2013/63

■

Continuous knapsack sets with divisible capacities

Laurence A. Wolsey and Hande Yaman

CORE

DISCUSSION PAPER

Continuous knapsack sets
with divisible capacities

Laurence A. WOLSEY [1] and Hande YAMAN[2]

Abstract

We study two continuous knapsack sets $Y^{\geq}$ and $Y^{\leq}$ with n integer, one unbounded continuous and m bounded continuous variables in either $\geq$ or $\leq$ form. When the coefficients of the integer variables are integer and divisible, we show in both cases that the convex hull is the intersection of the bound constraints and 2m polyhedra arising from a continuous knapsack set with a single unbounded continuous variable. The latter polyhedra are in turn completely described by an exponential family of partition inequalities. A polynomial size extended formulation is known in the $\geq$ case. We provide an extended formulation for the $\leq$ case. It follows that, given a specific objective function, optimization over both $Y^{\geq}$ and $Y^{\leq}$ can be carried out by solving a polynomial size linear program. A consequence of these results is that the coefficients of the continuous variables all take the values 0 or 1 (after scaling) in any non-trivial facet-defining inequality.

Keywords: continuous knapsack set, splittable flow arec set, divisible capacities, partition inequalities, convex hull.

Mathematical Subject Classification: 90C11, 90C26

[1] Université catholique de Louvain, CORE and IRES, B-1348 Louvain-la-Neuve, Belgium.
E-mail: Laurence.wolsey@uclouvain.be
[2] Bilkent University, Department of Industrial Engineering, Bilkent 06800 Ankara, Turkey.

1   Introduction

Let m and n be positive integers, $M = \{1, \ldots, m\}$, $M_0 = M \cup \{0\}$ and $N = \{1, \ldots, n\}$. The parameters a_i for $i \in M$, $c_1 \leq \ldots \leq c_n$ and b are positive integers. The multi-item continuous $\geq$-knapsack set is

$$\mathcal{Y}_\geq = \{(y, x) \in \mathbb{Z}_+^n \times \mathbb{R}_+^{m+1} : \sum_{j \in N} c_j y_j + \sum_{i \in M_0} x_i \geq b, x_i \leq a_i, i \in M\},$$

the multi-item continuous $\leq$-knapsack set is

$$\mathcal{Y}_\leq = \{(y, x) \in \mathbb{Z}_+^n \times \mathbb{R}_+^{m+1} : \sum_{j \in N} c_j y_j \leq b + \sum_{i \in M_0} x_i, \ x_i \leq a_i, i \in M\}$$

and the unbounded single item continuous knapsack sets are

$$\mathcal{Q}_\geq = \{(y, x) \in \mathbb{Z}_+^n \times \mathbb{R}_+^1 : \sum_{j \in N} c_j y_j + x \geq b\} \text{ and } \mathcal{Q}_\leq = \{(y, x) \in \mathbb{Z}_+^n \times \mathbb{R}_+^1 : \sum_{j \in N} c_j y_j \leq b + x\}.$$

These sets arise as relaxations of many mixed-integer programming problems and consequently strong valid inequalities for these sets can be used in solving more complicated problems. Indeed, many strong inequalities used in the literature can be obtained using such knapsack relaxations. For books and inequalities on general knapsack sets, see among others [2, 6, 12, 14, 17].

Here we consider a case in which there is special structure, specifically the coefficients of the integer variables are divisible $1|c_1|\cdots|c_n$. Generalizing results of Pochet and Wolsey [16] for the $\geq$-knapsack set, we show that $\mathcal{Q}_\geq$ and $\mathcal{Q}_\leq$ can be described by two closely related families of "partition" inequalities. This in turn leads to complete polyhedral descriptions of $\mathcal{Y}_\geq$ and $\mathcal{Y}_\leq$. Specifically we show that

$$\text{conv}(\mathcal{Y}_\geq) = \cap_{S \subseteq M} \text{conv}(\mathcal{Q}_\geq^S) \cap \{(y, x) : x_i \leq a_i, i \in M\}$$

where

$$\mathcal{Q}_\geq^S = \{(y, x) \in \mathbb{Z}_+^n \times \mathbb{R}^{m+1} : x(S \cup \{0\}) + cy \geq b - a(M \setminus S), x(S \cup \{0\}) \geq 0\}$$

with a similar result for $\mathcal{Y}_\leq$ (where $v(A) = \sum_{a \in A} v_a$ for a vector v and a set A).

For $\mathcal{Y}^\geq$, this generalizes a result of Magnanti et al. [10] concerning the "single arc flow set" in which they show (modulo complementation of the continuous variables) that when $n = 1$, the convex hull of $\mathcal{Y}_\geq \cap \{(y, x) :$

$x_0 = 0\}$ is completely described by adding the "residual capacity" or mixed integer rounding (MIR) inequalities one for each of the relaxations

$$c_1 y_1 + x(S) \geq b - a(M \setminus S), x(S) \geq 0, y \in \mathbb{Z}_+^1,$$

where $S \subseteq M$. Atamtürk and Rajan [3] give a polynomial time separation algorithm for the residual capacity inequalities. Magnanti et al. [11] generalize the residual capacity inequalities for the two facility splittable flow arc set when $n = 2$, $c_1 = 1$ and state without proof that addition of the two MIR inequalities arising for each choice of $S \subseteq M$ suffices to give the convex hull.

Other work on divisible knapsack sets includes a convex hull description of the integer $\leq$-knapsack set $\{y \in \mathbb{Z}_+^n : \sum_{j \in N} c_j y_j \leq b\}$ consisting of n Chvatal-Gomory rounding inequalities by Marcotte [13] and a study of Pochet and Weismantel [15] of the case with bounded variables. Other (continuous) knapsack sets with special structure whose polyhedral structure has been studied include the set $\mathcal{Q}_\leq$ with $n = 2$ and c_1, c_2 arbitrary positive integers (Agra and Constantino [1] and Dash et al. [4]), as well as 0-1 knapsack sets with super-increasing coefficients (Laurent and Sassano [8]) and more recently a generalization with bounded integer variables (Gupta [5]).

The rest of the paper is organized as follows. In Section 2, we review some results on knapsack sets with divisible capacities. In Section 3 we study the convex hull of the multi-item continuous $\geq$-knapsack set and prove that the original constraints and the so-called "partition inequalities" are sufficient to describe the convex hull when the capacities are divisible. A result on the convex hull of the two-sided integer knapsack is an immediate corollary. In Section 4 we show that a new, but related, family of partition inequalities are valid for the continuous $\leq$-knapsack set with one unbounded continuous variable $\mathcal{Q}_\leq$. We also give a polynomial size extended formulation for $\mathcal{Q}_\leq$. In Section 5 we provide a convex hull description for the case of m bounded continuous variables and one unbounded continuous variable $\mathcal{Y}_\leq$. We conclude in Section 6.

2 The $\geq$-knapsack set and partition inequalities

Throughout the paper, we assume that the capacities are divisible. We use the notation $cy = \sum_{j \in N} c_j y_j$. Below we present results from Pochet and Wolsey [16] that will be used in Section 3.

Consider the integer $\geq$-knapsack set

$$\mathcal{C} = \{y \in \mathbb{Z}_+^n : cy \geq b\}$$

with $c_1 = 1$. Let $\gamma(b)$ be the index with $c_{\gamma(b)} \leq b < c_{\gamma(b)+1}$ if such an index exists and be n otherwise.

Let $\{i_1 = 1, \ldots, j_1\}, \ldots, \{i_p, \ldots, j_p = n\}$ be a partition of $\{1, \ldots, n\}$ such that $i_p \leq \gamma(b)$ and $i_t = j_{t-1} + 1$ for $t = 2, \ldots, p$. Compute

$$\beta_p = b, \ \kappa_t = \left\lceil \frac{\beta_t}{c_{i_t}} \right\rceil, \ \mu_t = (\kappa_t - 1)c_{i_t} \text{ and } \beta_{t-1} = \beta_t - \mu_t \text{ for } t = p, \ldots, 1.$$

The partition inequality is

$$\sum_{t=1}^{p} \prod_{l=1}^{t-1} \kappa_l \sum_{j=i_t}^{j_t} \min\left\{\frac{c_j}{c_{i_t}}, \kappa_t\right\} y_j \geq \prod_{t=1}^{p} \kappa_t. \tag{1}$$

Pochet and Wolsey [16] establish the following:

Theorem 2.1. *i) The partition inequality (1) is valid for the integer $\geq$-knapsack set $\mathcal{C}$.*

ii) If $c_{\gamma(b)}$ divides b, then the convex hull of $\mathcal{C}$ is $\{y \in \mathbb{R}_+^n : \sum_{j=1}^{\gamma(b)} c_j y_j + \sum_{j=\gamma(b)+1}^{n} b y_j \geq b\}$. Otherwise, the convex hull is described by the nonnegativity constraints and the partition inequalities (1).

iii) Let $g \in \mathbb{R}^n$ and $\{i_1, \ldots, j_1\}, \{i_2, \ldots, j_2\}, \ldots, \{i_p, \ldots, j_p\}$ be a partition of $\{1, \ldots, n\}$ such that $i_p \leq \gamma(b)$. If $g > 0$, $\frac{g_j}{c_j}$ is constant for $j = i_t, \ldots, j_t$ and $\frac{g_{i_t}}{c_{i_t}} > \frac{g_{i_{t+1}}}{c_{i_{t+1}}}$ for $t = 1, \ldots, p-1$, $\frac{g_{i_p}}{c_{i_p}} = \frac{g_j}{c_j}$ for $j = i_p, \ldots, \gamma(b)$ and $g_{\gamma(b)+1} = g_j$ for $j = \gamma(b) + 1, \ldots, n$, then

> *a. all optimal solutions of $\min\{\sum_{j=1}^{n} g_j y_j : cy \geq b, y \in \mathbb{Z}_+^n\}$ satisfy (1) at equality,*

> *b. all optimal solutions of $\min\{\sum_{j=i_2}^{n} g_j y_j : \sum_{j=i_2}^{n} c_j y_j \geq \left\lceil \frac{b}{c_{i_2}} \right\rceil c_{i_2}, y \in \mathbb{Z}_+^n\}$ satisfy*

$$\sum_{t=2}^{p} \prod_{l=2}^{t-1} \kappa_l \sum_{j=i_t}^{j_t} \min\left\{\frac{c_j}{c_{i_t}}, \kappa_t\right\} y_j \geq \prod_{t=2}^{p} \kappa_t$$

> *at equality ([16], Theorem 8),*

c. all optimal solutions of $\min\{\sum_{j=i_2}^n g_j y_j : \sum_{j=i_2}^n c_j y_j = \left\lfloor \frac{b}{c_{i_2}} \right\rfloor c_{i_2}, y \in \mathbb{Z}_+^n\}$ *satisfy*

$$\sum_{t=2}^p \prod_{l=2}^{t-1} \kappa_l \sum_{j=i_t}^{j_t} \min\left\{ \frac{c_j}{c_{i_t}}, \kappa_t \right\} y_j \geq \prod_{t=2}^p \kappa_t - 1$$

at equality ([16], proof of Theorem 16).

3 The multi-item continuous $\geq$-knapsack set

In this section, we study the convex hull of the multi-item continuous $\geq$-knapsack set $\mathcal{Y}_\geq$ when the capacities are divisible.

Our goal now is to show that

$$\operatorname{conv}(\mathcal{Y}_\geq) = \cap_{S \subseteq M} \operatorname{conv}(\mathcal{Q}_\geq^S) \cap \{(y, x) : x_i \leq a_i, i \in M\}$$

where

$$\mathcal{Q}_\geq^S = \{(y, x) \in \mathbb{Z}_+^n \times \mathbb{R}^{m+1} : x(S_0) + cy \geq b - a(M \setminus S), x(S_0) \geq 0\},$$

where $S_0 = S \cup \{0\}$.

We first use the results of Section 2 to obtain valid inequalities for set $\mathcal{Q}_\geq$. Then we prove that these valid inequalities and the original constraints are sufficient to describe the convex hull of the continuous $\geq$-knapsack set with divisible capacities.

Given $S \subseteq M$, consider the relaxation

$$\{(y_0, y, x) \in \mathbb{R}_+^1 \times \mathbb{Z}_+^n \times \mathbb{R}^{m+1} : y_0 + cy \geq b - a(M \setminus S), y_0 = x(S_0)\}.$$

As the data is integral, y_0 takes an integer value in every extreme point of the convex hull of the above set. Setting y_0 integer, we obtain the divisible capacity knapsack cover set

$$\{(y_0, y, x) \in \mathbb{Z}_+^{n+1} \times \mathbb{R}^{m+1} : \sum_{j \in N_0} c_j y_j \geq B(S), y_0 = x(S_0)\}$$

where $N_0 = N \cup \{0\}$, $c_0 = 1$ and $B(S) = b - a(M \setminus S)$.

Proposition 3.1. *Let* $\{i_1 = 0, \ldots, j_1\}, \ldots, \{i_p, \ldots, j_p = n\}$ *be a partition of* $\{0, 1, \ldots, n\}$ *such that* $i_p \leq \gamma(B(S))$ *and* $i_t = j_{t-1} + 1$ *for* $t = 2, \ldots, p$. *Let*

6

$\beta_p = B(S)$, $\kappa_t = \left\lceil \frac{\beta_t}{c_{i_t}} \right\rceil$, $\mu_t = (\kappa_t - 1)c_{i_t}$ and $\beta_{t-1} = \beta_t - \mu_t$ for $t = p, \ldots, 1$. *Then the partition inequality*

$$x(S_0) + \sum_{j=1}^{j_1} \min\{c_j, \kappa_1\} y_j + \sum_{t=2}^{p} \prod_{l=1}^{t-1} \kappa_l \sum_{j=i_t}^{j_t} \min\left\{ \frac{c_j}{c_{i_t}}, \kappa_t \right\} y_j \geq \prod_{t=1}^{p} \kappa_t \qquad (2)$$

is valid for $\mathcal{Y}_\geq$.

Note that in the extreme points of conv($\mathcal{Y}_\geq$), x_0 takes integer values. We give the convex hull proof for $\mathcal{Y}'_\geq = \mathcal{Y}_\geq \cap \{(y, x) : x_0 = 0\}$ since x_0 can be considered an integer variable with coefficient 1.

Theorem 3.2. conv($\mathcal{Y}'_\geq$) *is described by the initial constraints and the partition inequalities* (2).

Proof. We use the technique of Lovász [9]. Suppose that we minimize $\sum_{i \in M} h_i x_i + \sum_{j \in N} g_j y_j$ over $\mathcal{Y}'_\geq$. We need $g \geq 0$ for the problem to be bounded. Suppose that $g \geq 0$ and let $\Omega(h, g)$ be the set of optimal solutions. If $h_i < 0$ for some $i \in M$, then $\Omega(h, g) \subseteq \{(y, x) : x_i = a_i\}$.
If $g_j = 0$ and $g_{j'} > 0$ for some pair $j, j' \in N$, $\Omega(h, g) \subseteq \{(y, x) : y_{j'} = 0\}$.
Thus we are left with $h \geq 0$ and $g > 0$. We will investigate this in two cases:

Case 1 $h \neq 0$: Let $S = \{i \in M : h_i > 0\} \neq \emptyset$. If $B(S) \leq 0$, then $\Omega(h, g) \subseteq \{(y, x) : x_i = 0\}$ for all $i \in S$.

Now suppose that $B(S) > 0$. If $\frac{c_{j_2}}{c_{j_1}} g_{j_1} < g_{j_2}$ for $j_1 < j_2 \leq \gamma(B(S))$, then $y_{j_2} = 0$ for all $(y, x) \in \Omega(h, g)$. If $g_{j_1} < g_{j_2}$ for j_1 and j_2 in $\{\gamma(B(S)) + 1, \ldots, n\}$, then $y_{j_2} = 0$ for all $(y, x) \in \Omega(h, g)$.

Now we have $S \neq \emptyset$, $h_i > 0$ for all $i \in S$, $B(S) > 0$, $\frac{g_1}{c_1} \geq \ldots \geq \frac{g_{\gamma(B(S))}}{c_{\gamma(B(S))}} > 0$ and $g_{\gamma(B(S))+1} = \ldots = g_n > 0$. One possibility is that $\Omega(h, g) \subseteq \{(y, x) : x(M) + cy = b\}$.

The last case to be considered is that in which there exists an optimal solution (y, x) with $x(M) + cy > b$. Let q be the smallest index such that there exists an optimal solution (x^*, y^*) at which the knapsack cover constraint is not tight and $y_q^* > 0$. Then we know the following:

- $x_i = 0$ for all $i \in S$ in any optimal solution (y, x) at which the knapsack cover constraint is not tight.

- $\frac{g_{q-1}}{c_{q-1}} > \frac{g_q}{c_q}$.

  For $j \in N$, define e_j to be the j-th unit vector of size n. If $\frac{g_{q-1}}{c_{q-1}} = \frac{g_q}{c_q}$, then $(x^*, y^* - e_q + \frac{c_q}{c_{q-1}} e_{q-1})$ is also optimal, contradicting the definition of q.

- c_q does not divide $B(S)$.

  Suppose on the contrary that c_q divides $B(S)$. We have $\sum_{j=q}^{n} c_j y_j^* > B(S)$. As $\sum_{j=q}^{n} c_j y_j^*$ is a multiple of c_q, it follows that $\sum_{j=q}^{n} c_j y_j^* \geq B(S) + c_q$. But now $(x^*, y^* - e_q)$ is feasible and cheaper since $g_q > 0$, contradicting the optimality of (x^*, y^*).

- $cy \geq \left\lfloor \frac{B(S)}{c_q} \right\rfloor c_q$ in any optimal solution.

  Define $\phi(\sigma) = \min\{\sum_{i \in S} h_i x_i : x(S) \geq B(S) - \sigma, 0 \leq x_i \leq a_i \ i \in S\}$. Optimality of (x^*, y^*) implies $g_q \leq \phi(c(y^* - e_q)) - \phi(cy^*)$ and the fact that the knapsack cover constraint is not tight implies that $cy^* \geq \left\lfloor \frac{B(S)}{c_q} \right\rfloor c_q + c_q$. Suppose that (x', y') is a feasible solution with $cy' < \left\lfloor \frac{B(S)}{c_q} \right\rfloor c_q$. Now ϕ is a piecewise linear convex function with $\phi(\sigma) > 0$ for $\sigma < B(s)$ and $\phi(\sigma) = 0$ for $\sigma \geq B(S)$. It is strictly decreasing on the interval $[0, B(S)]$. Therefore, as $cy' < c(y^* - e_q) < B(S)$ and $cy^* > B(S)$, one has $\phi(cy') - \phi(c(y' + e_q)) > \phi(c(y^* - e_q)) - \phi(cy^*)$. It follows that $\phi(cy') - \phi(c(y' + e_q)) > g_q$ and thus $g(y' + e_q) + \phi(c(y' + e_q)) < gy' + \phi(cy')$. So increasing y_q' by 1 and picking the best x improves the objective function value. Hence (x', y') cannot be optimal.

Now let $y_0 = x(S)$ and q be as defined above. Let $\{i_1, \ldots, j_1\}, \{i_2, \ldots, j_2\}, \ldots, \{i_p, \ldots, j_p\}$ be a partition of $\{0, \ldots, n\}$ such that $i_1 = 0$, $j_1 = q - 1$, $i_p \leq \gamma(B(S))$, $\frac{g_j}{c_j}$ is constant for $j = i_t, \ldots, j_t$ and $\frac{g_{i_t}}{c_{i_t}} > \frac{g_{i_{t+1}}}{c_{i_{t+1}}}$ for $t = 2, \ldots, p - 1$, $\frac{g_{i_p}}{c_{i_p}} = \frac{g_j}{c_j}$ for $j = i_p, \ldots, \gamma(B(S))$ and $g_{\gamma(B(S))+1} = g_j$ for $j = \gamma(B(S))+1, \ldots, n$. We claim that all optimal solutions satisfy the corresponding partition inequality (2) at equality.

Take an arbitrary point $(y, x) \in \Omega(h, g)$. If the knapsack cover constraint is not tight at (y, x), then $x_i = 0$ for all $i \in S$, $y_j = 0$ for $j = 1, \ldots, q - 1$ and the restriction of y to entries $q, \ldots, n$ is optimal for the problem of minimizing $\sum_{j=q}^{n} g_j y_j$ subject to $\sum_{j=q}^{n} c_j y_j \geq \left\lceil \frac{B(S)}{c_q} \right\rceil c_q$ and $y \in \mathbb{Z}_+^n$. Then $\sum_{t=2}^{p} \prod_{l=2}^{t-1} \kappa_l \sum_{j=i_t}^{j_t} \min\left\{\frac{c_j}{c_{i_t}}, \kappa_t\right\} y_j = \prod_{t=2}^{p} \kappa_t$ using (iii b) of Theorem 2.1. Hence (y, x) satisfies (2) at equality.

Now suppose that the knapsack cover constraint is tight at (y, x). There are two subcases.

a) $x(M \setminus S) < a(M \setminus S)$.

   Then $x_i = 0$ for all $i \in S$ and $cy > B(S)$. Now the set of optimal points have the same y values as in the above case, completed by $x_i = 0$ for $i \in S$ and x_i for $i \in M \setminus S$ satisfying $0 \leq x_i \leq a_i$, $x(M \setminus S) = b - cy$.

8

b) $x_i = a_i$ for $i \in M \setminus S$.

Since $B(S)$ is not a multiple of c_q, we have $\sum_{j=q}^{n} c_j y_j \leq \left\lfloor \frac{B(S)}{c_q} \right\rfloor c_q$.
Then the fact that $\frac{g_{q-1}}{c_{q-1}} > \frac{g_q}{c_q}$ together with $\sum_{j=1}^{n} c_j y_j \geq \left\lfloor \frac{B(S)}{c_q} \right\rfloor c_q$
implies that $\sum_{j=q}^{n} c_j y_j = \left\lfloor \frac{B(S)}{c_q} \right\rfloor c_q$, and $y_0 + \sum_{j=1}^{q-1} \min\{c_j, \kappa_1\} y_j = B(S) - \left\lfloor \frac{B(S)}{c_q} \right\rfloor c_q = \kappa_1$. Using (iii c) of Theorem 2.1, any optimal solution to the problem of minimizing $\sum_{j=q}^{n} g_j y_j$ subject to $\sum_{j=q}^{n} c_j y_j = \left\lfloor \frac{B(S)}{c_q} \right\rfloor c_q$ and $y \in \mathbb{Z}_+^n$ satisfies $\sum_{t=2}^{p} \prod_{l=2}^{t-1} \kappa_l \sum_{j=i_t}^{j_t} \min\left\{ \frac{c_j}{c_{i_t}}, \kappa_t \right\} y_j = \prod_{t=2}^{p} \kappa_t - 1$. Now

$$\sum_{t=1}^{p} \Pi_{l=1}^{t-1} \kappa_l \sum_{j=i_t}^{j_t} \min\left\{ \frac{c_j}{c_{i_t}}, \kappa_t \right\} y_j$$

$$= x(S) + \sum_{j=1}^{q-1} \min\{c_j, \kappa_1\} y_j + \sum_{t=2}^{p} \prod_{l=1}^{t-1} \kappa_l \sum_{j=i_t}^{j_t} \min\left\{ \frac{c_j}{c_{i_t}}, \kappa_t \right\} y_j$$

$$= x(S) + \sum_{j=1}^{q-1} \min\{c_j, \kappa_1\} y_j + \kappa_1 \left(\sum_{t=2}^{p} \prod_{l=2}^{t-1} \kappa_l \sum_{j=i_t}^{j_t} \min\left\{ \frac{c_j}{c_{i_t}}, \kappa_t \right\} y_j \right)$$

$$= \kappa_1 + \kappa_1 \left(\prod_{t=2}^{p} \kappa_t - 1 \right) = \prod_{t=1}^{p} \kappa_t.$$

Thus all optimal solutions satisfy this partition inequality (2) at equality.

Case 2 $S = \emptyset$: The argument is the same setting $y_0 = 0$. $\square$

As $\mathrm{conv}(\mathcal{Q}_{\geq}^S)$ is described by the trivial inequalities and the partition inequalities, we get $\mathrm{conv}(\mathcal{Y}_{\geq}) = \cap_{S \subseteq M} \mathrm{conv}(\mathcal{Q}_{\geq}^S) \cap \{(y, x) : x_i \leq a_i, i \in M\}$.

As a corollary, one obtains a simple result concerning the intersection of two parallel divisible knapsack sets.

Theorem 3.3. $\mathrm{conv}(\{y \in \mathbb{Z}_+^n : b - a \leq cy \leq b\}) = \mathrm{conv}(\{y \in \mathbb{Z}_+^n : b - a \leq cy\}) \cap \mathrm{conv}(\{y \in \mathbb{Z}_+^n : cy \leq b\})$.

Proof. Consider the case of Theorem 3.2 when $m = 1$, i.e., when there is a single continuous variable. Then we have

$$\text{conv}(\mathcal{Y}'_{\geq}) = \text{conv}(\{(y,x) \in \mathbb{Z}^n_+ \times \mathbb{R} : cy \geq b-a\}) \cap \text{conv}(\{(y,x) \in \mathbb{Z}^n_+ \times \mathbb{R}_+ : x+cy \geq b\})$$

$$\cap \{(y,x) \in \mathbb{R}^{n+1} : x \leq a\}.$$

The intersection of the set on the right hand side with the set $\{(y,x) \in \mathbb{R}^{n+1} : x = b-cy\}$ is equal to $\text{conv}(\{(y,x) \in \mathbb{Z}^n_+ \times \mathbb{R} : cy \geq b-a\}) \cap \{(y,x) \in \mathbb{R}^{n+1} : x = cy-b\} \cap \text{conv}(\{(y,x) \in \mathbb{Z}^n_+ \times \mathbb{R} : cy \leq b\})$. If we project this onto the y space, we obtain $\text{conv}(y \in \mathbb{Z}^n_+ : cy \geq b-a) \cap \text{conv}(y \in \mathbb{Z}^n_+ : cy \leq b)$.

If we intersect the set on the left hand side with $\{(y,x) \in \mathbb{R}^{n+1} : x = b-cy\}$ and project onto the y space, we get $\text{conv}(\{y \in \mathbb{Z}^n_+ : b-a \leq cy \leq b\})$. The claim follows. $\square$

4   The continuous $\leq$-knapsack set

Now we study the convex hull of the continuous $\leq$-knapsack set, namely the set

$$\mathcal{Q}_{\leq} = \{(y,x) \in \mathbb{Z}^n_+ \times \mathbb{R}^1_+ : \sum_{j=1}^n c_j y_j \leq b + x\},$$

where again the data are integer and $c_1 | \cdots | c_n$. Initially we suppose that c_1 does not divide b. Below we will define a new family of "$\leq$-partition" inequalities.

Given a partition $\{i_1, \ldots, j_1\}, \cdots, \{i_p, \ldots, j_p\}$ of $\{1, \ldots, n\}$ into intervals, let $\beta_p = b, \kappa_t = \left\lceil \frac{\beta_t}{c_{i_t}} \right\rceil, \mu_t = (\kappa_t - 1)c_{i_t}, \beta_{t-1} = \beta_t - \mu_t$ and $s_t = c_{i_t} - \beta_{t-1}$ for $t = p, \ldots, 1$.

We first consider the partition into singletons $(\{1\}, \cdots, \{n\})$.

Observation 4.1. *The following $n + 1$ points*

$$z^k = (y^k, x^k) = (0, 0, \ldots, 0, \kappa_k, \kappa_{k+1} - 1 \ldots, \kappa_n - 1, s_k) \quad k = 1, \ldots, n$$

$$z^0 = (\kappa_1 - 1, \ldots, \kappa_k - 1, \kappa_{k+1} - 1 \ldots, \kappa_n - 1, 0)$$

are in $\mathcal{Q}_{\leq}$ and are linearly independent.

Observation 4.2. *The $(\{1\}, \cdots, \{n\})$ $\leq$-partition inequality*

$$\sum_{j=1}^n \pi_j y_j \leq \pi_0 + x \tag{3}$$

passes through these $n + 1$ points, where $\pi_1 = s_1$, $\pi_j = \kappa_{j-1}\pi_{j-1} + (s_j - s_{j-1})$ for $j = 2, \ldots, n$ and $\pi_0 = \kappa_n \pi_n - s_n$.

Proof. Let e'_j be the jth unit vector of size $n+1$. $z^1 - z^0 = e'_1 + s_1 e'_{n+1}$ and thus $\pi_1 = s_1$. For $j = 2, \ldots, n$, $z^j - z^{j-1} = e'_j - \kappa_{j-1} e'_{j-1} + (s_j - s_{j-1}) e'_{n+1}$ and thus $\pi_j = \kappa_{j-1} \pi_{j-1} + (s_j - s_{j-1})$. Finally $\pi y^n = \pi_0 + s_n$ implies $\pi_0 = \kappa_n \pi_n - s_n$. $\square$

Observation 4.3. *Let $\sum_{j=1}^n \alpha_j y_j \geq \alpha_0$ denote the $(\{1\}, \{2\}, \cdots, \{n\})$ partition inequality (1) for the $\geq$-knapsack set $\mathcal{C}$. Then the $\leq$-partition inequality (3) for $\mathcal{Q}_\leq$ can also be viewed as a lifting of this inequality, and can be written in the form*

$$\sum_{j=1}^n (c_j - (c_1 - s_1)\alpha_j) y_j \leq (b - (c_1 - s_1)\alpha_0) + x.$$

Proof. The points y^k are precisely the roots of the facet $\alpha y \geq \alpha_0$ of $\mathcal{C}$, and $cy^k = b + s_k$. So the points z^k satisfy (3) at equality. In addition $y^0 = y^1 - e_1$, so $cy^0 = cy^1 - c_1 = b + s_1 - c_1$ and $\alpha y^0 = \alpha y^1 - \alpha_1 = \alpha_0 - 1$. So z^0 also lies on the inequality and the inequalities must be identical. $\square$

4.1 Validity of partition inequalities

We need a refinement of the notation in this subsection. We denote the continuous $\leq$-knapsack set with n integer variables by $\mathcal{Q}_\leq^n$ and the set with the first $n-1$ integer variables by $\mathcal{Q}_\leq^{n-1}$. Similarly π_0^n is the right hand side of the partition inequality for $\mathcal{Q}_\leq^n$ and π_0^{n-1} for $\mathcal{Q}_\leq^{n-1}$. In particular we will use validity of $\sum_{j=1}^{n-1} \pi_j y_j \leq \pi_0^{n-1} + x$ for $\mathcal{Q}_\leq^{n-1}$ to show the validity of $\sum_{j=1}^n \pi_j y_j \leq \pi_0^n + x$ for $\mathcal{Q}_\leq^n$.

We first establish some basic properties. We still assume that c_1 does not divide b. First note that

$$\beta_j = b - \left\lfloor \frac{b}{c_{j+1}} \right\rfloor c_{j+1} \text{ for } j = 1, \ldots, n-1$$

and

$$s_j - s_{j-1} = \left\lfloor \frac{s_j}{c_{j-1}} \right\rfloor c_{j-1} \text{ for } j = 2, \ldots, n.$$

Lemma 4.1. *i) $\pi_j \leq c_j$ for $j = 1, \ldots, n$,*

ii) $\frac{\pi_j}{c_j} \geq \frac{\pi_{j-1}}{c_{j-1}}$ for $j = 2, \ldots, n$,

iii) $\pi_0^n = \pi_0^{n-1} + (\kappa_n - 1)\left(\pi_n - \pi_{n-1}\frac{c_n}{c_{n-1}}\right)$,

iv) $\pi_0^n = b - (c_n - \pi_n)\kappa_n$.

Proof. We use induction to prove part (i). For $j = 1$, we have $\pi_1 = s_1 \leq c_1$. Suppose that $\pi_{j-1} \leq c_{j-1}$. Then $\pi_j = \pi_{j-1}\kappa_{j-1} + s_j - s_{j-1} \leq c_{j-1}\kappa_{j-1} + s_j - s_{j-1}$. The right hand side is equal to $c_{j-1}\left\lceil \dfrac{b - \left\lfloor \frac{b}{c_j}\right\rfloor c_j}{c_{j-1}}\right\rceil + c_j -$

$c_{j-1} + \left\lfloor \dfrac{b}{c_j}\right\rfloor c_j - \left\lfloor \dfrac{b}{c_{j-1}}\right\rfloor c_{j-1} = c_j$. Hence $\pi_j \leq c_j$.

To prove (ii),

$$\pi_j = \kappa_{j-1}\pi_{j-1} + s_j - s_{j-1} = \left\lceil \frac{c_j - s_j}{c_{j-1}}\right\rceil \pi_{j-1} + s_j - s_{j-1} = \frac{c_j}{c_{j-1}}\pi_{j-1} - \left\lfloor \frac{s_j}{c_{j-1}}\right\rfloor \pi_{j-1} + s_j - s_{j-1}$$

$$\geq \frac{c_j}{c_{j-1}}\pi_{j-1} - \left\lfloor \frac{s_j}{c_{j-1}}\right\rfloor c_{j-1} + s_j - s_{j-1} = \frac{c_j}{c_{j-1}}\pi_{j-1},$$

where the inequality is obtained using (i) and the last equality is obtained using $s_j - s_{j-1} = \left\lfloor \frac{s_j}{c_{j-1}}\right\rfloor c_{j-1}$.

Next we prove part (iii). First, $\pi_0^n = \pi_n\kappa_n - s_n$ and

$$\pi_0^{n-1} = \pi_{n-1}\left\lceil \frac{b}{c_{n-1}}\right\rceil - s_{n-1} = \pi_{n-1}\left(\kappa_{n-1} + (\kappa_n - 1)\frac{c_n}{c_{n-1}}\right) - s_{n-1}.$$

Now

$$\pi_0^n - \pi_0^{n-1} = (\kappa_n - 1)(\pi_n - \frac{c_n}{c_{n-1}}) + \pi_n - \pi_{n-1}\kappa_{n-1} - s_n + s_{n-1} = (\kappa_n - 1)(\pi_n - \frac{c_n}{c_{n-1}}),$$

using the definition of π_n.

Finally, since $\pi_0^n = \kappa_n\pi_n - s_n$ and $s_n = c_n - \beta_{n-1} = \kappa_n c_n - b$, we have $\pi_0^n = b - (c_n - \pi_n)\kappa_n$, which proves part (iv). $\square$

Lemma 4.2. *If* $\sum_{j=1}^{n-1}\pi_j y_j \leq \pi_0^{n-1} + x$ *is valid for* $\mathcal{Q}_{\leq}^{n-1}$, *then*

$$\sum_{j=1}^{n-1}\pi_j y_j + \pi_{n-1}\frac{c_n}{c_{n-1}}y_n \leq \pi_0^{n-1} + x$$

is valid for $\mathcal{Q}_{\leq}^n$.

Proof. If $(y_1, \ldots, y_n, x) \in \mathcal{Q}_{\leq}^n$, then $\sum_{j=1}^n c_j y_j = \sum_{j=1}^{n-2} c_j y_j + c_{n-1}(y_{n-1} + \frac{c_n}{c_{n-1}}y_n) \leq b + x$. Hence $(y_1, \ldots, y_{n-2}, y_{n-1} + \frac{c_n}{c_{n-1}}y_n, x) \in \mathcal{Q}_{\leq}^{n-1}$ and $\sum_{j=1}^{n-2}\pi_j y_j + \pi_{n-1}(y_{n-1} + \frac{c_n}{c_{n-1}}y_n) \leq \pi_0^{n-1} + x$ as claimed. $\square$

12

Theorem 4.3. *The* $(\{1\}, \cdots, \{n\})$ *partition inequality is valid for* $\mathcal{Q}^n_{\leq}$.

Proof. The proof is by induction.

For $n = 1$, the MIR inequality is:

$$c_1(1 - f)y_1 \leq \left\lfloor \frac{b}{c_1} \right\rfloor c_1(1 - f) + x,$$

where $(1 - f) = \left\lceil \frac{b}{c_1} \right\rceil - \frac{b}{c_1} = \frac{s_1}{c_1}$. This is precisely the $(\{1\})$ partition inequality $s_1 y_1 \leq (\kappa_1 s_1 - s_1) + x$.

Now suppose that $\sum_{j=1}^{n-1} \pi_j y_j \leq \pi_0^{n-1} + x$ is valid for $\mathcal{Q}^{n-1}_{\leq}$. We consider two cases:

Case 1. $y_n \geq \kappa_n$.

Suppose that $(y, x) \in \mathcal{Q}^n_{\leq}$, so that $\sum_{j=1}^{n} c_j y_j \leq b + x$, or rewriting $\sum_{j=1}^{n-1} c_j y_j + c_n(y_n - \kappa_n) + c_n \kappa_n \leq b + x$. As $c_j \geq \pi_j$ by Lemma 4.1(i), $y_j \geq 0$ for $j = 1, \ldots, n-1$ and $y_n - \kappa_n \geq 0$, we have $\sum_{j=1}^{n-1} \pi_j y_j + \pi_n(y_n - \kappa_n) + c_n \kappa_n \leq b + x$, or equivalently $\sum_{j=1}^{n-1} \pi_j y_j + \pi_n y_n \leq b - (c_n - \pi_n)\kappa_n + x$. Using Lemma 4.1(iv), we obtain $\sum_{j=1}^{n} \pi_j y_j \leq \pi_0^n + x$.

Case 2. $y_n \leq \kappa_n - 1$.

From Lemma 4.2, (y, x) satisfies $\sum_{j=1}^{n-1} \pi_j y_j + \pi_{n-1} \frac{c_n}{c_{n-1}} y_n \leq \pi_0^{n-1} + x$. Adding $\pi_n - \pi_{n-1} \frac{c_n}{c_{n-1}} \geq 0$ (from Lemma 4.1(ii)) times $y_n \leq \kappa_n - 1$ gives $\sum_{j=1}^{n} \pi_j y_j \leq \pi_0^{n-1} + (\kappa_n - 1)(\pi_n - \pi_{n-1} \frac{c_n}{c_{n-1}}) + x = \pi_0^n + x$ where the last equality is obtained using Lemma 4.1(iii).

Therefore by a disjunctive argument, the inequality is valid for $\mathcal{Q}^n_{\leq}$. $\square$

Example 4.1. Consider the continuous $\leq$-knapsack set defined by the constraints

$$5y_1 + 10y_2 + 30y_3 \leq 72 + x, \ y \in \mathbb{Z}^3_+, \ x \in \mathbb{R}^1_+.$$

The coefficients of the $(\{1\}\{2\}\{3\})$ partition inequality are given by

t	β	κ	μ	s
3	72	3	60	18
2	12	2	10	8
1	2	1	0	3

Then $\pi_1 = 3$, $\pi_2 = 1 \times 3 + (8 - 3) = 8$, $\pi_3 = 8 \times 2 + (18 - 8) = 26$ and $\pi_0 = 26 \times 3 - 18 = 60$ giving the inequality

$$3y_1 + 8y_2 + 26y_3 \leq 60 + x.$$

Note that the $(\{1\}\{2\}\{3\})$ $\geq$-partition inequality for

$$5y_1 + 10y_2 + 30y_3 \geq 72, \ y \in \mathbb{Z}^3_+$$

13

is the inequality
$$y_1 + y_2 + 2y_3 \geq 6.$$
Now $5y_1 + 10y_2 + 30y_3 \leq 72 + x$ plus $c_1 - s_1 = 2$ times the latter inequality again gives $3y_1 + 8y_2 + 26y_3 \leq 60 + x$.

Now we describe the inequality associated with an arbitrary partition and we drop the assumption that c_1 does not divide b. Thus we suppose that $c_{r-1}|b$, but c_r does not divide b.

For the partition $\{i_1, \ldots, j_1\}, \ldots, \{i_p, \ldots, j_p\}$ of $\{r, \ldots, n\}$, we construct the partition inequality
$$\sum_{t=1}^{p} \pi_{i_t} y_{i_t} \leq \pi_0 + x$$
for the set
$$\sum_{t=1}^{p} c_{i_t} y_{i_t} \leq b + x, \ y \in \mathbb{Z}_+^p, \ x \in \mathbb{R}_+^1.$$

Proposition 4.4. *The* $\{i_1, \ldots, j_1\}, \ldots, \{i_p, \ldots, j_p\}$ *partition inequality*
$$\sum_{t=1}^{p} \pi_{i_t} \sum_{j=i_t}^{j_t} \frac{c_j}{c_{i_t}} y_j \leq \pi_0 + x. \tag{4}$$
is valid for $\mathcal{Q}_{\leq}$.

The set of points of $\mathcal{Q}_{\leq}$ *that are tight for such inequalities is the union of the sets*

$$Z^k = \{(y, x) \in \mathbb{Z}_+^n \times \mathbb{R}^1 : y_j = 0 \ j < i_k, \ \sum_{j=i_k}^{j_k} \frac{c_j}{c_{i_k}} y_j = \kappa_k, \ \sum_{j=i_t}^{j_t} \frac{c_j}{c_{i_t}} y_j = \kappa_t - 1 \ t = k+1, \ldots, p, x = s_k\}$$

$$Z^0 = \{(y, x) \in \mathbb{Z}_+^n \times \mathbb{R}^1 : y_j = 0 \ j < r-1, \sum_{j=i_t}^{j_t} \frac{c_j}{c_{i_t}} y_j = \kappa_t - 1 \ t = 1, \ldots, p, x = 0\}$$

$$W^j = Z^0 + \{(y, x) \in \mathbb{Z}_+^n \times \mathbb{R}^1 : \sum_{j=1}^{r-1} c_j y_j = \delta, y_j = 0 \ j \geq r, \ x = 0\}$$

where $\delta = b - \sum_{t=1}^{p} c_{i_t} (\kappa_t - 1)$.

The proof that the inequality is valid for $\sum_{j=r}^{n} c_j y_j \leq b + x, y \in \mathbb{Z}_+^{n-r+1}, x \in \mathbb{R}_+^1$ is as in Lemma 4.2. The structure of the tight points follows from that of the tight points $\{z^k\}_{k=0}^{p}$ in Observation 4.1.□

It is easily seen that some partition inequalities are not facet defining.

4.2 Decomposition and Extended Formulation for $\mathcal{Q}_\leq$

Here we show how the set $\mathcal{Q}_\leq$ can be decomposed allowing one to derive a polynomial size extended formulation for $\mathrm{conv}(\mathcal{Q}_\leq)$. First we look at the simple cases.

Observation 4.4. *If $c_1|\cdots|c_n|b$, the polyhedron*

$$\{(y, x) \in \mathbb{R}_+^n \times \mathbb{R}_+^1 : \sum_j c_j y_j \leq b + x\}$$

is integral and describes $\mathrm{conv}(\mathcal{Q}_\leq)$.

Observation 4.5. *If $c_j|b$ for $j < r$ and $c_j > b$ for $j \geq r$, then $\mathrm{conv}(\mathcal{Q}_\leq)$ is described by the original constraints $cy \leq b + x, y, x \geq 0$ and one additional constraint*

$$\sum_{j=1}^n (c_j - b)^+ \leq x.$$

4.2.1 Decomposition of $\mathrm{conv}(\mathcal{Q}_\leq)$

Let t be the index with $c_t \leq b$ and $c_{t+1} > b$. To avoid the case covered in Observation 4.5, we assume $r \leq t$.

Let

$$\mathcal{U} = \{(y, x) \in \mathbb{Z}_+^n \times \mathbb{R}_+^1 : \sum_{j=1}^t c_j y_j + \sum_{j=t+1}^n (c_j - \mu) y_j \leq b - \mu + x\}$$

where $\mu = \left\lfloor \frac{b}{c_t} \right\rfloor c_t$.

Let $\mathcal{R}$ be the set of vectors $(y, x, \alpha, \gamma, \delta)$ that satisfy

$$\sum_{j=1}^t c_j \alpha_j + \mu \sum_{j=t+1}^n \alpha_j \leq \mu$$

$$(\gamma, \gamma_0) \in \mathrm{conv}(\mathcal{U})$$

$$y_j = \alpha_j + \gamma_j \qquad\qquad j = 1, \ldots, t$$

$$y_j = \gamma_j + \delta_j, \ \ \gamma_j = \alpha_j \qquad\qquad \text{for } j = t+1, \ldots, n$$

$$x \geq \gamma_0 + \sum_{j=t+1}^n c_j \delta_j$$

$$\alpha, \gamma, \delta \in \mathbb{R}_+^n.$$

Proposition 4.5. $proj_{y,x}\mathcal{R} = \text{conv}(\mathcal{Q}_{\leq})$.

Proof. The extreme rays (e_j, c_j) for $j = 1, \ldots, t$ and $(0, 1)$ of $\text{conv}(\mathcal{U})$ also become extreme rays of $proj_{y,x}\mathcal{R}$. The variables δ_j provide additional rays (e_j, c_j) for $j > t$. Thus the rays of the two sets are the same. In addition it is straightforward to check that the points $(0, 0)$ and the maximal points for the partition inequalities given in Proposition 4.4 lie in $proj_{y,x}\mathcal{R}$.

To show that $proj_{y,x}\mathcal{R} \subseteq \text{conv}(\mathcal{Q}_{\leq})$, consider a point $(y, x) \in proj_{y,x}\mathcal{R}$. Thus there exist (α, γ, δ) such that $(y, x, \alpha, \gamma, \delta) \in \mathcal{R}$. Let I be the set of extreme points of $\text{conv}(\mathcal{U})$ with $\sum_{j=t+1}^{n} \gamma_j = 0$. As $(\gamma, \gamma_0) \in \text{conv}(\mathcal{U})$, we can write

$$(\gamma, \gamma_0) = \sum_{i \in I} (\gamma^i, \gamma_0^i)\lambda_i + \sum_{j=t+1}^{n} (e_j, c_j - b)\epsilon_j + \sum_{j=1}^{t} (e_j, c_j)\delta_j + \sum_{j=t+1}^{n} (e_j, c_j - \mu)\phi_j + (0, 1)\phi_0$$

where $\sum_{i \in I} \lambda_i + \sum_{j=t+1}^{n} \epsilon_j = 1$, $\lambda, \epsilon, \phi, \delta \geq 0$ with (γ^i, γ_0^i) for $i \in I$ and $(e_j, c_j - b)$ for $j > t$ the extreme points of $\text{conv}(\mathcal{U})$.

Also

$$\alpha = \sum_{j=0}^{n} \alpha^j \nu_j, \sum_{j=0}^{n} \nu_j = 1, \nu \geq 0,$$

where α^j for $j = 0, \ldots, n$ are the extreme points of $\{\alpha \in \mathbb{R}_+^n : \sum_{j=1}^{t} c_j \alpha_j + \mu \sum_{j=t+1}^{n} \alpha_j \leq \mu\}$ ($\alpha^0 = 0$, $\alpha^j = \frac{\mu}{c_j} e_j$ for $j = 1, \ldots, t$ and $\alpha^j = e_j$ for $j = t+1, \ldots, n$).

Then

$$(y, x) = \sum_{i \in I} (\gamma^i, \gamma_0^i)\lambda_i + \sum_{j=0}^{t} (\alpha^j, 0)\nu_j + \sum_{j=t+1}^{n} (e_j, c_j - b)\epsilon_j + \sum_{j=t+1}^{n} (e_j, c_j - \mu)\phi_j + \sum_{j=1}^{t} (e_j, c_j)\delta_j + (0, 1)\phi_0.$$

Let $\rho = \sum_{i \in I} \lambda_i$ and $\sigma = \sum_{j=0}^{t} \nu_j$. For $j = t+1, \ldots, n$, since $\gamma_j = \alpha_j$, we have $\epsilon_j + \phi_j = \nu_j$. Also $\sum_{j=t+1}^{n} \phi_j = \sum_{j=t+1}^{n} (\nu_j - \epsilon_j) = 1 - \sigma - (1 - \sum_{i \in I} \lambda_i) = \rho - \sigma$.

Let $(y^{ij}, y_0^{ij}) = (\gamma^i, \gamma_0^i) + (\alpha^j, 0)$ for $i \in I$ and $j = 0, \ldots, t$. Clearly $(y^{ij}, y_0^{ij}) \in \mathcal{Q}_{\leq}$. Also let $(z^{ij}, z_0^{ij}) = (\gamma^i, \gamma_0^i) + (e_j, c_j - \mu)$ for $i \in I$ and $j = t+1, \ldots, n$. As $c\gamma^i \leq b - \mu + \gamma_0^i$, we have $cz^{ij} = c\gamma^i + c_j \leq b + (\gamma_0^i + c_j - \mu)$ and thus $(z^{ij}, z_0^{ij}) \in \mathcal{Q}_{\leq}$. Also $(e_j, c_j - b) \in \mathcal{Q}_{\leq}$ for $j > t$.

Now since

$$\frac{\sigma}{\rho} \sum_{i \in I} (\gamma^i, \gamma_0^i)\lambda_i + \sum_{j=0}^{t} (\alpha^j, 0)\nu_j = \sum_{i \in I} \sum_{j=0}^{t} ((\gamma^i, \gamma_0^i) + (\alpha^j, 0)) \frac{1}{\rho} \lambda_i \nu_j = \sum_{i \in I} \sum_{j=0}^{t} (y^{ij}, y_0^{ij}) \frac{1}{\rho} \lambda_i \nu_j$$

16

and

$$\frac{\rho-\sigma}{\rho}\sum_{i\in I}(\gamma^i,\gamma_0^i)\lambda_i + \sum_{j=t+1}^{n}(e_j,c_j-\mu)\phi_j = \sum_{i\in I}\sum_{j=t+1}^{n}((\gamma^i,\gamma_0^i)+(e_j,c_j-\mu))\frac{1}{\rho}\lambda_i\phi_j$$

$$= \sum_{i\in I}\sum_{j=t+1}^{n}(z^{ij},z_0^{ij})\frac{1}{\rho}\lambda_i\phi_j,$$

we can rewrite (y,x) as a convex combination of extreme points plus extreme rays

$$(y,x) = \left(\sum_{i\in I}\sum_{j=0}^{t}(y^{ij},y_0^{ij})\frac{1}{\rho}\lambda_i\nu_j + \sum_{i\in I}\sum_{j=t+1}^{n}(z^{ij},z_0^{ij})\frac{1}{\rho}\lambda_i\phi_j + \sum_{j=t+1}^{n}(e_j,c_j-b)\epsilon_j\right)$$
$$+ \sum_{j=1}^{n}(e_j,c_j)\delta_j + (0,1)\phi_0,$$

as $\sum_{i\in I}\sum_{j=0}^{t}\frac{1}{\rho}\lambda_i\nu_j+\sum_{i\in I}\sum_{j=t+1}^{n}\frac{1}{\rho}\lambda_i\phi_j+\sum_{j=t+1}^{n}\epsilon_j = \sigma+(\rho-\sigma)+(1-\rho) = 1$ and all multipliers are nonnegative. Thus (y,x) lies in $\text{conv}(\mathcal{Q}_\le)$.$\square$

4.2.2 An Extended Formulation for $\text{conv}(\mathcal{Q}_\le)$

As before, we assume that $c_j|b$ for $j < r$, c_r does not divide b, $c_t < b$ and $c_j > b$ for $j > t$. Repeating the decomposition a maximum of $t - r$ times, one terminates with a set $\mathcal{U}$ of the form

$$\{(y,x) \in \mathbb{Z}_+^n \times \mathbb{R}_+^1 : \sum_{j=1}^{r-1}c_j y_j + \sum_{j=r}^{n}\tilde{c}_j y_j \le b - \lfloor\frac{b}{c_r}\rfloor c_r + x\}$$

where $c_j|(b-\lfloor\frac{b}{c_r}\rfloor c_r)$ for $j < r$ and $\tilde{c}_j > (b-\lfloor\frac{b}{c_r}\rfloor c_r)$ for $j \ge r$, so Observation 4.5 gives $\text{conv}(\mathcal{U})$ completing the polynomial size extended formulation.

Example 4.2. Consider a set $\mathcal{Q}_\le$ with $n = 5$, $c = (3,6,18,90,180)$ and $b = 737$.
This decomposes into
$3a_{11}+6a_{12}+18a_{13}+90a_{14}+180a_{15} \le 720$ and $3\gamma_1+6\gamma_2+18\gamma_3+90\gamma_4+180\gamma_5 \le 17 + \gamma_0$.
The latter decomposes into
$3a_{21} + 6a_{22} + 12a_{23} + 12a_{24} + 12a_{25} \le 12$ and $3\gamma_1 + 6\gamma_2 + (18 - 12)a_{23} + 78a_{24} + 168a_{25} \le 5 + \gamma_0$.
The latter decomposes into

$3a_{31} + 3a_{32} + 3a_{23} + 3a_{24} + 3a_{25} \leq 3$ and $3\gamma_1 + (6-3)a_{32} + (6-3)a_{23} + 75a_{24} + 165a_{25} \leq 2 + \gamma_0$.

To complete the convex hull of the latter, we add

$(3-2)a_{41} + 1a_{32} + 1a_{23} + 73a_{24} + 163a_{25} \leq \gamma_0$.

The complete formulation is:

$3a_{11} + 6a_{12} + 18a_{13} + 90a_{14} + 180a_{15} \leq 720$

$3a_{21} + 6a_{22} + 12a_{23} + 12a_{24} + 12a_{25} \leq 12$

$3a_{31} + 3a_{32} + 3a_{23} + 3a_{24} + 3a_{25} \leq 3$

$3a_{41} + 3a_{32} + 3a_{23} + 75a_{24} + 165a_{25} \leq 2 + \gamma_0.$

$1a_{41} + 1a_{32} + 1a_{23} + 73a_{24} + 163a_{25} \leq \gamma_0$

$y_1 = a_{11} + a_{21} + a_{31} + a_{41} + \delta_1$

$y_2 = a_{12} + a_{22} + a_{32} + \delta_2$

$y_3 = a_{13} + a_{23} + \delta_3$

$y_4 = a_{14} + a_{24} + \delta_4$

$y_5 = a_{15} + a_{25} + \delta_5$

$x \geq \gamma_0 + 3\delta_1 + 6\delta_2 + 18\delta_3 + 90\delta_4 + 180\delta_5$

$a_{ij} \geq 0 \ \forall \ i, j, \ \delta \geq 0$

5 The multi-item continuous $\leq$-knapsack set

Finally, we study the multi-item continuous $\leq$-knapsack set

$$\mathcal{Y}_{\leq} = \{(y, x) \in \mathbb{Z}_+^n \times \mathbb{R}_+^{m+1} : \sum_{j \in N} c_j y_j \leq b + \sum_{i \in M_0} x_i, \ x_i \leq a_i, i \in M\}.$$

Proposition 5.1. *Let $S \subseteq M$, $B(S) = b + a(M \setminus S)$ and $r(S)$ be the smallest index j such that c_j does not divide $B(S)$. Let $q \in \{r(S), \ldots, n\}$ and $\{i_1, \ldots, j_1\}, \cdots, \{i_p, \ldots, j_p\}$ be a partition of $\{q, \ldots, n\}$. Define $\beta_p = B(S), \kappa_t = \left\lceil \frac{\beta_t}{c_{i_t}} \right\rceil, \mu_t = (\kappa_t - 1)c_{i_t}, \beta_{t-1} = \beta_t - \mu_t$ and $s_t = c_{i_t} - \beta_{t-1}$ for $t = p, \ldots, 1$. Also let $\pi_1 = s_q, \pi_t = \kappa_{t-1}\pi_{t-1} + (s_t - s_{t-1})$ for $t = 2, \ldots, p$ and $\pi_0 = \kappa_p \pi_p - s_p$. The partition inequality*

$$\sum_{t=1}^{p} \pi_{i_t} \sum_{j=i_t}^{j_t} \frac{c_j}{c_{i_t}} y_j \leq \pi_0 + \sum_{i \in S^0} x_i \tag{5}$$

is valid for $\mathcal{Y}_{\leq}$.

Proof. Let

$$\mathcal{Q}_{\leq}^S = \{(y, x) \in \mathbb{Z}_+^n \times \mathbb{R}^{m+1} : cy \leq B(S) + x(S_0), x(S_0) \geq 0\}.$$

18

The partition inequality is valid for the set $\mathcal{Q}_{\leq}^{S}$ and this set is a relaxation of $\mathcal{Y}_{\leq}$. $\square$

Theorem 5.2. $\mathrm{conv}(\mathcal{Y}_{\leq})$ *is described by the initial constraints and the partition inequalities* (5). *Equivalently,*

$$\mathrm{conv}(\mathcal{Y}_{\leq}) = \cap_{S \subseteq M} \mathrm{conv}(\mathcal{Q}_{\leq}^{S}) \cap \{(y, x) : x_i \leq a_i \ i \in M\}.$$

Proof. Let $\Omega(g, h)$ be the set of optimal solutions to the problem of maximizing $\sum_{j=1}^{n} g_j y_j - \sum_{i=0}^{m} h_i x_i$ over $\mathcal{Y}_{\leq}$.

If $h_0 < 0$ or if there exists $j \in N$ with $g_j > c_j h_0$, then the problem is unbounded. If $h_i < 0$ for some $i \in M$, then all optimal solutions satisfy $x_i = a_i$. If $g_j < 0$ or if there exists $i < j$ with $\frac{c_j}{c_i} g_i > g_j$, then $\Omega(g, h) \subseteq \{(y, x) : y_j = 0\}$.

Now suppose that $h_0 \geq \frac{g_n}{c_n} \geq \ldots \geq \frac{g_1}{c_1} \geq 0$ and $h \geq 0$. If $h_0 = 0$ then $g = 0$ and $\Omega(g, h) \subseteq \{(y, x) : x_i = 0\}$ for $i \in M$ with $h_i > 0$. If $h_0 > 0$ and $h_i = 0$ for all $i \in M$, then $(y, x) \in \Omega(g, h)$ implies that (y, x_0) is optimal for $\max\{gy - h_0 x_0 : cy \leq b + x_0 + a(M), x_0 \geq 0, y \in \mathbb{Z}_{+}^{n}\}$. Note that the feasible set of the latter optimization problem is $\mathcal{Q}_{\leq}^{\emptyset}$.

In the remaining case, we have $h \geq 0$, $h_0 > 0$, $S = \{i \in M : h_i > 0\} \neq \emptyset$ and $h_0 \geq \frac{g_n}{c_n} \geq \ldots \geq \frac{g_1}{c_1} \geq 0$.

Suppose that there exists $j \in \{1, \ldots, r(S) - 1\}$ with $g_j > 0$. Take the smallest such j. Suppose that there exists an optimal solution (y, x) with $cy < b + x(M_0)$. Now consider the point (y', x) where $y'_k = 0$ for $k < j$, $y'_j = y_j + 1$ and $y'_k = y_k$ for $k > j$. As $c(y' - e_j) \leq cy < B(S)$ and c_j divides $B(S) - c(y' - e_j)$, it follows that $c(y' - e_j) \leq B(S) - c_j$. Thus (y', x) is feasible and its objective value exceeds that of (y, x) by $c_j > 0$. Hence when there exists $j \in \{1, \ldots, r(S) - 1\}$ with $g_j > 0$, $\Omega(g, h) \subseteq \{(y, x) : cy = b + x(M_0)\}$. From now on, we take $g_j = 0$ for $j \in \{1, \ldots, r(S) - 1\}$.

We look at three cases. In one case, $\Omega(g, h) \subseteq \{(y, x) : x(S_0) = 0\}$. In the second case, $\Omega(g, h) \subseteq \{(y, x) : cy = b + x(M_0)\}$. In the third case, there exists an optimal solution (y, x) with $x(S_0) > 0$ and there exists an alternative optimal solution with $cy < b + x(M_0)$. Let q be the smallest index for which there exists such an optimal solution (y, x) with $x(S_0) > 0$ and $y_q > 0$. This implies that $q \geq r(S)$ and $\frac{g_q}{c_q} > \frac{g_{q-1}}{c_{q-1}}$ or else $q = r(S)$ and $g_q > 0$.

Define $\phi(\sigma) = \min\{\sum_{i \in S_0} h_i x_i : x(S_0) \geq \sigma - B(S), x \in \mathbb{R}_{+}^{|S_0|}, x_i \leq a_i, i \in S\}$. If $\sigma \leq B(S)$, then $\phi(\sigma) = 0$. For $B(S) < \sigma$, $\phi(\sigma)$ is piecewise linear, strictly increasing and convex. Let (y^*, x^*) be an an optimal solution with

19

$x^*(S_0) > 0$ and $y_q^* > 0$. Then since cy^* is divisible by c_q, we have $cy^* \geq \left\lceil \frac{B(S)}{c_q} \right\rceil c_q$. In addition, optimality of (y^*, x^*) implies that $g_q \geq \phi(cy^*) - \phi(c(y^* - e_q))$.

Suppose that $(y, x) \in \mathcal{Y}_\leq$ and $cy < \left\lfloor \frac{B(S)}{c_q} \right\rfloor c_q$. Let $y' = y + e_q$. Then $cy' = cy + c_q < \left\lfloor \frac{B(S)}{c_q} \right\rfloor c_q + c_q = \left\lceil \frac{B(S)}{c_q} \right\rceil c_q$ (since c_q does not divide $B(S)$). Now as $cy^* \geq \left\lceil \frac{B(S)}{c_q} \right\rceil c_q$, we have $cy' < cy^*$. Then $g_q \geq \phi(cy^*) - \phi(c(y^* - e_q)) > \phi(cy') - \phi(c(y' - e_q))$ where the strict inequality follows from the form of the function ϕ and the fact that $cy^* > B(S)$. Hence (y, x) cannot be optimal. As a result, every optimal solution (y, x) satisfies $cy \geq \left\lfloor \frac{B(S)}{c_q} \right\rfloor c_q$.

Now consider the partition $\{i_1, \ldots, j_1\}, \ldots, \{i_p, \ldots, j_p\}$ of $\{q, \ldots, n\}$ with $\frac{g_j}{c_j} = \frac{g_{i_t}}{c_{i_t}}$ for $t = 1, \ldots, p$ and $j \in \{i_t, \ldots, j_t\}$ and $\frac{g_{i_t}}{c_{i_t}} > \frac{g_{i_{t-1}}}{c_{i_{t-1}}}$ for $t = 2, \ldots, p$.

We first consider optimal solutions with $x(S_0) = 0$ and then optimal solutions with $x(S_0) > 0$. In an optimal solution with $x(S_0) = 0$, we have $\sum_{j=q}^{n} c_j y_j = \left\lfloor \frac{B(S)}{c_q} \right\rfloor c_q$ as $\frac{g_q}{c_q} > \frac{g_{q-1}}{c_{q-1}}$ or else $q = r(S)$ and $g_q > 0$, and $\sum_{j=1}^{q-1} c_j y_j \leq B(S) - \left\lfloor \frac{B(S)}{c_q} \right\rfloor c_q$. Since $\frac{g_{i_t}}{c_{i_t}} > \frac{g_{i_{t-1}}}{c_{i_{t-1}}}$ for $t = 2, \ldots, p$, we have $\sum_{j=i_t}^{j_t} \frac{c_j}{c_{i_t}} y_j = \kappa_t - 1$ for $t = 1, \ldots, p$. So the point lies on the partition inequality.

Now we consider an optimal solution (y, x) with $x(S_0) > 0$. Note that if (y', x') is an optimal solution with $cy' < b + x'(M_0)$, then $x'(S_0) = 0$ and $cy' < B(S)$. Suppose that there exists j with $y_j > 0$ and $c(y - e_j) \geq B(S)$. Then as $c(y - e_j) > cy'$, we have $g_j \leq \phi(c(y' + e_j)) - \phi(cy') < \phi(cy) - \phi(c(y - e_j))$. This contradicts the optimality of (y, x). Hence, in any optimal solution (y, x) with $x(S_0) > 0$, we have $c(y - e_j) < B(S)$ for all j with $y_j > 0$. Then for $(y, x) \in \Omega(g, h)$ with $x(S_0) > 0$, we have $\sum_{j=i_k}^{n} c_j y_j = \left\lceil \frac{B(S)}{c_{i_k}} \right\rceil c_{i_k}$ for some $k \in \{1, \ldots, p\}$, $y_j = 0$ for $j \in \{1, \ldots, i_k - 1\}$, $\sum_{j=i_t}^{j_t} \frac{c_j}{c_{i_t}} y_j = \kappa_t - 1$ for $t = k+1, \ldots, p$, $\sum_{j=i_k}^{j_k} \frac{c_j}{c_{i_k}} y_j = \kappa_k$ and $x(S_0) = s_k$. So again the point lies on the partition inequality. $\square$

6   Conclusion

In this paper, we have studied the polyhedra associated with knapsack sets with integer and continuous variables and divisible capacities.

In particular, we have studied the continuous $\geq$-knapsack set (equivalently the splittable flow arc set) with multiple capacities (facilities) and

given a description of the convex hull when the capacities are divisible. We have shown that $conv(\mathcal{Y}_\geq) = \cap_{S \subseteq M} conv(\mathcal{Q}_\geq^S) \cap \{(y,x) : x \leq a\}$ where $\mathcal{Q}_\geq^S$ is a continuous $\geq$-knapsack set for each $S \subseteq M$.

Consider the optimization problem $\min\{\sum_{i \in M} h_i x_i + \sum_{j \in N} g_j y_j : (y,x) \in \mathcal{Y}_\geq\}$. If $h_i < 0$, $x_i = a_i$ in every optimal solution and it suffices to solve a smaller problem. Thus we can assume that $0 \leq h_1 \leq \ldots \leq h_m$. Now there exists an optimal solution (x,y) with $x_j = a_j$ for $j < i$ and $x_j = a_j$ for $x_j = 0$ for $j > i$ for some i. Thus it suffices to solve the m problems $z_i = \sum_{j:j<i} p_j a_j + \min\{p_i x_i + \sum_{j \in N} g_j y_j : x_i + cy \geq b - \sum_{j:j<i} a_j, 0 \leq x_i \leq a_i, y \in \mathbb{Z}_+^n\}$ and take the best solution. Each of these can be represented by a polynomial size linear program so the optimization problem is in $\mathcal{P}$. Thus separation of $conv(\mathcal{Y}_\geq)$ is polynomial using the ellipsoid algorithm.

Though polynomial time combinatorial separation algorithms are known both for the partition inequalities for the integer $\geq$-knapsack set and the residual capacity inequalities for the single facility splittable flow arc set (see Pochet and Wolsey [16] and Atamtürk and Rajan [3], respectively), we do not know such an efficient combinatorial algorithm to separate the exponential family of partition inequalities (2). As a corollary it follows that, in any non-trivial facet-defining inequality for $conv(\mathcal{Y}_\geq)$, the coefficients of the continuous variables all take the values 0 or 1 (after scaling).

We have shown in Theorem 5.2 that a result similar to that of Theorem 3.2 holds for the corresponding multi-item continuous $\leq$-knapsack set $\mathcal{Y}_\leq$. It is natural to ask if similar results hold for other continuous knapsack sets with some special structure. Recently Dash et al. have shown that such results hold when there are just $n = 2$ integer variables [4].

Kianfar [7] has shown how the partition inequalities for the integer $\geq$-knapsack set with divisible capacities can be viewed as a special case of n-step MIR inequalities and thus generalized. It seems likely that a similar approach can be taken for the $\leq$-knapsack set.

References

[1] A. Agra, M. Constantino, Description of 2-integer continuous knapsack polyhedra, Discrete Optimization 3 (2006) 95–110.

[2] A. Atamtürk, On the facets of the mixed integer knapsack polyhedron, Mathematical Programming B 98 (2003) 145-175.

[3] A. Atamtürk, D. Rajan, On splittable and unsplittable flow capacitated network design arc-set polyhedra, Mathematical Programming 92 (2002) 315-333.

[4] S. Dash, O. Günlük, L.A. Wolsey, Notes on the continuous knapsack set, IBM, Yorktown Heights, in preparation.

[5] A. Gupta, n-dimensional superincreasing knapsack polytopes have n-facets, Optimization Online 2012

[6] H. Kellerer, U. Pferschy and D. Pisinger, Knapsack Problems. Springer. (2004).

[7] K. Kianfar, On n-step MIR and partition inequalities for integer knapsack and single-node capacitated flow sets, Discrete Applied Mathematics 160 (2012) 1567-1582.

[8] M.Laurent and A.Sassano, A characterization of knapsacks with the max-flow min-cut property, Operations Research Letters 11 (1992) 105110.

[9] L. Lovász, Graph theory and integer programming, Annals of Discrete Mathematics 4 (1979) 141-158.

[10] T.L. Magnanti, P. Mirchandani, R. Vachani, The convex hull of two core capacitated network design polyhedra, Mathematical Programming 60 (1993) 233-250.

[11] T.L. Magnanti, P. Mirchandani, R. Vachani, Modeling and solving the two-facility capacitated network loading problem, Operations Research 43 (1995) 142-157.

[12] H. Marchand, L.A. Wolsey, The 0-1 knapsack problem with a single continuous variable, Mathematical Programming 85 (1999) 15-33.

[13] O. Marcotte, The cutting stock problem and integer rounding, Mathematical Programming 33 (1985) 82-92.

[14] S.Martello and P.Toth, Knapsack problems: Algorithms and computer interpretations. Wiley-Interscience. (1990).

[15] Y. Pochet, R. Weismantel, The sequential knapsack polytope, SIAM Journal on Optimization 8 (1998) 248-264.

[16] Y. Pochet, L.A. Wolsey, Integer knapsack and flow covers with divisible coefficients: polyhedra, optimization and separation, Discrete Applied Mathematics 59 (1995) 57-74.

[17] J.-P.P. Richard, I.R. de Farias Jr, G.L. Nemhauser, Lifted inequalities for 0-1 mixed integer programming: Basic theory and algorithms, Mathematical Programming B 98 (2003) 89-113.

Recent titles

CORE Discussion Papers

2013/24   Axel GAUTIER and Jean-Christophe POUDOU. Reforming the postal universal service.
2013/25   Fabian Y.R.P. BOCART and Christian M. HAFNER. Fair re-valuation of wine as an investment.
2013/26   Yu. NESTEROV. Universal gradient methods for convex optimization problems.
2013/27   Gérard CORNUEJOLS, Laurence WOLSEY and Sercan YILDIZ. Sufficiency of cut-generating functions.
2013/28   Manuel FORSTER, Michel GRABISCH and Agnieszka RUSINOWSKA. Anonymous social influence.
2013/29   Kent WANG, Shin-Huei WANG and Zheyao PAN. Can federal reserve policy deviation explain response patterns of financial markets over time?
2013/30   Nguyen Thang DAO and Julio DAVILA. Can geography lock a society in stagnation?
2013/31   Ana MAULEON, Jose SEMPERE-MONERRIS and Vincent VANNETELBOSCH. Contractually stable alliances.
2013/32   Jean-François CAULIER, Ana MAULEON and Vincent VANNETELBOSCH. Allocation rules for coalitional network games.
2013/33   Georg KIRCHSTEIGER, Marco MANTOVANI, Ana MAULEON and Vincent VANNETELBOSCH. Limited farsightedness in network formation.
2013/34   Ana MAULEON and Vincent VANNETELBOSCH. Relative concerns and delays in bargaining with private information.
2013/35   Kristof BOSMANS, Koen DECANCQ and Erwin OOGHE. What do normative indices of multidimensional inequality really measure?
2013/36   Alain PHOLO BALA, Dominique PEETERS and Isabelle THOMAS. Spatial issues on a hedonic estimation of rents in Brussels.
2013/37   Lionel ARTIGE, Antoine DEDRY and Pierre PESTIEAU. Social security and economic integration.
2013/38   Nicolas BOUCKAERT and Erik SCHOKKAERT. Differing types of medical prevention appeal to different individuals.
2013/39   Pierre M. PICARD. Trade, economic geography and the choice of product quality.
2013/40   Tanja B. MLINAR and Philippe CHEVALIER. Pooling in manufacturing: do opposites attract?
2013/41   Chiara CANTA and Marie-Louise LEROUX. Public and private hospitals, congestion, and redistribution.
2013/42   Mathieu LEFEBVRE, Pierre PESTIEAU and Gregory PONTHIERE. FGT poverty measures and the mortality paradox: Theory and evidence.
2013/43   Nada BELHADJ, Jean J. GABSZEWICZ and Ornella TAROLA. Social awareness and duopoly competition.
2013/44   Volker BRITZ, P. Jean-Jacques HERINGS and Arkadi PREDTETCHINSKI. On the convergence to the Nash bargaining solution for action-dependent bargaining protocols.
2013/45   Pasquale AVELLA, Maurizio BOCCIA and Laurence WOLSEY. Single item reformulations for a vendor managed inventory routing problem: computational experience with benchmark instances.
2013/46   Alejandro LAMAS, Tanja MLINAR, Liang LU and Philippe CHEVALIER. Revenue management for operations with urgent orders.
2013/47   Helmuth CREMER, Firouz GAHVARI and Pierre PESTIEAU. Uncertain altruism and the provision of long term care.
2013/48   Claire DUJARDIN, Vincent LORANT and Isabelle THOMAS. Self-assessed health of elderly people in Brussels: does the built environment matter?
2013/49   Marc FLEURBAEY, Marie-Louise LEROUX, Pierre PESTIEAU and Grégory PONTHIERE. Fair retirement under risky lifetime.
2013/50   Manuel FÖRSTER, Ana MAULEON and Vincent VANNETELBOSCH. Trust and manipulation in social networks.

Recent titles

CORE Discussion Papers - continued

2013/51   Anthony PAPAVASILIOU, Yi HE and Alva SVOBODA. Self-commitment of combined cycle units under electricity price uncertainty.

2013/52   Ana MAULEON, Elena MOLIS, Vincent VANNETELBOSCH and Wouter VERGOTE. Dominance invariant one-to-one matching problems.

2013/53   Jean GABSZEWICZ and Skerdilajda ZANAJ. (Un)stable vertical collusive agreements.

2013/54   François MANIQUET and Massimo MORELLI. Approval quorums dominate participation quorums.

2013/55   Mélanie LEFÈVRE and Joe THARAKAN. Intermediaries, transport costs and interlinked transactions.

2013/56   Gautier M. KRINGS, Jean-François CARPANTIER and Jean-Charles DELVENNE. Trade integration and the trade imbalances in the European Union: a network perspective.

2013/57   Philip USHCHEV, Igor SLOEV and Jacques-François THISSE. Do we go shopping downtown or in the 'burbs'? Why not both?

2013/58   Mathieu PARENTI. Large and small firms in a global market: David vs. Goliath.

2013/59   Paul BELLEFLAMME and Francis BLOCH. Dynamic protection of innovations through patents and trade secrets.

2013/60   Christian HAEDO and Michel MOUCHART. Specialized agglomerations with areal data: model and detection.

2013/61   Julien MARTIN and Florian MAYNERIS. High-end variety exporters defying distance: micro facts and macroeconomic implications.

2013/62   Luca G. DEIDDA and Dimitri PAOLINI. Wage premia, education race, and supply of educated workers.

2013/63   Laurence A. WOLSEY and Hande YAMAN. Continuous knapsack sets with divisible capacities.

Books

V. GINSBURGH and S. WEBER (2011), *How many languages make sense? The economics of linguistic diversity*. Princeton University Press.

I. THOMAS, D. VANNESTE and X. QUERRIAU (2011), *Atlas de Belgique – Tome 4 Habitat*. Academia Press.

W. GAERTNER and E. SCHOKKAERT (2012), *Empirical social choice*. Cambridge University Press.

L. BAUWENS, Ch. HAFNER and S. LAURENT (2012), *Handbook of volatility models and their applications*. Wiley.

J-C. PRAGER and J. THISSE (2012), *Economic geography and the unequal development of regions*. Routledge.

M. FLEURBAEY and F. MANIQUET (2012), *Equality of opportunity: the economics of responsibility*. World Scientific.

J. HINDRIKS (2012), *Gestion publique*. De Boeck.

M. FUJITA and J.F. THISSE (2013), *Economics of agglomeration: cities, industrial location, and globalization.* (2nd edition). Cambridge University Press.

J. HINDRIKS and G.D. MYLES (2013). *Intermediate public economics.* (2nd edition). MIT Press.

J. HINDRIKS, G.D. MYLES and N. HASHIMZADE (2013). *Solutions manual to accompany intermediate public economics.* (2nd edition). MIT Press.

CORE Lecture Series

R. AMIR (2002), Supermodularity and complementarity in economics.

R. WEISMANTEL (2006), Lectures on mixed nonlinear programming.

A. SHAPIRO (2010), Stochastic programming: modeling and theory.