Aspects of Statistical Analysis in DEA-Type Frontier Models*

by

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Abstract

In Grosskopf (1995) and Banker (1995) different approaches and problems of statistical inference in DEA frontier models are presented. This paper focuses on the basic characteristics of DEA models from a *statistical* point of view. It arose from comments and discussions on both papers above. The framework of DEA models is deterministic (all the observed points lie on the same side of the frontier) nevertheless a stochastic statistical model can be constructed once a data generating process is defined. So statistical analysis may be performed and sampling properties of DEA estimators can be established. However, practical statistical inference (like test of hypothesis, confidence intervals,...) still needs artifacts like the bootstrap to be performed. A consistent bootstrap relies also on a clear definition of the data generating process and on a consistent estimator of it: the approach of Simar and Wilson (1995) is described. Finally, some trails are proposed for introducing stochastic noise in DEA models, in the spirit of the Kneip-Simar (1995) approach.

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1 Introduction

The papers of Grosskopf (1995) and Banker (1995) are very important and contribute both to develop reflections on the crucial but hard problem of how to conduct statistical inference in DEA models.

A nice job is done in Grosskopf's survey. It was not an easy task to summarize the different approaches of statistical inference in nonparametric efficiency analysis: the result is good, the survey is stimulating and raises good questions for future research.

Banker's paper devotes attention to one important aspect of statistical analysis: hypotheses testing. The paper is a summary of his own several papers on the subject adapting a very good simple argument to various testing problems. He then analyzes their performance through Monte-Carlo simulations. His ideas are very original and will certainly motivate new developments although their practical uses have still to be refined (see below).

This paper is organized as follows: section 2 presents the basic features of so-called deterministic frontiers, showing clearly that a statistical (stochastic) model for the deterministic frontiers can be defined once the Data Generating Process (DGP) is correctly provided. A way of defining a DGP in the DEA framework is proposed in section 3. A statistical analysis can then be conducted, but due to the complexity of the process, the bootstrap seems to be, till now, an attractive solution to approximate the desired inference. In particular, it provides an alternative to Banker's approximations. Section 4 then proposes some trails for introducing stochastic frontiers in the framework of DEA techniques and clarifies the differences between two proposed "two-step" procedures quoted in Grosskopf's survey.

2 Are DEA models deterministic?

The DEA type models are extensions of the very simple ideas developed in Farrell (1957). They were popularized in a modern (OR) fashion by Charnes, Cooper and Rhodes (1978) and extended to possible nonconvex attainable sets by Deprins, Simar and Tulkens (1984) (the FDH models). All those models are often classified in the literature as being (nonparametric) **deterministic** models as opposed to

econometric models as if the former had no statistical backgrounds*. This impression is confirmed by the fact that, till now, many of the applications using DEA provided only point estimates of the efficiency measures.

However this way of looking at deterministic models is certainly too narrow and the survey of Grosskopf (1995) and the paper of Banker (1995) will certainly contribute to change this (general) state of mind. It is now clear that statistical inference (in the sense of a stochastic appreciation of statistical estimators) is possible in deterministic frontiers although it might be still complicated to perform. In a certain sense the word "deterministic" is inappropriate, although I will continue to use it to avoid confusion.

Deterministic is in fact used to characterize the property that all the observations generated by some DGP lie on one side of the frontier. It is a very old idea whose main interest lies in its apparent simplicity for estimating the models.

Let us illustrate that first in a very simple model: this will be useful to understand the more complex case of section 3. In the output oriented case, if the deterministic frontier is parametric and if the output is one dimensional, the model is easily defined as *e.g.* in Greene (1980)

$$y_i = x_i'\beta - t_i \quad i = 1, \ldots, n \tag{1}$$

where $y_i \in \mathbb{R}$ is the output, $x_i \in \mathbb{R}^p$ is the vector of inputs and $t_i \geq 0$ denotes the random (one-sided) departure from the frontier. This term t_i represents a combination of random elements (including eventually some elements of noise) which explain why an observed firm is below the frontier and so is apparently inefficient.

Clearly, in Greene's approach, statistical inference can be done within the limit of the statistical model (1) which precludes that some observations could be above the frontier. Once a particular probability law is chosen for t_i (say $F(t)$) the DGP is completely defined: conditional on the observed x_i, the firm i produces a random level of production y_i at a random distance t_i from the frontier $x_i'\beta$. The unknown parameters of the models are (β, F) to be estimated from

*The amusing thing is that the pioneer paper of Farrell appeared in one of the most famous statistical journals.

the sample $\{(x_i, y_i),\ i = 1, \ldots, n\}$. The DGP is thus completely characterized by (β, F).

The basic problem is that, due to a lack of modeling and identification, the estimated values of $\hat{t}_i$ are used as a first proxy for estimating the inefficiency of the firms. Thus the error term is in fact interpreted as capturing inefficiency only, as if no noise or no other random elements were present. The stochastic model of Aigner, Lovell and Schmidt (1977) provides a solution to this: a random term $\epsilon_i \in \mathbb{R}$ capturing noise is convoluted with t_i capturing inefficiency[†].

In a nonparametric setup, it may be much more delicate to define the DGP but this can be done (see the next section). The construction will be in the spirit of what has been done for the simple model (1) but taking into account the multidimensional and nonparametric framework of the DEA. The DGP will again be characterized by a set of (functional) unknown parameters: the stochastic nature of the sample can be defined. So, conceptually, in "deterministic" nonparametric frontier models, statistical inference can be conducted. However, due to the functional nature of the parameters, the statistical properties of the estimators are still hard to derive.

As pointed out above, the **basic issue**, and the basic limitation, of the deterministic frontier (parametric or not), is the one sided nature of the residuals. This point cannot be skipped and should never be forgotten. In some situations this may be realistic (when *e.g.* variables are averages over several periods and so, as a first approximation, the noise can be considered as being averaged out) but in many situations this is not natural (randomness of economic behavior, noise, errors of measurement on the variables, ...). The use of so-called deterministic models may be misleading in such cases. In particular, the presence of possible outliers should lead the researcher, either to identify and eliminate them (see Grosskopf's survey for references), or use stochastic frontier models if they cannot be identified[‡].

But this basic issue is not concerned with statistical inference, it is

[†]This is the elegant way to introduce noise but it raises identification problems even in the simple parametric setup and complicates somewhat the mathematics of the analysis (it is well known that a convolution of a real random variable with a nonnegative one is hard to handle analytically).

[‡]A procedure proposed in section 4 may be viewed as one way to clean-up the data for noise and for outliers.

the issue of chosing the appropriate model for analyzing a problem[§]. Once a "deterministic" model has been chosen, the estimates can be analysed by statistical tools.

3 A data generating process for DEA

No statistical inference can be conducted without a clear description of a statistical model which contains the description of the DGP[¶]. This will be particularly important in this framework since we often will need to use the bootstrap to approximate the desired sampling distributions. Those approximations are indeed obtained by mimicing the DGP; it may thus be very hazardous (and even inconsistent) to use bootstrap if this DGP is not clearly defined.

In the case of a one dimensional output (in the output efficiency oriented case) this is not too complicated[‖]. Banker (1993) introduces the randomness of the data exactly as in model (1) above except that now the DEA frontier is characterized by some unknown function $g(x)$ sharing some properties (concavity, monotonicity, ...). Roughly speaking we have:

$$y_i = g(x_i) - t_i, \quad t_i \geq 0 \tag{2}$$

with some restriction on $F(t)$. Under this statistical model, observation of the sample (x_i, y_i) $i = 1, \ldots, n$ allows one to make inference on the unknown functional parameters (g, F). The DGP is now completely characterized by (g, F).

Banker shows that the DEA estimator $\hat{g}$ is the MLE of the unknown g. He proves also its consistency.

Korostelev, Tsybakov and Simar (1995a) and (1995b) consider a related but more general setup where $(y, x^1, \ldots, x^p)$ are distributed according some unknown d.f. $H(y, x^1, \ldots, x^p)$ which gives probability

[§]The confusion there explain maybe why DEA models have occasionally been criticized by some econometricians.

[¶]See Kneip and Simar (1995), section 2.1, for an exhaustive description of a statistical model (including the DGP) in a very general setup of stochastic frontiers with panel data.

[‖]The same could be said for a one dimensional input in the input efficiency oriented case.

one to the set $y \leq g(x^1, \ldots, x^p)$. Here the DGP is completely characterized by (g, H). Again, under restricted models for g (concavity and monotonicity or monotonicity alone), DEA and FDH estimators $\hat{g}$ are the MLE of g. However the achieved rates of convergence, although optimal, are very low as p increases (*e.g.* of order $\mathcal{O}_p(n^{-\frac{2}{2+p}})$ for DEA, and of order $\mathcal{O}_p(n^{-\frac{1}{1+p}})$ in the less restricted case of FDH)**.

It is a little more complicated to define carefully the DGP in the more general multi-input and multi-output case considered by DEA models. Simar and Wilson (1995) propose to extend the ideas of the simplest model (1) taking Farrell's ideas of efficiency into account. Consider for instance the input efficiency oriented case with p inputs and q outputs. The process may be summarized as follows. Let Ψ be the production set of physically attainable points (x, y):

$$\Psi = \{(x, y) \in \mathbb{R}_+^{p+q} \mid x \text{ can produce } y\} \tag{3}$$

The input requirement set $X(y)$ is defined $\forall\ y \in \Psi$ by:

$$X(y) = \{x \in \mathbb{R}_+^p \mid (x, y) \in \Psi\} \tag{4}$$

The Farrell input efficient boundary is denoted by $\partial X(y)$, it is the minimal attainable boundary in a radial sense:

$$\partial X(y) = \{x \mid x \in X(y), \theta x \notin X(y),\ \forall\ 0 < \theta < 1\} \tag{5}$$

and thus, for a given point (x_0, y_0), its input efficiency is defined as:

$$\theta_0 = \min\{\theta \mid \theta x_0 \in X(y_0)\} \tag{6}$$

The efficient level of input corresponding to the output level y_0 is then given by:

$$x^{\partial}(y_0) = \theta_0 x_0 \tag{7}$$

Now a DGP can be described as follows: for a given value of y, we know that $x \in X(y)$. Due to the presence of inefficiency, x may not be equal to $x^{\partial}(y)$, but is generated along a fixed ray (*i.e.* with a fixed

**These low rates of convergence for large p are specific to nonparametric techniques because the parameter space is a functional space: it is the "curse of dimensionality" (see Kneip and Simar (1995) for more details).

proportion of inputs) away from the frontier. Therefore a particular unit (x_i, y_i) may be considered as being generated, conditionally on y_i and on the observed proportion of inputs by the random variables $\theta_i \in [0,1]$ such that $x_i = x^{\partial}(y_i)/\theta_i$. Note that, *mutatis mutandis*, θ_i plays exactly the same role as t_i in the simple model (1) above. If the θ_i are distributed according to some probability law $F(\theta)$, the DGP $\mathcal{P}_i$, generating x_i conditionally on the observed value of y_i and on the observed proportion of inputs is completely characterized by $x^{\partial}(y_i)$ and F:

$$\mathcal{P}_i = (x^{\partial}(y_i), F). \tag{8}$$

The whole DGP is now completely characterized by $\mathcal{P}_i$; $i = 1, \ldots, n$.

The set $\mathcal{P}_1, \ldots, \mathcal{P}_n$ has to be estimated, as in all statistical models, from a sample (x_i, y_i) $i = 1, \ldots, n$. In particular, if the DEA estimator is used, $\widehat{x^{\partial}(y_i)} = \hat{\theta}_i x_i$ where $\hat{\theta}_i$ is the input efficiency estimate obtained by the usual linear program:

$$\hat{\theta}_i = \min\{\theta | y_i \leq \sum_{k=1}^{n} \gamma_k y_k; \theta x_i \geq \sum_{k=1}^{n} \gamma_k x_k; \theta > 0;$$
$$\sum_{k=1}^{n} \gamma_k = 1; \gamma_k \geq 0, k = 1, \ldots, n\}$$

Finally, an estimator $\hat{F}$ of F might be provided by the empirical d.f. of $\hat{\theta}_i$, $i = 1, \ldots, n$. It is known that this is a poor estimator of F on the boundaries of its domain $[0,1]$; it will be improved in order to build a consistent bootstrap (see below).

From a conceptual point of view, we are now in a position to conduct statistical inference on the unknown quantities of the model (sampling distribution of the efficiency estimates $\hat{\theta}_i$, tests of hypotheses on θ_i, test on F, ...).

However, from a practical point of view, the sampling distributions of the obtained estimators are unknown. As far as consistency is concerned, the results of Banker and of Korostelev, Simar and Tsybakov have still to be extended to this general multidimensional case. In particular, the speed of convergence of $\hat{\theta}_i$ to θ_i have not yet been investigated although, as a conjecture, the results of Korostelev, Simar and Tsybakov (1995a and b) could be applied.

Some attempts have been proposed for making inference on the efficiency measures θ_i. For instance, in Banker (1995)[††], a procedure is proposed for testing hypotheses on the efficiency measures. Three types of problems are analyzed: testing the equality of the mean efficiency of two subgroups of firms, testing hypotheses on returns to scale and testing for input substitutability. The test statistics proposed in each problem are very attractive and certainly intuitively appealing.

Consider for example the comparison of the mean efficiency of two groups of firms. The test statistic is roughly a (rescaled) ratio of the within group averages of the estimates $\hat{\theta}_i$. Under the null (no difference between the two groups) we expect indeed a standard value for this statistic (like 1). The problem is that due *a.o.* to the dependence structure of the estimators $\hat{\theta}_i$, it is almost impossible to derive their exact sampling distribution under the null hypotheses, and so no critical values or $p-$values can be computed. This is true even in the simpler cases, considered by Banker, where F belongs to some known parametric family (exponential or half normal).

Banker argues, comparing the results of a great variety of simulation scenarios, that the sampling distributions of the (appropriately rescaled) test statistics can be approximated by well known usual distributions (like χ^2 or F distributions in some cases), and that the procedure seems to behave better than some other appropriate parametric tests. Unfortunately, the Monte-Carlo simulations of Banker (1995) do not provide any support to this. Indeed, the number of replications of the Monte-Carlo is definitely too small to draw any conclusions of the quality of the approximations or on the virtues of the procedure (comparisons of type I and type II errors are based on comparisons of proportions of successes based only on 10 experiments in the worst case and on 30 in the best case)[‡‡].

Nevertheless, the ideas are very attractive and some work has still to be done to show the practical usefulness of the procedures.

As pointed out above, in this complex framework, sampling distri-

[††]Note that Banker does not make explicit the DGP in terms of (x, y) but concentrates his attention on $F(\theta)$ only. The description of the DGP above is thus implicit in Banker's approach which is restricted to one-dimensional output.

[‡‡]See also Kittlesen (1995) for more substantial Monte-Carlo experiments confirming my fears on the validity of the approximations.

butions are not easily obtainable: this is a situation where the bootstrap can help. In Simar and Wilson (1995), a procedure is proposed to conduct, in a consistent way, the bootstrap. The basic idea is to construct a consistent estimator of the DGP (a crucial step) and then to draw random samples from this estimated DGP in order to mimic the original unknown DGP. As pointed in Grosskopf's survey, many have used the bootstrap to analyse the sensitivity of the efficiency measures to sampling variations. Unfortunately, in many cases, the DGP is not clearly defined and so it is not clear which process the bootstrap is mimicing; consequently the proposed approximations are very hazardous. Moreover, in most cases, inconsistent estimators of the distribution F are used, so the bootstrap is inconsistent.

In Simar and Wilson (1995), both issues are addressed. The ideas can be summarized as follows: the DGP $\mathcal{P}_1, \ldots, \mathcal{P}_n$ has to be defined as above, a consistent estimator of the frontier is provided by the DEA estimator, and a nonparametric kernel density estimator $\hat{F}$ has to be used for F, (it is a smoothed version of the empirical distribution of the $\hat{\theta}_i$, $i = 1, \ldots, n$ corrected by a reflection method, in order to provide a consistent estimator of F over its domain $[0, 1]$ and particularly at the upper bound 1).

From a practical point of view, a pseudo-sample (x_i^*, y_i^*); $i = 1, \ldots, n$ is generated as follows: starting from the original cloud of points, we estimate (consistently by DEA) the frontier. This gives a set of "estimated efficient" points $(x^{\widehat{\partial(y_i)}}, y_i)$ $i = 1, \ldots, n$, which will stay fixed during all the process (they provide the estimated frontier). The DGP says that the sample is generated by **radial** random deviation (characterized by F) from the unknown frontier; the pseudo-sample is thus generated according to the estimated DGP in the same way:

$$(x_i^*, y_i^*) = (\frac{x^{\widehat{\partial(y_i)}}}{\theta_i^*}, y_i), \tag{9}$$

where θ_i^* is generated from the consistent estimator $\hat{F}$.

We can now, for one given pseudo-sample (x_i^*, y_i^*); $i = 1, \ldots, n$, compute the corresponding value $\hat{\theta}_i^*$ by using the same DEA method. The usefulness of the bootstrap appears in the following key approxi-

mation:

$$\hat{\theta}_i^* - \hat{\theta}_i \mid \hat{\mathcal{P}}_1, \ldots, \hat{\mathcal{P}}_n \sim \hat{\theta}_i - \theta_i \mid \mathcal{P}_1, \ldots, \mathcal{P}_n. \tag{10}$$

The right-hand side of (10) is the desired unknown quantity, in the left-hand side everything is observable: it provides the bootstrap approximation to the sampling distribution.

For practical purposes, it is easily obtained by repeating the whole process (*i.e.* drawing pseudo-samples (x_i^*, y_i^*) $i = 1, \ldots, n$ and computing $\hat{\theta}_i^*$) a very large number of times: the empirical d.f. of $\hat{\theta}_i^*$ over the repetitions provides an estimator of the left-hand side of (10). This algorithm is easy to implement and requires reasonable computer time (a few minutes to a few hours depending on the size of the problem: see Wilson and Simar (1995) for a large set of applications, and for a discussion on the consistency of (10)).

This idea can also be used to simulate the sampling distribution of any regular statistics computed from $\hat{\theta}_i$. Wilson and Simar (1996) show how to adapt the bootstrap algorithm in order to solve some testing problems.

In conclusion, DGPs can be defined in a DEA context; statistical properties of DEA have been established, but practical statistical computations need still to use artifacts like the bootstrap.

4 How to use DEA in a stochastic frontier framework?

As pointed in section 2, there are situations where pure deterministic frontiers are not natural, or at least, situations where we would be more robust w.r.t. possible outliers. We would then analyze stochastic nonparametric models. In the case of a one dimensional output, it could be written as:

$$y_i = g(x_i) + \varepsilon_i - t_i; \tag{11}$$

where $\varepsilon_i \in \mathbb{R}$ represents the noise, $t_i \in \mathbb{R}^+$ is the inefficiency and g is the production frontier. The aim of this section is to discuss two "two-step procedures" mentioned in Grosskopf's survey allowing to mix DEA and noise, in order to clarify the differences between the two approaches. We propose also some trails for future research in that direction.

A first two-step procedure suggested by Thiry and Tulkens (1992) is provided in Simar (1992) as a "semiparametric" approach. The main objective there is to obtain a **parametric** frontier representing the best practice production function of efficient production units. This goal is achieved by deleting, from the initial sample, and in a nonparametric way, all the production units which are "clearly" inefficient. The tool used for this first filtering step is the FDH technique (or the DEA, although the latter relies on more stringent assumptions). Then, on the remaining sample points, a standard procedure (OLS, ...) is used to estimate the parametric production frontier.

Clearly, this first two-step procedure is only valid in a "**deterministic**" framework and **does not** provide a stochastic version of DEA. If the real DGP implies some two-sided noise, the probability of observing an outlier tends to one as the sample size increases. So this procedure is **not valid** (inconsistent) in the case of an implicit "stochastic" frontier model like (11). However, this procedure remains valid in a "deterministic" framework where a parametric function is wanted to model the best practice frontier.

It is clear that the filtering step remains, as for any deterministic method, sensitive to outliers. One way to alleviate the role of possible outliers is provided by Simar (1992). It is there recommended to keep for the second step procedure all the firms achieving at least 0.90 (or 0.95) efficiency levels.

This way of robust nonparametric filtering (using $p-$percent efficient units) could be extended in a more probabilistic setup by using new results of Fan, Hu and Truong (1994). They propose a nonparametric percentile curve estimation. For a given p (say $p = 0.90$ or 0.95), the $p-$percentile curve $f_p(x)$ is defined as $P(y \leq f_p(x)|x) = p$. The paper proposes a nonparametric estimation of $f_p(x)$.

Another two step procedure is suggested by Kneip and Simar (1995), and looks at the problem the other way around. The idea is to estimate the frontier using DEA type estimators but with a DGP allowing for noise. Since DEA is deterministic, the noise, in a first step, has to be averaged out. One way to achieve this is to use a robust method, *e.g.* a nonparametric smoothing technique.

Indeed, those techniques provide fitted values $\hat{y}_i$ of the output for given x_i, $i = 1, \ldots, n$ and, roughly speaking, the fitted values $\hat{y}_i$ are

the values of y_i given by (11), after cleaning the noise ϵ_i out.

Then, keeping the cloud of points $(x_i, \hat{y}_i)$; $i = 1, \ldots, n$, we are back to a "deterministic" framework. So that, in a second step, the frontier may be estimated by using DEA techniques. The problem still remains of how to chose the nonparametric smoothing technique: Kneip and Simar (1995) propose a procedure based on a kernel smoother in the presence of panel data. This technique could be adapted to more general data structures.

The procedure is an attempt for proposing the use of DEA techniques in a real stochastic frontier framework. However the statistical properties of the procedure have still to be explored. As a conjecture, I believe that the rates of convergence for this stochastic DEA model will be of the same order those obtained in the pure deterministic case (*i.e.* $\mathcal{O}_p(n^{-\frac{2}{p+q+1}})$). I expect that, very soon, new results will give the theoretical background to this appealing stochastic DEA approach. Once the consistency is proved, an (adapted) bootstrap algorithm could serve as a basis for analyzing the sampling variations of the obtained estimators.

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