Front dynamics in a two-species competition model driven by Lévy flights

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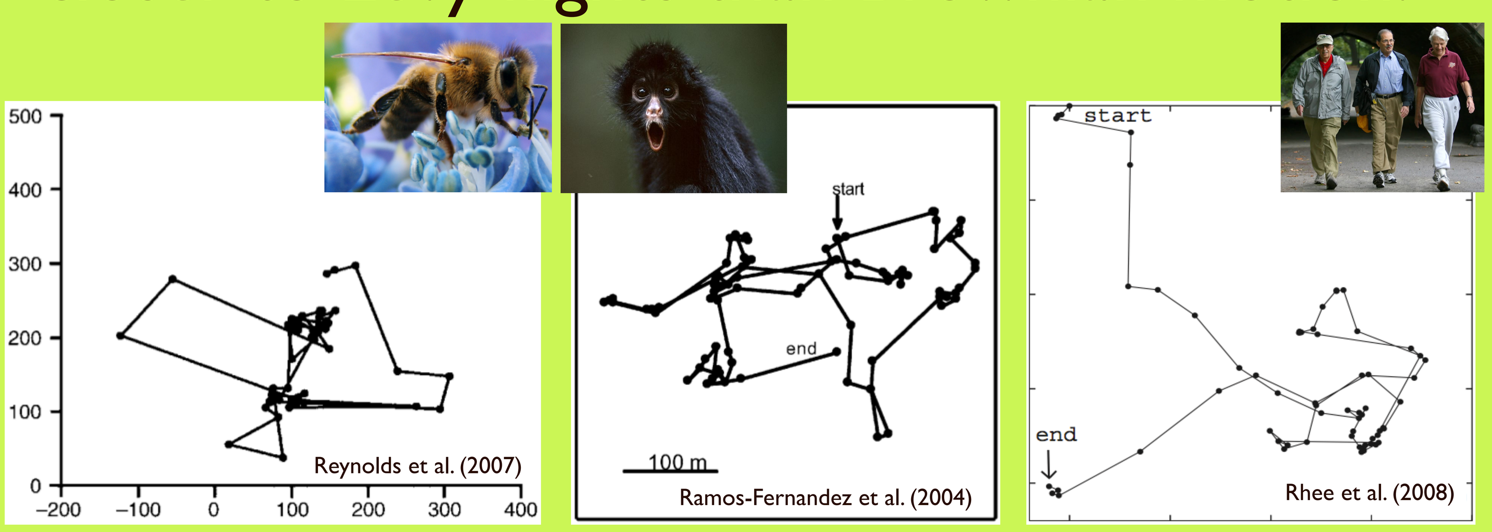
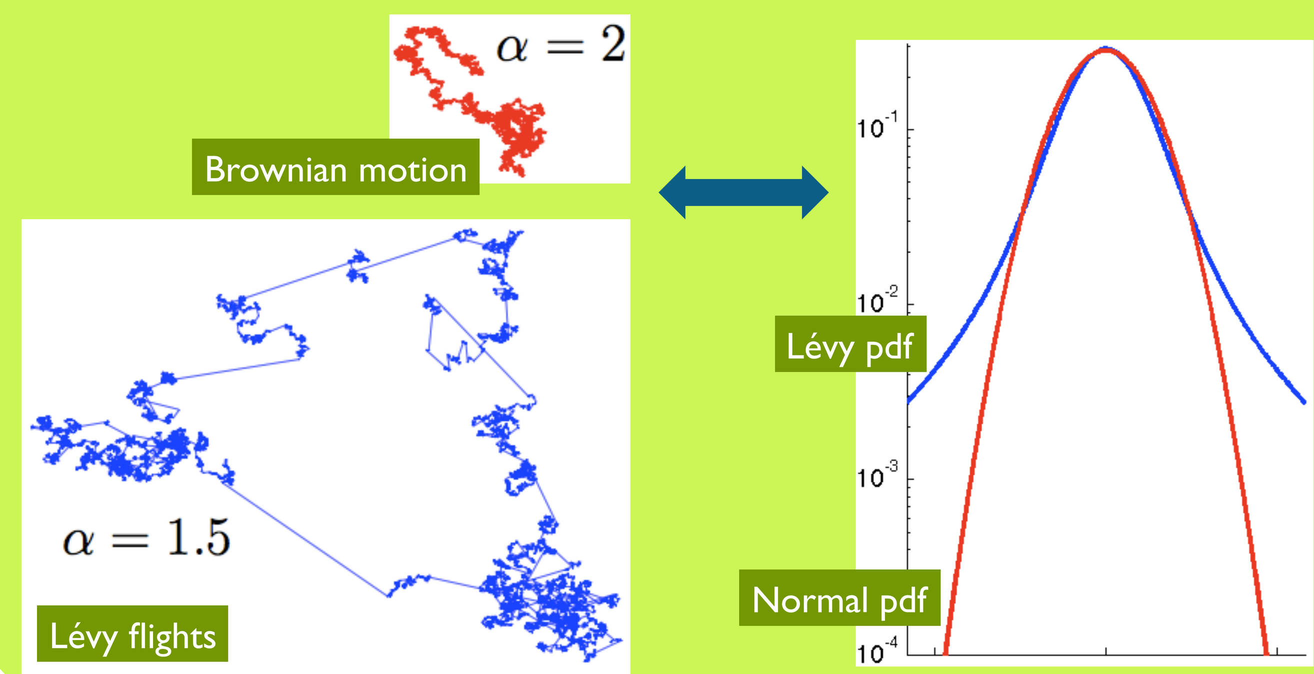
1. Empirical data suggests that the search patterns of many animals, including humans, is closer to Lévy flights than Brownian motion.



Performing Lévy flights maximizes the chance of encountering randomly distributed targets when their concentration is low.

2. Unlike Brownian motion that has a typical length scale, Lévy flights are scale-free. The corresponding probability distribution function (pdf) is not a Normal pdf but a "heavy-tailed" Lévy pdf, which is solution of a fractional-order diffusion equation.

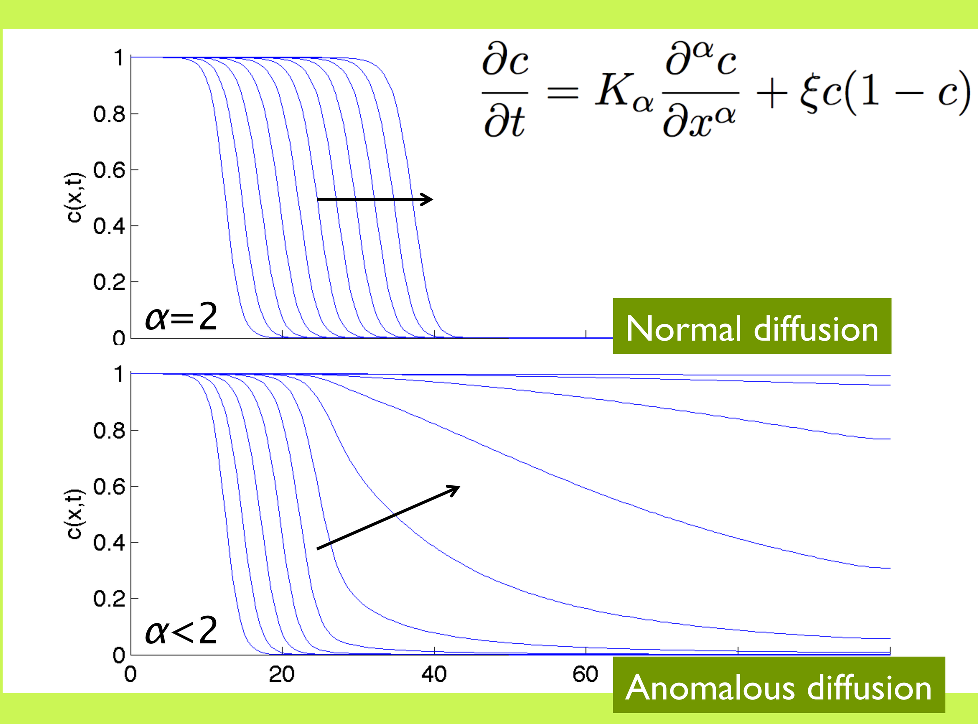


$$\frac{\partial c}{\partial t} = K_\alpha \frac{\partial^\alpha c}{\partial x^\alpha}$$

α = 2 : Normal second-order diffusion.
α < 2 : Anomalous fractional-order diffusion, *ie* superdiffusion.

$\langle x^2 \rangle \sim t^{2/\alpha}$

3. In a simple fractional-order reaction diffusion model, traveling waves no longer propagate at a constant speed but accelerate.



4. Here we consider a 1D competition model between 2 species, represented by density functions $S_1(x,t)$ and $S_2(x,t)$, that both perform Lévy flights with indices α_1 and α_2.

$$\frac{\partial S_1}{\partial t} = K_{\alpha_1\,-\infty}D_x^{\alpha_1} S_1 + a_1 S_1(1 - b_{11}S_1 - b_{12}S_2)$$
$$\frac{\partial S_2}{\partial t} = K_{\alpha_2\,-\infty}D_x^{\alpha_2} S_2 + a_2 S_2(1 - b_{21}S_1 - b_{22}S_2)$$

For simplicity, we assume that the fractional-order diffusion term is only "left-sided".

$$_{-\infty}D_x^{\alpha_i} S_i(x,t) = \mathcal{F}_k^{-1}\left[(ik)^{\alpha_i}\hat{S}_i(k,t)\right] = \frac{1}{\Gamma(2-\alpha_i)}\frac{\partial^2}{\partial x^2}\int_{-\infty}^{x}\frac{S_i(y,t)}{(x-y)^{\alpha_i-1}}\,\mathrm{d}y$$

What is the impact of Lévy-flight foraging patterns on both species dynamics? Can we derive a "catch-up" condition if one species is chasing the other?

5. Let us assume that S_2 is introduced in the wake of S_1 to control it. Initially, S_1 is at its carrying capacity in the region where S_2 is introduced. The asymptotic solution then reads:

$$S_1(x,t) \sim (K_{\alpha_1}t)e^{a_1 t}x^{-(\alpha_1+1)},$$
$$S_2(x,t) \sim (K_{\alpha_2}t)e^{a_2(1-b_{21}/b_{11})t}x^{-(\alpha_2+1)}$$

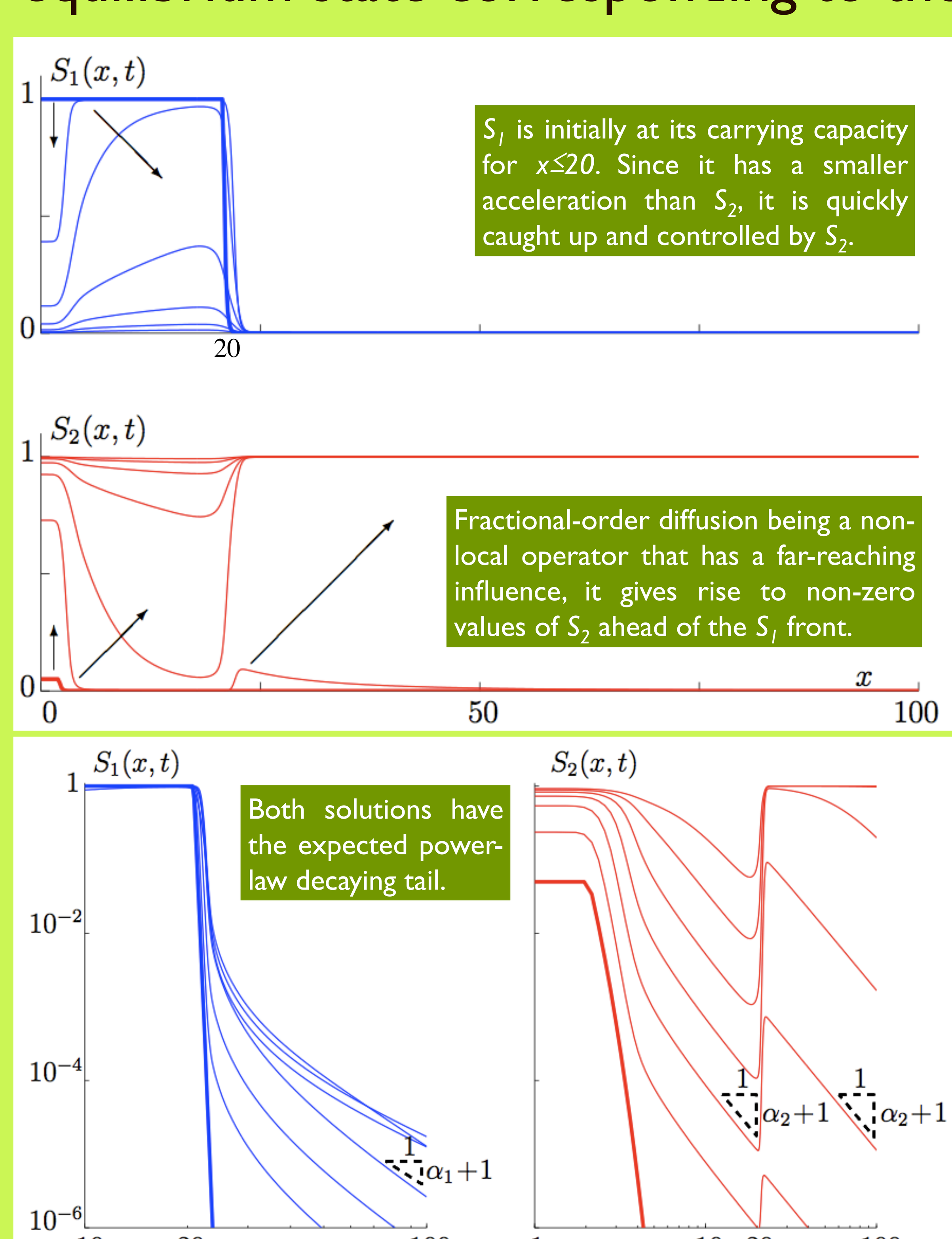
Both solutions have a power-law decaying tail and lead to exponentially-accelerating traveling waves. By computing the Lagrangian trajectory of a point at the leading edge, one obtains the following expressions for the front speed:

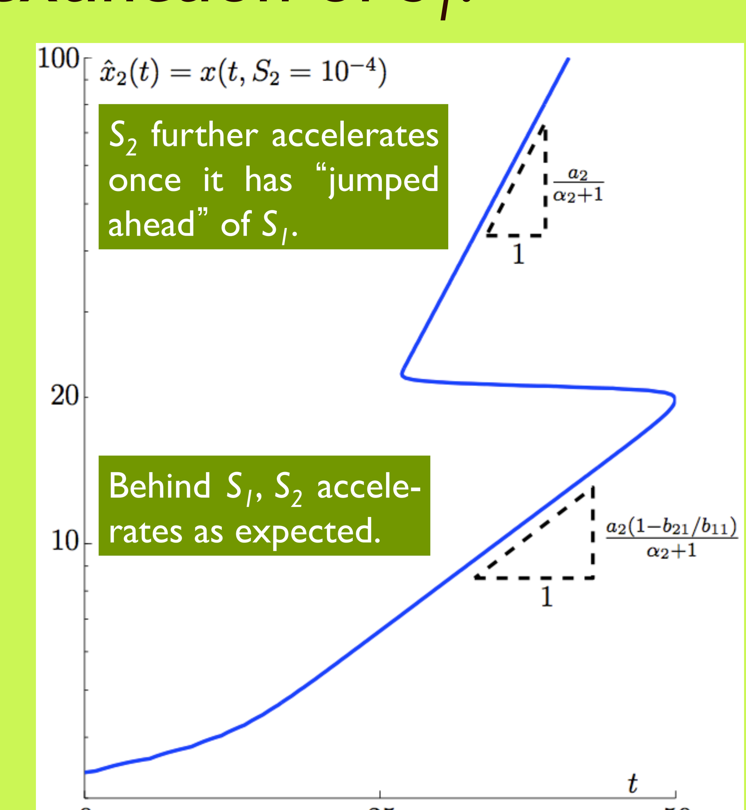
$$c_1(t) \sim \frac{1}{\alpha_1+1}\left(\frac{K_{\alpha_1}t}{\hat{S}_1}\right)^{\frac{1}{\alpha_1+1}} e^{\frac{a_1}{\alpha_1+1}t}\left(a_1 + \frac{1}{t}\right),$$
$$c_2(t) \sim \frac{1}{\alpha_2+1}\left(\frac{K_{\alpha_2}t}{\hat{S}_2}\right)^{\frac{1}{\alpha_2+1}} e^{\frac{a_2(1-b_{21}/b_{11})}{\alpha_2+1}t}\left(a_2\left(1 - \frac{b_{21}}{b_{11}}\right) + \frac{1}{t}\right)$$

S_2 will therefore catch up with S_1 if

$$\frac{a_2(1-b_{21}/b_{11})}{\alpha_2+1} > \frac{a_1}{\alpha_1+1}$$

6. In the example below, S_2 is both quicker (α_1 = 1.95, α_2 = 1.8) and stronger than S_1. The system evolves towards the equilibrium state corresponding to the extinction of S_1.





The system dynamics is thus strongly influenced by Lévy flights as they give rise to accelerating rather than constant-speed traveling waves, which are predicted by 2nd-order models.

References:
Hanert E. et al. (2011) Front dynamics in fractional-order epidemics models, *Journal of Theoretical Biology*, **279**, 9-16.
Hanert E. (2012) Front dynamics in a two-species competition model driven by Lévy flights, *Journal of Theoretical Biology*, **300**, 134-142.

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