l_t)$$
(8)

each period  $t \ge 0$ , given  $k_0, y_{-1}$ .

From the characterizations in Propositions 1 and 2 of, respectively, the market allocation given some initial conditions and the one the planner would choose instead, it can be seen that the market remunerates the agents too little for their savings —and hence consume too early— as well as for their labor —and hence work too little— as the next proposition establishes.

**Proposition 3.** In the infinitely-lived agents economy with output-driven TFP, given any initial condition, the planner's allocation is not a competitive equilibrium allocation. More specifically, the agents get at any equilibrium allocation

(1) a too low return on their savings —compared to the implicit return the planner would choose at the equilibrium allocation— and

$$c_t + k_{t+1} \le A(y_{t-1})F(k_t, l_t)$$

<sup>&</sup>lt;sup>6</sup>Note that the constrained set in (7) is *not* the same as the one represented by the (sequence of) feasibility constraint(s) resulting form collapsing the two in (7) into a single one as in

given  $k_0, y_{-1}$ . The reason is that the latter does not constrain  $y_0$  (resulting in the maximization not having a solution) while this is not the case for the constrained set above given  $k_0, y_{-1}$ . A crucial bound on  $y_0$  is inadvertently dropped when the two (sequences of) constraints in (7) are collapsed into the single one above.

(2) a too low wage —compared to the implicit wage the planner would choose at the equilibrium allocation.

*Proof.* Since a competitive equilibrium allocation satisfies, at any given t,

$$\frac{1}{\beta} \frac{u'(\tilde{c}_t)}{u'(\tilde{c}_{t+1})} = A(\tilde{y}_t) F_K(\tilde{k}_{t+1}, \tilde{l}_{t+1})$$
(9)

in (6), if A' > 0, then

$$\frac{1}{\beta} \frac{u'(\tilde{c}_t)}{u'(\tilde{c}_{t+1})} < A(\tilde{y}_t) F_K(\tilde{k}_{t+1}, \tilde{l}_{t+1}) \left[ 1 + \frac{A'(\tilde{y}_{t+1}) F(\tilde{k}_{t+2}, \tilde{l}_{t+2})}{A(\tilde{y}_{t+1}) F_K(\tilde{k}_{t+2}, \tilde{l}_{t+2})} \right]$$
(10)

so that it cannot satisfy the conditions (8) characterizing the planner's allocation, which establishes the main result.

Moreover, the left-hand side in (10) is the equilibrium return to savings at t + 1, while the right-hand side is the implicit return the planner would choose at the equilibrium allocation, which establishes (1) in Proposition 3. Similarly, since at the competitive equilibrium allocation it holds

$$\frac{v'(\tilde{l}_t)}{u'(\tilde{c}_t)} = A(\tilde{y}_{t-1})F_L(\tilde{k}_t, \tilde{l}_t)$$
(11)

in (6), if A' > 0, then

$$\frac{v'(\tilde{l}_t)}{u'(\tilde{c}_t)} < A(\tilde{y}_{t-1})F_L(\tilde{k}_t, \tilde{l}_t) \left[ 1 + \frac{A'(\tilde{y}_t)F(\tilde{k}_{t+1}, \tilde{l}_{t+1})}{A(\tilde{y}_t)F_K(\tilde{k}_{t+1}, \tilde{l}_{t+1})} \right]$$
(12)

so that the equilibrium wage rate, equal to the left-hand side in (12), is lower than the implicit wage the planner would choose at the equilibrium allocation, in the right-hand side, which establishes (2) in Proposition 3. QED

In a nutshell, Proposition 3 establishes that the factors of production are priced too cheaply by the market, which distorts their supply by the agents. I provide next a policy that undoes these distortions by subsidizing labor and capital in a balancedbudget way (by means of a lump-sum tax) and allows, therefore, to decentralize the planner's allocation.

# 2.2 Decentralization of the planner's allocation as a competitive equilibrium.

Consider now the same infinitely-lived agents economy with output-driven TFP, but under a period-by-period balanced policy of linear subsidies or taxes (depending on the sign) on capital and labor incomes —possibly at different rates  $\tau_t^l$  and  $\tau_t^k$  across time and factors— and lump-sum taxes or transfers  $T_t$ , so that the representative agent faces the problem

$$\max_{c_t, l_t, k_{t+1} \ge 0} \sum_{t=0}^{\infty} \beta^t \left[ u(c_t) - v(l_t) \right]$$

$$c_t + k_{t+1} \le (1 + \tau_t^l) w_t l_t + (1 + \tau_t^k) r_t k_t - T_t$$
(13)

given an initial capital endowment  $k_0$  and some sequences  $w_t, k_t, \tau_t^l, \tau_t^k$ , and  $T_t$  of factors prices, tax or subsidy rates, and lump-sum taxes or transfers. Under standard assumptions, the agent's choice is characterized by the first-order conditions

$$\begin{pmatrix} \beta^t u'(c_t) \\ -\beta^t v'(l_t) \\ 0 \end{pmatrix} = \lambda_t \begin{pmatrix} 1 \\ -(1+\tau_t^l)w_t \\ 1 \end{pmatrix} + \lambda_{t+1} \begin{pmatrix} 0 \\ 0 \\ -(1+\tau_{t+1}^k)r_{t+1} \end{pmatrix}$$
(14)

for some multipliers  $\lambda_t > 0$ , and the binding constraints

$$c_t + k_{t+1} = (1 + \tau_t^l) w_t l_t + (1 + \tau_t^k) r_t k_t - T_t$$
(15)

for all  $t \ge 0$ . For the government budget to be balanced period by period it must hold, at every t,

$$T_t = \tau_t^l w_t l_t + \tau_t^k r_t k_t \tag{16}$$

which along with the budget constraints implies the feasibility of the allocation.

The competitive equilibria under this policy are then necessarily and sufficiently characterized by conditions (14) through (16) above, with the factor prices being equal to the marginal productivities, from which the next proposition follows straightforwardly.

**Proposition 4.** A competitive equilibrium allocation of the infinitely-lived agents economy with output-driven TFP —under a period-by-period balanced policy of linear subsidies/taxes on capital and labor incomes at rates  $\tau_t^l$  and  $\tau_t^k$  respectively,

and lump-sum taxes/transfers  $T_t$ — is characterized by sequences for consumption  $c_t$ , labor supply  $l_t$ , capital savings  $k_{t+1}$ , and output  $y_t$ , such that for each period  $t \ge 0$ 

$$\frac{v'(l_t)}{u'(c_t)} = (1 + \tau_t^l) A(y_{t-1}) F_L(k_t, l_t)$$

$$\frac{1}{\beta} \frac{u'(c_t)}{u'(c_{t+1})} = (1 + \tau_{t+1}^k) A(y_t) F_K(k_{t+1}, l_{t+1})$$

$$c_t + k_{t+1} = A(y_{t-1}) F(k_t, l_t)$$

$$y_t = A(y_{t-1}) F(k_t, l_t)$$
(17)

given initial conditions  $k_0, y_{-1}$ .

From the comparison of the conditions characterizing the planner's choice in (8) and the competitive equilibria under this policy in (17), it follows that the planner's choice is an equilibrium outcome for the subsidy/tax rates provided in the next proposition.

**Proposition 5.** The planner's allocation, for given initial conditions  $k_0, y_{-1}$ , of the infinitely-lived agents economy with output-driven TFP, is the competitive equilibrium allocation under common linear subsidies on capital and labor incomes at rates satisfying<sup>7</sup>

$$\tau_t = \frac{A'(y_t)F(k_{t+1}, l_{t+1})}{A(y_t)F_K(k_{t+1}, l_{t+1})} > 0 \tag{18}$$

and lump-sum taxes

$$T_t = \tau_t A(y_{t-1}) F(k_t, l_t).$$
(19)

From the viewpoint of providing a policy decentralizing the planners allocation for any given initial conditions, it should be noted that the common subsidy rate  $\tau_t$ in (18) depends on the agent's capital savings and labour supply one period later. This might be considered problematic since it would require perfect foresight from the policy-maker.<sup>8</sup> The next proposition establishes, nonetheless, that making the

<sup>&</sup>lt;sup>7</sup>Note that it follows from the assumption A' > 0 that the rate  $\tau_t$  in (16) is positive and corresponds therefore to a subsidy.

<sup>&</sup>lt;sup>8</sup>Although the representative agent himself is supposed to have perfect foresight of factor prices, and is choosing all his future capital savings and labor supplies at the outset of the economy, rather than sequentially.

policy depend on observed past values of labor and savings still succeeds in making the economy converge to the planner's allocation, in the case of log utilities, a power function for the total factor productivity, and a neoclassical production function. The proof is provided in the Appendix.

**Proposition 6.** In the infinitely-lived representative agent economy with

$$u(c) - v(l) = \ln c + \ln(1 - l)$$
  

$$A(y)F(k, l) = y^{\gamma}k^{\alpha}l^{1-\alpha}$$
(20)

and  $\gamma - \alpha < \frac{1}{\beta} < \gamma + \alpha$ , the competitive equilibrium converges to the planner's allocation if, at each period t, labor and capital incomes are linearly subsidized at the common rate

$$\tau_t = \frac{A'(y_{t-2})F(k_{t-1}, l_{t-1})}{A(y_{t-2})F_K(k_{t-1}, l_{t-1})}$$
(21)

by means of a lump-sum tax

$$T_t = \frac{A'(y_{t-2})F(k_{t-1}, l_{t-1})}{A(y_{t-2})F_K(k_{t-1}, l_{t-1})}A(y-1)F(k_t, l_t).$$
(22)

Moreover, the planner's allocation growth rate is  $\gamma + \alpha > 1$ .

In order to establish the robustness of these results and enlarge their scope, we extend them in the next section to economies of heterogeneous overlapping generations.

# 3. Output externalities on TFP with overlapping generations

# 3.1 Competitive equilibria and planner's allocations.

Consider now overlapping generations of heterogeneous, two-period lived agents deriving utility from consumption and disutility from working when young. Population grows by a factor n > 0 every period, but the distribution of agents across a finite number of types remains constant within each generation,<sup>9</sup> with  $\mu_i > 0$  being

 $<sup>^9{\</sup>rm For}$  the sake of simplicity, I assume no impact from the allocation of resources on the distribution of types.

the proportion of agents of type *i* within each generation, so that  $\sum_i \mu_i = 1$ . In order to finance their consumptions when young and old, the agents choose how much to work when young and how much to save (and how, i.e. whether in capital or in money). Specifically, an agent of type *i* born at date *t* chooses a labor supply  $l^{it}$ , consumptions  $c_0^{it}, c_1^{it}$ , and savings in capital and money  $k^{it}, M^{it}$ , that solve

$$\max_{0 \le c_0^{it}, c_1^{it}, k^{it}, M^{it}, l^{it}} u_0^i(c_0^{it}) + u_1^i(c_1^{it}) - v^i(l^{it})$$

$$c_0^{it} + k^{it} + \frac{M^{it}}{p_t} \le w_t l^{it}$$

$$c_1^{it} \le r_{t+1}k^{it} + \frac{M^{it}}{p_{t+1}}$$
(23)

given the monetary prices of output  $p_t, p_{t+1}$  and the real wage and rental rate of capital  $w_t, r_{t+1}$  during his lifetime. Under standard assumptions guaranteeing the concavity of the objective and the interiority of the solution, his choice is then characterized by the first-order conditions

$$\begin{pmatrix} u_0^{i\prime}(c_0^{it})\\ u_1^{i\prime}(c_1^{it})\\ 0\\ 0\\ -v^i(l^{it}) \end{pmatrix} = \lambda^t \begin{pmatrix} 1\\ 0\\ 1\\ \frac{1}{p_t}\\ -w_t \end{pmatrix} + \mu^t \begin{pmatrix} 0\\ 1\\ -r_{t+1}\\ -\frac{1}{p_{t+1}}\\ 0 \end{pmatrix}$$
(24)

for some  $\lambda^t, \mu^t > 0$ , and the budget constraints in (23) binding.

At the outset of the economy, an agent of type i born old at date 0 and endowed with initial savings in capital and money  $k^{i0}$ ,  $M^{i0}$ , trivially chooses a consumption  $c_1^{i0}$ , that solves

$$\max_{0 \le c_1^{i0}} u_1^i(c_1^{i0})$$

$$c_1^{i0} \le r_1 k^{i0} + \frac{M^{i0}}{p_1}$$
(25)

given the initial monetary price of output  $p_1$  and the initial real rental rate of capital  $r_1$ .

As before, output at t is produced out of capital and labor through a constant returns to scale neoclassical production function whose total factor productivity depends on period t - 1 output per worker,<sup>10</sup> so that the output per worker at t is given by

$$y_t = A(y_{t-1})F(\frac{k^{t-1}}{n}, l^t)$$
(26)

where  $k^{t-1} = \sum_{i} \mu^{i} k^{it-1}$  is the average/aggregate capital savings of generation t-1and  $l^{t} = \sum_{i} \mu^{i} l^{it}$  is the average/aggregate labor supply of generation t. I will again assume A' > 0 and A'' < 0. Under the assumption of perfect competition in the capital and labor markets, the factor prices are, therefore,

$$w_{t} = A(y_{t-1})F_{L}(\frac{k^{t-1}}{n}, l^{t})$$

$$r_{t+1} = A(y_{t})F_{K}(\frac{k^{t}}{n}, l^{t+1})$$
(27)

It is straightforward to see from the budget constraints that the feasibility of the allocation of resources is equivalent to

$$\frac{M^t}{M^{t+1}} = n \tag{28}$$

where  $M^t = \sum_i \mu^i M^{it}$  is, similarly, the average/aggregate monetary savings of generation t.

From conditions (24) through (28) and the binding budget constraints follows straightforwardly the characterization of the competitive equilibria of this economy provided in the next proposition.

**Proposition 7.** A competitive equilibrium allocation of the heterogeneous overlapping generations economy with output-driven TFP, given an initial old-born generation endowed with  $k^{i0}$ ,  $M^{i0}$  for each agent of type *i*, and an initial total factor productivity  $A(y_0)$ , is characterized by a consumption profile  $\tilde{c}_0^{it}, \tilde{c}_1^{it}$ , capital

<sup>&</sup>lt;sup>10</sup>Now that population is growing, the fact that it is past output per worker rather than past output that has an impact on total factor productivity intends to convey the idea that increases in total factor productivity are obtained through resource-consuming organizational, institutional, and human capital improvements that are all the more efficient when they permeate the entire society, hence the need to put it in relation to population size. Period t output per worker  $\frac{Y_t}{n^t}$  is proportional to period t output per capita  $\frac{Y_t}{n^{t-1}+n^t}$  by a factor  $\frac{n}{1+n}$  so that it is equivalent to assume total factor productivity to depend on any of the two.

savings  $\tilde{k}^{it}$ , monetary savings  $\tilde{M}^{it}$ , and labor supply  $\tilde{l}^{it}$ , for each type *i* in each period  $t \geq 1$ , along with prices  $p_t$ , satisfying

$$\frac{u_{0}^{i\prime}(c_{0}^{it})}{u_{1}^{i\prime}(c_{1}^{it})} = \frac{p_{t}}{p_{t+1}} = A(y_{t})F_{K}(\frac{k^{t}}{n}, l^{t+1}) 
\frac{v^{i\prime}(l^{it})}{u_{0}^{i\prime}(c_{0}^{it})} = A(y_{t-1})F_{L}(\frac{k^{t-1}}{n}, l^{t}) 
c_{0}^{it} + k^{it} + \frac{M^{it}}{p_{t}} = A(y_{t-1})F_{L}(\frac{k^{t-1}}{n}, l^{t})l^{it} 
c_{1}^{it} = A(y_{t})F_{K}(\frac{k^{t}}{n}, l^{t+1})k^{it} + \frac{M^{it}}{p_{t+1}} 
y_{t} = A(y_{t-1})F(\frac{k^{t-1}}{n}, l^{t}) 
\frac{M^{t}}{M^{t+1}} = n$$
(29)

—where  $k^t = \sum_i \mu^i k^{it}$ ,  $M^t = \sum_i \mu^i M^{it}$ , and  $l^t = \sum_i \mu^i l^{it}$  are, respectively, the aggregate capital, monetary savings and labor supply— at each period  $t \ge 1$ , and

$$c_1^{i0} = A(y_0)F_K(\frac{k^0}{n}, l^1)k^{i0} + \frac{M^{i0}}{p_1}$$

given  $k^{i0}, M^{i0}, y_0$ .

As for the planner's choice, given the stationary distribution of different types of agents, a utilitarian planner would choose to allocate resources in such a way that a(n exponentially) weighted sum of the utility of the average agent across all generations (including the initial old), is maximized, that is to say,

$$\max_{0 \le c_0^{it}, c_1^{it}, l^{it}, k^{it}, y^t}} \eta^{-1} \sum_i \mu^i u_1^i(c_1^{i0}) + \sum_{t=1}^{+\infty} \left\{ \eta^{t-1} \sum_i \mu^i \left[ u_0^i(c_0^{it}) + u_1^i(c_1^{it}) - v^i(l^{it}) \right] \right\}$$

$$\sum_i \mu^i \left[ c_0^{it} + \frac{c_1^{it-1}}{n} + k^{it} \right] \le y_t$$

$$y_t \le A(y_{t-1}) F(\frac{k^{t-1}}{n}, l^t)$$
(30)

—where  $k^t = \sum_i \mu^i k^{it}$  and  $l^t = \sum_i \mu^i l^{it}$ —given  $k^{i0}, y_0$ .

The planner's choice is characterized by the conditions provided in the following proposition, whose proof can be found in the Appendix.

**Proposition 8.** The planner's allocation for the heterogeneous overlapping generations economy with output-driven TFP, given an initial old-born generation endowed with  $k^{i0}, y_0$ , is characterized by a consumption profile  $\bar{c}_0^{it}, \bar{c}_1^{it}$ , capital savings  $\bar{k}^{it}$ , and labor supply  $\bar{l}^{it}$ , for each agent of each type *i* in each period  $t \geq 1$ , and a consumption  $\bar{c}_1^{i0}$  for each initial old of each type *i*, satisfying

$$\frac{u_0^{i\prime}(c_0^{it})}{u_1^{i\prime}(c_1^{it})} = n \frac{\lambda^t}{\lambda^{t+1}} = A(y_t) F_K(\frac{k^t}{n}, l^{t+1}) \left[ 1 + \frac{A'(y_{t+1})F(\frac{k^{t+1}}{n}, l^{t+2})}{A(y_{t+1})F_K(\frac{k^{t+1}}{n}, l^{t+2})} n \right] 
\frac{v^{i\prime}(l^{it})}{u_0^{i\prime}(c_0^{it})} = A(y_{t-1})F_L(\frac{k^{t-1}}{n}, l^t) \left[ 1 + \frac{A'(y_t)F(\frac{k^t}{n}, l^{t+1})}{A(y_t)F_K(\frac{k^t}{n}, l^{t+1})} n \right] 
\sum_i \mu^i \left[ c_0^{it} + \frac{c_1^{it-1}}{n} + k^{it} \right] = y_t 
y_t = A(y_{t-1})F(\frac{k^{t-1}}{n}, l^t)$$
(31)

for positive multipliers  $\lambda^t, \rho^t$ , with  $t \ge 1$ , satisfying

$$\lambda^{t} = \rho^{t+1} A(y_{t}) F_{K}(\frac{k^{t}}{n}, l^{t+1}) \frac{1}{n}$$

$$\rho^{t} = \rho^{t+1} \left[ A'(y_{t}) F(\frac{k^{t}}{n}, l^{t+1}) + A(y_{t}) F_{K}(\frac{k^{t}}{n}, l^{t+1}) \frac{1}{n} \right]$$
(32)

and

$$\begin{split} \lambda^1 &= \eta^{-1} u_1^{i\prime}(c_1^{i0}) n \\ \text{--where } k^t &= \sum_i \mu^i k^{it} \text{ and } l^t = \sum_i \mu^i l^{it} \text{--given } k^{i0}, y_0 \end{split}$$

From the second equations in (29) and (31) in Proposition 7 and 8 respectively, it follows immediately that the planner's allocation cannot be a laissez-faire competitive equilibrium outcome. In particular, as in the infinitely-lived agents setup, the overlapping generations get paid too little for their labor in a competitive equilibrium, and get remunerated too little for their savings as well, and as established in the next proposition.

**Proposition 9.** In a heterogeneous overlapping generations economy with outputdriven TFP starting from any given any initial condition, the planner's allocation is not a competitive equilibrium allocation. More specifically, the agents get, at any equilibrium allocation,

(1) a too low return on their savings —compared to the implicit return the planner would choose at the equilibrium allocation— and

(2) a too low wage —compared to the implicit wage the planner would choose under the same conditions.

*Proof.* Since a competitive allocation satisfies, at any given t,

$$\frac{u_0^{i\prime}(\tilde{c}_0^{it})}{u_1^{i\prime}(\tilde{c}_1^{it})} = A(\tilde{y}_{t-1})F_K(\frac{\tilde{k}^{t-1}}{n}, \tilde{l}^t)$$
(33)

in (29), if A' > 0, then

$$\frac{u_0^{i\prime}(\tilde{c}_0^{it})}{u_1^{i\prime}(\tilde{c}_1^{it})} < A(\tilde{y}_{t-1})F_K(\frac{\tilde{k}^{t-1}}{n}, \tilde{l}^t) \left[ 1 + \frac{A'(\tilde{y}_t)F(\frac{k^t}{n}, \tilde{l}^{t+1})}{A(\tilde{y}_t)F_K(\frac{\tilde{k}^t}{n}, \tilde{l}^{t+1})} n \right]$$
(34)

so that it cannot satisfy the conditions in (31) characterizing the planner's allocation.

Moreover, the left-hand side in (34), is the equilibrium return to savings at t + 1, while the right-hand side is the implicit return the planner would choose at the equilibrium allocation, which establishes (1) in Proposition 9.

Similarly, since a competitive allocation satisfies, at any given t,

$$\frac{v^{i\prime}(\tilde{l}^{it})}{u_0^{i\prime}(\tilde{c}_0^{it})} = A(\tilde{y}_{t-1})F_L(\frac{\tilde{k}^{t-1}}{n}, \tilde{l}^t)$$
(35)

in (29), if A' > 0, then

$$\frac{v_0^{i\prime}(\tilde{l}^{it})}{u_0^{i\prime}(\tilde{c}_0^{it})} < A(\tilde{y}_{t-1})F_L(\frac{\tilde{k}^{t-1}}{n}, \tilde{l}^t) \left[ 1 + \frac{A'(\tilde{y}_t)F(\frac{\tilde{k}^t}{n}, \tilde{l}^{t+1})}{A(\tilde{y}_t)F_K(\frac{\tilde{k}^t}{n}, \tilde{l}^{t+1})} n \right]$$
(36)

so that the equilibrium wage rate, equal to the left-hand side in (36), is lower than the implicit wage the planner would choose at the equilibrium allocation, in the right-hand side, which establishes (2) in Proposition 9. QED

In the next section I will address the issue of whether there is a policy intervention that allows for the decentralization of the planner's allocation as a competitive equilibrium outcome.

# 3.2 Decentralization of the planner's allocation as a competitive equilibrium.

Assume the agents face labor and capital income subsidies (or taxes, depending on the sign of the rate) and lump-sum tax or subsidy as needed to balance each period the government budget. Specifically, assume generation t is subject to labor and capital subsidy or income tax rates  $\tau_0^t, \tau_1^t$  and a first-period lump-sum tax or transfer  $T_0^t$ .<sup>11</sup> Then the problem faced by an agent of type i and born at date t is

$$\max_{\substack{0 \le c_0^{it}, c_1^{it}, k, M^{it}, l^{it} \\ 0 \le c_0^{it}, c_1^{it}, k, M^{it}, l^{it}}} u_0^i(c_0^{it}) + u_1^i(c_1^{it}) - v^i(l^{it})$$

$$c_0^{it} + k^{it} + \frac{M^{it}}{p_t} \le (1 + \tau_0^t)w_t l^{it} - T_0^t$$

$$c_1^{it} \le (1 + \tau_1^t)r_{t+1}k^{it} + \frac{M^{it}}{p_{t+1}}$$
(37)

given the prices  $p_t, p_{t+1}, w_t, r_{t+1}$ , the rates  $\tau_0^t, \tau_1^t$ , and the lump-sums  $T_0^t$ . The competitive equilibria of the economy under a period-by-period balanced policy of this type are then characterized by

$$\frac{u_{0}^{i\prime}(c_{0}^{it})}{u_{1}^{i\prime}(c_{1}^{it})} = \frac{p_{t}}{p_{t+1}} = A(y_{t})F_{K}(\frac{k^{t}}{n}, l^{t+1})(1+\tau_{1}^{t}) 
\frac{v^{i\prime}(l^{it})}{u_{0}^{i\prime}(c_{0}^{it})} = A(y_{t-1})F_{L}(\frac{k^{t-1}}{n}, l^{t})(1+\tau_{0}^{t}) 
c_{0}^{it} + k^{it} + \frac{M^{it}}{p_{t}} = A(y_{t-1})F_{L}(\frac{k^{t-1}}{n}, l^{t})(1+\tau_{0}^{t})l^{it} - T_{0}^{t} 
c_{1}^{it} = A(y_{t})F_{K}(\frac{k^{t}}{n}, l^{t+1})(1+\tau_{1}^{t})k^{it} + \frac{M^{it}}{p_{t+1}} 
y_{t} = A(y_{t-1})F(\frac{k^{t-1}}{n}, l^{t}) 
\frac{M^{t}}{M^{t+1}} = n$$
(38)

—where  $k^t = \sum_i \mu^i k^{it}$ ,  $M^t = \sum_i \mu^i M^{it}$ , and  $l^t = \sum_i \mu^i l^{it}$  — at each period  $t \ge 1$ , and

$$c_1^{i0} = A(y_0) F_K(\frac{k^0}{n}, l^1)(1 + \tau_1^0) k^{i0} + \frac{M^{i0}}{p_1}$$
(39)

<sup>&</sup>lt;sup>11</sup>As a matter of fact, the lump-sum tax needs not be on the young agents labor income as expressed above. Any distribution of the amount to be raised between young and old, i.e. between labor and capital income, would do the job.

given  $k^{i0}, M^{i0}, y_0$ , with

$$A(y_{t-1})\sum_{i}\mu^{i}\left[F_{L}(\frac{k^{t-1}}{n},l^{t})l^{it}\tau_{0}^{t}+F_{K}(\frac{k^{t-1}}{n},l^{t})k^{it}\tau_{1}^{t-1}\right]=T_{0}^{t}$$
(40)

for all  $t \ge 1$ .

From the comparison of the conditions characterizing the planner's choice in (31) and the competitive equilibria under this policy above, it follows that the planner's choice is an equilibrium outcome for the subsidy/tax rates provided in the next proposition.

**Proposition 10.** In the heterogeneous overlapping generations economy with output-driven TFP, the planner's allocation is the competitive equilibrium allocation under the policy subsidizing labor and capital income at the rates

$$\tau_0^t = \frac{A'(y_t)F(\frac{k^t}{n}, l^{t+1})}{A(y_t)F_K(\frac{k^t}{n}, l^{t+1})} n = \tau_1^{t-1}$$
(41)

—where  $k^t = \sum_i \mu^i k^{it}$ ,  $l^t = \sum_i \mu^i l^{it}$ — by means of the lump-sum tax on the first period income in (\*) above.

As it was the case in the infinitely-lived agents setup, the subsidy rates  $\tau_0^t, \tau_1^{t-1}$  at any given period t in (41) depend, in particular, on the labour supplied by the next generation one period later. That a policy depending instead on observed past values still succeeds in making the economy converge to the planner's allocation in a specific case — like in the previous case of log utilities, a power function for the total factor productivity, and a neoclassical production— can be established along the lines of the argument presented in the Appendix for the infinitely-lived agents case. The decentralization as a competitive equilibrium of the allocation chosen by an egalitarian planner treating all generations identically is shown next instead. It reveals

A planner treating all generations equally would choose the steady state that is the limit to the solution to the problem in (30) as the positive rate  $\eta < 1$  with which the planner weights less and less future generations converges to 1—so that he becomes increasingly egalitarian. That is to say, the planner would choose the consumptions

 $\bar{c}_0^i, \bar{c}_1^i$ , capital savings  $\bar{k}^i$ , and labor supply  $\bar{l}^i$ , for all agents of each type *i*, along with a level of output per worker  $\bar{y}$ , satisfying<sup>12</sup>

$$\frac{u_0^{i\prime}(c_0^i)}{u_1^{i\prime}(c_1^i)} = n = \frac{A(y)F_K(\frac{k}{n},l)}{1 - A'(y)F(\frac{k}{n},l)} 
\frac{v^{i\prime}(l^i)}{u_0^{i\prime}(c_0^i)} = \frac{A(y)F_L(\frac{k}{n},l)}{1 - A'(y)F(\frac{k}{n},l)} 
\sum_i \mu^i [c_0^i + \frac{c_1^i}{n} + k^i] = y 
y = A(y)F(\frac{k}{n},l)$$
(42)

where  $k = \sum_{i} \mu^{i} k^{i}$  and  $l = \sum_{i} \mu^{i} l^{i}$ .

From the conditions characterizing the egalitarian planner's steady state in (42) and the competitive equilibrium steady state under the policy above follows the next proposition establishing the factor prices subsidy rates and lump-sum tax that decentralize the planner's choice. Specifically, Proposition 11 establishes that the subsidy rate supporting the steady state of an egalitarian planner treating all generations equally is determined by the ratio of the population growth to the marginal productivity of capital at the planner's steady state. Interestingly enough, it establishes at the same time that an egalitarian steady state requires therefore the population growth factor to exceed the return to capital. The proof is provided in Appendix.

**Proposition 11.** In the heterogeneous overlapping generations economy with output-driven TFP, an egalitarian planner's steady state requires a population growth rate higher than the return to capital, and is the competitive equilibrium steady state  $\bar{c}_0^i, \bar{c}_1^i, \bar{k}^i, \bar{l}^i, \bar{y}$  resulting from the policy subsidizing labor and capital income at the common rate

$$\tau = \frac{n}{A(\bar{y})F_K(\frac{\bar{k}}{n},\bar{l})} - 1 > 0 \tag{43}$$

—where  $\bar{k} = \sum_{i} \mu^{i} \bar{k}^{i}$ ,  $\bar{l} = \sum_{i} \mu^{i} \bar{l}^{i}$ — by means of the following lump-sum tax on the first period income

$$T_0 = \tau A(\bar{y}) F(\frac{\bar{k}}{n}, \bar{l}) \tag{44}$$

<sup>&</sup>lt;sup>12</sup>Note that at a planner's steady state necessarily  $1 - A'(y)F(\frac{k}{n}, l) > 0$  since  $y \leq A(y)F(\frac{k}{n}, l)$  needs to be binding at the planner choice. The details can be found in the Appendix.

# 4. TFP increases: past output, past consumption or past investment?

The previous sections show the impact of a positive externality of past levels of output, without distinction, on the productivity of factors. Nonetheless, different components of output can be thought of as possible channels for such an externality, from the learning-by-doing effect associated to new investments and capital accumulation, to the effect of consumption on the (cognitive) skills and health of the labor force, for instance, on top of the impact of publicly provided infrastructure funded by taxes.

This section establishes that it makes a difference whether the externality is exerted through one or several of this channels. As a consequence, implementing the planner's choices requires qualitatively different policies depending on the channel through which the externality operates. This is done in the overlapping generations framework with a representative agent for the sake of brevity but, as before, the results extend to the case of heterogeneous generations and the infinitely-lived agents setup too.

### 4.1. Past investment only.

Consider the same overlapping generations economy as in Section 3 except for the fact that the output per worker at t is now given by

$$y_t = A(k^{t-1})F(\frac{k^{t-1}}{n}, l^t)$$
(45)

It is straightforward to derive (following the analogous steps in the previous section) the system characterizing the competitive equilibrium steady state in this case

$$\frac{u_0'(c_0)}{u_1'(c_1)} = n = A(k)F_K(\frac{k}{n}, l)$$

$$\frac{v'(l)}{u_0'(c_0)} = A(k)F_L(\frac{k}{n}, l)$$

$$c_0 + k + m = A(k)F_L(\frac{k}{n}, l)l$$

$$\frac{c_1}{n} = A(k)F_K(\frac{k}{n}, l)\frac{k}{n} + m$$
(46)

As in previous sections, a utilitarian planner would choose, at a steady state, to allocate resources as if he was to maximize the utility of the representative agent at the steady state, i.e.

$$\max_{0 \le c_0, c_1, k, l} u_0(c_0) + u_1(c_1) - v(l)$$

$$c_0 + \frac{c_1}{n} + k \le A(k)F(\frac{k}{n}, l)$$
(47)

and his choice would therefore be characterized by the first-order conditions

$$\begin{pmatrix} u_0'(c_0) \\ u_1'(c_1) \\ 0 \\ -v'(l) \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ \frac{1}{n} \\ 1 - A'(k)F(\frac{k}{n}, l) - A(k)F_K(\frac{k}{n}, l)\frac{1}{n} \\ -A(k)F_L(\frac{k}{n}, l) \end{pmatrix}$$
(48)

and the feasibility constraint binding. That is to say, the planner's steady state is a solution to

$$\frac{u_0'(c_0)}{u_1'(c_1)} = n = \frac{A(k)F_K(\frac{k}{n}, l)}{1 - A'(k)F(\frac{k}{n}, l)} 
\frac{v(l)}{u_0'(c_0)} = A(k)F_L(\frac{k}{n}, l) 
c_0 + \frac{c_1}{n} + k = A(k)F(\frac{k}{n}, l)$$
(49)

Note that in this case only the intertemporal trade-off between first and second period consumptions is distorted in (46) compared to the planer's conditions in (49) by the lack on internalization of the externality by competitive agents. As a consequence, a subsidy to the return to savings at a rate

$$\tau_1 = \frac{A'(\bar{k})F(\frac{k}{n},\bar{l})}{1 - A'(\bar{k})F(\frac{\bar{k}}{n},\bar{l})}$$
(50)

financed by a first-period lump-sum tax

$$T_0 = \tau_1 A(y) F_K(\frac{k}{n}, l) \frac{k}{n}$$
(51)

suffices to decentralize the planner's choice.

### 4.2 Past consumption only.

Consider the same overlapping generations economy as in Section 3 except for the fact that the output per worker at t is now given by

$$y_t = A(c_0^{t-1} + \frac{c_1^{t-2}}{n})F(\frac{k^{t-1}}{n}, l^t)$$
(52)

It is straightforward to derive (following the analogous steps in the previous section) the system characterizing the competitive equilibrium steady state in this case

$$\frac{u_0'(c_0)}{u_1'(c_1)} = n = A(c_0 + \frac{c_1}{n})F_K(\frac{k}{n}, l)$$

$$\frac{v'(l)}{u_0'(c_0)} = A(c_0 + \frac{c_1}{n})F_L(\frac{k}{n}, l)$$

$$c_0 + k + m = A(c_0 + \frac{c_1}{n})F_L(\frac{k}{n}, l)l$$

$$\frac{c_1}{n} = A(c_0 + \frac{c_1}{n})F_K(\frac{k}{n}, l)\frac{k}{n} + m$$
(53)

A utilitarian planner would again choose, at a steady state, to allocate resources so as to maximize the utility of the representative agent at the steady state, i.e.

$$\max_{\substack{0 \le c_0, c_1, k, l}} u_0(c_0) + u_1(c_1) - v(l)$$

$$c_0 + \frac{c_1}{n} + k \le A(c_0 + \frac{c_1}{n})F(\frac{k}{n}, l)$$
(54)

and his choice would therefore be characterized by the first-order conditions

$$\begin{pmatrix} u_0'(c_0) \\ u_1'(c_1) \\ 0 \\ -v'(l) \end{pmatrix} = \lambda \begin{pmatrix} 1 - A'(c_0 + \frac{c_1}{n}) \\ \frac{1}{n} [1 - A'(c_0 + \frac{c_1}{n})] \\ 1 - A(c_0 + \frac{c_1}{n}) F_K(\frac{k}{n}, l) \frac{1}{n} \\ -A(c_0 + \frac{c_1}{n}) F_L(\frac{k}{n}, l) \end{pmatrix}$$
(55)

and the feasibility constraint binding. That is to say, the planner's steady state is a solution to  $\frac{1}{2}$ 

$$\frac{u_0'(c_0)}{u_1'(c_1)} = n = A(c_0 + \frac{c_1}{n})F_K(\frac{k}{n}, l)$$

$$\frac{v(l)}{u_0'(c_0)} = \frac{A(c_0 + \frac{c_1}{n})F_L(\frac{k}{n}, l)}{1 - A'(c_0 + \frac{c_1}{n})F(\frac{k}{n}, l)}$$

$$c_0 + \frac{c_1}{n} + k = A(k)F(\frac{k}{n}, l)$$

$$22$$
(56)

Note that in this case only the trade-off between first period consumptions and leisure is distorted in (53) compared to the planer's conditions in (56) by the lack on internalization of the externality by competitive agents. As a consequence, a subsidy to the return to savings at a rate

$$\tau_0 = \frac{A'(\bar{c}_0 + \frac{\bar{c}_1}{n})F(\frac{\bar{k}}{n}, \bar{l})}{1 - A'(\bar{c}_0 + \frac{\bar{c}_1}{n})F(\frac{\bar{k}}{n}, \bar{l})}$$
(57)

financed by a first-period lump-sum tax

$$T_0 = \tau_0 A(\bar{c}_0 + \frac{\bar{c}_1}{n}) F_L(\frac{\bar{k}}{n}, l)$$
(58)

suffices to decentralize the planner's choice.

Therefore, in the case the only positive externality on productivity comes from learning by doing, only capital income must be subsidized at the expense of a lump-sum tax on labor income. If the positive externality were to work its way towards TFP through channels other than direct investment in capital, but spills over from economic activity in general through output or total income, labor income must be linearly subsidized as well as capital income at the expense of a lump sum tax on labor income. It is thus important to know what the empirical evidence says about the dependence of TFP on k and y.

### Appendix

Proof of Proposition 2. Under standard assumptions (along with A'' < 0), the planner's choice is characterized by the binding constraints, which are the two last equations in the characterization (8)

$$c_t + k_{t+1} = y_t$$

$$y_t = A(y_{t-1})F(k_t, l_t)$$
23
(59)

and the first-order conditions

$$\begin{pmatrix} \beta^{t} u'(c_{t}) \\ -\beta^{t} v'(l_{t}) \\ 0 \\ 0 \end{pmatrix} = \lambda_{t-1} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \lambda_{t} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

$$+ \mu_{t} \begin{pmatrix} 0 \\ -A(y_{t-1})F_{L}(k_{t}, l_{t}) \\ -A(y_{t-1})F_{K}(k_{t}, l_{t}) \\ 1 \end{pmatrix} + \mu_{t+1} \begin{pmatrix} 0 \\ 0 \\ -A'(y_{t})F(k_{t+1}, l_{t+1}) \end{pmatrix}$$

$$(60)$$

for all t and some sequences of positive multipliers  $\lambda_t$  and  $\mu_t$ . Equivalently, the first-order conditions are

$$\beta^{t} u'(c_{t}) = \lambda_{t}$$

$$\beta^{t} v'(l_{t}) = \mu_{t} A(y_{t-1}) F_{L}(k_{t}, l_{t})$$

$$\lambda_{t-1} = \mu_{t} A(y_{t-1}) F_{K}(k_{t}, l_{t})$$

$$\lambda_{t} = \mu_{t} - \mu_{t+1} A'(y_{t}) F(k_{t+1}, l_{t+1})$$
(61)

so that, dividing the first equation by the second,

$$\frac{v'(l_t)}{u'(c_t)} = \frac{\mu_t}{\lambda_t} A(y_{t-1}) F_L(k_t, l_t)$$
(62)

while the fourth can be written as

$$\frac{\mu_t}{\lambda_t} = 1 + \frac{\mu_{t+1}}{\lambda_t} A'(y_t) F(k_{t+1}, l_{t+1})$$
(63)

and the third as

$$\frac{\mu_{t+1}}{\lambda_t} = \frac{1}{A(y_t)F_K(k_{t+1}, l_{t+1})}$$
(64)

so that

$$\frac{v'(l_t)}{u'(c_t)} = A(y_{t-1})F_L(k_t, l_t) \left[ 1 + \frac{A'(y_t)F(k_{t+1}, l_{t+1})}{A(y_t)F_K(k_{t+1}, l_{t+1})} \right]$$
(65)

which is the first equation in the characterization (8) above.

Similarly,

$$\frac{1}{\beta} \frac{u'(c_t)}{u'(c_{t+1})} = \frac{\lambda_t}{\lambda_{t+1}} = \frac{\lambda_t}{\mu_{t+1}} \frac{\mu_{t+1}}{\lambda_{t+1}}$$
(66)

but, from the third equation,

$$\frac{\lambda_t}{\mu_{t+1}} = A(y_t) F_K(k_{t+1}, l_{t+1}) \tag{67}$$

and, from the fourth,

$$\frac{\mu_{t+1}}{\lambda_{t+1}} = 1 + \frac{\mu_{t+2}}{\lambda_{t+1}} A'(y_{t+1}) F(k_{t+2}, l_{t+2})$$
(68)

i.e.

$$\frac{\mu_{t+1}}{\lambda_{t+1}} = 1 + \frac{A'(y_{t+1})F(k_{t+2}, l_{t+2})}{A(y_{t+1})F_K(k_{t+2}, l_{t+2})}$$
(69)

so that

$$\frac{1}{\beta} \frac{u'(c_t)}{u'(c_{t+1})} = A(y_t) F_K(k_{t+1}, l_{t+1}) \left[ 1 + \frac{A'(y_{t+1})F(k_{t+2}, l_{t+2})}{A(y_{t+1})F_K(k_{t+2}, l_{t+2})} \right]$$
(70)

which is the second equation in the characterization (8) above. QED

. .

Proof of Proposition 6. Consider the case

$$u(c) = \ln c$$

$$v(l) = -\ln(1-l)$$

$$A(y) = y^{\gamma}$$

$$F(k,l) = k^{\alpha} l^{1-\alpha}$$
(71)

with l being the share of a unit of time supplied each period as labor, and  $\alpha,\gamma\in(0,1).^{13}$ 

The allocations chosen by the planner or the market under the policy (based on sufficiently lagged variables) are then:

(1) the planner's choice solves

$$\frac{c_t}{1-l_t} = y_{t-1}^{\gamma} (1-\alpha) \left(\frac{k_t}{l_t}\right)^{\alpha} \left[1 + \frac{\gamma}{\alpha} \frac{k_{t+1}}{y_t}\right]$$

$$\frac{1}{\beta} \frac{c_{t+1}}{c_t} = y_t^{\gamma} \alpha \left(\frac{k_{t+1}}{l_{t+1}}\right)^{\alpha-1} \left[1 + \frac{\gamma}{\alpha} \frac{k_{t+2}}{y_{t+1}}\right]$$

$$c_t + k_{t+1} = y_{t-1}^{\gamma} k_t^{\alpha} l_t^{1-\alpha}$$

$$y_t = y_{t-1}^{\gamma} k_t^{\alpha} l_t^{1-\alpha}$$
(72)

<sup>&</sup>lt;sup>13</sup>Note that the period utility  $u(c) - v(l) = \ln c + \ln(1 - l)$  is concave in both consumption c and labor l (as well as in consumption c and leisure 1 - l).

from where, lagging one period the second equation and dividing it by the first, one gets that

$$k_t = \beta \frac{\alpha}{1-\alpha} \cdot \frac{l_t}{1-l_t} c_{t-1}.$$
(73)

This allows to express the planner's choice in  $c_t$ ,  $l_t$ , and  $y_t$  as

$$c_{t} = y_{t-1}^{\gamma} (1-\alpha) \left(\beta \frac{\alpha}{1-\alpha}\right)^{\alpha} \left(\frac{l_{t}}{1-l_{t}}\right)^{\alpha} c_{t-1}^{\alpha} \frac{1-l_{t}}{l_{t}^{\alpha}} \left[1+\beta \frac{\gamma}{1-\alpha} \frac{l_{t+1}}{1-l_{t+1}} \frac{c_{t}}{y_{t}}\right]$$

$$c_{t} + \beta \frac{\alpha}{1-\alpha} \cdot \frac{l_{t}}{1-l_{t}} c_{t} = y_{t-1}^{\gamma} \left(\beta \frac{\alpha}{1-\alpha}\right)^{\alpha} \left(\frac{l_{t}}{1-l_{t}}\right)^{\alpha} c_{t-1}^{\alpha} l_{t}^{1-\alpha}$$

$$y_{t} = y_{t-1}^{\gamma} \left(\beta \frac{\alpha}{1-\alpha}\right)^{\alpha} \left(\frac{l_{t}}{1-l_{t}}\right)^{\alpha} c_{t-1}^{\alpha} l_{t}^{1-\alpha}$$
(74)

which implies (from the first and third equations)

$$\frac{c_t}{y_t} \frac{l_t}{1 - l_t} = (1 - \alpha) \left[ 1 + \beta \frac{\gamma}{1 - \alpha} \cdot \frac{c_t}{y_t} \frac{l_{t+1}}{1 - l_{t+1}} \right]$$
(75)

(2) the competitive equilibria under the proposed policy, but sufficiently lagged in order to make it depend on observed variables at t, namely

$$\tau_t^k = -\frac{A'(y_{t-2})F(k_{t-1}, l_{t-1})}{A(y_{t-2})F_K(k_{t-1}, l_{t-1})} = \tau_t^l$$
(76)

are characterized by

$$\frac{c_t}{1-l_t} = y_{t-1}^{\gamma} (1-\alpha) \left(\frac{k_t}{l_t}\right)^{\alpha} \left[1 + \frac{\gamma}{\alpha} \frac{k_{t-1}}{y_{t-2}}\right]$$

$$\frac{1}{\beta} \frac{c_{t+1}}{c_t} = y_t^{\gamma} \alpha \left(\frac{k_{t+1}}{l_{t+1}}\right)^{\alpha-1} \left[1 + \frac{\gamma}{\alpha} \frac{k_t}{y_{t-1}}\right]$$

$$c_t + k_{t+1} = y_{t-1}^{\gamma} k_t^{\alpha} l_t^{1-\alpha}$$

$$y_t = y_{t-1}^{\gamma} k_t^{\alpha} l_t^{1-\alpha}$$
(77)

from where (lagging the second equation one period and dividing it by the first) one gets equation (73) above again, which allows to express the competitive equilibria in  $c_t$ ,  $l_t$ , and  $y_t$  as

$$c_{t} = y_{t-1}^{\gamma} (1-\alpha) \left(\beta \frac{\alpha}{1-\alpha}\right)^{\alpha} \left(\frac{l_{t}}{1-l_{t}}\right)^{\alpha} c_{t-1}^{\alpha} \frac{1-l_{t}}{l_{t}^{-\alpha}} \left[1+\beta \frac{\gamma}{1-\alpha} \frac{l_{t-1}}{1-l_{t-1}} \frac{c_{t-2}}{y_{t-2}}\right]$$

$$c_{t} + \beta \frac{\alpha}{1-\alpha} \frac{l_{t}}{1-l_{t}} c_{t} = y_{t-1}^{\gamma} \left(\beta \frac{\alpha}{1-\alpha}\right)^{\alpha} \left(\frac{l_{t}}{1-l_{t}}\right)^{\alpha} c_{t-1}^{\alpha} l_{t}^{1-\alpha}$$

$$y_{t} = y_{t-1}^{\gamma} \left(\beta \frac{\alpha}{1-\alpha}\right)^{\alpha} \left(\frac{l_{t}}{1-l_{t}}\right)^{\alpha} c_{t-1}^{\alpha} l_{t}^{1-\alpha}$$
(78)

implying (from the first and third equations)

$$\frac{c_t}{y_t} \frac{l_t}{1 - l_t} = (1 - \alpha) \left[ 1 + \beta \frac{\gamma}{1 - \alpha} \frac{c_{t-2}}{y_{t-2}} \frac{l_{t-1}}{1 - l_{t-1}} \right]$$
(79)

In both cases, from the feasibility condition

$$c_t + k_{t+1} = y_{t-1}^{\gamma} k_t^{\alpha} l_t^{1-\alpha} = y_t \tag{80}$$

and (73) one can obtain

$$\frac{l_t}{1 - l_t} = \frac{1 - \alpha}{\beta \alpha} \left( \frac{1}{c_{t-1}/y_{t-1}} - 1 \right)$$
(81)

so that the dynamics in (75) and (81) can be expressed in the share of consumption in output as follows

(1) for the planner

$$\frac{c_t}{y_t} \frac{1}{\beta \alpha} \left( 1 - \frac{c_{t-1}}{y_{t-1}} \right) = \left[ 1 + \frac{\gamma}{\alpha} \left( 1 - \frac{c_t}{y_t} \right) \right] \frac{c_{t-1}}{y_{t-1}}$$
(82)

# (2) for the competitive equilibrium under the policy

$$\frac{c_t}{y_t} \frac{1}{\beta \alpha} \left( 1 - \frac{c_{t-1}}{y_{t-1}} \right) = \left[ 1 + \frac{\gamma}{\alpha} \left( 1 - \frac{c_{t-2}}{y_{t-2}} \right) \right] \frac{c_{t-1}}{y_{t-1}}$$
(83)

Notice that, while the market's consumption share of output dynamics is a difference equation of order 1 and the planner's is of order 2, the share nonetheless the same balanced growth split of output between consumption and savings, i.e.

$$1 - \frac{c}{y} = \frac{\beta\alpha}{1 - \beta\gamma} \tag{84}$$

as the steady state capital-output ratio and

$$\frac{c}{y} = \frac{1 - \beta(\gamma + \alpha)}{1 - \beta\gamma} \tag{85}$$

as the steady state consumption-output ratio, which are in (0, 1) as long as  $\beta(\gamma + \alpha) < 1$ .

Moreover, that balanced growth path is globally stable in the market dynamics under the given policy, so that in the limit the policy implements the planner's balanced growth path. In effect, the second-order market dynamics can be made into a first-order one writing it as

$$\begin{pmatrix} \frac{c_{t+1}}{y_{t+1}} \\ \frac{c_t}{y_t} \end{pmatrix} = G\begin{pmatrix} \frac{c_t}{y_t} \\ \frac{c_{t-1}}{y_{t-1}} \end{pmatrix} = \begin{pmatrix} \beta \left[ \alpha + \gamma \left( 1 - \frac{c_{t-1}}{y_{t-1}} \right) \right] \frac{\frac{c_t}{y_t}}{1 - \frac{c_t}{y_t}} \\ \frac{c_t}{y_t} \end{pmatrix}$$
(86)

Its linearization around the steady state consumption-output ratio

$$\begin{pmatrix} \frac{c_{t+1}}{y_{t+1}} - \frac{c}{y} \\ \frac{c_t}{y_t} - \frac{c}{y} \end{pmatrix} = \begin{pmatrix} \frac{1-\beta\gamma}{\beta\alpha} & [\beta(\gamma+\alpha)-1]\frac{\gamma}{\alpha} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{c_t}{y_t} - \frac{c}{y} \\ \frac{c_{t-1}}{y_{t-1}} - \frac{c}{y} \end{pmatrix}$$
(87)

has eigenvalues of modulus smaller than one if  $\left|\frac{1-\beta\gamma}{\beta\alpha}\right| < 1$  (or equivalently  $\gamma - \alpha < \frac{1}{\beta} < \gamma + \alpha$ ), which is satisfied for reasonable parameters like, for instance,  $\alpha = .3$  and  $\beta = .9 = \gamma .^{14}$ 

Finally, the rate of growth along the planner's balance-growth path follows from the feasibility constraint divided by output

$$\frac{c_t}{y_t} + \frac{k_{t+1}}{y_t} = \frac{y_{t-1}^{\gamma+\alpha}}{y_t} \frac{k_t^{\alpha}}{y_{t-1}^{\alpha}} l_t^{1-\alpha}$$
(88)

since (i) the consumption-output and capital-output ratios are constant, which implies  $l_t$  constant too from (81), so that the first factor on the right-hand in (88) side must be constant too. That is to say,

$$y_t = C y_{t-1}^{\gamma + \alpha} \tag{89}$$

for some constant C, where  $\gamma + \alpha > 1$  under the sufficient condition for global stability above. This means that the balanced-growth path can either exhibit unboundedly increasing levels of output consumption and investment, or monotonically decreasing ones collapsing to zero in the limit, depending on whether the initial  $y_0$  is above or below 1. QED

<sup>&</sup>lt;sup>14</sup>The condition on parameters is not necessary but just sufficient, and hence unduly restrictive. Many more profiles of parameters still make the market dynamics under the policy converge to the planner's balanced growth split of output between consumption and savings. In effect, since the sum of the eigenvalues coincides with the trace of the Jacobian at the steady state  $\frac{1-\beta\gamma}{\beta\alpha}$  and their product with its determinant  $[1 - \beta(\alpha + \gamma)]\frac{\gamma}{\alpha}$ , and the latter is positive (so that both eigenvalues have the same sign), then a trace smaller than one guarantees that both of them have a modulus smaller than one.

Proof of Proposition 8. The planner's choice is characterized by the first-order conditions, for  $t \ge 1$ ,

$$\begin{pmatrix} \begin{pmatrix} \eta^{t-1}\mu^{i}u_{0}^{i\prime}(c_{0}^{it}) \\ \eta^{t-1}\mu^{i}u_{1}^{i\prime}(c_{1}^{tt}) \\ -\eta^{t-1}\mu^{i}v^{i\prime}(l^{it}) \\ 0 \end{pmatrix}_{i} \end{pmatrix} = \lambda^{t} \begin{pmatrix} \begin{pmatrix} \mu^{i} \\ 0 \\ 0 \\ \mu^{i} \end{pmatrix}_{i} \\ -1 \end{pmatrix} + \lambda^{t+1} \begin{pmatrix} \begin{pmatrix} \mu^{i} \frac{1}{n} \\ 0 \\ 0 \end{pmatrix}_{i} \\ 0 \end{pmatrix} \\ + \rho^{t} \begin{pmatrix} \begin{pmatrix} 0 \\ 0 \\ -\mu^{i}A(y_{t-1})F_{L}(\frac{k^{t-1}}{n}, l^{t}) \\ 0 \\ 1 \end{pmatrix}_{i} \end{pmatrix}$$
(90)
$$+ \rho^{t+1} \begin{pmatrix} \begin{pmatrix} 0 \\ 0 \\ -\mu^{i}A(y_{t})F_{K}(\frac{k^{t}}{n}, l^{t+1}) \frac{1}{n} \\ -A'(y_{t})F(\frac{k^{t}}{n}, l^{t+1}) \end{pmatrix} \\ \end{pmatrix}$$

and

$$\lambda^1 = \eta^{-1} u_1^{i\prime}(c_1^{i0}) n \tag{91}$$

for some  $\lambda^t, \lambda^{t+1}, \rho^t, \rho^{t+1} > 0$ , and

$$\sum_{i} \mu^{i} \left[ c_{0}^{it} + \frac{c_{1}^{it-1}}{n} + k^{it} \right] = y_{t}$$

$$y_{t-1} = A(y_{t-1}) F(\frac{k^{t-1}}{n}, l^{t})$$
(92)

with  $k^t = \sum_i \mu^i k^{it}$  and  $l^t = \sum_i \mu^i l^{it}$ , for all  $t \ge 1$ , from where (31,32) follow. QED

Solution to the egalitarian planner's steady state problem (footnote 12). It can be straightforwardly checked from the first-oder conditions that, as  $\eta \to 1$ , the ratio's of the planner's multipliers  $\frac{\lambda^t}{\lambda^{t+1}}$  and  $\frac{\rho^t}{\rho^{t+1}}$  in (31,32) converge to 1 too, in which case

$$1 + \frac{A'(y)F(\frac{k}{n},l)}{A(y)F_K(\frac{k}{n},l)}n = \frac{1}{1 - A'(y)F(\frac{k}{n},l)}$$
(93)

so that the right-hand sides in (31) and (42) coincide.<sup>15</sup> Equivalently, the egalitarian planner's allocation can be found as the allocation solving the problem

$$\max_{0 \le c_0^i, c_1^i, k^i, l^i, y} \sum_i \mu^i \left[ u_0^i(c_0^i) + u_1^i(c_1^i) - v^i(l^i) \right]$$

$$\sum_i \mu^i \left[ c_0^i + \frac{c_1^i}{n} + k^i \right] \le y \qquad (95)$$

$$y \le A(y) F(\frac{k}{n}, l)$$

—where  $k = \sum_i \mu^i k^i$  and  $l = \sum_i \mu^i l^i$ — given  $k^i$  and y, which is characterized by the first-order conditions

$$\begin{pmatrix} \begin{pmatrix} \mu^{i} u_{0}^{i\prime}(c_{0}^{i}) \\ \mu^{i} u_{1}^{i\prime}(c_{1}^{i}) \\ 0 \\ -\mu^{i} v^{i\prime}(l^{i}) \end{pmatrix}_{i} \end{pmatrix} = \lambda \begin{pmatrix} \begin{pmatrix} \mu^{i} \\ \mu^{i} \frac{1}{n} \\ \mu^{i} \\ 0 \\ -1 \end{pmatrix}_{i} \end{pmatrix} + \rho \begin{pmatrix} \begin{pmatrix} 0 \\ 0 \\ -\mu^{i} A(y) F_{K}(\frac{k}{n}, l) \frac{1}{n} \\ -\mu^{i} A(y) F_{L}(\frac{k}{n}, l) \end{pmatrix}_{i} \end{pmatrix}$$
(96)

for some  $\lambda, \rho > 0$ , and

$$\sum_{i} \mu^{i} \left[ c_{0}^{i} + \frac{c_{1}^{i}}{n} + k^{i} \right] = y$$

$$y = A(y)F(\frac{k}{n}, l)$$
(97)

from which —eliminating the multipliers  $\lambda > 0$  and  $\rho > 0$  — (42) follows. QED

*Proof of Proposition 11.* The agent's choice under the stated policy solution to (37) is characterized by the first-order conditions

$$\begin{pmatrix} u_0^{i\prime}(c_0^{it}) \\ u_1^{i\prime}(c_1^{it}) \\ 0 \\ 0 \\ -v^{i\prime}(l^{it}) \end{pmatrix} = \lambda^{it} \begin{pmatrix} 1 \\ 0 \\ 1 \\ \frac{1}{p_t} \\ -(1+\tau_0^t)w_t \end{pmatrix} + \mu^{it} \begin{pmatrix} 0 \\ 1 \\ -(1+\tau_1^t)r_{t+1} \\ -\frac{1}{p_{t+1}} \\ 0 \end{pmatrix}$$
(99)

for some  $\lambda^{it}, \mu^{it} > 0$ , and the binding budget constraints. The output per worker at t is given by (26), as before, and similarly the real wage and rental rate of capital

<sup>&</sup>lt;sup>15</sup>Note that the existence of a egalitarian planner's steady state requires it to satisfy  $A'(y)F(\frac{k}{n},l) < 1$ .

by (27), so that a competitve equilibrium under a period-by-period balanced policy of taxes and transfers  $\tau_0^t, \tau_1^t, T_0^t$  is characterized by the following dynamics<sup>16</sup>

$$\frac{u_0^{i\prime}(c_0^{it})}{u_1^{i\prime}(c_1^{it})} = \frac{p_t}{p_{t+1}} = (1+\tau_1^t)A(y_t)F_K(\frac{k^t}{n}, l^{t+1})$$

$$\frac{v^{i\prime}(l^{it})}{u_0^{i\prime}(c_0^{it})} = (1+\tau_0^t)A(y_{t-1})F_L(\frac{k^{t-1}}{n}, l^t)$$

$$c_0^{it} + k^{it} + \frac{M^{it}}{p_t} = (1+\tau_0^t)A(y_{t-1})F_L(\frac{k^{t-1}}{n}, l^t)l^{it} - T_0^t$$

$$c_1^{it} = (1+\tau_1^t)A(y_t)F_K(\frac{k^t}{n}, l^{t+1})k^{it} + \frac{M^{it}}{p_{t+1}} \qquad (100)$$

$$y_t = A(y_{t-1})F(\frac{k^{t-1}}{n}, l^t)$$

$$\frac{M^t}{M^{t+1}} = n$$

$$\sum_j \mu^j A(y_t) \left[\tau_0^{t+1}F_L(\frac{k^t}{n}, l^{t+1})l^{jt} + \tau_1^tF_K(\frac{k^t}{n}, l^{t+1})\frac{k^{jt}}{n}\right] = T_0^{t+1}$$

with  $k^t = \sum_j \mu^j k^{jt}$ ,  $l^t = \sum_j \mu^j l^{jt}$ , and  $M^t = \sum_i \mu^i M^{it}$ , —where the last two conditions are, respectively, equivalent to the feasibility of the allocation, and the period by period balance of the government budget— and whose steady state characterized by

$$\frac{u_0^{i'}(c_0^i)}{u_1^{i'}(c_1^i)} = n = (1 + \tau_1)A(y)F_K(\frac{k}{n}, l) 
\frac{v^{i'}(l^i)}{u_0^{i'}(c_0^i)} = (1 + \tau_0)A(y)F_L(\frac{k}{n}, l) 
c_0^i + k^i + m^i = (1 + \tau_0)A(y)F_L(\frac{k}{n}, l)l^i - T_0$$
(101)
$$\frac{c_1^i}{n} = (1 + \tau_1)A(y)F_K(\frac{k}{n}, l)\frac{k^i}{n} + m^i 
y = A(y)F(\frac{k}{n}, l)$$

is, by direct substitution, the egalitarian planner's steady state solution to (30) in

<sup>&</sup>lt;sup>16</sup>For a period-by-period balanced policy, i.e. such that  $T_0^t = \sum_i \mu_i \left( \tau_0^t w_t l^{it} + \tau_1^{t-1} r_t \frac{k^{it-1}}{n} \right)$  the feasibility of the allocation of resources is again equivalent to  $\frac{M^t}{M^{t+1}} = n$ , where  $M^t = \sum_i \mu^i M^{it}$ .

(42) if the subsidy rates are constant and equal to the positive rate

$$\tau = \frac{1}{1 - A'(\bar{y})F(\frac{\bar{k}}{n}, \bar{l})} - 1 > 0 \tag{102}$$

from identifying the right-hand sides of any of the first two lines in (42) and (101) above, or equivalently —from the second equation in the first line in (42)—

$$\tau = \frac{n}{A(\bar{y})F_k(\frac{\bar{k}}{n},\bar{l})} - 1 > 0 \tag{103}$$

and the policy is period by period balanced by a labor income lump-sum tax

$$T_0 = A(\bar{y}) \left[ \tau_0 F_L(\frac{\bar{k}}{n}, \bar{l})\bar{l} + \tau_1 F_K(\frac{\bar{k}}{n}, \bar{l})\frac{\bar{k}}{n} \right]$$
(104)

which by the homogeneity of degree 1 of F amounts to

$$T_0 = \tau A(\bar{y}) F(\frac{\bar{k}}{n}, \bar{l}) \tag{105}$$

QED

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