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A sufficient condition of crossing-type for the bivariate orthant convex order

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A SUFFICIENT CONDITION OF CROSSING-TYPE FOR THE BIVARIATE ORTHANT CONVEX ORDER

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Abstract

In this paper, we derive a sufficient condition for the orthant convex order based on the single crossing of their respective joint survival functions. This condition is expressed in terms of the generators for archimedean copulas. Numerical examples show that this condition is valid for members of standard copula families (including Clayton and Frank).

Key words and phrases: copula, dependence, stochastic order.

Classification: 60E15 (Inequalities; stochastic orderings)

1 Introduction

Consider two random variables X and Y with respective distribution functions $F(x) = \Pr[X \leq x]$ and $G(x) = \Pr[Y \leq x]$ and survival functions $\overline{F} = 1 - F$ and $\overline{G} = 1 - G$, such that $\mathbb{E}[X] \leq \mathbb{E}[Y] < +\infty$. If there exists a constant c such that

$$\begin{cases} \overline{F}(x) \geq \overline{G}(x) \text{ for all } x < c, \\ \overline{F}(x) \leq \overline{G}(x) \text{ for all } x \geq c, \end{cases} \tag{1.1}$$

then the inequality $\mathbb{E}[v(X)] \leq \mathbb{E}[v(Y)]$ holds for all the non-decreasing convex functions v such that the expectations exist (which is henceforth denoted as $X \preceq_{\text{icx}} Y$). This result is known as the Karlin-Novikoff cut-criterion following the work of Karlin and Novikoff (1963). We refer the interested reader to the books of Shaked and Shanthikumar (2007) for a general presentation of the stochastic order relations and of Denuit et al. (2005) for their applications in risk theory.

This paper aims to extend the Karlin-Novikoff cut-criterion to the orthant convex order among bivariate random vectors. Recall that, given non-negative random vectors (X_1, X_2) and (Y_1, Y_2) with respective marginal survival functions $\overline{F}_i(x_i) = \Pr[X_i > x_i]$ and $\overline{G}_i(x_i) = \Pr[Y_i > x_i]$, and joint survival functions $\overline{F}(x_1, x_2) = \Pr[X_1 > x_1, X_2 > x_2]$ and $\overline{G}(x_1, x_2) = \Pr[Y_1 > x_1, Y_2 > x_2]$, $i = 1, 2$, (X_1, X_2) is smaller than (Y_1, Y_2) in the orthant convex order (which is henceforth denoted as $(X_1, X_2) \preceq_{\text{uo-cx}} (Y_1, Y_2)$) if the inequalities

$$\int_{x_1}^{\infty} \overline{F}_1(u)\, du \leq \int_{x_1}^{\infty} \overline{G}_1(u) \quad \text{holds for all } x_1,$$

$$\int_{x_2}^{\infty} \overline{F}_2(u)\, du \leq \int_{x_2}^{\infty} \overline{G}_2(u) \quad \text{holds for all } x_2,$$

$$\int_{x_1}^{\infty} \int_{x_2}^{\infty} \overline{F}(u_1, u_2)\, du_1 du_2 \leq \int_{x_1}^{\infty} \int_{x_2}^{\infty} \overline{G}(u_1, u_2)\, du_1 du_2 \text{ holds for all } (x_1, x_2).$$

Equivalently, $(X_1, X_2) \preceq_{\text{uo-cx}} (Y_1, Y_2)$ holds if, and only if, the inequality

$$E\big[g_1(X_1) g_2(X_2)\big] \leq E\big[g_1(Y_1) g_2(Y_2)\big]$$

is valid for every univariate non-negative non-decreasing convex functions g_1 and g_2.

In this paper, we aim to show that a suitable bivariate extension of (1.1) suffices to ensure that a pair of nonnegative random vectors are ordered with respect to $\preceq_{\text{uo-cx}}$. This is done in Section 2. Section 3 is devoted to bivariate distributions built from archimedean copulas. The sufficient condition of Section 2 is expressed in terms of properties of the respective generators of these copulas. We also apply these results to compare random vectors with different archimedean copulas. Whereas most of the literature compares members of a given copula family when the dependence parameter varies, this paper considers copulas from different families and provide a criterion to rank them with respect to the orthant convex order. The final Section 4 concludes the paper with a brief discussion of the results and provides several possible extensions.

2 Crossing condition

Consider a set $\mathcal{C} \subset \mathbb{R}^+ \times \mathbb{R}^+$ delimited by the axis $\mathcal{D}_x = \{(x,0), x \in \mathbb{R}^+\}$ and $\mathcal{D}_y = \{(0,y), y \in \mathbb{R}^+\}$. Suppose that the upper boundary curve $\partial\mathcal{C}$ of $\mathcal{C}$ is a continuous and decreasing function f. Let $c_1 > 0$ and $c_2 > 0$ and consider the following cases:

1. $\partial\mathcal{C} \cap \mathcal{D}_x = \{c_1\}$ and $\partial\mathcal{C} \cap \mathcal{D}_y = \{c_2\}$;
2. $\partial\mathcal{C} \cap \mathcal{D}_x = \{c_1\}$ and $\partial\mathcal{C} \cap \mathcal{D}_y = \emptyset$;
3. $\partial\mathcal{C} \cap \mathcal{D}_x = \emptyset$ and $\partial\mathcal{C} \cap \mathcal{D}_y = \{c_2\}$;
4. $\partial\mathcal{C} \cap \mathcal{D}_x = \emptyset$ and $\partial\mathcal{C} \cap \mathcal{D}_y = \emptyset$.

We denote by $\overline{\mathcal{C}}$ the complement of $\mathcal{C}$ in $\mathbb{R}^+ \times \mathbb{R}^+$. We are now ready to state the main result of this section.

Proposition 2.1. *Let (X_1, X_2) and (Y_1, Y_2) be two nonnegative random vectors with continuous marginal distribution functions $F_i(x) = \Pr[X_i \leq x]$ and $G_i(x) = \Pr[X_i \leq x]$, $i = 1, 2$. Denote the corresponding survival functions $\bar{F}_i = 1 - F_i$ and $\bar{G}_i = 1 - G_i$, $i = 1, 2$. Assume that the inequalities*

$$E[X_1 \mathbb{I}_{(X_2 > y)}] \leq E[Y_1 \mathbb{I}_{(Y_2 > y)}] \text{ and } E[X_2 \mathbb{I}_{(X_1 > x)}] \leq E[Y_2 \mathbb{I}_{(Y_1 > x)}]$$

hold true for all $(x, y) \in \mathcal{C}$ and that

$$\begin{cases} \bar{G}(x,y) - \bar{F}(x,y) \leq 0 & \text{for all} \quad (x,y) \in \mathcal{C}, \\ \bar{G}(x,y) - \bar{F}(x,y) \geq 0 & \text{for all} \quad (x,y) \in \overline{\mathcal{C}}. \end{cases}$$

Then, $(X_1, X_2) \preceq_{uo\text{-}cx} (Y_1, Y_2)$.

Proof. Let us establish the result in case 1. The proof is similar for cases 2,3 and 4. First, we see that $(x, 0) \in \mathcal{C}$ if $x \leq c_1$ and $(x, 0) \in \overline{\mathcal{C}}$ if $x \geq c_1$. Similarly, $(0, y) \in \mathcal{C}$ if $y \leq c_2$ and $(0, y) \in \overline{\mathcal{C}}$ if $y \geq c_2$. It follows that $\bar{G}_1(x) - \bar{F}_1(x) \leq 0$ if $x \leq c_1$ and $\bar{G}_1(x) - \bar{F}_1(x) \geq 0$ if $x \geq c_1$, as well as $\bar{G}_2(y) - \bar{F}_2(y) \leq 0$ if $y \leq c_2$ and $\bar{G}_2(y) - \bar{F}_2(y) \geq 0$ if $y \geq c_2$. Since $E[X_1] \leq E[Y_1]$ and $E[X_2] \leq E[Y_2]$, we then have $X_i \preceq_{\text{icx}} Y_i$, $i = 1, 2$. Now, we show that for all $(x, y) \in \mathbb{R}^+ \times \mathbb{R}^+$

$$D(x,y) = \int_x^\infty \int_y^\infty \Big(\bar{G}(u,v) - \bar{F}(u,v)\Big) du dv \geq 0.$$

First, since the curve $x \mapsto f(x)$ is decreasing, then for all $(x, y) \in \overline{\mathcal{C}}$, $[x, \infty) \times [y, \infty) \subset \overline{\mathcal{C}}$. Thus, $D(x, y) \geq 0$ for all $(x, y) \in \overline{\mathcal{C}}$. It remains to show that for all $(x, y) \in \mathcal{C}$, $D(x, y) \geq 0$. Clearly, for fixed $x \leq c_1$, the function

$$\phi_x : y \mapsto \frac{\partial}{\partial x} D(x,y) = \int_y^\infty \Big(\bar{F}(x,v) - \bar{G}(x,v)\Big) dv$$

defined on $[0, f(x)]$ is decreasing, because $\phi'_x(y) = \bar{G}(x,y) - \bar{F}(x,y) \leq 0$ for $(x,y) \in [0,c_1] \times [0, f(x)] \subset \mathcal{C}$. Thus for all $x \in [0, c_1]$, one has

$$\phi_x(y) \leq \phi_x(0) = E[X_2 \mathbb{I}_{(X_1>x)}] - E[Y_2 \mathbb{I}_{(Y_1>x)}] \leq 0$$

which implies that $x \to D(x,y)$ is decreasing for all $(x,y) \in \mathcal{C}$. The same arguments show, under the condition $E[X_1 \mathbb{I}_{(X_2>y)}] \leq E[Y_1 \mathbb{I}_{(Y_2>y)}]$, that $y \mapsto D(x,y)$ is also decreasing for all $(x,y) \in \mathcal{C}$. Consequently $(x,y) \mapsto D(x,y)$ is decreasing in $\mathcal{C}$. It follows that

$$D(x, f(x)) \leq D(x,y) \quad \text{for all} \quad (x,y) \in [0,c_1] \times [0, f(x)]$$

and since $[x, \infty) \times [f(x), \infty) \subset \overline{\mathcal{C}}$, then $D(x, f(x)) \geq 0$ which ensure $D(x,y) \geq 0$ for all $(x,y) \in \mathcal{C}$. □

Remark 2.2. When $F_i = G_i$, $i = 1,2$, the condition $E[X_1 \mathbb{I}_{(X_2>y)}] \leq E[Y_1 \mathbb{I}_{(Y_2>y)}]$ is equivalent to $E[X_1 | X_2 > y] \leq E[Y_1 | Y_2 > y]$. Similarly $E[X_2 \mathbb{I}_{(X_1>x)}] \leq E[Y_2 \mathbb{I}_{(Y_1>x)}]$ is equivalent to $E[X_2 | X_1 > x] \leq E[Y_2 | Y_1 > x]$.

Example 2.3. Let (U_1, V_1) and (U_2, V_2) be random vectors with unit uniform marginals and joint distributions (or copulas) C_1 and C_2, respectively. Assume that C_1 is a Clayton copula with parameter $\theta_1 = 1$ and C_2 is a Frank copula with parameter $\theta_2 = 2.1$, that is

$$C_1(u,v) = \left(\frac{1}{u} + \frac{1}{v} - 1 \right)^{-1}$$

and

$$C_2(u,v) = -\frac{1}{2.1} \ln \left(1 + \frac{(\exp(-2.1u) - 1)(\exp(-2.1v) - 1)}{\exp(-2.1) - 1} \right).$$

Consider the set $\mathcal{C} = \{(u,v) \in [0,1] \times [0,1] : v \leq f(u)\}$ for some decreasing function $f : [0,1] \to [0,1]$. If (U_1, V_1) and (U_2, V_2) are such that $E[U_1 | V_1 > v] \leq E[U_2 | V_2 > v]$ and $E[V_1 | U_1 > u] \leq E[V_2 | U_2 > u]$ for all $u, v \in \mathcal{C}$, and if

$$\begin{cases} C_2(u,v) - C_1(u,v) \leq 0 & \text{for all} \quad (u,v) \in \mathcal{C}, \\ C_2(u,v) - C_1(u,v) \geq 0 & \text{for all} \quad (u,v) \in \overline{\mathcal{C}} \end{cases} \tag{2.1}$$

then from Proposition 2.1, the stochastic inequality $(U_1, V_1) \preceq_{\text{uo-cx}} (U_2, V_2)$ is valid.

Figure 2.1 displays the difference $C_1(u,v) - C_2(u,v)$ over the unit square. We see that there exists a function f such that $C_1(u,v) - C_2(u,v) \leq 0$ if $v \leq f(u)$ and $C_1(u,v) - C_2(u,v) \geq 0$ if $v \geq f(u)$. The level curve $v = f(u)$ such that $C_1(u, f(u)) - C_2(u, f(u)) = 0$ is described in Figure 2.2. This graph shows that the domain $\mathcal{C}$ of this example corresponds to situation 4 as described in Section 2.

It remains to verify that $E[U_1 | V_1 > v] \leq E[U_2 | V_2 > v]$ for all $v \in [0,1]$, which is equivalent to show that

$$g(v) = \int_0^1 \Big(C_1(u,v) - C_2(u,v) \Big) du \leq 0 \text{ for all } v \in [0,1].$$

The function $v \mapsto g(v)$ displayed in Figure 2.3 is indeed negative over $[0,1]$. We can thus conclude that $(U_1, V_1) \preceq_{\text{uo-cx}} (U_2, V_2)$.

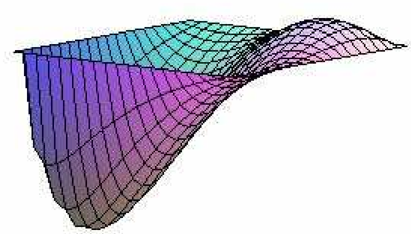

Figure 2.1: Graph of $(u, v) \mapsto C_1(u, v) - C_2(u, v)$

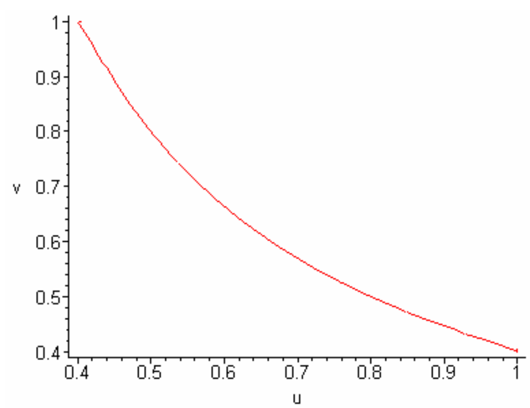

Figure 2.2: Graph of the level curve $v = g(u)$

3 Archimedean copulas

In this section, we consider random couples (X_1, X_2) with archimedean copula generated by a continuous, possibly infinite, strictly decreasing convex function $\phi : [0, 1] \to \mathbb{R}^+$ such that $\phi(1) = 0$. Specifically, define the pseudo-inverse $\phi^{[-1]}$ of the generator ϕ as

$$\phi^{[-1]}(t) = \begin{cases} \phi^{-1}(t) \text{ for } 0 \leq t \leq \phi(0), \\ 0 \text{ for } \phi(0) \leq t \leq +\infty. \end{cases} \tag{3.1}$$

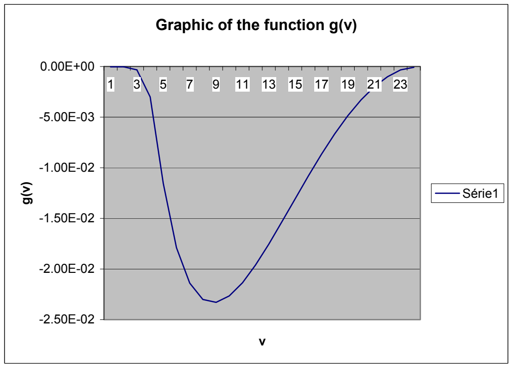

Figure 2.3: Graph of the function $v = g(u)$

The archimedean copula with generator ϕ is defined as

$$C_\phi(u_1, u_2) = \phi^{[-1]}\big(\phi(u_1) + \phi(u_2)\big) \tag{3.2}$$

for $0 \leq u_1, u_2 \leq 1$.

Archimedean copulas (3.2) enjoy numerous convenient mathematical properties and are therefore appreciated for modelling or simulating bivariate data. See, e.g., Nelsen (2006, Chapter 4) for a review. In particular, archimedean copulas naturally appear in relation with frailty models for the joint distribution of two survival times depending on the same latent factor (the generator being then the inverse of the Laplace transform of this latent factor).

Our aim in this section is to study the conditions (2.1) in the class of archimedean copulas. In particular, an alternative formulation of these conditions in terms of generators of archimedean copulas is obtained. To this end, recall that a function h defined on $[0, \infty[$ is subadditive in the set $\mathcal{A} \subset \mathbb{R}^+ \times \mathbb{R}^+$ if the inequality

$$h(x + y) \leq h(x) + h(y) \text{ for all } x, y \in \mathcal{A}.$$

Now, let ϕ_1 and ϕ_2 be two generators and define the sets

$$\mathcal{C}_{\phi_2} = \{(\phi_2(u), \phi_2(v)) : (u, v) \in \mathcal{C}\}$$

and $\overline{\mathcal{C}}_{\phi_2}$, the complement $\mathcal{C}_{\phi_2}$ in $\mathbb{R}^+ \times \mathbb{R}^+$.

Proposition 3.1. *Let (U_1, V_1) and (U_2, V_2) be random vectors with joint distribution functions C_1 and C_2. Assume that C_1 and C_2 are archimedean copulas with generators ϕ_1 and ϕ_2. Assume that $E[U_1|V_1 > v] \leq E[U_2|V_2 > v]$ for all v. If $-\phi_1 \circ \phi_2^{-1}$ is subadditive in $\mathcal{C}_{\phi_2}$ and $\phi_1 \circ \phi_2^{-1}$ is subadditive in $\overline{\mathcal{C}}_{\phi_2}$, then $(U_1, V_1) \preceq_{uo\text{-}cx} (U_2, V_2)$.*

Proof. For all $(u,v) \in \mathcal{C}$, put $x = \phi_2(u)$ and $y = \phi_2(v)$. Since the generator ϕ_2 is decreasing, we have

$$\begin{aligned} & C_2(u,v) \leq C_1(u,v) \quad \text{for all} \quad (u,v) \in \mathcal{C} \\ \Leftrightarrow \quad & \phi_1 \circ \phi_2^{-1}\left[\phi_2(u) + \phi_2(v)\right] \geq \phi_1(u) + \phi_1(v) \quad \text{for all} \quad (u,v) \in \mathcal{C} \\ \Leftrightarrow \quad & \phi_1 \circ \phi_2^{-1}(x+y) \geq \phi_1 \circ \phi_2^{-1}(x) + \phi_1 \circ \phi_2^{-1}(y) \quad \text{for all} \quad (x,y) \in \mathcal{C}_{\phi_2} \\ \Leftrightarrow \quad & -\phi_1 \circ \phi_2^{-1} \quad \text{is subadditive in} \quad \mathcal{C}_{\phi_2}. \end{aligned}$$

The same arguments show that $C_1(u,v) \leq C_2(u,v)$ for all $(u,v) \in \overline{\mathcal{C}}$ is equivalent to $\phi_1 \circ \phi_2^{-1}$ being subadditive in $\overline{\mathcal{C}}_{\phi_2}$. Hence the result. □

Example 3.2. Let us consider archimedean copulas in Families 2 and 6 from Table 4.1 in Nelsen (2006). Let (U_1, V_1) and (U_2, V_2) be random vectors with joint distribution functions C_1 and C_2. Assume that C_1 and C_2 are archimedean copulas with generators $\phi_{2,\theta}(t) = -\ln\left(1-(1-t)^\theta\right)$ and to $\phi_{1,\theta}(t) = (1-t)^\theta$, $\theta \in [1,\infty)$. It's easy to see for $\theta_1, \theta_2 \in [1,\infty)$ that $h_\alpha(t) = \phi_{1,\theta_1} \circ \phi_{2,\theta_2}^{-1}(t) = -\ln(1-t^\alpha)$ where $\alpha = \theta_1/\theta_2$. It follows that

$$h_\alpha(x+y) - h_\alpha(x) - h_\alpha(y) = \ln\left(\frac{(1-x^\alpha)(1-y^\alpha)}{1-(x+y)^\alpha}\right).$$

Consider the set $\mathcal{C}_\alpha = \{(x,y) \in [0,1]^2 : (1-x^\alpha)(1-y^\alpha) + (x+y)^\alpha - 1 \geq 0\}$. Then, $-h_\alpha$ is subadditive in $\mathcal{C}_\alpha$ and h_α is subadditive $\overline{\mathcal{C}}_\alpha$.

Consider now for instance $\alpha = \theta_1/\theta_2 = 0.2$. Numerical computations show that the condition $E[U_1|V_1 > v] \leq E[U_2|V_2 > v]$ is verified for all v so that we have $(U_1, U_2) \preceq_{\text{uo-cx}} (U_1, U_2)$ in this particular case. For $\alpha = 0.2$, Figure 3.1 displays the graph of $\psi_\alpha(x,y) = h_\alpha(x+y) - h_\alpha(x) - h_\alpha(y)$. We see there that their exists a certain level curve $f(x)$ given in Figure 3.2 such that $\psi_\alpha(x,y)$ is negative (h_α subadditive) under $f(x)$ and positive over $f(x)$ ($-h_\alpha$ subadditive). Note that the level curve $x \mapsto f(x)$ is unique and can be obtained as the solution of the implicit equation $(1-x^\alpha)(1-f(x)^\alpha) + (x+f(x))^\alpha - 1 = 0$.

4 Discussion

In this paper, we have established sufficient conditions for the orthant convex order to hold between bivariate distribution functions with special emphasis to archimedean copulas. The conditions derived in this paper are easy to verify (at least numerically) and are satisfied by standard copulas including Clayton and Frank families for some values of the dependence parameter.

Of course, there are situations where the sufficient condition derived in the present paper does not apply. Consider for instance the Fréchet copulas $C_\alpha(u,v) = \alpha \max\{u+v-1, 0\} + (1-\alpha)\min\{u,v\}$. To compare C_α to the independence copula $\Pi(u,v) = uv$, consider the difference $D_\alpha(u,v) = C_\alpha(u,v) - \Pi(u,v)$ and define the $\mathcal{A} = \{(u,v) \in [0,1]^2 : u+v-1 \leq 0\}$.

One can see that $D_\alpha(u,v) = \min\{u,v\}\left(1-\alpha-\max\{u,v\}\right)$ if $(u,v) \in \mathcal{A}$ and $D_\alpha(u,v) = \left(1-\max\{u,v\}\right)\left(\min\{u,v\}-\alpha\right)$ if $(u,v) \in \bar{\mathcal{A}}$ (the complement of $\mathcal{A}$ in $[0,1]^2$). Take $\alpha > 1/2$, then one has $D_\alpha(u,v) \geq 0$ if $(u,v) \in \mathcal{S}_{1,\alpha} = [0, 1-\alpha] \times [0, 1-\alpha]$ and $D_\alpha(u,v) \leq 0$ in $\mathcal{A} \setminus \mathcal{S}_{1,\alpha}$

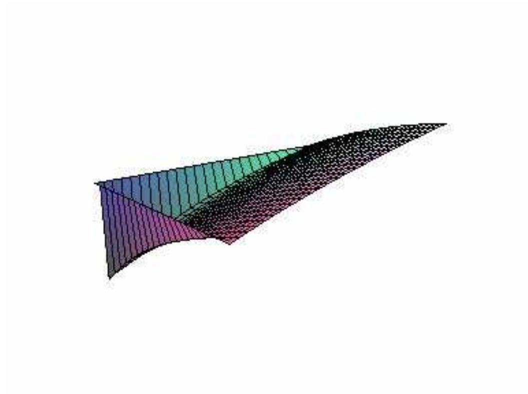

Figure 3.1: Graph of $\psi_\alpha : (x,y) \mapsto h_\alpha(x+y) - h_\alpha(x) - h_\alpha(y)$

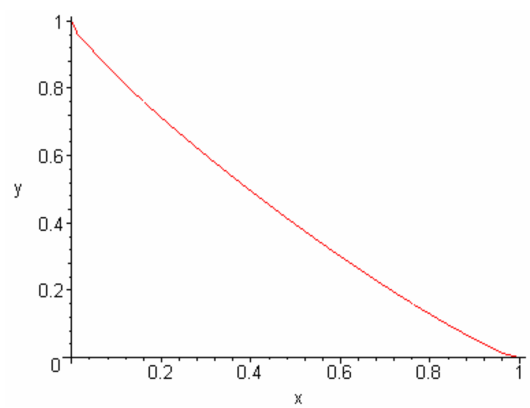

Figure 3.2: Graph of the level curve $x \mapsto f(x)$

(notice that $\mathcal{S}_{1,\alpha} \subset \mathcal{A}$, since $\alpha > 1/2$). Also, $D_\alpha(u,v) \geq 0$ if $(u,v) \in \mathcal{S}_{2,\alpha} = [\alpha,1] \times [\alpha,1]$ and $D_\alpha(u,v) \leq 0$ in $\bar{\mathcal{A}} \setminus \mathcal{S}_{2,\alpha}$ (notice that $\mathcal{S}_{2,\alpha} \subset \bar{\mathcal{A}}$, since $\alpha > 1/2$). Finally, we have

$$\begin{cases} D_\alpha(u,v) \geq 0 & \text{for all} \quad (u,v) \in \mathcal{S}_{1,\alpha}, \\ D_\alpha(u,v) \leq 0 & \text{for all} \quad (u,v) \in [0,1]^2 \setminus \mathcal{S}_{1,\alpha} \cup \mathcal{S}_{2,\alpha}, \\ D_\alpha(u,v) \geq 0 & \text{for all} \quad (u,v) \in \mathcal{S}_{2,\alpha}. \end{cases} \tag{4.1}$$

In this case, we thus see that there are two level curves $L_{1,\alpha} = \{(u, 1-\alpha), u \in [0, 1-\alpha]\} \cup \{(1-\alpha, v), v \in [0, 1-\alpha]\}$ and $L_{2,\alpha} = \{(u,\alpha), u \in [\alpha,1]\} \cup \{(\alpha, v), v \in [\alpha, 1]\}$ such that $D_\alpha(u,v)$ is positive under $L_{1,\alpha}$, negative between $L_{1,\alpha}$ and $L_{2,\alpha}$ and positive over $L_{2,\alpha}$. This shows that we cannot establish a comparaison between $C_\alpha(u, v)$ and $\Pi(u, v) = uv$ with respect to the upper orthant convex order by means of our sufficient condition.

This example also suggests to derive similar sufficient conditions for the weaker bivariate s-increasing convex order (see Denuit and Mesfioui (2010) for a definition). The extension of the crossing condition to these weaker stochastic orderings is under investigation.

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