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Single item reformulations for a vendor managed inventory routing problem: computational experience with benchmark instances

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Abstract

The Inventory Routing Problem (IRP) involves the distribution of one or more products from a supplier to a set of customers over a discrete planning horizon. The version treated here, the so- called Vendor Managed Inventory Routing Problem (VMIRP), is the Inventory Routing problem arising when the replenishment policy is decided a priori. We consider two replenishment policies. The first is known as Order-Up (OU): if a client is visited in a period, then the amount shipped to the client must bring the stock level up to the upper bound. The latter is called Maximum Level (ML): the maximum stock level in each period cannot be exceeded. The objective is to find replenishment decisions minimizing the sum of the storage and distribution costs.

VMIRP contains two important subproblems: a lot-sizing problem for each customer and a classical vehicle routing problem for each time period. In this paper we present a-priori reformulations of VMIRP-OU and VMIRP-ML derived from the single-item lot-sizing substructure. In addition we introduce two new cutting plane families - the Cut Inequalities - deriving from the interaction between the Lot-Sizing and the Routing substructures.

A Branch-and-Cut algorithm has been implemented to demonstrate the effectiveness of Single-Item reformulations. Computational results on the benchmark instances with 50 customers and 6 periods with a single product and a single vehicle are presented.

Keywords: mixed integer programming, inventory routing, lot-sizing, reformulations, vendor management.

Mathematics Subject Classification: 90C11, 90C57, 90C90

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1 Introduction

The Inventory Routing Problem (IRP) arises from the integration of two basic components of the logistic supply chain, namely Inventory Management and Vehicle Routing. IRP involves the distribution of one or more products from a supplier to a set of customers over a discrete planning horizon. Each customer has a known demand to be met in each period and can hold a limited amount of stock. The product is shipped through a distribution network by one or more vehicles of limited capacity. IRP has found applications in several contexts, such as maritime logistics and the distribution of gas, perishable items, groceries, etc. We refer to Federgruen and Simchi-Levi [12], Campbell et. [7], Bertazzi et al. [5], Bertazzi and Speranza [6], Coelho [8] for surveys on IRP.

The version treated here, the so-called Vendor Managed Inventory Routing Problem (VMIRP) [3], is the Inventory Routing problem arising when the replenishment policy is decided a priori, and the supplier must select which customers to serve in each period, the order in which they are visited and the amount of good to be delivered as a function of the replenishment policy.

We consider two replenishment policies, both assuming that a stock upper bound is given for each customer. The first is known as Order-Up (OU): if a client is visited in a period, then the amount shipped to the client must bring the stock level up to the upper bound. The latter is called Maximum Level (ML): the maximum stock level in each period cannot be exceeded. The objective is to find replenishment decisions minimizing the sum of the storage and distribution costs.

1.1 Literature review

Archetti et al. [3] considered three different replenishment policies for VMIRP, namely VMIRP-OU, VMIRP-ML and VMIRP-Uncapacitated, i.e. without stock upper bounds. They introduced several families of valid inequalities and reported computational results with a Branch-and-Cut algorithm for a set of benchmark instances up to 50 clients with 3 periods and 30 clients with a time horizon of 6 periods. Solyali and Süral [16] presented a reformulation of VMIRP-OU based on a shortest-path network representation of the OU policy at each customer and a Branch-and-Cut approach. They could solve to optimality, within four hours of computation time, instances up to 60 customers with 3 periods, and 15 customers with 12 periods.

Adulyasak et al. [1] proposed reformulations using vehicle indices for the multiple vehicle case and introduced some symmetry breaking constraints. They also extended some of the inequalities introduced by Archetti et al. [3] to the multi-vehicle case.

Coelho and Laporte [11] present a unified model and a Branch-and-Cut algorithm able to address several variants of IRP, including the case with multiple vehicles. Particularly in [9] they report a detailed computational experience on VMIRP-ML benchmark instances with a single vehicle, improving the results shown in [3] and extending the test-bed with "large" benchmark instances introduced in [2], solving to optimality those with 50 customers and 6 periods and providing lower and upper bounds for the larger ones. In a very recent paper Coelho and Laporte [10] introduced some valid inequalities based on the minimum number of visits each client must receive. They report average results showing significant improvements in computation times.

Valid inequalities for single item Lot-Sizing with upper bounds on stocks have been studied by Atamturk [4] and Pochet and Wolsey [15, ?] among others. However the only tight description of the convex hull of solutions is that based on dynamic programming that is too large to be practically useful. The single-item model with an Order-Up policy is studied in Solyali and Süral [16].

1.2 Outline of the paper

VMIRP naturally decomposes into two important subproblems: a lot-sizing problem for each customer and a classical vehicle routing problem for each time period. In this paper we present a-priori reformulations of VMIRP-OU and VMIRP-ML based on extended formulations for the single-item lot-sizing sets arising for each client. We then project these formulations into the original variable space, leading to new valid inequalities/constraints, but avoiding the introduction of additional variables. The reformulations proposed have been somewhat influenced by the benchmark instances introduced in Archetti et. [3] and [2], which are the standard test-bed for VMIRP-OU and VMIRP-ML (see also [16, 1, 11]). Such instances have a special structure, namely they have time-constant customer demands and stock capacities which are small integer multiples of the respective demands. Besides the single-item reformulations, we introduce two new cutting plane families - the Cut Inequalities - derived from the interaction between the lot-sizing and the routing substructures.

A basic Branch-and-Cut algorithm, without any special-purpose primal heuristics, has been implemented to demonstrate the strength of the lower (dual) bounds obtained using the single-item reformulations. Computational results on the benchmark instances with 50 customers and 6 periods with a single product and a single vehicle are presented. All ther VMIRP-OU instances appear to have been solved to optimality for the first time, improving the best known upper bounds. For VMIRP-ML, we solved three new instances to optimality, significantly improving computation times reported in [9] for several others.

The remainder of the paper is organized as follows. In Section 2 we give a more formal definition of VMIRP and an initial mixed integer programming formulation. In sections 3 and 4 we present a-priori single-item reformulations of VMIRP-OU and VMIRP-ML, respectively. In Section 5 we introduce two new cutting plane families for VMIRP-OU and VMIRP-ML, called Cut Inequalities. Finally in Section 6 we report on a computational experience validating the effectiveness of the proposed reformulations.

2 Problem definition and formulation

Let $T = \{1, 2, ..., T_{max}\}$ be a discrete time horizon. In each period $t \in T$, D^0 units of a single item are delivered to the supplier 0, and the supplier then uses a vehicle of capacity C to supply a set of customers $I = \{1, 2, \ldots, n\}$. The deliveries must be planned so that the demand D^i of each customer $i \in I$ in each period is satisfied and his stock capacity is not exceeded. In addition, in each period t, the vehicle delivering to the clients leaves before the arrival of the D^0 units at the supplier.

Let $sinit^0$ be the initial stock of the supplier and let $sinit^i$ and U^i be the initial stock and the stock upper bound of the customer i, respectively. Let h^0 be the storage cost of the supplier. Two different replenishment policies are considered:

Order-up (OU): if customer $i \in I$ is visited in the period $t \in T$, the amount x^i_t shipped to i is such that the stock of i reaches its upper bound U^i ($x^i_t = U^i - s^i_{t-1} - D^i$).

Maximum Level (ML): if customer $i \in I$ is visited in the period $t \in T$, then the amount x^i_t shipped to i is such that the stock of i is not greater than the upper bound U^i ($x^i_t \leq U^i - s^i_{t-1} - D^i$).

Besides defining stock levels, the vehicle routes in each period $t \in T$ must be determined. The distribution network in each period $t \in T$ is represented by a directed graph $G(I \cup \{0, n+1\}, A)$ where I are the customers, and the supplier is splitted into the nodes 0 (starting depot) and $n+1$ (ending depot), and $A = \{(0, j),\ j \in I\} \cup \{(i, j),\ i, j \in I\} \cup \{(i, n+1),\ i \in I\}$. A cost c^{ij} is associated with each arc $ij \in A$. VMIRP consists of determining which customers must be served at time t, the order in which they are served (i.e. the route at time t) and the delivery amounts in order to minimize the sum of the storage and of the routing costs.

The following variables are used:

x^i_t is the amount shipped to customer $i \in I$ in period $t \in T$;

s^i_t is the stock of customer $i \in I$ at the end of time period $t \in T \cup \{0\}$;

s^0_t is the stock level of the supplier at the end of time period $t \in T \cup \{0\}$;

z^i_t is a binary variable which is 1 if the customer $i \in I$ is visited at time $t \in T$, 0 otherwise;

z_t^0 is a binary variable which is 1 if the vehicle delivers to some customers in period $t \in T$, 0 otherwise;

y_t^{ij} is a binary variable which is 1 if the arc $(ij) \in A$ belongs to the route of the vehicle at time $t \in T$, 0 otherwise.

A formulation of VMIRP-OU is:

$$\min \sum_{i \in I} \sum_{t \in T \cup \{0\}} h_t^i s_t^i + \sum_{t \in T} \sum_{(i,j) \in A} c^{ij} y_t^{ij}$$

$$s_0^0 = sinit^0, \tag{1}$$

$$s_t^0 = s_{t-1}^0 + D_t^0 - \sum_{i \in I} x_t^i, \quad t \in T \tag{2}$$

$$s_{t-1}^0 \geq \sum_{i \in I} x_t^i, \quad t \in T \tag{3}$$

$$s_0^i = sinit^i, \quad i \in I \tag{4}$$

$$s_i^t = s_i^{t-1} + x_i^t - D_i^t, \quad i \in I, \ t \in T \tag{5}$$

$$x_t^i \leq (\sum_{i \in I} D^i) z_i^t \quad i \in I, \ t \in T \tag{6}$$

$$x_i^t \leq U^i - s_{t-1}^i, \quad i \in I, \ t \in T \tag{7}$$

$$x_i^t \geq D^i + (U^i - D^i) z_t^i - s_{t-1}^i, \quad i \in I, \ t \in T \tag{8}$$

$$z_t^0 \geq z_t^i, \quad i \in I, t \in T \tag{9}$$

$$\sum_{i \in I} x_t^i \leq C z_t^0, \quad t \in T \tag{10}$$

$$\sum_{j \in I} y_t^{0j} = z_t^0, \quad t \in T \tag{11}$$

$$\sum_{j \in I \cup \{n+1\}} y_t^{ij} = z_t^i, \quad i \in I, \ t \in T \tag{12}$$

$$\sum_{j \in I \cup \{0\}} y_t^{ji} = z_t^i, \quad i \in I \cup, \ t \in T \tag{13}$$

$$\sum_{i \in S \cup \{0\}} \sum_{j \in I \setminus S} y_t^{ij} \geq z_t^i, \quad i \in I, \ t \in T \tag{14}$$

$$\sum_{t \in T} z_t^0 \geq \left\lceil \frac{\sum_{i \in I} (T_{max} D^i - sinit^i)}{C} \right\rceil \tag{15}$$

$$z_t^i \in \{0,1\}, \quad i \in I \cup \{0\}, \ t \in T \tag{16}$$

$$y_t^{ij} \in \{0,1\}, \quad i,j \in I, \ t \in T \tag{17}$$

$$s_t^i \geq 0, \quad i \in I \cup \{0\}, \ t \in T \cup \{0\} \tag{18}$$

$$x_t^i \geq 0, \quad i \in I, \ t \in T \tag{19}$$

Constraints (1) and (2) define the stock levels for the supplier 0. Constraints (3) impose that the supplier stock level at the end of period $t-1$ must be greater than or equal to the total amount shipped to the customers at time t. Constraints (4) and (5) define the stock levels for the customers.

The variable upper bounds (6) enforce $z_t^i = 1$ if the customer i is served at time t. Constraints (7) define stock upper bounds. Constraints (8) impose that the amount x_t^i shipped to i is such that the stock of i reaches the upper bound U^i if customer i is visited at time t (i.e. if $z_t^i = 1$).

Constraints (9) are variable upper bounds enforcing $z_j^0 = 1$ if at least a customer is served in the period t.

Constraints (10) impose that the total amount shipped from the supplier to customers at t cannot exceed the vehicle capacity C.

Constraints (11) impose that the vehicle leaves the depot if at least a customer is served in the period t (i.e. if $z_j^0 = 1$). Constraints (12) and (13) impose that the vehicle visits the customer j iff $z_t^j = 1$.
Constraints (14) are subtour elimination constraints [14], which are added dynamically to the formulation. The separation algorithm consists of solving a min-cut problem between 0 and i, for each $i \in V$ and $t \in T$ on the graph $G(I \cup \{0, n+1\}, A)$, where the arcs A are weighted with the fractional values attained by the variables y_t^{ij} in the current LP relaxation.
Constraints (15) provide a lower bound on the number of periods in which at least a customer must be visited.

We will denote by X^{OU} the set of the (s, x, z, y) solutions which are feasible in $(1) - (19)$.

Observation 1 *A formulation of VMIRP-ML is easily derived by dropping the constraints (8). We will denote by X^{ML} the set of the (s, x, z, y) solutions which are feasible in* (1)-(7) *and* (10)-(19).

3 Single-Item reformulations of VMIRP-OU

Here we take into account the fact that the initial stock levels $sinit^i$ and upper bound levels U^i are all integer multiples of the client demand D^i. Thus we suppose that $U^i = D^i V^i$ where V^i is a small positive integer, $V^i \in \{2, 3\}$ in the test instances. Now the delivery quantities x_t^i and the stock levels s_t^i are measured in units of D^i.
Dropping the superscript i, we have that:

if $s_{t-1} = k \leq V - 1$ and the customer is served in t $(z_t = 1)$, then $x_t = V - k$;

if $s_{t-1} = k$ and the customer is not served in t $(z_t = 0)$, then $s_t = k - 1$.

The formulation of the Single-Item Lot-Sizing problem with OU constraints is then:

$$\begin{aligned}
& s_{t-1} + x_t = 1 + s_t, \quad t \in T,\ t \geq 2 \\
& x_t \leq V z_t, \quad t \in T \\
& s_t \leq V - 1, \quad t \in T \\
& s_t \geq (V-1) z_t, \quad t \in T \\
& s_t \geq 0, \quad t \in T \\
& x_t \geq 0, \quad t \in T \\
& z_t \in \{0, 1\}, \quad t \in T
\end{aligned}$$

Note that s_t and x_t take integral values in the extreme points of the convex hull of solutions.
We first examine two extended formulations valid for all values of V, and then the projections into the original (s, z) space for the values $V \in \{2, 3\}$.

The Unit Stock Formulation

Let $w_{ut} = 1$ if $s_t = u \in \{0, 1, \ldots, V-1\}$. The resulting formulation is:

$$s_t = \sum_{u=1}^{V-1} u w_{ut}, \quad t \in T \tag{20}$$

$$z_t = w_{V-1,t}, \quad t \in T \tag{21}$$

$$\sum_{u=0}^{V-1} w_{ut} = 1, \quad t \in T \tag{22}$$

$$w_{u,t} \geq w_{u-1,t+1}, \quad u \in \{1, \ldots, V-1\},\ t \in T \tag{23}$$

$$w_{ut} \geq 0, \quad u \in \{0, 1, \ldots, V-1\},\ t \in T \tag{24}$$

$$w_{ut} \in \{0, 1\}, \quad u \in \{0, 1, \ldots, V-1\},\ t \in T \tag{25}$$

where (22) forces the stock to take one of the values in $\{0, \ldots, V-1\}$ and (23) indicates that if $s_{t+1} = u-1 < V-1$, then $z_t = x_t = 0$ and thus $s_t = u$.

The Unit Flow Formulation

Let $q_t^{uv} = 1$ if $s_{t-1} = u$ and $s_t = v$ where $v \in \{u-1, V-1\}$. Now one has the *unit flow formulation*:

$$w_{ut} = q_t^{u+1,u}, \quad u \in \{0, 1, \ldots, V-2\}, \quad t \in T, \tag{26}$$

$$w_{V-1,t} = \sum_{u=0}^{V-1} q_t^{u,V-1}, \quad t \in T \tag{27}$$

$$q_t^{u+1,u} = q_{t+1}^{u,u-1} + q_{t+1}^{u,V-1}, \quad u \in \{1, \ldots, V-2\},\ t \in T,\ t \leq T_{max} - 1 \tag{28}$$

$$\sum_{u=0}^{V-1} q_t^{u,V-1} = q_{t+1}^{V-1,V-1} + q_{t+1}^{V-1,V-2}, \quad t \in T,\ t \leq T_{max} - 1 \tag{29}$$

$$\sum_{u=0}^{V-1} q_t^{u,V-1} + \sum_{u=1}^{V-1} q_t^{u,u-1} = 1, \quad t \in T \tag{30}$$

$$q_t^{uv} \geq 0, \quad u, v \in \{0, 1, \ldots, V-1\},\ t \in T \tag{31}$$

$$q_t^{uv} \in \{0, 1\} \quad u, v \in \{0, 1, \ldots, V-1\},\ t \in T \tag{32}$$

Note that (29) is implied by (28) and (30) and is thus redundant. An example of the unit flow network for the OU problem is shown in Figure 1.

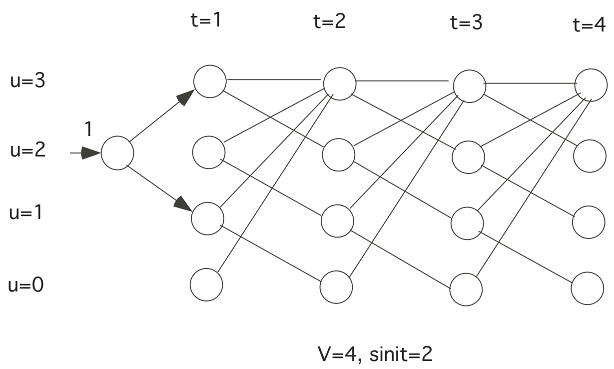

Figure 1: Unit Flow/Stock Formulation

Theorem 1 *The unit flow polyhedron (26)-(31) and the unit stock polyhedron(20)-(24) are integral polyhedra.*

Proof The unit flow polyhedron corresponds to a unit flow in a network and so the corresponding matrix is totally unimodular. We then project this polyhedron onto the w space using Fourier-Motzkin elimination. Specifically from (26), (28) and (29), we have that $q_{t+1}^{u-1,V-1} = w_{ut} - w_{u-1,t+1} \geq 0$ giving (23). Also (30) gives (22) and the equations (20) and (21) follow from the definition of the variables.

Corollary 1 *The linear programs* $\min\{px + hs + fz : (x, s, z, w) \in Q^{US}\}$ *and* $\min\{px + hs + fz : (x, s, z, w, q) \in Q^{UF}\}$ *have solutions with x, s, z integer, where Q^{US} denotes the polytope (20)-(24)) and Q^{UF} the polytope (26)-(31).*

Note that Solyali and Süral [16] have presented a shortest path extended formulation that is integral for time-varying demands. Our unit stock formulation is very closely related to their path formulation and also extends easily to handle time dependent demands.

3.1 Formulation in the Original Variables

Now we fix the value of V and consider the projections into the original (x, s, z) space. First we consider the case with $V = 3$. We take the extended formulation

$$
\begin{aligned}
& s_t = w_{1t} + 2w_{2t}, \quad t \in T \\
& z_t = w_{2t}, \quad t \in T \\
& w_{0t} + w_{1t} + w_{2t} = 1, \quad t \in T \\
& w_{1t} \geq w_{0,t+1}, \quad t \in T,\ t \leq T_{max} - 1 \\
& w_{2t} \geq w_{1,t+1}, \quad t \in T,\ t \leq T_{max} - 1 \\
& w_{ut} \geq 0, \quad u \in \{0, 1, 2\},\ t \in T
\end{aligned}
$$

and project out the w variables. We eliminate the variables w_{0t} and then use the equations $z_t = w_{2t}$ and $w_{1t} = s_t - 2z_t$ to complete the projection. The result is a description of the convex hull of solutions in the original space

$$
\begin{aligned}
& s_{t-1} + s_t \geq 1 + 2z_{t-1} + z_t, \quad t \in T,\ t \geq 2 && (33) \\
& s_t \leq z_{t-1} + 2z_t, \quad t \in T,\ t \geq 2 && (34) \\
& s_t \leq 1 + z_t, \quad t \in T && (35) \\
& s_t \geq 0, \quad t \in T && (36) \\
& z_t \in \{0, 1\}, \quad t \in T
\end{aligned}
$$

as well as the original constraints: $x_t = 1 + s_t - s_{t-1} \geq 0$ and $s_t \geq 2z_t$.
For $V = 2$, it is much simpler and one obtains

$$
\begin{aligned}
& s_t = z_t, \quad t \in T && (37) \\
& z_t + z_{t+1} \geq 1, \quad t \in T,\ t \leq T_{max} - 1 && (38) \\
& z_t \in \{0, 1\}, \quad t \in T
\end{aligned}
$$

again with $x_t = 1 + s_t - s_{t-1} \geq 0$.

4 Single-Item Reformulation of VMIRP-ML

We again treat D^i as the basic unit for customer i. The single item model with stock upper bounds is:

$$
\begin{aligned}
& s_{t-1} + x_t = 1 + s_t, \quad t \in T \\
& s_t \leq V - 1, \quad t \in T \\
& x_t \leq V z_t, \quad t \in T \\
& x_t \geq 0, \quad t \in T \\
& s_t \geq 0, \quad t \in T \cup \{0\} \\
& z_t \in \{0, 1\}, \quad t \in T
\end{aligned}
$$

We first examine two extended formulations valid for all values of V, and then the projections into the original (s, z) space for the values $V \in \{2, 3\}$.

The Unit Flow Formulation

Let $q_t^{uv}=1$ if $s_{t-1}=u$ and $s_t=v$ where $v\in\{u-1,u,\ldots,V-1\}$. Now one has the *unit flow formulation* (with added node variables $w_{ut}=1$ if $s_t=u\in\{0,1,\ldots,V-1\}$):

$$\sum_u q_{t-1}^{uv}=w_{vt},\quad v,\ t\in T,\ t\geq 2 \tag{39}$$

$$\sum_v q_{t-1}^{uv}=w_{u,t-1},\quad u,\ t\in T,\ t\geq 2 \tag{40}$$

$$\sum_{u,v} q_t^{uv}=1,\quad t\in T \tag{41}$$

$$q_t^{uv}\geq 0,\quad u,v\in\{0,1,\ldots,V-1\},\ t\in T \tag{42}$$

$$z_t\geq \sum_{u,v:u\leq v} q_{t-1}^{uv},\quad t\in T,\ t\geq 2 \tag{43}$$

$$z_t\leq 1,\quad t\in T \tag{44}$$

$$q_t^{uv}\in\{0,1\},\quad u,v\in\{0,1,\ldots,V-1\},\ t\in T \tag{45}$$

$$z_t\in\{0,1\},\quad t\in T \tag{46}$$

The Unit Stock Formulation

Fix t and consider just the bipartite graph with nodes $N_1=\{u\}_{u=0}^V$ for period $t-1$ and nodes $N_2=\{v\}_{v=0}^V$ for period t. We will also need the family $\mathcal{S}$ of pairs of sets $S_1\subset N_1, S_2\subset N_2$ that provide a minimal node cover for the edges $\{(u,u-1)\}_{u=1}^{V-1}$ corresponding to the arcs used in the unit flow formulation when there is no production. Specifically for each arc, either $u\in S_1$ or $u-1\in S_2$ and $|S_1\cup S_2|=V-1$, Also let $\mathcal{T}$ be the set of pairs in $\mathcal{S}$ with the additional property that $S_1\neq\emptyset, S_2\neq\emptyset$ and all the edges $\{(u,v):u\in S_1, v\in\delta(u)\}$ and $\{(u,v):v\in S_2, u\in\delta(v)\}$ are distinct.

Proposition 2 *The following inequalities are valid:*

$$\sum_{u\notin S_1} w_{u,t-1}+\sum_{v\notin S_2} w_{vt}\geq 1,\quad (S_1,S_2)\in\mathcal{T},\ t\in T,\ t\geq 2\quad\text{and} \tag{47}$$

$$z_t+\sum_{u\in S_1} w_{u,t-1}+\sum_{v\in S_2} w_{ut}\geq 1,\quad (S_1,S_2)\in\mathcal{S},\ t\in T,\ t\geq 2. \tag{48}$$

Proof. We show the validity of the inequalities that can obtained by Fourier-Motzkin elimination. For $(S_1,S_2)\in\mathcal{T}$, LHS$=2-\sum_{u\in S_1} w_{u,t-1}-\sum_{v\in S_2} w_{vt}\geq 2-\sum_{u\in S_1}\sum_v q_{t-1}^{uv}-\sum_{v\in S_2}\sum_u q_{t-1}^{uv}\geq 1$, where the last inequality follows from the disjointness property.
For $(S_1,S_2)\in\mathcal{S}$, LHS $\geq(1-\sum_{u=1}^{V-1} q_{t-1}^{u,u-1})+\sum_{u=1}^{V-1} q_{t-1}^{u,u-1})\geq 1$. Note these are just cut inequalities.

The resulting unit stock formulation consists of

$$\sum_u w_{ut}=1,\quad t\in T \tag{49}$$

$$w_{ut}\geq 0,\quad u\in\{0,1,\ldots,V-1\},\ t\in T \tag{50}$$

$$0\leq z_t\leq 1,\quad t\in T \tag{51}$$

$$\text{Inequalities (47) and (48)} \tag{52}$$

Theorem 3 *The unit flow polyhedron (39)-(44) and the unit stock polyhedron consisting of (49)-(52) are integral polyhedra.*

Proof The unit flow polyhedron corresponds to a unit flow in a network and is integral. The additional variables z_t either become equalities or inactive with z_t=1 in each extreme point and do not affect integrality. For fixed t, the unit stock polyhedron is obtained by projecting the constraints of the unit flow polyhedron by Fourier-Motzkin elimination (or minimum cuts). It follows that the flow polyhedron itself is integral because the values $w_{u,t-1}, w_{vt}, z_t$ suffice to reconstruct a feasible q_t vector for each t such that $(q_1,\cdots,q_n)$ is feasible for the unit flow polyhedron.

4.1 $V \in \{2, 3\}$

If $V = 3$, $\mathcal{T} = \{\ (\{2\}, \{0\})\ \}$, and $\mathcal{S} = \{\ (\{1, 2\}, \emptyset),\ (\emptyset, \{0, 1\}),\ (\{1\}, \{1\})\ \}$, we have the following formulation:

$$\sum_{u=0}^{2} w_{ut} = 1, \quad t \in T$$
$$w_{0,t-1} + w_{1,t-1} + w_{1t} + w_{2t} \geq 1, \quad t \in T,\ t \geq 2$$
$$z_t + w_{1,t-1} + w_{2,t-1} \geq 1, \quad t \in T,\ t \geq 2$$
$$z_t + w_{0t} + w_{1t} \geq 1, \quad t \in T$$
$$z_t + w_{1,t-1} + w_{1t} \geq 1, \quad t \in T,\ t \geq 2$$
$$w_{ut} \geq 0, \quad u \in \{0, \ldots, V-1\},\ t \in T$$
$$0 \leq z_t \leq 1, \quad t \in T$$

Projecting into the s, z space, the resulting polyhedron appears to be more complicated. Below we generate several families of valid inequalities, but they do not suffice to generate the convex hull for instances even for $n = 3$ periods.

Proposition 4 *With $V = 3$, the following inequalities are valid:*

$$s_t \leq 1 + z_t, \quad t \in T \tag{53}$$
$$s_{t-1} + z_t \geq 1, \quad t \in T,\ t \geq 2 \tag{54}$$
$$s_{t-1} + 3z_t \geq 1 + s_t, \quad t \in T,\ t \geq 2 \tag{55}$$
$$z_{t-1} + 2z_t \geq s_t, \quad t \in T,\ t \geq 2 \tag{56}$$
$$z_{t-2} + z_{t-1} + z_t \geq 1, \quad t \in T,\ t \geq 3 \tag{57}$$
$$s_{t-2} + 2z_{t-1} + z_t \geq 2, \quad t \in T,\ t \geq 3 \tag{58}$$
$$s_{t+1} + z_{t-1} + z_t \geq s_t, \quad t \in T, 2 \leq t \leq T_{max} - 1 \tag{59}$$
$$s_{t-1} + s_{t+1} + 2z_t \geq 1 + s_t, \quad t \in T, 2 \leq t \leq T_{max} - 1 \tag{60}$$
$$1 + s_t + 2z_{t+1} \geq s_{t-1} + s_{t+1}, \quad t \in T, 2 \leq t \leq T_{max} - 1. \tag{61}$$

Proof
$s_t \leq 1 + z_t$. If $z_t = 0$, then as $s_{t-1} \leq 2$, $s_t \leq 1$. Alternatively is $z_t = 1$, then valid as $s_t \leq 2$.
$s_{t-1} + z_t \geq 1$. If $s_{t-1} = 0$, then $z_t = 1$.
$s_{t-1} + 3z_t \geq 1 + s_t$. As $s_t \leq 2$, the inequality is satisfied when $z_t = 1$. When $z_t = 0$, $s_{t-1} = 1 + s_t$, and the inequality is satisfied.
$s_t \leq z_{t-1} + 2z_t$. If $z + t - 1 = z_t = 0$, then as $s_{t-2} \leq 2$ and the demands are 1, $s_t \leq 0$.. If $z_t = 1$, then validity follows as $s_t \leq 2$.If $z_{t-1} = 1$ and $z_t = 0$, one has $s_{t-1} \leq 2$ and thus $s_t = s_{t-1} - 1 \leq 1$.
$z_{t-2} + z_{t-1} + z_t \geq 1$. As $x_t \leq 3$ and the demand in each period is 1, one must produce at least once in every three consecutive periods.
$s_{t-2} + 2z_{t-1} + z_t \geq 2$. If $z_{t-1} = 1$, the inequality holds. If $z_{t-1} = z_t = 0$ and then necessarily $s_{t-2} = 2$. If $z_{t-1} = 0$ and $z_t = 1$, then necessarily $s_{t-2} \geq 1$, so the inequality holds.
The last three arguments are similar.

For $V = 2$, one has $\mathcal{S} = \{ (\{1\}, \emptyset),\ (\emptyset, \{0\}) \}$ and the unit stock formulation is

$$\begin{aligned}
& w_{0t} + w_{1t} = 1, \quad t \in T \\
& z_t + w_{1,t-1} \geq 1, \quad t \in T,\ t \geq 2 \\
& z_t + w_{0t} \geq 1, \quad t \in T \\
& w_{ut} \geq 0, \quad u \in \{0, \ldots, V-1\},\ t \in T \\
& 0 \leq z_t \leq 1, \quad t \in T
\end{aligned}$$

so finding the formulation in the s, z space using $s_t = w_{1t}$ is straightforward.

$$z_t + s_{t-1} \geq 1, \quad t \in T,\ t \geq 2 \tag{62}$$
$$z_t \geq s_t, \quad t \in T \tag{63}$$
$$s_t \geq 0, \quad t \in T$$
$$0 \leq z_t \leq 1, \quad t \in T$$

5 Cut inequalities

Here we introduce two cutting planes families having the role of preventing infeasible short tours in each period $t \in T$ and arising from the interaction between the lot-sizing and the routing substructures.

Definition 1 *Let $S \subseteq I$ and let $S = S^2 \cup S^3$, with $S^h = \{i \in S : V^i = h\}$. Let $t \in T$, $t \leq T_{max} - 2$. We call* Cover inequality*, any inequality of the form:*

$$\sum_{i \in S} z_t^i + \sum_{i \in S^3} z_{t+2}^i \geq 1 \tag{64}$$

We give sufficient conditions ensuring the validity of the Cover Inequalities (64) for X^{OU} and X^{ML}, respectively.

Proposition 5 *The Cover Inequality* (64) *is valid for X^{OU} if $\sum_{i \in S} 2D^i \geq C$.*

Proof. Suppose $\sum_{i \in S} z_t^i = \sum_{i \in S^3} z_{t+2}^i = 0$. Then all the customers in S must be served in the period $t+1$ and, due to the OU policy, $x_{t+1}^i = 2D^i$ for each $i \in S$. Then $\sum_{i \in S} x_{t+1}^i = \sum_{i \in S} 2D^i > C$ and the problem is infeasible.

Cover inequalities are valid for X^{ML} under stronger conditions.

Proposition 6 *The Cover Inequality* (64) *is valid for X^{ML} if $\sum_{i \in S^2} 2D^i + \sum_{i \in S^3} D^i \geq C$.*

Proof. Suppose $\sum_{i \in S} z_t^i = \sum_{i \in S^3} z_{t+2}^i = 0$. Then all the customers in S must be served in the period $t+1$ and, due to the ML policy, $x_{t+1}^i = 2D^i$ for each $i \in S^2$ and $x_{t+1}^i \geq D^i$ for each $i \in S^3$. Then $\sum_{i \in S^2} 2D^i + \sum_{i \in S^3} D^i > C$ and the problem is infeasible.

Let $\delta_t(0 : W)$ denote a cut between the node 0 and the nodes W on the graph $G_t(I, A_t)$.

Proposition 7 *Let $S \subset V$ and let $S^3 = \{i \in S : V^i = 3\}$. Let $t \in T$, $t \leq T_{max} - 2$. The Cut Inequality*

$$\sum_{ij \in \delta_t(0,S)} y_t^{ij} + \sum_{ij \in \delta_{t+2}(0,S^3)} y_{t+2}^{ij} \geq 1 \tag{65}$$

is valid for X^{OU} (X^{ML}) if the Cover Inequality (64) *is valid for X^{OU} (X^{ML}).*

Proof. From inequality (64) we get that either a customer in S in the period t or a customer in S^3 in the period $t+2$ must be visited, so either a path from 0 to one of the nodes in S on the graph G_t or a path between 0 and S^3 on the graph G_{t+2} must belong to any feasible solution. But then it follows that either the cut $\delta_t(0, S)$ or the cut $\delta_{t+2}(0, S^3)$ are crossed by a path.

Observation 1 *Using the same arguments, it can also be proved that the Cover Inequalities*

$$\sum_{i \in S} z_t^i + \sum_{i \in S^3} z_{t-2}^i \geq 1 \tag{66}$$

and the Cut Inequalities

$$\sum_{(i,j) \in \delta(0,S)} y_t^{ij} + \sum_{(i,j) \in \delta(0,S^3)} y_{t-2}^{ij} \geq 1 \tag{67}$$

are valid for X^{OU} and X^{ML}, respectively, under the same conditions.

5.1 2-Cut inequalities

Definition 2 *Let $S \subseteq V$ and let $S = S^2 \cup S^3$, with $S^h = \{i \in S : V^i = h\}$. Let $p \in S^2$ and let $t \in T$, $t \leq T_{max} - 2$. We call* 2-Cover Inequality, *any inequality of the form:*

$$\sum_{i \in S} z_t^i + z_{t+1}^p + \sum_{i \in S^3 \cup \{p\}} z_{t+2}^i \geq 2 \tag{68}$$

2-Cover Inequalities (68) are valid for X^{OU} and X^{ML} under the same conditions as in Propositions (5) and (6) respectively.

Proposition 8 *The 2-Cover inequality* (68) *is valid for X^{OU} if $\sum_{i \in S} 2D^i > C$.*

Proof. Case 1: $z_{t+1}^p = 0$. Since $V^p = 2$, feasibility implies $z_t^p = z_{t+2}^p = 1$, and the (68) is satisfied.
Case 2: $z_{t+1}^p = 1$. We get $\sum_{i \in S} z_t^i + \sum_{i \in S^3 \cup \{p\}} z_{t+2}^i \geq 1$ which is dominated by the cover inequality (64) and hence is valid for X^{OU} if $\sum_{i \in S} 2D^i \geq C$.

Proposition 9 *The 2-Cover inequality* (68) *is valid for X^{ML} if $\sum_{i \in S^2} 2D^i + \sum_{i \in S^3} D^i > C$.*

Proof. Case 1: $z_{t+1}^p = 0$. Since $V^p = 2$, feasibility implies $z_t^p = z_{t+2}^p = 1$, and the (68) is satisfied.
Case 2: $z_{t+1}^p = 1$. We get $\sum_{i \in S} z_t^i + \sum_{i \in S^3 \cup \{p\}} z_{t+2}^i \geq 1$ which is dominated by the (64) and hence is valid for X^{ML} if $\sum_{i \in S^2} 2D^i + \sum_{i \in S^3} D^i > C$.

As in Proposition (7), we can derive a 2-Cut inequality from the 2-Cover Inequality (68):

Proposition 10 *The 2-Cut Inequality*

$$\sum_{ij \in \delta(0,S)} y_t^{ij} + z_{t+1}^p + \sum_{ij \in \delta(0,S^3)} y_{t+2}^{ij} \geq 2 \tag{69}$$

is valid for X^{OU} (X^{ML} respectively) if the 2-Cover inequality (68) is valid for X^{OU} (X^{ML} respectively).

Proof. Case 1: $z_{t+1}^p = 0$. Since $V^p = 2$, customer p must be visited both in the periods t and $t+2$. So a path from 0 to p on the graph G_t and a path from 0 to p on the graph G_{t+2} must belong to any feasible solution. It follows that at least one arc has to traverse each of the cuts $\delta_t(0, S)$ and $\delta_{t+2}(0, S^3 \cup \{p\})$.
Case 2: $z_{t+1}^p = 1$. We get the $\sum_{i \in S} z_t^i + \sum_{i \in S^3 \cup \{p\}} z_{t+2}^i \geq 1$ and it follows that either a path from 0 to one of the nodes in S on the graph G_t or a path between 0 and $S^3 \cup \{p\}$ on the graph G_{t+2} must belong to any feasible solution and that either the cut $\delta_t(0, S)$ or the cut $\delta_{t+2}(0, S^3 \cup \{p\})$ is crossed by a path.

Observation 2 *Using the same arguments, it can also be proved that the 2-Cover Inequalities:*

$$\sum_{i\in S} z_t^i + z_{t-1}^p + \sum_{i\in S^3} z_{t-2}^i \geq 2 \tag{70}$$

and the 2-Cut Inequalities:

$$\sum_{ij\in\delta(0,S)} y_t^{ij} + z_{t-1}^p + \sum_{ij\in\delta(0,S^3\cup\{p\})} y_{t-2}^{ij} \geq 2 \tag{71}$$

are valid for X^{OU} *and* X^{ML} *under the same conditions as in Propositions* (8) *and* (9) *respectively.*

5.2 Separation algorithms for Cut inequalities

We use the Cut and the 2-Cut Inequalities with $S = I$ and $|S| = |I| - 1$ in the a-priori reformulation of VMIRP-OU and VMIRP-ML. Then we adopt a two-stage separation heuristic for Cut and 2-Cut Inequalities with $|S| \leq |I| - 2$.

We outline the separation heuristic for the OU case, putting into parenthesis the modifications for the ML case. For each period $t \in T$, $t \leq T_{max} - 2$, we first select by a greedy heuristic a subset of customers $S \subset I$ such that $\sum_{i\in S} 2D^i > C$ ($\sum_{i\in S^2} 2D^i + \sum_{i\in S^3} D^i > C$ in the ML case) and the sum of the fractional variables $\sum_{i\in S} \bar{z}_i$ is minimized. The greedy heuristic consists of sorting the customers I in ascending order of the fractional values $\bar{z}_t^i$ and then including them into S until $\sum_{i\in S} 2D^i > C$ ($\sum_{i\in S^2} 2D^i + \sum_{i\in S^3} D^i > C$ in the ML case).

Then we partition S into S^2 and S^3 and compute a minimum cut between 0 and S in the graph G_t, weighted with the fractional values $\bar{y}_t^{ij}$ and a minimum cut between 0 and S^3 in the graph G_{t+2}, weighted with the fractional values y_{t+2}^{ij}.

The separation heuristic for the 2-Cut inequalities is a slight modification of the heuristic for the Cut Inequalities. We need to iterate over $p \in S^2$, adding the fractional value of z_{t+1}^p and computing a minimum cut between between 0 and $S^3 \cup \{p\}$ in the graph G_{t+2}

6 Computational results

The reformulations outlined in sections 3, 4, 5 have been tested with a Branch-and-Cut algorithm based on FICO Xpress 7.3 [13]. The code is written in ANSI C. Computational experiments were carried out on a 64bit Pentium Quad-core 2.6 GHz processor Personal Computer with 4 Gb RAM and Microsoft Windows $XP64$ operating system. We set Xpress 7.3 parameters to run the code with a single thread. The node selection strategy was best bound and the branching variable selection rule was strong branching with priority on branching on the z_t^i variables. Xpress cut generation, primal heuristics and preprocessing were disabled and no initial upper bound was given.

The test bed consists of the instances with 50 customers and 6 periods introduced in [2] and available at L. Coelho's webpage *http://www.leandro-coelho.com/instances/*. They are partitioned in two main groups: those with "low" storage costs, namely $h_i \in [0.01, 0.05]$ and $h_o = 0.03$, and those with "high" storage costs, namely $h_i \in [0.1, 0.5]$ and $h_0 = 0.3$.

Each instance is labeled as nXX-TY-{low,high}-k, where XX is the number of customers, Y is the number of periods, the attribute {low, high} denotes the magnitude of the storage costs and k is a number identifying the instance.

6.1 Results for VMIRP-OU

Preliminary tests comparing the formulations, the unit flow polyhedron (28)-(31), the unit stock polyhedron (22)-(24) and the formulations in the original space (33)-(38) indicated that the tight original

space formulations gave the best results, presumably because of the smaller number of variables in the corresponding model.
We report computational results for VMIRP-OU in Table 1, which is organized as follows. Column “Name" shows the name of the instance and column LB_{Ini} shows the lower bounds returned by the initial formulation (1)-(19). Columns BC report on the results provided by the Branch-and-Cut algorithm based on the reformulations introduced in Sections 3 and 5: namely column LB_{LS} shows the lower bound returned by adding the single-item inequalities (33)-(38); column LB_{Cut} the lower bound returned by the single-item reformulation (33)-(38) and by the Cut (65) and the 2-Cut (69) inequalities; columns BLB and BUB show the best lower bound and the best upper bound found (bold means that the upper bound has been proven to be optimal), columns $Nodes$ and $Time$ show the number of tree nodes and the CPU seconds spent to produce the final BLB and BUB values.
VMIRP-OU instances with 50 customers and 6 periods do not appear to have been addressed by exact algorithms before, so for the sake of comparison, in the last column of Table 6.1 we show the best known upper bounds returned by the hybrid local search heuristic of Archetti et al. [2] after one hour of CPU time spent on an Intel Dual Core 1.86 GHz and 3.2 GB RAM Personal Computer.

Table 1: VMIRP-OU: computational results for the instances with $n = 50$ and $T_{max} = 6$

		BC						ABHS
Name	LB_{Ini}	LB_{LS}	LB_{Cut}	BLB	BUB	Nodes	Time	BUB
n50-T6-low-1	8374.46	9793.84	9868.16	10262.45	**10262.45**	1687	3255	10409.13
n50-T6-low-2	8952.24	10534.84	10555.93	10798.71	**10798.71**	1231	2771	10881.35
n50-T6-low-3	8732.15	10397.91	10412.18	10572.11	**10572.11**	661	1446	10767.39
n50-T6-low-4	8636.32	10285.43	10383.34	10546.36	**10546.36**	72	342	10656.21
n50-T6-low-5	8384.77	9890.79	9912.24	10166.25	**10166.25**	155	1118	10234.60
n50-T6-low-6	8408.77	9970.21	9991.24	10331.40	**10331.40**	341	2332	10533.63
n50-T6-low-7	8341.90	9801.22	9857.35	10327.51	**10327.51**	1069	2203	10460.82
n50-T6-low-8	8394.24	10029.70	10057.75	10363.20	**10363.20**	80	763	10411.20
n50-T6-low-9	8465.93	9915.72	9930.67	10243.16	**10243.16**	183	1494	10305.69
n50-T6-low-10	8019.13	9582.58	9616.24	9966.99	**9966.99**	603	2159	10470.63
n50-T6-high-1	28470.44	30113.27	30302.46	30613.81	**30613.81**	8679	8294	31147.82
n50-T6-high-2	27914.37	29790.68	29808.58	30068.28	**30068.28**	237	1369	30192.51
n50-T6-high-3	27892.29	29890.65	29910.66	30140.09	**30140.09**	1395	2315	30420.90
n50-T6-high-4	29493.88	31509.02	31556.38	31815.60	**31815.60**	179	567	31898.84
n50-T6-high-5	27436.20	29189.15	29211.27	29510.68	**29510.68**	49	343	29518.68
n50-T6-high-6	29939.68	31897.31	31908.24	32309.01	**32309.01**	966	1853	32394.50
n50-T6-high-7	27839.18	29592.91	29662.92	30146.64	**30146.64**	647	1057	30165.00
n50-T6-high-8	24085.85	25812.47	25847.67	26157.83	**26157.83**	77	529	26416.46
n50-T6-high-9	28264.44	30057.20	30092.06	30450.84	**30450.84**	606	1357	30671.88
n50-T6-high-10	29525.11	31388.97	31416.69	31832.59	**31832.59**	733	1993	32362.01

All the VMIRP-OU instances have been solved to optimality and computation times did not exceed one hour, except *n50-T6-high-1*. We could significantly improve the upper bounds of Archetti et al. [2] for all the benchmark instances.

6.2 Results for VMIRP-ML

We report computational results for VMIRP-ML in Table 2, which is organized as follows. Column “Name" shows the name of the instance and column LB_{Ini} shows the lower bounds returned by the initial formulation (1)-(19) except (8). Columns BC report on the results provided by the Branch-and-Cut algorithm based on the reformulations introduced in Sections 4 and 5: namely column LB_{LS} shows the lower bound returned by adding the single-item inequalities (53)-(63); column LB_{Cut} the lower bound returned by the single-item reformulation (53)-(63) and by the Cut (65) and the 2-Cut (69) inequalities; columns BLB and BUB show the best lower bound and the best upper bound found (bold means that the upper bound has been proven to be optimal), columns $Nodes$ and $Time$ show the number of tree nodes and the total CPU seconds.
In Table 2 we compare the results of our cutting plane algorithm with those provided by the Branch-and-Cut algorithm of Coelho and Laporte [9]. Columns BLB, BUB and $Time$ show the best lower bound and the best upper bound found in [9] respectively. Again bold means that the upper bound has been

proven to be optimal. For the sake of comparison we note that they ran their experiments on a GRID of Intel Xeon processors at 2.66 Ghz, using IBM Concert Technology and Cplex 12.3 with six threads.

Table 2: VMIRP-ML: computational results for the instances with $n = 50$ and $T_{max} = 6$

		BC						C&L		
Name	LB_{Ini}	LB_{LS}	LB_{Cut}	BLB	BUB	Nodes	Time	BLB	BUB	Time
n50-T6-low-1	8374.69	9754.38	9754.38	9966.14	**9966.14**	15123	1485	9901.42	9975.82	86400
n50-T6-low-2	8952.21	10515.99	10523.24	10632.04	**10632.04**	65	334	10632.0	**10632.0**	2536
n50-T6-low-3	8725.14	10375.98	10390.66	10510.72	**10510.72**	3972	1876	10510.7	**10510.7**	1355
n50-T6-low-4	8628.38	10242.90	10242.90	10513.43	**10513.43**	166667	18016	10513.4	**10513.4**	60289
n50-T6-low-5	8385.56	9859.51	9899.12	10113.05	**10113.05**	2500	2327	10113.0	**10113.0**	2416
n50-T6-low-6	8417.36	9944.55	9948.15	10148.02	**10148.02**	1900	2318	10113.6	10148.0	86400
n50-T6-low-7	8354.78	9775.95	9775.95	9982.20	**9982.20**	284288	28195	9982.2	**9982.2**	14698
n50-T6-low-8	8384.72	10015.36	10066.26	10299.13	**10299.13**	878	1360	10252.8	10229.1	86400
n50-T6-low-9	8483.51	9897.43	9904.03	10009.90	**10009.90**	819	801	10009.9	**10009.9**	6326
n50-T6-low-10	8014.26	9545.66	9545.66	9659.20	**9659.20**	2425	2081	9659.2	**9659.2**	3523
n50-T6-high-1	29508.20	29861.88	29905.99	30189.42	**30189.42**	1235	645	30189.4	**30189.42**	3036
n50-T6-high-2	27983.11	29600.59	29615.13	29790.05	**29790.05**	133	357	29790.0	**29790.0**	3334
n50-T6-high-3	27830.41	29633,56	29657.11	29790.91	**29790.91**	219	809	29790.9	**29790.9**	4020
n50-T6-high-4	29516.86	31240.62	31240.62	31518.26	**31518.26**	1424	1618	31518.3	**31518.3**	5737
n50-T6-high-5	27413.47	28992.54	29021.11	29240.42	**29240.42**	199	565	29240.4	**29240.4**	684
n50-T6-high-6	30007.78	31620.79	31630.26	31903.12	**31903.12**	367	1048	31903.1	**31903.1**	28320
n50-T6-high-7	27933.00	29396.58	29396.58	29734.48	**29734.48**	7988	1703	29734.5	**29734.5**	13561
n50-T6-high-8	23923.48	25692.27	25692.27	25709.61	**25954.19**	328	1202	25954.2	**25954.2**	21552
n50-T6-high-9	28467.02	29863.25	29884.01	30192.88	**30192.88**	390	822	30192.9	**30192.9**	20581
n50-T6-high-10	29508.20	31101.43	31101.43	31338.24	**31338.24**	83	488	31338.2	**31338.2**	1879

Although we must careful when comparing computation times obtained on different machines, we note that the Branch-and-Cut algorithm produces goods results for the instances with high storage costs, improving the results reported in [9]. We could also solve to optimality three instances still unsolved in [9].

7 Final Remarks

In the Tables we have started by reporting the values of the lower bounds obtained after the addition of subtour inequalities. This appears to be the natural base point given that all the other authors necessarily add such inequalities. This then allows us to measure the effect of the lot-sizing inequalities and the cuts described in Section 5. Other authors [3, 11, 9] have only added simple lot-sizing inequalities, and unfortunately have not reported comparable information on the lower bounds. Their work has concentrated more on the development of specialized branch-and-cut codes including problem-specific primal heuristics.

Though the test instances have a very special structure in which all data and decisions concern a small multiple of the demand D^i, it is easily seen that the extended reformulations for VMIRP-OU can easily be adapted to treat arbitrary time-dependent order up levels and demands without significantly changing in size. For VMIRP-ML the flow models will grow in size with the data, so for more arbitrary data it is probably preferable to use the valid inequalities proposed in Atamturk [4] and Pochet and Wolsey [15, **?**].

In a very recent paper Coelho and Laporte [10] add some valid inequalities based on the minimum number of visits each client must receive. They report average results showing significant improvements in computation times. It appears a priori that their improvements are complementary to our reformulations.

Preliminary tests on the 100 customer OU and ML instances that they tackled indicate that, though our approach produces lower bounds at the top node of the same quality as the lower bounds they achieve after several hours of computation, it is not able to solve the instances to optimality in reasonable computation times. This is presumably because of the larger duality gaps and of the growing size of the LP relaxation, which make re-optimization much slower. So to address large scale instances, the interaction between the lot-sizing and the routing substructure to derive new cutting planes families will have to be further investigated as well as primal heuristics, preprocessing and more efficient techniques to (re-)optimize LPs. Another research direction we plan to address is the extension to the multi-vehicle case.

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