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A conic optimization approach for SKU rationalization

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Abstract

Expanding variety and the number of offered products is attractive for a firm to fit customer needs. Nevertheless, the greater complexity and the proliferation of stock-keeping units (SKUs) without substantial differentiation may not substantially improve customer satisfaction while raising costs. Based on the principle that the product-line size involves operational implications and particularly manufacturing and holding costs, this paper develops a mixed-integer nonlinear mathematical program (MINLP) to support efficient product portfolio reductions. Basically, the fixed costs elimination and the risk-pooling effects must balance the demand contraction due to customer dissatisfaction. Off-the-shelve Mixed-Integer Quadratic Problem (MIQP) solver provides optimal solution to the proposed conic quadratic reformulation, and a real-life industrial case illustrates the program and the algorithm efficiency. Findings show that our mathematical programming subject to various assumptions and estimations is efficient to rationalize portfolios up to at least 400 SKUs.

Keywords: supply chain management, efficient portfolio reduction, joint product portfolio-inventory model, conic quadratic mixed-integer program.

JEL classification: C61, M11.

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1 Introduction

To satisfy customers, companies develop product lines and thereby increase the number of products. This allows capturing demand by offering different functionalities fitting the various customer needs. Consumers also experience variety as a service and this explains the regular product accumulation in many firms' portfolio. At some point however, the product proliferation and the portfolio complexity are so high that end-of-life or similar stock-keeping units (SKUs) need to be discontinued. This process is known as SKU rationalization. We develop in this paper a mathematical programming tool supporting this process for a two-echelon supply chain case composed of a supplier, a central warehouse and multiple customers. More specifically, the goal is to balance the savings provided by SKUs elimination and the loss of revenue from decreasing sales volume.

With this goal in mind, the main difficulty lies in the determination and in the modelling of various implications of a product portfolio simplification.

Morgan et al.[16] analyse the importance of operational cost implications of the product line design. They describe five different levers of cost savings (Table 1). Firstly, the elimination of SKUs allows shifting the customers towards products with higher average selling prices. Secondly, it improves the manufacturing productivity thanks to fewer changeovers and less scraping waste. Thirdly, the improvements in manufacturing and the reduction of changeovers free up capacity. Fourthly and as Alfaro and Corbett make use in 2003 [2], inventories are pooled and this SKU consolidation cuts holding costs. Finally, the portfolio simplification brings about a benefit regarding the operational and administrative product management, decreasing thereby the workload and improving the yield of employees.

Regarding the drawbacks, the elimination of SKUs surely impacts the demands of the whole set of products. The modelling difficulty actually lies in the forecast of the customer's behaviour following the portfolio reshape. How many of them will agree swapping? In which extent does the firm control the choice of a substitute? Which substitute would customers pick up if they can choose? Do they prefer moving to the competitor? All these questions need to be covered by the demands reshape modelling. Also, demanded quantities impact all the variable costs or benefits. Therefore, the model has to take care of the revenue and of the production, transportation and inventory costs to embody all the rationalization effects in the profit calculation.

The rest of the paper is divided into three main sections. The following reviews the relevant literature. Section 3 describes a mixed-integer conic optimization model in the case of corporate control of product substitution and independent demand. We then show how to adapt the model to deal with several variants: dependent demand, customer choice of substitution and non-homogeneous product substitution. Finally, section 4.1 illustrates the approach in a real-life industrial application and demonstrates that our procedures scale up to portfolios of at least 400 SKUs.

Levers	**Benefits**
Pricing	Shift of customers to products with higher average selling price
Raw Materials	Higher manufacturing productivity due to fewer changeovers and scraps
Manufacturing	Capacity freed up due to the diminution of changeover waste Capacity freed up due to the higher manufacturing productivity
Inventory	Reduction of required inventory due to consolidation and pooling of products
Management	Reduction of workload due to the simplification

Table 1: Benefits of Portfolio Simplification

2 Literature Review

Literature makes use of a diverse vocabulary to analyse the number of products that a firm wants to make available for her customers. This topic is generally described as variety management, product line management or product line design. However, a substantial fraction of this body of research including the present paper considers the process of reducing the number of SKUs offered by the company. In this regard, terms such as SKU rationalization, efficient product portfolio reduction, product range simplification or optimization of product variety are used. This is related to concepts like complexity reduction, product proliferation or cost of providing variety. In this article, all of those expressions will be used interchangeably.

However, assortment planning must be distinguished from variety management. Indeed, it occurs at the retailer level and involves an understanding of the cognitive consumer's behaviour in front of the shelf. In this paper, we focus on longer term decisions regarding the product portfolio management, for which the supporting analyses are carried out from a broader viewpoint, the market level.

Following Ramdas [18], we classify the studies intended to support product range simplification in two categories.

On the one hand, qualitative methodologies have been set up by various authors and consulting companies, proving the importance of SKU rationalization in modern businesses. This results in the elaboration of best practices based on the analysis of transactional data and key performance indices (sales volumes, profit, productivity, substitutability, etc.) and on the firm's objectives (targeted SKU number, priorities, etc.) to select the products to discontinue from an existing portfolio. Also, Martin, Hausman and Ishii [13] recommend to support product portfolio decisions the use of indices of commonality, of differentiation and of set up, or other qualitative tools such as the process sequence and commonality graphs.

On the other hand, and the present paper falls in this category, models can be built to try determining the optimal product range by means of mathematical programming. Fellini, Kokkolaras and Papalambros [10] use an extended commonality strategy to reduce the number of SKUs by efficiently grouping similar products. McBride and Zufryden [14] propose an integer programming approach to the optimal product line selection based on products attributes and consumer measurements. Dobson and Kalish ([8] and [9]) include in their programming a fixed and a variable cost related to each product profile.

The model that we will develop contains formal similarities to location-inventory problems. Therefore, we briefly review now the most relevant works in this area. Shen, Coullard and Daskin ([7] and [19]) develop a joint location-inventory model which embodies the inventory pooling effects on the facility location decision. This is done in the objective of shaping increasingly integrated supply chains. They consider independent and normally distributed demands, and solve the problem with a Lagrangian relaxation algorithm requiring additional distributional assumptions.

Different extensions of their facility-location model have been studied. On the one hand, Aydin, Kayis and Guker [4] consider in 2011 dependent or correlated demands and on the other hand, Corbett and Rajaram [6] cover the case of non-normal and dependent demands. Gerchak and He [12] analyse also the relation between the benefits of risk pooling and the demand variability to show that savings are lower when demands are more positively correlated. Shen and Daskin [20] find a way to serve the uncovered demand through the location decision. Shu, Song and Sun [21] and then Snyder, Daskin and Teo [22] transform the problem in a stochastic location-inventory model with risk pooling, using scenarios where long-term location and inventory allocation decisions are temporally separated. Finally, to solve the joint location-inventory model, Atamtürk, Berenguer and Shen [3] provide an innovative conic integer programming formulation which we will use in the next sections.

We contribute to this literature by describing a more comprehensive model that includes effects like product pooling or pricing. Using recent ideas for similar location-inventory models, we show how to reformulate as a MIQCP, and that commercial solvers can be efficient to solve the program in a reasonable amount of time.

3 Mathematical Models

This section is divided in three parts. The first one is dedicated to the model development, where we review the required assumptions and needed notations. The actual problem formulation, the objective function definition and a resolution procedure compose the three last subsections of this first part. The two last parts of the section discuss different sets of assumptions about customer behaviour. The model is adapted in the first one to correlated demand distributions, while it considers the customer behaviour out of firm's control in the second one.

3.1 Model with Company Control

3.1.1 Assumptions

Here is the list of basic assumptions that we make and we discuss next. The first two will be made throughout the paper, while H3, H4, H5 and H6 will be relaxed in the next subsections.

H1 Linear transportation cost function;

H2 EOQ policy to compute the transportation and working-inventory costs;

H3 Constant lead time respective to the SKU;

H4 Corporate control of the demand reallocation;

H5 Only one substitute SKU;

H6 Independent and normally distributed demands.

To model production cost savings thanks to SKU rationalization, we include in our model two fixed costs: one per product, and another per family of products. The per-product fixed cost will be representative of the following benefits associated with an SKU suppression: increase of capacity utilization (e.g. less setups and changeovers), decrease of scraps (e.g. less setups and changeovers), reduced supervision and administrative workload from simplification.

In addition to these per-SKU fixed costs, the production structure may justify additional savings if entire product families can be eliminated. Following Meyer and Utterback [15] we define a product family as "*a group of products sharing a common platform in terms of market understanding, distribution, manufacturing or service dimensions*". Park and Simpson [17] discuss a method to determine the costs shared by common components. To include this aspect in our model, a family fixed charge is introduced to illustrate that a whole family elimination entitles the suppression of an additional fixed cost. In other words, this charge occurs if at least one SKU from the family is produced. This also exemplifies how flexible the modelling process is and how it may be modified to deal with additional features.

Next, to model beneficial effects related to inventory pooling, we model safety stock using the classical safety stock formula that assumes normal and independent demands (H6) and constant lead times (H3). This is for ease of presentation only, as we will allow for correlated and random lead times in Section 3.2. Finally, to derive transportation cost we assume them to be linear (H1).

The main drawback associated with SKU elimination is of course a decrease of demand. This leads to classical recommendation as *products should only be dropped if they merely cannibalize demand from other products* [5]. To go beyond this, one needs to understand what will be the customer reaction to the elimination of their preferred product. To model this, we assume that some fraction of the demand will disappear, while the remainder of the demand will be reported to another, similar product in the offering. Let us define the substitution parameter $\delta_{ij} \geq 0$ as the positive number so that, in case product i would be eliminated from the portfolio and the only alternative would be product j, each unit of demand for product i would be replaced by δ_{ij} units of demand for product j. That is, δ_{ij} models both a partial substitution effect and a unit conversion effect (in case i and j are expressed in different units). When i and j are expressed in identical units, then δ_{ij} is the substitution rate and in particular $\delta_{ij} = 1$ means that products i and j are perfect substitutes.

We suppose that this substitution relation can be reliably estimated between each single pair of SKUs, by means of forecasting, surveys, customer consultations or other quantitative methods. We also assume that the company controls the substitution and decides how she wants to reallocate the customers among the products (H4). An alternative model of customer behaviour will be presented in Section 3.3 where we will assume that the company does not control the substitute SKU, but customers decide the substitution product, based on a ranking of these products. Finally we also assume for simplicity that there is only one substitution SKU, or that the residual demand of a cancelled item is not split up between different replacements (H5). Thereby, the customers of that deleted SKU are supposed to move homogeneously towards either one other SKU, or a competitor. This assumption can be relaxed by using customer individual information to reallocate demand, as we do in section 3.4.

3.1.2 Notation

Before moving to the model formulation, we introduce the notation used.

Sets

$i, j \in I$ set of SKUs composing the portfolio,

$k \in K$ set of SKU families,

I_k subsets of SKUs i belonging to the family k,

Parameters

Demand

Ω_i annual demand distribution assumed to be Normal (μ_i,$12\sigma_i^2$),

μ_i yearly demand for SKU i,

σ_i standard deviation of monthly demand for SKU i;

Revenue and costs

p_i unit average selling price of SKU i,

c_i variable production cost related to SKU i,

f_i fixed cost related to the presence of SKU i in the portfolio,

l_k fixed cost related to the presence of family k, i.e. if at least one SKU of I_k is in the final portfolio,

$t(x)$ annual transportation cost function for a yearly demand quantity of x,

$v(x)$ shipment cost function for x units from the supplier to the warehouse,

d unit transportation cost from the supplier to the warehouse,

g fixed cost of shipment,

$w(x)$ working inventory cost function for a yearly demand quantity of x,

h_j annual unit holding cost,

F fixed cost of ordering,

Other parameters

LT_i lead time for SKU i,

α desired service level, or the probability of not encountering stock out,

δ_{ij} substitution parameter between products i and j, see definition and discussion above,

Weights

$\theta \geq 0$ relative weight assigned to the inventory cost,

$\beta \geq 0$ relative weight assigned to the transportation cost.

Decision Variables

$Y_i \in \{0, 1\}$ equals 1 if the SKU i remains in the standard offering and 0 otherwise,

$W_k \in \{0, 1\}$ equals 1 if at least one SKU belonging the the family k remains in the offering, and 0 otherwise,

$X_{ij} \in \{0, 1\}$ equals 1 if SKU i is discontinued and if its demand is transferred to j, and 0 otherwise,

$D_j \geq 0$ demand for SKU j after rationalization,

The model is thus composed of $\mathcal{O}(|I|^2 + |K|)$ variables.

3.1.3 Problem Formulation

Based on the provided hypothesis and notations, we obtain the following mixed-integer optimization problem. At this point, we only give a very generic objective function that we will fully specify in Section 3.1.4.

$$\max \sum_{j \in I} K_j(D_j) - F_j(Y_j) - H_j(X_{.j}) - \sum_{k \in K} G_K(W_k)$$

subject to

$$X_{ij} \leq Y_j, \qquad \forall\, i, j \in I \tag{1}$$
$$\sum_{j \in I} X_{ij} \leq 1, \qquad \forall\, i \in I \tag{2}$$
$$D_j = \sum_{i \in I} \delta_{ij} \mu_i X_{ij}, \qquad \forall\, j \in I \tag{3}$$
$$X_{ii} = Y_i, \qquad \forall\, i \in I \tag{4}$$
$$Y_i \leq W_k, \qquad \forall\, i \in I_k, k \in K \tag{5}$$
$$X_{ij}, Y_i, W_k \in \{0, 1\}, \qquad \forall\, i, j \in I, k \in K \tag{6}$$

where $X_{.j}$ denotes the column j vector of matrix X. Constraint (1) ensures the demand of the discontinued SKUs is allocated to an SKU kept in the portfolio. The second constraint makes sure that the demand of an abandoned SKU is transferred to only one other product. The constraint (6) defines the existence domain of decision variables.

Clearly, this part of the model corresponds to the uncapacitated facility problem. In most of cases, these models attempt to minimize the involved fixed and transportation costs. Our formulation is partly analogous because the same kinds of decisions are expected. The initial set of SKUs corresponds to the potential facility sites and each product involves a fixed cost, just like the building of a plant. Then, the opening variables will determine if the specific SKU is confirmed and if the respective fixed cost has to be reckoned. Also, the distance from the plant to the customer is analogous to the discrepancy between the product and the customer's needs. While closing plants generates higher transportation costs, discontinuing SKUs increases the gap between the products and the preferences, which results in a loss of demand depending on the substitution parameter.

However, even if the framework of both problems seems similar, an important discrepancy arises in comparison with the basic facility location problem. As we will see, the objective function is non-linear because of other elements like the safety stock. Also, additional constraints are necessary to model SKU rationalization. The new demand needs to be defined as a function of the allocation variables and parameters δ_{ij}. This is ensured by (3). Constraint (4) guarantees that the demand of an SKU kept in the portfolio is not reallocated. Finally, constraint (5) ensures the family fixed cost is saved only if all SKUs in the family are discontinued. This generic model has $\mathcal{O}(|I|(|I|+|K|))$ variables and constraints.

3.1.4 Objective Function Definition

As explained in the introduction of the model, we are interested in the effects of pooling the demand of different SKUs together, which are the elimination of the set up costs per SKU and family of SKUs, and the benefits of pooling the safety stock. Moreover, the rationalization is going to affect the demand. Hence, we have to consider all the revenues and costs depending on the total demand. We will consider the revenue, the production, the transport and the working inventory and we'll try to maximize the benefits minus those different costs. Table 2 summarizes the components involved and provides a first mathematical formulation.

	Objective function	Formulation	Explanation
+	Revenue	$p_j D_j$	Price multiplied by the new demand
-	Production cost	$c_j D_j$	Production cost multiplied by the new demand
-	Set up cost related to the existence of a SKU	$f_j Y_j$	Set up cost to be considered when the SKU continues to exist (Y =1)
-	Set up cost related to the existence of a family	$l_k W_k$	Set up cost to be considered when the family exists (W =1)
-	Safety stock cost	$\theta h_j Z_\alpha \sqrt{LT_j}\sqrt{\sum_{i\in I}\delta_{ij}^2\sigma_i^2 X_{ij}}$	Weighted SS formula where the deviation is the sum of pooled items' standard deviations
-	Annual transportation cost	$t(D_j)$	Function to be determined hereafter
-	Working inventory cost	$w(D_j)$	Function to be determined hereafter

Table 2: Components of the objective function

When we aggregate the seven elements, we get a first formulation of the objective to maximize:

$$\max_{X_{i,j},Y_j,D_j,W_k} \sum_{j\in I}\left[(p_j-c_j)D_j - f_jY_j - \theta h_j Z_\alpha\sqrt{LT_j}\sqrt{\sum_{i\in I}\delta_{ij}^2\sigma_i^2X_{ij}} - \beta t(D_j) - w(D_j)\right] - \sum_{k\in K} l_k W_k$$

We still need to determine $t(D_j)$ and $w(D_j)$, the transportation and working inventory costs related to the annual demand D_j. Actually, both costs depend on the shipped quantities and on how often the orders are dropped. We use the approach developed by Shen et al. [19] based on the *Economic Order Quantity* policy (H2). The purpose of this model is to determine the optimal quantity to order (and thereby the number of orders), which balances and minimizes inventory and transportation costs. Once the expression of this number is known, a final formulation of the involved costs can be derived.

The development illustrated in Appendix A shows for a specific product the derivation of the optimal number of orders from the costs, and its insertion into the cost expressions. The final formulation for the transportation and working inventory costs obtained for every SKU j is:

$$\beta t(D_j) + w(D_j) = \sqrt{2\theta h D_j(F+\beta g)} + \beta d D_j$$

This expression can be injected in the objective function to reach the final version.

$$\max_{X_{i,j},Y_j,D_j,W_k} \sum_{j\in I} \left[(p_j - c_j)D_j - f_jY_j - \theta h_j Z_\alpha \sqrt{LT_j}\sqrt{\sum_{i\in I}\delta_{ij}^2\sigma_i^2 X_{ij}} - \sqrt{2\theta h D_j(F+\beta g)} - \beta d D_j \right] - \sum_{k\in K} l_k W_k$$

$$\max_{X_{i,j},Y_j,D_j,W_k} \sum_{j\in I} \left[(p_j - c_j - \beta d)D_j - f_jY_j - \theta h_j Z_\alpha \sqrt{LT_j}\sqrt{\sum_{i\in I}\delta_{ij}^2\sigma_i^2 X_{ij}} - \sqrt{2\theta h D_j(F+\beta g)} \right] - \sum_{k\in K} l_k W_k$$

$$\max_{X_{i,j},Y_j,D_j,W_k} \sum_{j\in I} \left[(p_j - c_j - \beta d)D_j - f_jY_j - q_j\sqrt{\sum_{i\in I}\delta_{ij}^2\sigma_i^2 X_{ij}} - s_j\sqrt{D_j} \right] - \sum_{k\in K} l_k W_k$$

where $q_j = \theta h_j Z_\alpha \sqrt{LT_j}$ and $s_j = \sqrt{2\theta h_j(F+\beta g)}$

3.1.5 Resolution Procedure: A Conic Quadratic Mixed Integer Reformulation

This problem is hard to solve because of the non-linearity of the objective function. Following Atamtürk et al.[3], we reformulate it as a mixed integer (convex) conic quadratic program. Indeed convexity is key to be able to efficiently solve the continuous relaxation and exploit it in a branch-and-bound procedure. For example commercial solvers like GUROBI [1] include now modern algorithms able to deal with conic quadratic formulations.

The objective function is linearised by replacing the two non-linear terms by auxiliary variables t_{1j} and t_{2j}. To be equivalent, the optimal solution of this program must satisfy $t_{1j} = \sqrt{\sum_{i\in I}\delta_{ij}^2\sigma_i^2 X_{ij}}$ and $t_{2j} = \sqrt{D_j} = \sqrt{\sum_{i\in I}\delta_{ij}\mu_i X_{ij}}$. Natural constraints are $t_{1j}^2 \geq \sum_{i\in I}\delta_{ij}^2\sigma_i^2 X_{ij}$ and $t_{2j}^2 \geq \sum_{i\in I}\delta_{ij}\mu_i X_{ij}$, with non-negativity conditions on t_{1j} and t_{2j}. To ensure convexity and since $X_{ij} = X_{ij}^2$ for binary variables, these inequalities are reformulated by the conic quadratic constraints (8) and (9):

$$\max_{Y_j,X_{ij},D_j,W_k} \sum_{j\in I} [(p_j - c_j - \beta d)D_j - f_jY_j - q_j t_{1j} - s_j t_{2j}] - \sum_{k\in K} l_k W_k$$

subject to

$$t_{1j}, t_{2j} \geq \quad 0 \quad \forall\, j\in I \tag{7}$$

$$\sum_{i\in I}\delta_{ij}^2\sigma_i^2 X_{ij}^2 \leq \quad t_{1j}^2 \quad \forall\, j\in I \tag{8}$$

$$\sum_{i\in I}\delta_{ij}\mu_i X_{ij}^2 \leq \quad t_{2j}^2 \quad \forall\, j\in I \tag{9}$$

(1) - (6)

3.2 Model with Dependent Demands and Stochastic Lead Times

Several assumptions made up to now are clearly unrealistic. In particular, it seems very natural for demands of substitutable products to be correlated. Moreover, the lead times are not always deterministic and may be characterized by a variance. Therefore, this section generalizes the model to integrate these two aspects.

Practically, to take these into account in the model, only the formulation for the safety stock cost needs to be modified. Instead of just taking the sum of the variances, we now need to take into account the correlation of demands and the variability of lead times.

Firstly, the aggregate variance of pooled correlated demands is basically $Var(\sum_i X_{ij}\delta_{ij}\Omega_i)$. Following classical rules from statistics, the variance of a sum of correlated variables multiplied by a constant parameter is:

$$Var(\sum_i X_{ij}\delta_{ij}\Omega_i) = \sum_{i,i'} \delta_{ij}\delta_{i'j}X_{ij}X_{i'j}Cov(\Omega_i, \Omega_{i'}), \quad \forall\ j\ \in\ I.$$

Let us denote V^j the adjusted variance-covariance matrix respective to each SKU j, where each element $V^j_{ii'}$ is $\delta_{ij}\delta_{i'j}Cov(\Omega_i, \Omega_{i'})$. Then, the variance term of demands becomes

$$X_{.j}^T V^j X_{.j}$$

where $X_{.j}$ denotes the column j vector of matrix X.

Secondly, the safety stock variation term now includes the variability of lead times. The deviation factor is equal to $\sqrt{LT\sigma_D^2 + \mu_D^2\sigma_{LT}^2}$, where LT and μ_D are the respective mean lead time and mean demand, and σ_{LT} and σ_D the respective lead time and demand deviations.

For a specific product pool, the mean lead time and the deviation of the initial product are used. The variance of the demand is computed as explained here above and the mean is simply equal to the sum of forecasts of assigned products, i.e. $\sum_i \mu_i X_{ij}$. As the variables X_{ij} are binary, the squared aggregate demand can be written $X_{.j}^T M X_{.j}$, where

$$M = \begin{pmatrix} \mu_1^2 & \mu_1\mu_2 & \cdots & \mu_1\mu_{|I|} \\ \mu_2\mu_1 & \mu_2^2 & \cdots & \mu_2\mu_{|I|} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{|I|}\mu_1 & \mu_{|I|}\mu_2 & \cdots & \mu_{|I|}^2 \end{pmatrix}$$

Hence, the derived safety stock cost term is

$$\begin{aligned} \theta Z_\alpha h_j \sqrt{LT_j\sigma_D^2 + \mu_D\sigma_{LT}^2} &= \theta Z_\alpha h_j \sqrt{LT_j(X_{.j}^T V^j X_{.j}) + (X_{.j}^T M X_{.j})\sigma_{LT}^2} \\ &= \theta Z_\alpha h_j \sqrt{X_{.j}^T(LT_j V^j + \sigma_{LT_j}^2 M)X_{.j}} \end{aligned}$$

and the generalized model becomes:

$$\max_{X_{i,j},Y_j,D_j,W_k} \sum_{j\in I} \left[(p_j - c_j - \beta d)D_j - f_jY_j - \bar{q_j}\sqrt{X_{.j}^T(LT_jV^j + \sigma_{LT_j}^2 M)X_{.j}} - s_j\sqrt{D_j}\right] - \sum_{k\in K} l_kW_k$$

where $\bar{q_j} = \theta h_j Z_\alpha$, $s_j = \sqrt{\theta h_j(F + \beta g)}$, and this is subject to constraints (1) to (6).

Following the same resolution procedure of section 3.1.5, we finally reformulate constraint (8) as shown in the resulting convex quadratic program:

$$\max_{Y_j,X_{ij},D_j,W_k} \sum_{j\in I} [(p_j - c_j - \beta d)D_j - f_jY_j - \bar{q_j}t_{1j} - s_jt_{2j}] - \sum_{k\in K} l_kW_k$$

subject to

$$X_{.j}^T(LT_jV^j + \sigma_{LT_j}^2 M)X_{.j} \leq t_{1j}^2, \qquad \forall\ j \in I \tag{10}$$

$$(1) - (7),\ (9)$$

3.3 Customer Behaviour Model

In the model of section 3.1, the demand reshape is assumed to be defined by the firm herself. It means she arbitrarily decides which substitute she provides to the customers, given the proportion who will agree. This is realistic when the marketing and the client relationship is strong enough.

However, in other cases, the firm does not pilot the reallocation but has to anticipate the reaction of the customers to the new product offering. To that end, we propose to model the consumer behaviour as follows: customers are supposed to have a preference ordering, and buyers of discontinued product will switch (partially according to the substitution parameter δ_{ij}) to their preferred product still in the offering (and not necessarily to the product most profitable for the firm). Consumers of an SKU are supposed to behave identically and rank the products in the same order. This assumption is relaxed in the following section where a customer-specific information is available.

Mathematically, we assume we have for each SKU i an ordering $\prec_i$ with $j \prec_i m$ meaning that if i is not available anymore, buyers of i would switch to j only if m is not available either. Again, this information has to be gathered based on forecasting, surveys, customer consultations or other quantitative methods. To incorporate this into model (1)–(9), we need to add the following constraints:

$$X_{ij} + Y_m \leq 1 \qquad \forall\ i, j, m \in I, j \prec_i m. \tag{11}$$

Indeed, this constraint ensures that when m is preferred over j as a substitute for i, X_{ij} cannot be 1 if m is still in the offering ($Y_m = 1$). To strengthen the formulation, the equivalent clique inequalities are preferable:

$$\sum_{j:j \prec_i m} X_{ij} + Y_m \leq 1 \qquad \forall\ i, m \in I \tag{12}$$

Indeed, these inequalities are stronger (i.e. constraints (11) are linearly implied by (12)), but there are also less clique inequalities than inequalities of type (11).

3.4 Model with Individual Customer Information

Conversely to the model with company control, the previous section provides a decision-making power to the consumers, but it still requires a segmental homogeneous behaviour. However, H5 may need to be relaxed if customers of a same product want to pick up different substitutes. If this information is available, the model can be modified to take into account individual behaviours. It could be particularly helpful in presence of big customers with different preference orderings.

The adaptation of section 3.1 model is straightforward. We create a set of customers C with index c. As we need an assignment decision for every single customer, the allocation variable becomes X_{ij}^c. Additionally, the distribution of demand, the substitution parameter δ_{ij}^c and the preference ordering $\succ_i^c$ all depend on the specific customer c.

Constraints (1), (2), (6) and (4) were using X_{ij} for all $i \in I$, while they are now considering the X_{ij}^c variables and are therefore also applied for every $c \in C$. Moreover, the new demand calculation has to sum the quantities of all the customers. Consequently, constraint (3) is replaced by the following equation (13). Constraint (5) about the family fixed cost definition remains identical.

$$D_j = \sum_{i \in I} \sum_{c \in C} \delta_{ij}^c \mu_i^c X_{ij}^c, \qquad \forall\, j \in\ I \tag{13}$$

As far as the objective function is concerned, the only difference lies in the computation of the safety stock cost. The aggregation of variations has to include all the customers. This is done by summing up on c.

$$\max_{X_{i,j}^c, Y_j, D_j, W_k} \sum_{j \in I} \left[(p_j - c_j - \beta d) D_j - f_j Y_j - q_j \sqrt{\sum_{i \in I} \sum_{c \in C} \delta_{ij}^{c2} \sigma_i^{c2} X_{ij}^c} - s_j \sqrt{D_j} \right] - \sum_{k \in K} l_k W_k$$

As a result, the modifications do not significantly transform the model, and the same solution procedure remains valid.

4 Computational Experiments

This section analyzes the benefits of the proposed rationalization approach by solving instances that differ in size, substitution levels and cost. Cost data directly come from a large company operating in the chemical sector in Belgium. The data is such that each individual product is profitable (i.e. with substitution levels at 0, it is optimal to keep the portfolio unchanged).

Considering uncorrelated and normally distributed demands, we first present in some detail the rationalization of a small product portfolio composed of 32 SKUs (which was a base case for the company considered). Afterwards, we treat portfolios of various sizes and subject to different substitution instances to analyse the algorithm efficiency. Finally, we perform a sensitivity analysis to highlight a couple of influential parameters. In each single part, the first model assuming company control on product substitution (M1) is compared to the customer behaviour model (M2).

Our real-life example provides all the parameters with exception of the substitution rates. The substitution data obtained from the company was not satisfactory for two reasons: it was incomplete, and applied to a portfolio much smaller than what we needed for our computational experiments. Moreover, our goal was also to experiment with different substitution levels. Consequently, we have decided to generate substitution levels ourselves. In the firm considered, all SKUs are expressed in identical units, so that our substitution parameters are between 0 and 1. To guarantee industrial consistency, we generate substitution levels based on the SKUs and product families similarity. It also takes the pricing differences into account, explaining the asymmetry of rates. This robust method fully described in Appendix B ensures that the substitution rates are mutually coherent among each other in terms of relative values. Finally, for the customer behaviour model of Section 3.3, we take the reasonable assumption that preference ordering is given by the substitution rates. That is, we define $m \prec_i j$ if and only if $\delta_{im} \leq \delta_{ij}$ (breaking ties arbitrarily). In other words, in case i is discontinued, customers will (partially) switch to the product j that has the highest substitution rate.

4.1 A Detailed Example

The tight product portfolio we are considering is composed of 4 product families, making a total of 32 SKUs. Tables 7, 8, 9 and 10 provided in Appendix C display the initial parameters needed in the rationalization program. As far as the inventory policies are concerned, the company guidelines state that 99% of orders are satisfied on-time.

The solver reaches an optimal solution for both problems in a few seconds and the substitution decisions are shown in Table 11 of Appendix C. The benefits are straightforward to depict. In the case of the company control model (M1) for instance, it suggests to reduce the portfolio size to 19 SKUs in order

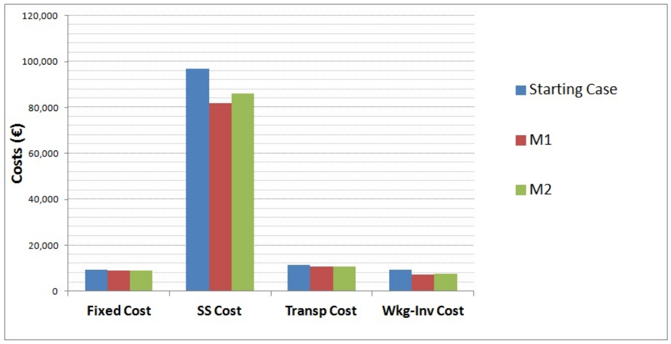

Figure 1: Comparison of Costs

to increase the total profit by 13,443€. Figure 1 and Table 3 respectively display for both models the different costs and some key performance indicators.

KPI	Initial	Company Control Model	Customer Behaviour Model
Number of SKUs	32	19	21
Profit	1,887,796	1,901,239	1,899,024
Fixed Cost	9,380	8,860	8,940
SS Cost	96,925	81,966	85,990
Transportation Cost	11,551	10,879	10,874
Working-Inventory Cost	9,335	7,389	7,670
Total Demand (units)	3,378,298	3,216,376	3,207,837
Gross Margin	2,004,177	2,000,040	2,002,233
Weighted Average Gross Margin	0.593	0.622	0.624

Table 3: Key Performance Indicators

The gross margin computed in the table is defined as the difference between the revenue and the costs of production and transportation. It actually provides an indication of the pricing effect on the global profit. One may think that the gains mostly come from substituting SKUs by more profitable products. However, results show that even if the weighted average gross margin increases, the gain arises mainly out of the pooling effect through the safety stock term. This can be easily observed in Figure 2 showing the breakdown of the profit increase.

As far as the customer behaviour model is concerned, the additional preference constraint makes the magnitude of the gain slightly smaller. The final number of SKUs is a bit higher and consequently, the gross margin loss is smaller, but the savings from product pooling are also weaker. This can be observed on Figures 1 and 2.

We can illustrate the relevance of the customer behaviour model through the following example. Let us consider the SKU 23. Model M1 recommends to discontinue this product and to swap 82% of the demand to the SKU 25. However, model M2 suggests to replace by SKU 27 to transfer 94% of the demand. This choice is more respectful of customer preferences if we assume that the most favourite product is the most substitutable SKU of the offering.

4.2 Efficiency of the Algorithmic Approach

In this section, we want to answer the following questions:

- What is the maximum size portfolio that we can handle using the solution approach proposed?

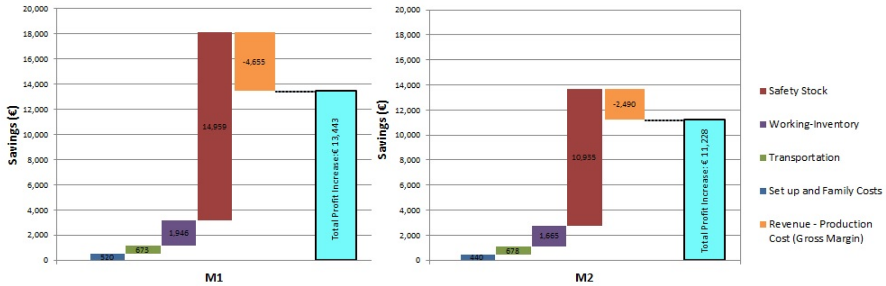

Figure 2: Profit Increase Decomposition. Each bar is related to one or two terms of the objective (see Table 2). Its height gives the difference between portfolios before and after rationalization. M1 assumes corporate control of customer reallocation, M2 does not.

- What is the impact of consumer models (with and without company control) at different substitution levels?

From the same database as the last section, we consider portfolios of 50, 100, 200, 300 and 400 SKUs. Using the estimating method described in Appendix B, we randomly generate 5 instances (i.e. differing by substitution rates) for each portfolio size. These medium rates are multiplied by 0.95 and 1.05 (with a maximum substitution rate of 1) to build respectively a low-level and a high-level substitution table. Hence, for each problem size we have 3 classes of substitution levels (low, medium, high) composed of 5 instances each. All instances are solved using Gurobi[1] version 5.5 on an Intel Quad-Core i5 platform, running at 3.40 GHz with 8 GB of RAM under Windows 7.

For each single portfolio and substitution level, average results of solving M1 and M2 are shown in Tables 4 and 5. The first table describes the solutions obtained in a maximum of 10 minutes, while the second gives the first feasible solution found within a gap of 1%.

For each run, we collect three solution values: $INIT$ is the objective value of the initial portfolio, $BEST$ is the objective value of the best solution found and UB is the solution upper bound returned by the solver.

In general, the solution process may terminate in three different states: the optimal solution is found, a feasible solution is found during the allocated time, or no feasible solution could be found. The solution status describes in both tables the state of five instances terminations. All other figures are the average of values related to the best solution found. Of course, when the optimal solution is found, $BEST = UB$.

The standard gap is traditionally defined as $(UB - BEST)/UB$, the relative distance between the upper bound and the objective value of the best solution found at the end of the solution process. However, for the instances we consider, by construction, this gap will always be small in absolute terms. This is because the starting portfolio is always substantially profitable (to cover all indirect and fixed cost not in the scope of the problem). This corresponds to adding a large constant to the objective function, mechanically reducing the gap.

To better analyze the quality of the solution found, we consider two other measures. The first one is the potential gain. It is defined as $(UB - INIT)/INIT$ and gives an upper bound on the relative gains the SKU rationalization can potentially deliver. The second measure is the realized potential gain. It is defined as (BEST-INIT)/(UB-INIT) and gives the fraction of this maximum possible gain achieved by the current best solution. For example, a realized potential gain of 90% means that the solution found realizes *at least* 90% of the gains the SKU rationalization can possibly deliver.

We can make the following observations based on these results:

- As expected, high substitution rates lead to discontinue more SKUs.

- The solution time to proven optimality increases very quickly with the number of SKUs. In particular, it becomes difficult for both consumer models to find the optimal solution within 10 minutes when

Model	Initial SKU Number	Substitution Level	Solution SKU Number	Solution Status opt / feas / none	Gap	Potential Gain	Realized Potential Gain	Solving Time (sec)
M1	**50**	**LOW**	42.6	5 / 0 / 0	0%	0.35%	100%	1.7
		MED	21.4	4 / 1 / 0	0.003%	1.68%	99.80%	123.5
		HIGH	12.2	5 / 0 / 0	0%	4.40%	100.00%	1.1
	100	**LOW**	89.6	5 / 0 / 0	0%	0.19%	100.00%	51.8
		MED	50.8	3 / 2 / 0	0.011%	1.23%	99.17%	367.8
		HIGH	34.8	5 / 0 / 0	0%	3.38%	100.00%	3.8
	200	**LOW**	154.6	5 / 0 / 0	0%	1.05%	100%	151.8
		MED	106.2	3 / 2 / 0	0.005%	1.92%	99.76%	419.3
		HIGH	77	5 / 0 / 0	0%	3.56%	100.00%	12.4
	300	**LOW**	208.4	0 / 5 / 0	0.054%	1.02%	94.69%	601
		MED	129.4	0 / 5 / 0	0.074%	2.20%	96.57%	601.2
		HIGH	88.2	0 / 5 / 0	0.031%	4.10%	99.21%	600.8
	400	**LOW**	266	0 / 5 / 0	0.091%	1.24%	92.52%	602.4
		MED	160	0 / 5 / 0	0.142%	2.62%	94.43%	601.8
		HIGH	105.6	0 / 5 / 0	0.065%	4.85%	98.59%	602.2
M2	**50**	**LOW**	42.2	5 / 0 / 0	0%	0.21%	100%	1.5
		MED	25.6	5 / 0 / 0	0%	1.50%	100%	4.7
		HIGH	16.4	5 / 0 / 0	0%	3.71%	100%	5.4
	100	**LOW**	89.4	5 / 0 / 0	0%	0.11%	100%	5.6
		MED	56.2	5 / 0 / 0	0%	1.12%	100%	18.9
		HIGH	40	5 / 0 / 0	0%	2.97%	100%	30.5
	200	**LOW**	155.6	5 / 0 / 0	0%	0.95%	100%	30.7
		MED	114.2	5 / 0 / 0	0%	1.81%	100%	81.6
		HIGH	88.6	5 / 0 / 0	0%	3.27%	100%	138.1
	300	**LOW**	212.4	4 / 1 / 0	0.004%	0.88%	99.56%	472.6
		MED	155	0 / 2 / 3	0.27%	2.20%	87.4%	600
		HIGH	129	0 / 2 / 3	0.656%	4.09%	83.43%	600
	400	**LOW**	259	0 / 5 / 0	0.277%	1.20%	75.79%	600
		MED	na	0 / 0 / 5	na	3.02%	na	600
		HIGH	na	0 / 0 / 5	na	7.73%	na	600

Table 4: Results within a 10 minutes time limit.

Model	Initial SKU Number	Substitution Level	Solution SKU Number	Solution Status opt / feas / none	Gap	Potential Gain	Realized Potential Gain	Solving Time (sec)
M1	**50**	**LOW**	44.6	0 / 5 / 0	0%	0.51%	58%	0.1
		MED	31.8	0 / 5 / 0	0.687%	2.09%	66.43%	0.1
		HIGH	15.4	0 / 5 / 0	1%	5.01%	85.23%	0.1
	100	**LOW**	93.4	0 / 5 / 0	0%	0.31%	53.68%	0.4
		MED	70.8	0 / 5 / 0	0.568%	1.54%	62.59%	0.4
		HIGH	40.8	0 / 5 / 0	1%	4.10%	78.14%	0.4
	200	**LOW**	157.4	0 / 5 / 0	0.13%	1.14%	88.60%	3.1
		MED	106.2	0 / 5 / 0	0.481%	2.21%	77.78%	3
		HIGH	79.2	0 / 5 / 0	0%	3.76%	92.98%	3.9
	300	**LOW**	227.6	0 / 5 / 0	0.382%	1.21%	68.04%	9.8
		MED	134.2	0 / 5 / 0	0.748%	2.64%	71.13%	10.6
		HIGH	96	0 / 5 / 0	0.470%	4.42%	89.12%	18.8
	400	**LOW**	299.2	0 / 5 / 0	0.497%	1.47%	65.60%	19.9
		MED	162.6	0 / 5 / 0	0.649%	2.92%	77.36%	26.4
		HIGH	115.4	0 / 5 / 0	0.385%	5.03%	92.12%	38.5
M2	**50**	**LOW**	46.8	0 / 5 / 0	0.23%	0.35%	34.93%	0.4
		MED	29	0 / 5 / 0	0.60%	1.88%	67.84%	1.7
		HIGH	16.6	0 / 5 / 0	0.08%	3.77%	97.80%	4.01
	100	**LOW**	98.4	0 / 5 / 0	0.17%	0.20%	12.97%	3.1
		MED	68	0 / 5 / 0	0.73%	1.58%	53.08%	5.5
		HIGH	45.6	0 / 5 / 0	0.57%	3.32%	82.47%	10
	200	**LOW**	160.4	0 / 5 / 0	0.12%	1.04%	87.83%	15.8
		MED	121.6	0 / 5 / 0	0.38%	2.03%	81.27%	40.2
		HIGH	94.2	0 / 5 / 0	0.32%	3.51%	90.75%	61.3
	300	**LOW**	237.8	0 / 5 / 0	0.286%	1.06%	72.80%	67.5
		MED	169	0 / 5 / 0	0.486%	2.28%	78.76%	556.3
		HIGH	121.6	0 / 5 / 0	0.557%	4.07%	85.86%	681.3
	400	**LOW**	259.4	0 / 5 / 0	0.394%	1.30%	68.71%	373.2
		MED	210.4	0 / 5 / 0	0.56%	2.65%	78.68%	2045.4
		HIGH	140.8	0 / 5 / 0	0.66%	4.79%	85.65%	2827.3

Table 5: Results of 1% gap-first solutions.

the number of SKUs exceeds 300 SKUs. This is not surprising as our problem generalizes the strongly NP-Hard uncapacitated facility location problem.

- By observing the realized part of the potential gain, we nevertheless observe that the feasible solutions found realize most of the gain the rationalization process could potentially achieve, even when one could not prove the optimality.
- Furthermore, the larger the potential gain, the larger the realized gain by the first feasible solution found. Conversely, it is only for instances where not much can be gained by portfolio rationalization that the first solution job is poor. This is of course good news.
- Within ten minutes, one obtains good quality solutions for portfolio up to 300 hundreds of products, while allowing for an hour of computation is enough to handle 400 products. For larger portfolios, it is likely that another solution approach would be necessary, as already solving the continuous relaxation of the problem is too time consuming.
- The solution time also seems to depend on the substitution level. It is likely to be more difficult to solve problems with high substitutability than with low substitutability, but somewhat surprisingly, it is still harder to solve problems with medium substitution levels (see columns "Gap" and "Time" in Table 4). This could be explained by a simple counting argument: there are more different portfolios with half the products retained, than there are portfolios with few, or many, products kept in the portfolio.
- Instances with low or medium substitution rates are easier to solve with M2 than with M1. Conversely, M1 is faster for problems with high substitution levels.
- Intuitively, one would think that more products will be discontinued in M1 (with company control) compared to M2, since the company could potentially gain more when discontinuing products (by reallocating demand where it is more profitable). The results we get confirm this intuition. Looking at Table 4, this intuition is correct for all portfolio sizes. Note that the results of Table 5 do not seem to support this conclusion, but this is only because the solutions are not optimal.

As a general conclusion, we believe that the size of instances we can handle is large enough for most industrial applications. Indeed, given the framework of the model, combining unsubstitutable products in the same portfolio is irrelevant. Therefore, only substitutable products should be gathered and submitted to the rationalization process. In this sense, clusters of 400 SKUs is a reasonable limitation for many industrial cases.

4.3 Sensitivity Analysis

Intuitively, the characteristics enhancing the suppression of an SKU can be derived from the model. One may indeed suppose that the substitution rates, the costs or the distribution of the demand highly influence the decision to keep a product in the portfolio. Table 6 presents those incentives and their intuitive justification. Nevertheless, all those effects have different impacts. They are combined and balanced in the problem optimization and it may be difficult to highlight them in the numerical example. While the impact of the substitution level was analysed in section 4.2, we investigate two additional assumptions on an initial portfolio composed of 100 SKUs. We vary first the inventory cost weight, and afterwards the standard deviation of demand distribution.

Weight of Inventory Cost In this part, we modulate the weight corresponding to the importance given to the inventory cost. Furthermore, this analysis stands for variations in annual holding cost values or in customer service, since the level of inventory cost is solely impacted. Initially equal to 1, our study makes use of weights stretched from 0.2 to 2. The total inventory cost (safety stock and working-inventory)

Incentives to discontinue	Justification
High Substitutability	Weak demand dead-weight loss
High Fixed Cost	Discontinuing the SKU and its fixed cost
High Holding Cost	Replacing the SKU by an SKU less costly to stock
Weak Initial Demand	Small demand loss even if low transfer rate
High Demand Deviation	Combining deviations to reduce the variability
Small Average Gross Margin (price - proportional costs)	Transferring demand to high-margin products despite the loss (to escape cannibalization)
Dependent Demands Case: Low Correlation	Discontinuing a highly correlated SKU has low pooling effects

Table 6: Incentives to discontinue SKUs

is shown in Figure 3. Figure 4 illustrates the solution indicators as a function of the inventory cost weight in the model. Before rationalization, the weight has a quasi-linear impact on the inventory cost, and therefore on the profit. After rationalization however, the lower concave curve shows that the cost reduction increases with the rising of holding cost. Hence, as it can be seen in Figure 4, the product rationalization and its pooling effect drive larger benefits and are more efficient in case of substantial holding cost.

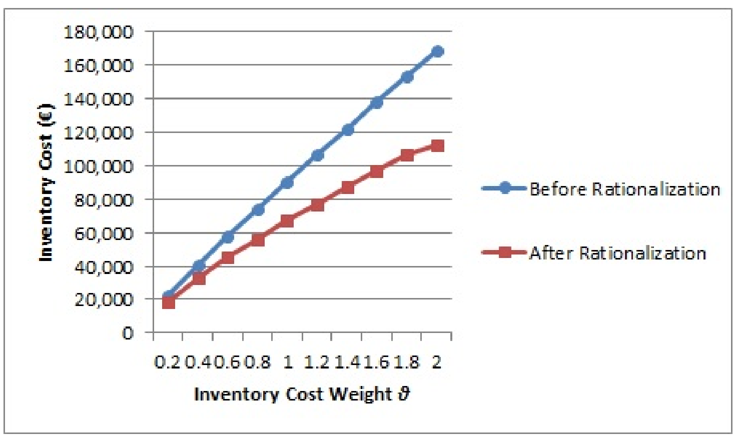

Figure 3: Evolution of inventory cost in function of the weight

Standard Deviation of the Demand Similarly, we analyse the impact of the demand variability on the rationalization benefits. We multiply the standard deviation by a factor varying from 0.2 to 2 in order to generate different levels of variability. Figure 5 shows the evolution of the safety stock cost, as a function of this multiplicative factor. It shows that the rationalization is increasingly efficient with a growing standard deviation. Indeed, high demand variability involves larger safety stock. Hence, the gains of pooling products are greater and the incentive to reduce the number of products consequently higher. These results can be observed in Figure 6 displaying the total profit and the optimal portfolio size in function of the variability level.

5 Conclusion

This paper contributes to the existing literature in three different ways. Firstly, the described model is comprehensive and includes major non-linear effects underestimated so far, like product pooling and pricing. Secondly, based on recent research on location-inventory problems, we show how to reformulate

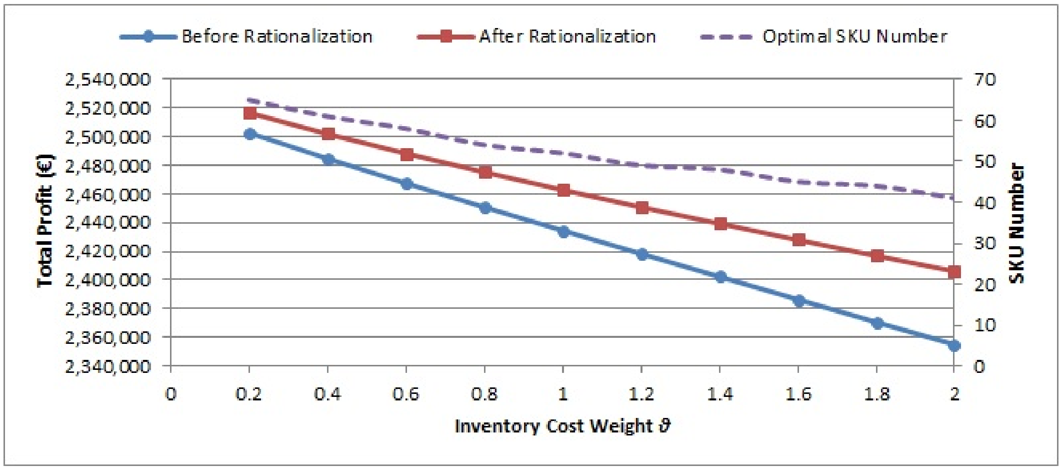

Figure 4: Profit and optimal SKU number in function of inventory cost weight

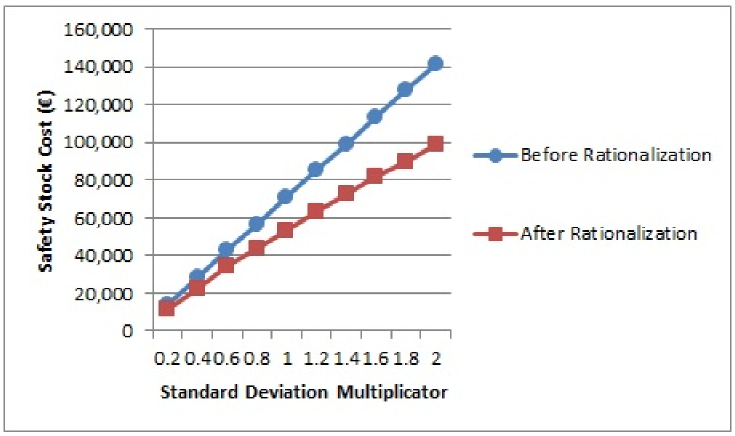

Figure 5: Evolution of safety stock cost in function of the demand deviation

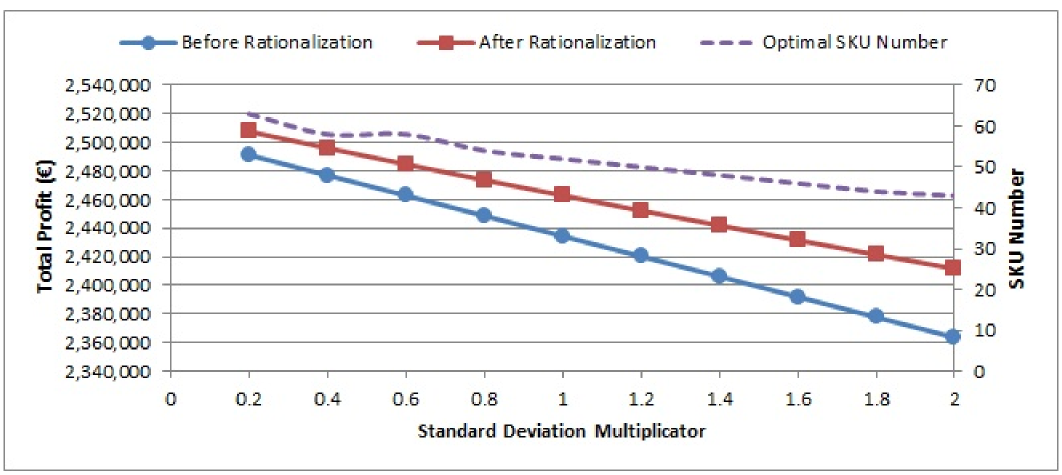

Figure 6: Profit and optimal SKU number in function of demand deviation

our model as a Mixed-Integer Quadratically Constrained Programming problem. Finally, we illustrate by means of a real-life industrial instance that commercial MIP solvers can efficiently solve problems of up to a few hundreds products.

Our model however suffers from several limitations. Firstly, it remains difficult to estimate essential parameters, like the substitution rate. However, avoiding this difficulty would require to use a completely different approach. Secondly, an SKU rationalization process may have other impacts which are not treated in our model, like improvements in productivity thanks to a more streamlined portfolio. Also, our model is essentially a one-period model that should be representative of a steady-state or average situation. When some parameters (e.g. demands) substantially vary over time, we may need to consider a multi-period setting. Finally, our proposed solution method does not seem to scale to thousands of products. Either a simplified model or an improved algorithmic approach is necessary in such a case. All these topics constitute interesting further research directions.

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Appendix A Derivation of optimal inventory and transportation costs

The goal of this development is to set up an optimal formulation for transportation and working inventory costs. As laid down in the Economic Order Quantity policy (H2), the cost function optimization determines the optimal number of orders (and consequently the optimal ordering quantities).

We currently drop the product index j to simplify the notation of demand, and we denote n the annual number of orders, still unknown at this point. The transportation and working-inventory cost function depending on n and on the annual demand is:

$$w(D) + \beta t(D) = Fn + \theta h \frac{D}{2n} + \beta v(\frac{D}{n})n$$

To define the costs, we assume the shipment cost function is linear (H1) and includes a fixed cost, i.e. $v(x) = dx + g$ and we have an annual transportation cost equal to $t(D) = n \times v(z)$, where n is the annual number of deliveries and z the quantity shipped per delivery. Thus, we suppose the fixed cost of each single delivery is composed by the fixed order cost and the fixed transportation cost. By splitting them up, one can give a different weight to the transportation cost through the parameter β for instance.

F is the fixed cost of placing an order and one just has to multiply it by the number of orders to get the annual order cost. The second term is the holding cost. Indeed, $\frac{D}{2n}$ is the average quantity in stock since $\frac{D}{n}$ is the quantity delivered by order or the inventory level at the beginning of the cycle. The average quantity is obtained by dividing it by 2 and this is multiplied by the unit holding cost h to get the total cost. The last term represents the transportation cost. It is equal to the number of deliveries per year n multiplied by the shipment cost which depends on the shipped quantity. We keep for the moment a cost function $v(x)$ where x corresponds to the quantity per delivery. Again, we use θ and β to modify the relative importance we want to give to the inventory and transportation costs.

From this formulation, first order conditions give:

$$\frac{\partial(w(D)+\beta t(D))}{\partial n} = F - \theta h\frac{D}{2n^2} + \beta v(\frac{D}{n}) - \beta v'(\frac{D}{n})\frac{D}{n^2}n$$

$$= F - \theta h\frac{D}{2n^2} + \beta v(\frac{D}{n}) - \beta\frac{D}{n}v'(\frac{D}{n}) = 0$$

We use now the assumption regarding the transportation cost function. As we suppose $v(x)$ is linear ($v(x) = dx + g$), it means a part of the transportation cost is fixed (g) and another one is variable (d). Under this condition, we have $v'(x) = d$ and we can pursue the determination of n:

$$F - \theta h\frac{D}{2n^2} + \beta d\frac{D}{n} + \beta g - \beta d\frac{D}{n} = 0$$

$$F - \theta h\frac{D}{2n^2} + \beta g = 0 \leftrightarrow 2n^2 = \frac{\theta hD}{F+\beta g}$$

$$n = \sqrt{\frac{\theta hD}{2(F+\beta g)}}$$

Of course, the number of orders is non-negative and there is only one solution for this optimization problem. The next step consists in replacing n into the working-inventory cost function. We start with inserting the linear transportation cost function in the equation, and we get:

$$w(D) + \beta t(D) = Fn + \theta h\frac{D}{2n} + \beta\left(d\frac{D}{n} + g\right)n = Fn + \theta h\frac{D}{2n} + \beta dD + \beta gn$$

$$= F\sqrt{\frac{\theta hD}{2(F+\beta g)}} + \theta h\frac{D}{2\sqrt{\frac{\theta hD}{2(F+\beta g)}}} + \beta dD + \beta g\sqrt{\frac{\theta hD}{2(F+\beta g)}}$$

$$= \sqrt{\frac{\theta hD}{2(F+\beta g)}}(F+\beta g) + \frac{\theta hD}{\sqrt{2\frac{\theta hD}{(F+\beta g)}}} + \beta dD$$

$$= \sqrt{\frac{\theta hD(F+\beta g)}{2}} + \sqrt{\frac{\theta hD(F+\beta g)}{2}} + \beta dD$$

$$w(D) + \beta t(D) = \sqrt{2\theta hD(F+\beta g)} + \beta dD$$

Appendix B Construction of the substitution rate table

Fisher [11] suggests to consider a product as a collection of attributes, and a family of products as the set of SKUs presenting approximatively the same attribute levels. As a result, each product is represented by a point in a multidimensional space and the distance between 2 points is a measure of (dis)similarity. Following this view, the similarity of products is a good predictor of their substitutability.

Practically, we generate 5 virtual attributes per family with level between 0 and 1. Then, we randomly determine a level per SKU within a 30%-interval around the family level. Moreover, the price is considered as a quality indicator and for each SKU the mean of 5 attributes is set to be equal to the price attribute. Based on this, the normalized spatial distance measures the similarity of products and 1 minus this distance provides a reasonable estimation of the substitution. To introduce asymmetry related to quality and prices aspects, we multiply this substitution rate by a factor between 0.5 and 1 depending on the price difference.

That is, the more expensive the substitute compared to the discontinued product, the fewer demand will be transferred to it.

As a result, we obtain coherent substitution rates that satisfy the following properties:

- products within a family are more similar to each other than across families,
- prices are consistent with quality attributes (to avoid generating cheap products with high substitution rates to them),
- asymmetric substitution rates: with equal attributes, the substitution rate from an expensive product to a cheaper product will be higher than in the opposite direction.

Tables 9 and 10 give the substitution parameters and prices for the instance discussed in Section 4.1 (remember that in this example, all SKU demands are expressed in identical units).

Appendix C Parameters and results of Section 4.1

Service Level	α	99%	Fixed Cost of Ordering	F	29
Inventory Cost Weight	θ	1	Fixed Cost of Transportation	g	5
Transportation Cost Weight	β	1	Variable Cost of Transportation	d	0.032

Table 7: Scalar Parameters

Family k	Family Fixed Cost l_k
1	1600
2	3000
3	1500
4	2000

Table 8: Family Fixed Costs

SKU	Family	Price	Demand	Standard Deviation	Lead Time	Fixed Cost	Production Cost	Inventory Cost
j	k	p_j	μ_j	σ_j	LT_j	f_j	c_j	h_j
1	1	1.43	156480	94678	2.58	0.648	40	0.0207
2	1	1.57	56307	18314	2.58	0.713	40	0.0228
3	1	1.45	121296	87606	2.58	0.658	40	0.0211
4	2	0.59	12398	6467	2.58	0.269	40	0.0086
5	2	0.65	159439	42682	2.58	0.296	40	0.0095
6	2	1.22	21057	5677	2.58	0.553	40	0.0177
7	2	1.05	182607	78594	2.58	0.476	40	0.0152
8	2	1.19	55706	31883	2.58	0.539	40	0.0173
9	2	0.84	97792	60406	2.58	0.381	40	0.0122
10	2	0.61	154868	88104	2.58	0.278	40	0.0089
11	2	0.51	102306	56309	2.58	0.234	40	0.0075
12	2	5.08	64967	23404	2.58	2.307	40	0.0738
13	2	1.20	114543	50989	2.58	0.546	40	0.0175
14	2	0.87	79483	43441	2.58	0.396	40	0.0127
15	2	0.76	23215	16751	2.58	0.346	40	0.0111
16	2	0.82	11629	3145	2.58	0.374	40	0.0120
17	3	0.78	19620	7537	2.58	0.357	40	0.0114
18	3	0.84	18430	8891	2.58	0.382	40	0.0122
19	3	0.75	183411	49383	2.58	0.342	40	0.0110
20	3	0.78	194499	85628	2.58	0.356	40	0.0114
21	3	1.09	123991	64971	2.58	0.497	40	0.0159
22	3	0.76	80519	33210	2.58	0.345	40	0.0110
23	3	0.97	113361	74155	2.58	0.441	40	0.0141
24	3	0.63	167628	56482	2.58	0.286	40	0.0092
25	3	1.13	166163	110681	2.58	0.514	40	0.0164
26	3	1.96	87095	27936	2.58	0.891	40	0.0285
27	3	0.80	191792	135089	2.58	0.363	40	0.0116
28	3	0.79	54616	33310	2.58	0.360	40	0.0115
29	4	1.35	138599	74594	2.58	0.613	40	0.0196
30	4	1.20	126349	61437	2.58	0.546	40	0.0175
31	4	1.04	121645	73784	2.58	0.470	40	0.0150
32	4	1.49	176487	53281	2.58	0.679	40	0.0217

Table 9: SKU Parameters

i \ j	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
1	1.00	0.81	0.92	0.35	0.36	0.33	0.34	0.33	0.35	0.35	0.36	0.00	0.33	0.35	0.35	0.35	0.40	0.42	0.41	0.41	0.43	0.41	0.40	0.40	0.41	0.39	0.41	0.40	0.39	0.40	0.39	0.38
2	0.94	1.00	0.96	0.36	0.36	0.34	0.35	0.34	0.36	0.36	0.36	0.00	0.34	0.36	0.36	0.36	0.40	0.42	0.41	0.41	0.44	0.41	0.41	0.39	0.41	0.43	0.41	0.40	0.41	0.41	0.40	0.40
3	0.94	0.85	1.00	0.36	0.36	0.34	0.35	0.34	0.36	0.36	0.37	0.00	0.34	0.36	0.36	0.36	0.41	0.43	0.42	0.42	0.45	0.42	0.41	0.40	0.42	0.42	0.42	0.41	0.42	0.42	0.41	0.40
4	0.13	0.03	0.11	1.00	0.89	0.32	0.49	0.35	0.70	0.97	0.94	0.00	0.33	0.66	0.79	0.72	0.44	0.42	0.43	0.43	0.41	0.43	0.43	0.44	0.39	0.00	0.43	0.44	0.20	0.33	0.44	0.09
5	0.17	0.07	0.16	0.95	1.00	0.37	0.54	0.41	0.78	0.96	0.96	0.00	0.39	0.74	0.86	0.79	0.45	0.42	0.43	0.43	0.41	0.43	0.44	0.44	0.43	0.00	0.44	0.45	0.24	0.37	0.43	0.13
6	0.33	0.34	0.34	0.84	0.84	1.00	0.94	0.93	0.88	0.84	0.80	0.00	0.92	0.88	0.87	0.88	0.39	0.37	0.37	0.37	0.37	0.37	0.39	0.38	0.39	0.17	0.37	0.39	0.39	0.39	0.41	0.41
7	0.34	0.34	0.35	0.88	0.89	0.78	1.00	0.81	0.93	0.89	0.86	0.00	0.77	0.92	0.92	0.93	0.41	0.39	0.39	0.39	0.39	0.39	0.41	0.40	0.41	0.07	0.40	0.41	0.40	0.40	0.43	0.41
8	0.33	0.34	0.34	0.85	0.86	0.91	0.94	1.00	0.91	0.85	0.83	0.00	0.94	0.92	0.89	0.91	0.40	0.38	0.38	0.38	0.38	0.38	0.40	0.39	0.41	0.16	0.39	0.40	0.39	0.40	0.41	0.40
9	0.29	0.20	0.29	0.92	0.95	0.55	0.73	0.60	1.00	0.93	0.91	0.00	0.58	0.95	0.97	0.98	0.43	0.41	0.42	0.42	0.40	0.42	0.43	0.42	0.43	0.00	0.42	0.43	0.39	0.41	0.43	0.28
10	0.14	0.04	0.13	0.99	0.92	0.34	0.51	0.37	0.73	1.00	0.95	0.00	0.35	0.69	0.82	0.75	0.44	0.42	0.43	0.43	0.41	0.43	0.43	0.44	0.41	0.00	0.43	0.44	0.22	0.34	0.44	0.10
11	0.07	0.00	0.06	0.87	0.83	0.25	0.41	0.28	0.62	0.86	1.00	0.00	0.26	0.59	0.70	0.64	0.45	0.43	0.44	0.44	0.35	0.44	0.43	0.45	0.33	0.00	0.44	0.45	0.14	0.26	0.42	0.03
12	0.15	0.17	0.16	0.37	0.39	0.52	0.48	0.52	0.43	0.38	0.35	1.00	0.52	0.44	0.41	0.43	0.17	0.16	0.16	0.16	0.17	0.16	0.18	0.15	0.20	0.20	0.17	0.17	0.20	0.20	0.20	0.24
13	0.33	0.34	0.34	0.83	0.85	0.90	0.91	0.95	0.90	0.84	0.81	0.00	1.00	0.90	0.88	0.89	0.39	0.37	0.38	0.38	0.37	0.38	0.40	0.38	0.41	0.17	0.38	0.39	0.38	0.39	0.40	0.39
14	0.32	0.22	0.31	0.91	0.94	0.58	0.76	0.63	0.99	0.92	0.91	0.00	0.61	1.00	0.96	0.97	0.43	0.41	0.42	0.42	0.40	0.41	0.43	0.42	0.43	0.00	0.42	0.43	0.39	0.41	0.43	0.31
15	0.24	0.15	0.23	0.95	0.96	0.48	0.66	0.52	0.90	0.96	0.93	0.00	0.50	0.85	1.00	0.93	0.44	0.42	0.42	0.42	0.40	0.42	0.43	0.43	0.42	0.00	0.42	0.44	0.33	0.41	0.44	0.22
16	0.28	0.19	0.28	0.94	0.95	0.54	0.73	0.58	0.97	0.94	0.92	0.00	0.56	0.92	0.98	1.00	0.43	0.41	0.42	0.42	0.40	0.42	0.43	0.43	0.43	0.00	0.42	0.43	0.38	0.41	0.44	0.27
17	0.30	0.18	0.28	0.44	0.45	0.39	0.41	0.40	0.43	0.44	0.45	0.00	0.39	0.43	0.44	0.43	1.00	0.90	0.97	0.97	0.64	0.96	0.77	0.96	0.61	0.00	0.95	0.98	0.37	0.43	0.45	0.24
18	0.35	0.24	0.34	0.42	0.42	0.37	0.39	0.38	0.41	0.42	0.43	0.00	0.37	0.41	0.42	0.41	0.95	1.00	0.96	0.97	0.71	0.98	0.81	0.93	0.65	0.00	0.97	0.95	0.42	0.44	0.45	0.29
19	0.28	0.16	0.26	0.43	0.43	0.37	0.39	0.38	0.42	0.43	0.44	0.00	0.38	0.42	0.42	0.42	0.94	0.88	1.00	0.95	0.61	0.96	0.73	0.95	0.57	0.00	0.93	0.93	0.34	0.43	0.44	0.21
20	0.30	0.19	0.29	0.43	0.43	0.37	0.39	0.38	0.42	0.43	0.44	0.00	0.38	0.42	0.42	0.42	0.97	0.92	0.98	1.00	0.65	0.97	0.76	0.96	0.60	0.00	0.95	0.96	0.37	0.43	0.44	0.24
21	0.43	0.44	0.45	0.41	0.41	0.37	0.39	0.38	0.40	0.41	0.41	0.00	0.37	0.40	0.40	0.40	0.92	0.94	0.92	0.93	1.00	0.93	0.92	0.89	0.88	0.12	0.92	0.92	0.43	0.45	0.44	0.41
22	0.28	0.16	0.27	0.43	0.43	0.37	0.39	0.38	0.42	0.43	0.44	0.00	0.38	0.41	0.42	0.42	0.93	0.90	0.97	0.95	0.62	1.00	0.73	0.94	0.57	0.00	0.93	0.93	0.35	0.44	0.44	0.22
23	0.40	0.33	0.41	0.43	0.44	0.39	0.41	0.40	0.43	0.43	0.43	0.00	0.40	0.43	0.43	0.43	0.95	0.93	0.93	0.93	0.81	0.93	1.00	0.91	0.82	0.02	0.94	0.95	0.42	0.44	0.44	0.39
24	0.17	0.06	0.16	0.44	0.44	0.32	0.40	0.35	0.42	0.44	0.45	0.00	0.33	0.42	0.43	0.43	0.81	0.74	0.84	0.81	0.49	0.82	0.60	1.00	0.45	0.00	0.78	0.80	0.23	0.37	0.43	0.11
25	0.41	0.41	0.42	0.42	0.43	0.39	0.41	0.41	0.43	0.42	0.42	0.00	0.41	0.43	0.42	0.43	0.92	0.91	0.91	0.91	0.92	0.90	0.97	0.88	1.00	0.14	0.92	0.93	0.42	0.44	0.44	0.41
26	0.41	0.43	0.43	0.33	0.33	0.33	0.33	0.34	0.34	0.33	0.33	0.00	0.34	0.34	0.33	0.34	0.74	0.76	0.74	0.75	0.81	0.75	0.77	0.71	0.80	1.00	0.75	0.74	0.41	0.41	0.38	0.39
27	0.31	0.20	0.30	0.43	0.44	0.37	0.40	0.39	0.42	0.43	0.44	0.00	0.38	0.42	0.42	0.42	0.96	0.93	0.97	0.97	0.65	0.97	0.78	0.93	0.62	0.00	1.00	0.97	0.38	0.44	0.45	0.25
28	0.30	0.19	0.29	0.44	0.45	0.39	0.41	0.40	0.43	0.44	0.45	0.00	0.39	0.43	0.44	0.43	0.99	0.91	0.96	0.96	0.64	0.96	0.79	0.95	0.62	0.00	0.96	1.00	0.37	0.44	0.45	0.25
29	0.39	0.41	0.42	0.39	0.39	0.39	0.40	0.39	0.39	0.39	0.39	0.00	0.38	0.39	0.40	0.40	0.41	0.42	0.41	0.41	0.43	0.42	0.42	0.40	0.42	0.32	0.42	0.41	1.00	0.94	0.91	0.80
30	0.40	0.41	0.42	0.41	0.41	0.39	0.40	0.40	0.41	0.41	0.40	0.00	0.39	0.41	0.41	0.41	0.43	0.44	0.43	0.43	0.45	0.44	0.44	0.42	0.44	0.21	0.44	0.44	0.80	1.00	0.92	0.63
31	0.39	0.38	0.41	0.44	0.43	0.41	0.43	0.41	0.43	0.44	0.43	0.00	0.40	0.43	0.44	0.44	0.45	0.45	0.44	0.44	0.44	0.44	0.44	0.43	0.44	0.07	0.45	0.45	0.63	0.77	1.00	0.49
32	0.38	0.40	0.40	0.39	0.39	0.41	0.41	0.40	0.40	0.39	0.38	0.00	0.39	0.40	0.40	0.40	0.40	0.40	0.39	0.39	0.41	0.40	0.40	0.38	0.41	0.39	0.40	0.40	0.93	0.89	0.89	1.00

Table 10: Substitution factor δ_{ij}

(a)

X i \ j	1	2	3	6	7	12	13	14	15	18	20	21	25	26	27	29	30	31	32
1	1																		
2		1																	
3			1																
4									1										
5									1										
6				1															
7					1														
8							1												
9								1											
10									1										
11									1										
12						1													
13							1												
14								1											
15									1										
16								1											
17											1								
18										1									
19											1								
20											1								
21												1							
22															1				
23													1						
24											1								
25													1						
26														1					
27															1				
28															1				
29																1			
30																	1		
31																		1	
32																			1

(b)

X i \ j	1	2	3	6	7	8	9	12	13	14	18	20	21	23	25	26	27	29	30	31	32
1	1																				
2		1																			
3			1																		
4							1														
5							1														
6				1																	
7					1																
8						1															
9							1														
10							1														
11							1														
12								1													
13									1												
14										1											
15							1														
16							1														
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19												1									
20												1									
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23														1							
24												1									
25															1						
26																1					
27																	1				
28												1									
29																		1			
30																			1		
31																				1	
32																					1

Table 11: Substitution decisions for models with company control (a) and with preference ordering (b)

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