MONOTONICITY RESULTS FOR PORTFOLIOS WITH HETEROGENEOUS CLAIMS ARRIVAL PROCESSES

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Abstract

Loosely speaking, actuaries believe that the heterogeneity of the risks tends to increase dangerousness. This is turn leads to requiring more economic capital. This paper aims to formalize this intuitive idea. More specifically, vectors of compound sums will be considered, with different claim frequency distributions and/or different claim severity distributions. The effect of increasing the heterogeneity will be studied with the help of majorization, allowing for comparing the dispersion of the components of two vectors of real numbers. Various multivariate integral stochastic orderings will be used to compare situations according to their level of heterogeneity.

Key words and phrases: Majorization, Schur-increasingness, univariate and multivariate stochastic orders, risk measures.

1 Introduction and motivation

Consider an insurance company with n lines of business. The number of claims of business i in a certain period of time (one year, say) is a random variable N_i, where the random variables $N_1, N_2, \ldots, N_n$ are independent. Claim amount of business i is represented as a compound sum $X_i = \sum_{j=1}^{N_i} C_{i,j}$ where the claim severities $C_{i,j}$ are independent and identically distributed, and independent of N_i. Moreover, $X_1, X_2, \ldots, X_n$ are assumed to be independent. Note that we could also consider N_i as the number of claims filed by policyholder i, the $C_{i,j}$'s being the corresponding claim costs. In this case, X_i is the total amount of claim for policy i.

Assume that the distribution of N_i belongs to some parametric family, that is, $\Pr[N_i \leq t] = F_{\theta_i}(t)$, $t \in \mathbb{R}$, $i = 1, \ldots, n$, where $\mathcal{F} = \{F_\theta,\ \theta \in \Theta\}$ is a given parametric family of distribution functions. Henceforth, we will sometimes denote N_i as N_{θ_i} to emphasize the dependence on the parameter θ_i.

Now, consider another portfolio, where the claim amount of business i is $Y_i = \sum_{j=1}^{N_{\gamma_i}} D_{i,j}$ where N_{γ_i} has distribution function F_{γ_i} in $\mathcal{F}$. If we assume that the $C_{i,j}$'s and $D_{i,k}$'s are identically distributed then the two portfolios differ in the vector parameters $\boldsymbol{\theta} = (\theta_1, \cdots, \theta_n)$ and $\boldsymbol{\gamma} = (\gamma_1, \cdots, \gamma_n)$ of claim counts. The more dispersed the θ_i's, the more heterogeneous the portfolios. More precisely, switching from $\boldsymbol{\theta}$ to $\boldsymbol{\gamma}$ larger in the sense of majorization, that is increasing the heterogeneity, leads to an increase in dangerousness, as it will be demonstrated in this paper. The dangerousness will be measured with the aid of reinsurance premium (associated with stop-loss and largest claims reinsurance treaties) and economic capital computed on the basis of classical risk measures (like TVaR and other coherent Yaari-Wang distorted expectations).

Let us keep assumption that the $C_{i,j}$'s and $D_{i,j}$'s are identically distributed for all the values of i and j. If $\mathcal{F}$ is closed under convolution (or satisfies the semi-group property), in the sense that the convolution $F_{\theta_1} \star F_{\theta_2}$ of F_{θ_1} and F_{θ_2} is equal to $F_{\theta_1+\theta_2}$, then it is clear that

$$\sum_{i=1}^n \theta_i = \sum_{i=1}^n \gamma_i \Rightarrow \sum_{i=1}^n X_i =_d \sum_{i=1}^n Y_i,$$

where $=_d$ stands for the equality in distribution and means "is distributed as". Provided $\sum_{i=1}^n \theta_i = \sum_{i=1}^n \gamma_i$, the heterogeneity has thus no effect on the aggregate claim amount. This is the case for instance with Poisson distributed claim frequencies. In such a case, the heterogeneity has no effect on the amount of economic capital. Nevertheless, we will see that it does matter if we do not compare aggregate claims, but well functions $f : \mathbb{R}^n \to \mathbb{R}$ of the X_i's. In that respect, we will establish various multivariate stochastic inequalities between the random vectors $\boldsymbol{X}$ and $\boldsymbol{Y}$.

We will also consider the case where the $C_{i,j}$'s are identically distributed for fixed i, but are allowed to have different distributions for different values of i. Then the distributional equality between $\sum_{i=1}^n X_i$ and $\sum_{i=1}^n Y_i$ no more holds and the heterogeneity will affect the amount of economic capital and/or reinsurance premiums.

The paper is organized as follows. In Section 2, we recall the definition and some basic properties of the tools used in this paper. The concept of majorization, central in this paper, is related to several classes of functions $\mathbb{R}^n \to \mathbb{R}$. These functions generate multivariate stochastic orderings that appear to be useful in risk theory. Section 3 considers claim severities $C_{i,j}$ identically distributed for all i and j. The heterogeneity is at the frequency level. The Poisson case is first considered, because of its importance in actuarial applications. Then, other count distributions with the semi-group property are examined. Section 4 allows for heterogeneity at the severity level. More precisely, the $C_{i,j}$'s are now assumed to be independent and identically distributed for fixed i, but may have different distributions for different values of i. The claim frequencies for the two portfolios are now identically distributed. Two cases are discussed: first, $C_{i,j}$'s with the semi-group property and then $C_{i,j}$'s in the same location-scale family. Section 5 considers that the heterogeneity is present at both the frequency and the severity levels. To this end, a bivariate version of majorization will be needed. The final Section 6 concludes.

To end with, let us stress the original point of view developed in this paper. Many papers have recently been devoted to the increase in economic capital induced by the positive dependence between the X_i's. The main result is that the more the X_i's are positively dependent (in the sense of supermodular order, for instance), the more capital the insurer needs (for any risk measure in agreement with the stop-loss order). In this paper, we keep the mutual independence assumption about the X_i's, and we examine the impact of the heterogeneity of the portfolio on the economic capital required for the insurer or on various reinsurance treaties. Departing from the homogeneous case turns out to be as dangerous as departing from mutual independence.

2 Majorization, related classes of functions and integral stochastic orders

2.1 Majorization

In this section, we describe the concept of majorization arising as a measure of diversity of the components of a n-dimensional vector. This concept will be used to compare $\boldsymbol{\theta}$ and $\boldsymbol{\gamma}$ involved in the distributions of the claim counts. Majorization has been comprehensively treated by Marshall & Olkin (1979); a brief introduction is given by Arnold (1987) and Denuit et al. (2005).

We aim to formalize the idea that the components of a vector $\boldsymbol{x}$ are "less spread out" or "more nearly equal" than the components of $\boldsymbol{y}$. In many cases, the appropriate precise statement is "$\boldsymbol{y}$ majorizes $\boldsymbol{x}$". Majorization is a partial order defined on the positive orthant $\mathbb{R}^n_+$. For a vector $\boldsymbol{x} \in \mathbb{R}^n_+$ we denote its elements ranked in descending order as

$$x_{(1:n)} \geq x_{(2:n)} \geq \ldots \geq x_{(n:n)}.$$

Thus $x_{(1:n)}$ is the largest of the x_i's, while $x_{(n:n)}$ is the smallest.

Definition 2.1. Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n_+$. The vector $\boldsymbol{y}$ is said to majorize $\boldsymbol{x}$, which is denoted as $\boldsymbol{x} \preceq_{\text{maj}} \boldsymbol{y}$, if

$$\sum_{i=1}^{k} x_{(i:n)} \leq \sum_{i=1}^{k} y_{(i:n)} \text{ for } k = 1, 2, \ldots, n-1 \text{ and } \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$$

or, equivalently, if

$$\sum_{i=k}^{n} x_{(i:n)} \geq \sum_{i=k}^{n} y_{(i:n)} \text{ for } k = 1, 2, \ldots, n \text{ and } \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i.$$

2.2 Schur-increasingness and related properties

Let us now recall several properties for functions $f : \mathbb{R}^n \to \mathbb{R}$. To this end, remember that an $n \times n$ matrix $\mathbf{\Pi}$ is said to be a permutations matrix if each row and column has a single unit, and all other entries are zero. For a vector $\boldsymbol{x} \in \mathbb{R}^n$, the vector $\mathbf{\Pi}\boldsymbol{x}$ is obtained by a permutation of the elements of $\boldsymbol{x}$.

Definition 2.2. Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$, and f be a function $\mathbb{R}^n \to \mathbb{R}$:

(i) f is increasing if $\boldsymbol{x} \leq \boldsymbol{y}$ coordinatewise, implies $f(\boldsymbol{x}) \leq f(\boldsymbol{y})$; f is decreasing if $-f$ is increasing.

(ii) f is convex if for all $\alpha \in [0, 1]$, the inequality $f(\alpha\boldsymbol{x} + (1-\alpha)\boldsymbol{y}) \leq \alpha f(\boldsymbol{x}) + (1-\alpha)f(\boldsymbol{y})$ holds. This condition is equivalent to the Hessian matrix $(\partial^2 f/\partial x_i \partial x_j)$ being positive semidefinite if f has second partial derivatives. f is concave if $-f$ is convex.

(iii) f is convex in each variable, or componentwise convex, if the function $f_i : \mathbb{R} \to \mathbb{R}$ defined by

$$f_i(\xi) = f(x_1, \ldots, x_{i-1}, \xi, x_{i+1}, \ldots, x_n)$$

is convex for every choice of $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n$ and for each $i = 1, \cdots, n$. If f has second partial derivatives this condition is equivalent to $\partial^2 f/\partial x_i^2 \geq 0$ for each $i = 1, \ldots, n$. f is componentwise concave if $-f$ is componentwise convex.

(iv) f is supermodular if for any $1 \leq i < j \leq n$ and any nonnegative ε, δ the following inequality holds:

$$\begin{aligned} & f(x_1, \cdots, x_i + \varepsilon, \cdots, x_j + \delta \cdots, x_n) + f(x_1, \cdots, x_i, \cdots, x_j, \cdots, x_n) \\ \geq \; & f(x_1, \cdots, x_i + \varepsilon, \cdots, x_j, \cdots, x_n) + f(x_1, \cdots, x_i, \cdots, x_j + \delta \cdots, x_n). \end{aligned}$$

The condition is equivalent to $\partial^2 f/\partial x_i \partial x_j \geq 0$ for all $i \neq j$ if f has second partial derivatives. f is submodular if $-f$ is supermodular.

(v) f is symmetric if for any permutation matrix $\mathbf{\Pi}$, $f(\boldsymbol{x}) = f(\boldsymbol{x}\mathbf{\Pi})$.

(vi) f is Schur-increasing if $f(\boldsymbol{x}) \leq f(\boldsymbol{y})$ for every pair $\boldsymbol{x}, \boldsymbol{y}$ for which $\boldsymbol{x} \preceq_{\text{maj}} \boldsymbol{y}$. f is Schur-decreasing if $-f$ is Schur-increasing.

Schur-increasing functions thus preserve majorization. Note that a Schur-increasing (decreasing) function must be a symmetric function in its arguments.

2.3 Characterization of majorization

Let us now consider the following classes of functions:

$\mathcal{C}_1 = \{f : \mathbb{R}^n \to \mathbb{R} | f \text{ is Schur increasing}\}.$

$\mathcal{C}_2 = \{f : \mathbb{R}^n \to \mathbb{R} | f \text{ is convex and symmetric}\}.$

$\mathcal{C}_3 = \left\{f : \mathbb{R}^n \to \mathbb{R} | f(\boldsymbol{x}) = \sum_{j=1}^n g(x_i) \text{ for some convex function } g \text{ on } \mathbb{R}\right\}.$

$\mathcal{C}_4 = \left\{f : \mathbb{R}^n \to \mathbb{R} | f(\boldsymbol{x}) = \sum_{i=1}^j x_{(i:n)} \text{ for some } j = 1, \ldots, n, \text{ or } f(\boldsymbol{x}) = -\sum_{i=1}^n x_{(i:n)}\right\}.$

$\mathcal{C}_5 = \{f : \mathbb{R}^n \to \mathbb{R} | f \text{ is symmetric, submodular and componentwise convex}\}.$

The first four classes have been introduced in Marshall & Olkin (1979, Chapter 11); class $\mathcal{C}_5$ has been introduced by Chang (1992). Marshall & Olkin (1979, p. 316) established the following chain of inclusions:

$$\mathcal{C}_1 \supset \mathcal{C}_2 \supset \mathcal{C}_3 \text{ and } \mathcal{C}_2 \supset \mathcal{C}_4. \tag{2.1}$$

Chang (1992) supplemented this result with

$$\mathcal{C}_3 \subset \mathcal{C}_5 \text{ and } \mathcal{C}_4 \subset \mathcal{C}_5. \tag{2.2}$$

The following result is stated and proved in Chang (1992).

Property 2.3. *For any $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$, $\boldsymbol{x} \preceq_{maj} \boldsymbol{y} \Leftrightarrow f(\boldsymbol{x}) \leq f(\boldsymbol{y})$ for all $f \in \mathcal{C}_1$ (resp. $\mathcal{C}_2$, $\mathcal{C}_3$, $\mathcal{C}_4$, $\mathcal{C}_5$).*

The concept of majorization can thus be alternatively defined with respect to any of the 5 classes of functions introduced above.

2.4 Univariate integral stochastic orders

2.4.1 Definition

Integral stochastic orders are defined with respect to some given classes of functions $\mathbb{R} \to \mathbb{R}$. More specifically, given a class $\mathcal{D}$ of functions $f : \mathbb{R} \to \mathbb{R}$, the integral stochastic order $\preceq_{\mathcal{D}}$ generated by the class $\mathcal{D}$ is defined as follows: having two random variables X and Y, X is smaller than Y in the $\preceq_{\mathcal{D}}$-sense if $\mathbb{E}[f(X)] \leq \mathbb{E}[f(Y)]$ for al the functions f in $\mathcal{D}$ for which the expectations exist. Integral stochastic orders $\preceq_{\mathcal{D}}$ have been studied, e.g., by Whitt (1986) and Marshall (1991) for various choices of $\mathcal{D}$. Stochastic dominance, stop-loss and convex orders can be obtained in this way, as it will be seen below.

This section gives the definitions of the univariate stochastic orderings used in this paper, as well as some intuitive interpretations. For more details about stochastic orderings, we refer the reader, e.g., to Denuit et al. (2005).

2.4.2 Stochastic dominance

Taking for $\mathcal{D}$ the class of the increasing functions yields the well-known stochastic dominance. More specifically, given two random variables X and Y, X is said to be smaller than Y in the stochastic dominance, written as $X \preceq_{\rm st} Y$, if the inequality $\mathbb{E}[f(X)] \leq \mathbb{E}[f(Y)]$ holds true for all the increasing functions f, provided the expectations exist. In the framework of von Neumann-Morgenstern expected utility theory, $\preceq_{\rm st}$ thus expresses the common preferences of all the profit-seeking decision-makers.

2.4.3 Stop-loss order

Taking for $\mathcal{D}$ the class of the increasing convex functions yields the increasing convex order, better known as the stop-loss order in the actuarial community. More specifically, considering two risks X and Y, X is said to be smaller than Y in the stop-loss order, henceforth denoted by $X \preceq_{\rm sl} Y$, if $\mathbb{E}[f(X)] \leq \mathbb{E}[f(Y)]$ holds for all the increasing and convex functions f for which the expectations exist. Intuitively, the claim amount X is thus "less risky" than Y, since all decision makers with non-decreasing and concave utility functions, i.e. those being profit seeking and risk averse, will prefer a loss of amount X to a loss of amount Y.

2.4.4 Convex order

Taking for $\mathcal{D}$ the class of the convex functions yields the convex order. This order can be seen as a strengthening of the stop-loss order, obtained by requiring in addition that the means of the risks to be compared are equal. More precisely, if X and Y are two risks, X is said to be smaller than Y in the convex order, henceforth denoted by $X \preceq_{\rm cx} Y$ (or sometimes by $X \preceq_{\rm sl,=} Y$ in the actuarial literature), if $\mathbb{E}[X] = \mathbb{E}[Y]$ and $X \preceq_{\rm sl} Y$. The term "convex" is used since $X \preceq_{\rm cx} Y \Leftrightarrow \mathbb{E}[f(X)] \leq \mathbb{E}[f(Y)]$ for all convex functions f for which the expectations exist, so that $\preceq_{\rm cx}$ expresses the common preferences of all the risk-averse decision-makers.

2.5 Multivariate integral stochastic orders

2.5.1 Definition

The definition of the multivariate integral stochastic orders is a direct extension of the univariate case, considering $\mathcal{D}$ as a class of functions $\mathbb{R}^n \to \mathbb{R}$. In this paper, we will consider the orderings obtained in this way with the classes $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$, $\mathcal{C}_4$ and $\mathcal{C}_5$ introduced above. More precisely, given two n-dimensional random vectors $\boldsymbol{X}$ and $\boldsymbol{Y}$, if

$$\mathbb{E}[f(\boldsymbol{X})] \leq \mathbb{E}[f(\boldsymbol{Y})] \text{ for all } f \in \mathcal{C}_i, \tag{2.3}$$

provided the expectations exist, then $\boldsymbol{X}$ is said to be smaller than $\boldsymbol{Y}$ in the $\preceq_{\mathcal{C}_i}$-order, which is denoted as $\boldsymbol{X} \preceq_{\mathcal{C}_i} \boldsymbol{Y}$. Applications of the $\preceq_{\mathcal{C}_i}$-orders in reliability can be found in Chang (1992).

The actuary can take advantage of the fact that a stochastic inequality of the type $\boldsymbol{X} \preceq_{\mathcal{C}_i} \boldsymbol{Y}$ holds true, as it is shown in the next couple of examples.

Example 2.4 (Excess-of-loss reinsurance and $\preceq_{\mathcal{C}_3}$). Let us assume that X_i represents the total claim amount for policy i, $i = 1, 2, \ldots, n$. If $\boldsymbol{X} \preceq_{\mathcal{C}_3} \boldsymbol{Y}$ is valid then the inequality

$$\sum_{i=1}^{n} \mathbb{E}[(X_i - d_i)_+] \leq \sum_{i=1}^{n} \mathbb{E}[(Y_i - d_i)_+] \tag{2.4}$$

holds true for every set of retentions $d_1, \ldots, d_n$. The pure premium for an excess-of-loss treaty with retention d_i for policy i will be higher for the insurer facing $\boldsymbol{Y}$ than for the insurer facing $\boldsymbol{X}$.

Example 2.5 (Largest claims reinsurance and $\preceq_{\mathcal{C}_4}$). Ammeter (1964) proposed the largest claims reinsurance treaty. In such a case, the reinsurer covers the claims for the j policies with the highest costs. The reinsurer's payout is then $\sum_{i=1}^{j} X_{(i:n)}$. If $\boldsymbol{X} \preceq_{\mathcal{C}_4} \boldsymbol{Y}$ is valid then the inequality

$$\sum_{i=1}^{j} \mathbb{E}[X_{(i:n)}] \leq \sum_{i=1}^{j} \mathbb{E}[Y_{(i:n)}] \tag{2.5}$$

holds true for any $j = 1, \ldots, n$.

We will see below that some of the $\preceq_{\mathcal{C}_i}$-orders are well-known in the literature.

2.5.2 Stochastic majorization: (2.3) with $\mathcal{C}_1$

Nevius, Proschan & Sethuraman (1977) introduced the notion of stochastic majorization as a tool to compare random vectors. Basically, $\boldsymbol{Y}$ is said to stochastically majorize $\boldsymbol{X}$ when (2.3) holds with $\mathcal{C}_1$. Note that when $\boldsymbol{X} \preceq_{\mathcal{C}_1} \boldsymbol{Y}$, we have that $\sum_{i=1}^{n} X_i =_d \sum_{i=1}^{n} Y_i$, since both functions $\boldsymbol{x} \mapsto \sum_{i=1}^{n} x_i$ and $\boldsymbol{x} \mapsto -\sum_{i=1}^{n} x_i$ are Schur-increasing.

The order $\preceq_{\mathcal{C}_1}$ is used in the next example, related to the results established in the next sections of this paper.

Example 2.6. Recall that the family $\{f_\theta,\ \theta \in \Theta\}$ of (discrete or continuous) probability density functions possesses the semi-group property in θ if the convolution $f_{\theta_1} \star f_{\theta_2}$ equals $f_{\theta_1+\theta_2}$. Equivalently, having independent random variables X_{θ_1} and X_{θ_2} with respective densities f_{θ_1} and f_{θ_2}, $X_{\theta_1} + X_{\theta_2}$ has density $f_{\theta_1+\theta_2}$.

Let us consider a family $\{f_\theta,\ \theta \in \Theta\}$ satisfying the semi-group property in θ as well as the TP2-inequality

$$f_{\theta_1}(x_1) f_{\theta_2}(x_2) \geq f_{\theta_2}(x_1) f_{\theta_1}(x_2) \text{ for all } \theta_1 \leq \theta_2 \text{ and } x_1 \leq x_2.$$

This condition means that the corresponding densities increase in θ in the likelihood ratio order. A prominent example is furnished by the Poisson distribution.

Having $g \in \mathcal{C}_1$, let us now consider the auxiliary function $g^\star : \Theta^n \to \mathbb{R}$ defined as

$$g^\star(\boldsymbol{\theta}) = \mathbb{E}[g(X_{\theta_1}, \ldots, X_{\theta_n})]$$

for independent random variables $X_{\theta_1}, \ldots, X_{\theta_n}$ with respective probability density functions $f_{\theta_1}, \ldots, f_{\theta_n}$. Proschan & Sethuraman (1977) proved that the family $\{f_\theta,\ \theta \in \Theta\}$ possesses the stochastic Schur-increasing property, in the sense that the Schur-increasingness is transmitted from g to $g^\star$, that is $g \in \mathcal{C}_1 \Rightarrow g^\star \in \mathcal{C}_1$. This in turn implies

$$\boldsymbol{\theta} \preceq_{\text{maj}} \boldsymbol{\gamma} \Rightarrow (X_{\theta_1}, \ldots, X_{\theta_n}) \preceq_{\mathcal{C}_1} (X_{\gamma_1}, \ldots, X_{\gamma_n}).$$

See also NEVIUS, PROSCHAN & SETHURAMAN (1977, Theorem 3.3). Thus a deterministic property (majorization) of the vector of risk parameters $\boldsymbol{\theta}$ is transformed into a corresponding stochastic property (stochastic majorization) of the random vector $(X_{\theta_1}, \ldots, X_{\theta_n})$.

This example contains all the ingredients of the reasonings held in this paper. For g in one of the classes $\mathcal{C}_i$ defined above, we will study the properties of the associated functions $g^\star$. This in turn leads to stochastic inequalities involving $\preceq_{\mathcal{C}_i}$.

2.5.3 Symmetric convex order: (2.3) with $\mathcal{C}_2$

If $\boldsymbol{X} \preceq_{\mathcal{C}_2} \boldsymbol{Y}$ holds then $\boldsymbol{X}$ is said to be smaller than $\boldsymbol{Y}$ in the symmetric convex order. Clearly,

$$\boldsymbol{X} \preceq_{\mathcal{C}_2} \boldsymbol{Y} \Rightarrow \sum_{i=1}^{n} X_i \preceq_{\text{cx}} \sum_{i=1}^{n} Y_i.$$

A discussion regarding the order $\preceq_{\text{sym-cx}}$ can be found in Chapter 7 by Tong in Shaked and Shanthikumar (1994).

2.5.4 Relationships between the multivariate integral orders $\preceq_{\mathcal{C}_i}$

From the inclusions (2.1) and (2.2), we deduce that

$$\boldsymbol{X} \preceq_{\mathcal{C}_1} \boldsymbol{Y} \Rightarrow \boldsymbol{X} \preceq_{\mathcal{C}_2} \boldsymbol{Y} \Rightarrow \left\{ \begin{array}{l} \boldsymbol{X} \preceq_{\mathcal{C}_3} \boldsymbol{Y} \\ \boldsymbol{X} \preceq_{\mathcal{C}_4} \boldsymbol{Y} \end{array} \right.$$

as well as

$$\boldsymbol{X} \preceq_{\mathcal{C}_5} \boldsymbol{Y} \Rightarrow \left\{ \begin{array}{l} \boldsymbol{X} \preceq_{\mathcal{C}_3} \boldsymbol{Y} \\ \boldsymbol{X} \preceq_{\mathcal{C}_4} \boldsymbol{Y}. \end{array} \right.$$

2.6 Risk measures and economic capital

2.6.1 Definition

Since risks are modelled as non-negative random variables, measuring risk is equivalent to establishing a correspondence ϱ between the space of random variables and non-negative real numbers $\mathbb{R}^+$. The real number denoting a general risk measure associated with the risk X will henceforth be denoted as $\varrho[X]$. Thus, a risk measure is nothing but a functional that assigns a non-negative real number to a risk. See DENUIT ET AL. (2005) for an overview.

In this paper, we will stick to the following meaning of the risk measure: we focus on risk measures that can be used for determining provisions and capital requirements in order to avoid insolvency. Specifically, if X is a possible loss of some financial portfolio over a time horizon, we interpret $\varrho[X]$ as the amount of capital that should be added as a buffer to this portfolio so that it becomes acceptable to an internal or external risk controller. In such a case, $\varrho[X]$ is the risk capital of the portfolio.

Example 2.7. Prominent examples of risk measures include value-at-risk

$$\text{VaR}[X;\alpha] = \inf\{x \in \mathbb{R} |\, \Pr[X \leq x] \geq \alpha\},$$

tail value-at-risk

$$\mathrm{TVaR}[X;\alpha] = \frac{1}{1-\alpha}\int_{\alpha}^{1} \mathrm{VaR}[X;\epsilon]d\epsilon$$

and other Yaari-Wang risk measures

$$\varrho_g[X] = \int_0^{+\infty} g\big(\Pr[X > t]\big)dt$$

with the distortion function g non-decreasing with $g(0) = 0$ and $g(1) = 1$.

2.6.2 Risk measures and stochastic orderings

Risk measures agree with appropriate stochastic order relations. Specifically, VaR and all the Yaari-Wang risk measures agree with $\preceq_{\mathrm{st}}$ in the sense that

$$\begin{aligned} X \preceq_{\mathrm{st}} Y \quad &\Leftrightarrow \quad \mathrm{VaR}[X;\alpha] \leq \mathrm{VaR}[Y;\alpha] \text{ for all } \alpha \in [0,1] \\ &\Leftrightarrow \quad \varrho_g[X] \leq \varrho_g[Y] \text{ for all the distortions } g. \end{aligned}$$

Analogously, TVaR and Yaari-Wang risk measures associated with concave distortions agree with $\preceq_{\mathrm{sl}}$ (and thus also with $\preceq_{\mathrm{st}}$ and $\preceq_{\mathrm{cx}}$) in the sense that

$$\begin{aligned} X \preceq_{\mathrm{sl}} Y \quad &\Leftrightarrow \quad \mathrm{TVaR}[X;\alpha] \leq \mathrm{TVaR}[Y;\alpha] \text{ for all } \alpha \in [0,1] \\ &\Leftrightarrow \quad \varrho_g[X] \leq \varrho_g[Y] \text{ for all the concave distortions } g. \end{aligned}$$

2.6.3 Economic capital derived from risk measure

Insurance companies as well as banks should hold some capital cushion against unexpected losses. The most common way to quantify risk capital is the concept of economic capital (EC): EC is defined with respect to some risk measure ϱ as

$$\mathrm{EC}[S] = \varrho[S] - \mathbb{E}[S]$$

where S is the total loss.

The reason for reducing the risk measure $\varrho[S]$ by the expected loss $\mathbb{E}[S]$ is due to the "best practice" of decomposing the total risk capital $\varrho[S]$ into a first part $\mathbb{E}[S]$ covering expected losses and a second part $\mathrm{EC}[S]$ meant as a cushion against unexpected losses.

3 Heterogeneity in the arrival processes

In this section, we assume that the $C_{i,j}$'s are independent and identically distributed for all the values of i and j.

3.1 Poisson arrivals

Considering independent Poisson random variables $N_{\lambda_1}, \cdots, N_{\lambda_n}$ with respective means $\lambda_1, \ldots, \lambda_n$, Example 2.6 shows that, defining the function $g^\star$ as $g^\star(\boldsymbol{\lambda}) = \mathbb{E}[g(N_{\lambda_1}, \cdots, N_{\lambda_n})]$, the implication $g \in \mathcal{C}_1 \Rightarrow g^\star \in \mathcal{C}_1$ holds true. Furthermore, Marshall & Olkin (1979) mentioned that $g \in \mathcal{C}_2 \Rightarrow g^\star \in \mathcal{C}_2$ also holds in this case. The next result considers compound Poisson distributions (and shows that the implication $g \in \mathcal{C}_5 \Rightarrow g^\star \in \mathcal{C}_5$ is valid).

Proposition 3.1. *Consider $\boldsymbol{X} = (X_1, \cdots, X_n)$ and $\boldsymbol{Y} = (Y_1, \cdots, Y_n)$, where $X_i = \sum_{j=1}^{N_{\lambda_i}} C_{i,j}$, and $Y_i = \sum_{j=1}^{N_{\nu_i}} C_{i,j}$. Let $N_{\lambda_1}, \cdots, N_{\lambda_n}, N_{\gamma_1}, \cdots, N_{\gamma_n}$ be independent Poisson random variables with respective means $\lambda_1, \ldots, \lambda_n, \gamma_1, \ldots, \gamma_n$. Suppose that the $C_{i,j}$'s are independent and identically distributed for all $i = 1, \cdots, n$ and $j = 1, 2, \cdots$, and independent of $N_{\lambda_1}, \cdots, N_{\lambda_n}$, $N_{\gamma_1}, \cdots, N_{\gamma_n}$. Then,*

$$\boldsymbol{\lambda} \preceq_{maj} \boldsymbol{\nu} \Rightarrow \boldsymbol{X} \preceq_{\mathcal{C}_5} \boldsymbol{Y}.$$

Proof. Let us define $g^\star$ as

$$g^\star(\boldsymbol{\lambda}) = \mathbb{E}\left[g\left(\sum_{j=1}^{N_{\lambda_1}} C_{1,j}, \cdots, \sum_{j=1}^{N_{\lambda_n}} C_{n,j}\right)\right].$$

The proof consists in establishing the implication

$$g \in \mathcal{C}_5 \Rightarrow g^\star \in \mathcal{C}_5. \tag{3.1}$$

To see that (3.1) gives the announced result, it suffices to note that for any function $g \in \mathcal{C}_5$,

$$\mathbb{E}[g(\boldsymbol{X})] = g^\star(\lambda) \leq g^\star(\nu) = \mathbb{E}[g(\boldsymbol{Y})]$$

where the inequality follows from $\boldsymbol{\lambda} \preceq_{\text{maj}} \boldsymbol{\nu}$ considering Property 2.3.

Let us now establish (3.1). To this end, let us first write

$$g^\star(\boldsymbol{\lambda}) = \sum_{z_1=0}^{+\infty} \cdots \sum_{z_n=0}^{+\infty} \mathbb{E}\left[g\left(\sum_{j=1}^{z_1} C_{1,j}, \cdots, \sum_{j=1}^{z_n} C_{n,j}\right)\right] \prod_{i=1}^{n} \frac{e^{-\lambda_i}\lambda_i^{z_i}}{z_i!}.$$

Let $C_{i,0}$ be distributed as $C_{i,1}$ and independent of X_i, $i = 1, \ldots, n$. If we differentiate $g^\star$ with respect to λ_k we obtain

$$\begin{aligned}
\frac{\partial}{\partial \lambda_k} g^\star(\boldsymbol{\lambda}) &= \frac{\partial}{\partial \lambda_k} \sum_{z_k=0}^{\infty} \frac{e^{-\lambda_k}\lambda_k^{z_k}}{z_k!} \mathbb{E}\left[g\left(X_1, \cdots, X_{k-1}, \sum_{j=1}^{z_k} C_{k,j}, X_{k+1}, \cdots, X_n\right)\right] \\
&= -\sum_{z_k=0}^{\infty} \frac{e^{-\lambda_k}\lambda_k^{z_k}}{z_k!} \mathbb{E}\left[g\left(X_1, \cdots, X_{k-1}, \sum_{j=1}^{z_k} C_{k,j}, X_{k+1}, \cdots, X_n\right)\right] \\
&\quad + \sum_{z_k=1}^{\infty} \frac{e^{-\lambda_k}\lambda_k^{z_k-1}}{(z_k-1)!} \mathbb{E}\left[g\left(X_1, \cdots, X_{k-1}, \sum_{j=1}^{z_k} C_{k,j}, X_{k+1}, \cdots, X_n\right)\right] \\
&= \mathbb{E}\left[g\left(X_1, \cdots, X_{k-1}, C_{k,0} + X_k, X_{k+1}, \cdots, X_n\right) - g\left(\boldsymbol{X}\right)\right].
\end{aligned}$$

Considering $k \neq h$, the same reasoning gives

$$\begin{aligned}\frac{\partial^2}{\partial\lambda_k\partial\lambda_h}g^\star(\boldsymbol{\lambda}) &= \mathbb{E}\Big[g\left(X_1,\cdots,X_{h-1},C_{h,0}+X_h,X_{h+1},\cdots,X_{k-1},C_{k,0}+X_k,X_{k+1},\cdots,X_n\right) \\ &\qquad -g\left(X_1,\cdots,X_{h-1},X_h+C_{h,0},X_{h+1},\cdots,X_n\right) \\ &\qquad -g\left(X_1,\cdots,X_{k-1},X_k+C_{k,0},X_{k+1},\cdots,X_n\right)+g\left(\boldsymbol{X}\right)\Big] \leq 0\end{aligned}$$

where the last inequality comes from the submodularity of g and the fact that $C_{k,0}$ and $C_{h,0}$ are non-negative random variables independent of $X_1, \cdots, X_n$. Thus $g^\star$ is also submodular. Similarly, since g is convex in each variable we obtain that

$$\begin{aligned}\frac{\partial^2}{\partial^2\lambda_k}g^\star(\boldsymbol{\lambda}) &= \mathbb{E}\left[g\left(X_1,\cdots,X_{k-1},C_{k,0}+\tilde{C}_{k,0}+X_k,X_{k+1},\cdots,X_n\right)\right. \\ &\qquad \left.-2g\left(X_1,\cdots,X_{k-1},X_k+\tilde{C}_{k,0},X_{k+1},\cdots,X_n\right)+g\left(\boldsymbol{X}\right)\right] \geq 0 \qquad (3.2)\end{aligned}$$

where $C_{k,0}$ and $\tilde{C}_{k,0}$ are independent and identically distributed, and independent of $X_1, \cdots, X_n$. Inequality (3.2) ensures that $g^\star$ is convex in each variable. The symmetry of $g^\star$ follows from the symmetry of g. □

Proposition 3.1 indicates that an increase in the heterogeneity among Poisson distributed claim counts yields more dangerousness. Since $\sum_{i=1}^n X_i =_d \sum_{i=1}^n Y_i$ under the assumptions of Proposition 3.1 $\rho[\sum_{i=1}^n X_i] = \rho[\sum_{i=1}^n Y_i]$ for any risk measure in Example 2.7. Nevertheless, we have in particular that the inequalities (2.4) and (2.5) hold, which shows that any excess-of-loss or largest claims reinsurance cover will be more expensive for the insurer facing $\boldsymbol{Y}$ than for the one facing $\boldsymbol{X}$.

3.2 Arrival processes with the semi-group property

Let us now consider claim count distributions with the semi group property. Thus we assume that given independent random variables N_λ and N_ν, $N_\lambda + N_\nu$ is distributed as $N_{\lambda+\nu}$. In addition to the Poisson distribution, prominent examples of such families include the Binomial distribution with parameter (ν, p), for fixed p, with

$$\Pr[N_\nu = n] = \binom{\nu}{n} p^n (1-p)^{\nu-n}, \;\; n = 0, 1, \cdots, \nu,$$

and the Negative Binomial distribution with parameter (ν, p), for fixed p, with

$$\Pr[N_\nu = n] = \binom{n+\nu-1}{\nu-1} p^\nu (1-p)^n, \;\; n = 0, 1, \cdots.$$

We keep the homogeneity assumption on the claim sizes. The next result is similar to Proposition 3.1, but with $\preceq_{\mathcal{C}_5}$ replaced with $\preceq_{\mathcal{C}_2}$.

Proposition 3.2. *Consider $\boldsymbol{X} = (X_1, \cdots, X_n)$ and $\boldsymbol{Y} = (Y_1, \cdots, Y_n)$, where $X_i = \sum_{j=1}^{N_{\lambda_i}} C_{i,j}$ and $Y_i = \sum_{j=1}^{N_{\nu_i}} C_{i,j}$. Assume that the N_λ's are independent non-negative integer valued random variables with the semi-group property. The random variables $C_{i,j}$ are independent and* *identically distributed for all the values of i and j, and independent of* $N_{\lambda_1}, \ldots, N_{\lambda_n}, N_{\nu_1}, \ldots, N_{\nu_n}$. *Then,*

$$\boldsymbol{\lambda} \preceq_{maj} \boldsymbol{\nu} \Rightarrow \boldsymbol{X} \preceq_{\mathcal{C}_2} \boldsymbol{Y},$$

that is, $\boldsymbol{X}$ *is smaller than* $\boldsymbol{Y}$ *in the symmetric convex order.*

Proof. Without loss of generality we assume that $\boldsymbol{\lambda}$ and $\boldsymbol{\nu}$ differs only in two components, components i and j say. To simplify notation we consider only the case $n = 2$ and we assume that $(\lambda_1, \lambda_2) \preceq_{\text{maj}} (\nu_1, \nu_2)$ holds. The proof is similar to the proof of proposition F.6 in Marshall & Olkin (1979, p. 314). Without loss of generality we may assume that $(\lambda_1, \lambda_2) \preceq_{\text{maj}} (\nu_1, \nu_2)$ implies that $\nu_1 > \lambda_1 \geq \lambda_2 > \nu_2$ and $\lambda_1 + \lambda_2 = \nu_1 + \nu_2$. We can then write

$$\begin{aligned} \nu_1 &= \nu_2 + (\lambda_2 - \nu_2) + (\lambda_1 - \lambda_2) + (\nu_1 - \lambda_1) \\ \lambda_1 &= \nu_2 + (\lambda_2 - \nu_2) + (\lambda_1 - \lambda_2) \\ \lambda_2 &= \nu_2 + (\lambda_2 - \nu_2). \end{aligned} \tag{3.3}$$

Let us define

$$\begin{aligned} &A = \sum_{j=1}^{N_{(\nu_1-\lambda_1)}} C_{1,j}, \quad B = \sum_{j=1}^{N_{(\lambda_1-\lambda_2)}} C_{2,j}, \quad C = \sum_{j=1}^{N_{(\lambda_2-\nu_2)}} C_{3,j}, \\ &D = \sum_{j=1}^{N'_{(\lambda_2-\nu_2)}} C_{4,j}, \quad E = \sum_{j=1}^{N_{\nu_2}} C_{5,j}, \quad G = \sum_{j=1}^{N'_{\nu_2}} C_{6,j}, \end{aligned} \tag{3.4}$$

where $N'_{(\lambda_2-\nu_2)}$ and N'_{ν_2} are distributed as $N_{(\lambda_2-\nu_2)}$ and N_{ν_2}, respectively, all the random variables being independent. Note that A, B, C, D, E, G are independent non-negative random variables. Moreover, E and G are independent and identically distributed.

The semi-group property and equations (3.3) and (3.4) imply that

$$\begin{aligned} \mathbb{E}[g(X_1, X_2)] &= \mathbb{E}\left[g\left(\sum_{j=1}^{N_{\lambda_1}} C_{1,j}, \sum_{j=1}^{N_{\lambda_2}} C_{2,j}\right)\right] \\ &= \mathbb{E}[g\,(B + C + E, D + G)] \\ &= \mathbb{E}[g\,((B + C, D) + (E, G))] \\ \mathbb{E}[g(Y_1, Y_2)] &= \mathbb{E}\left[g\left(\sum_{j=1}^{N_{\nu_1}} Z_{1,j}, \sum_{j=1}^{N_{\nu_2}} Z_{2,j}\right)\right] \\ &= \mathbb{E}[g\,(A + B + C + D + E, G)] \\ &= \mathbb{E}[g\,((A + B + C + D, 0) + (E, G))]. \end{aligned}$$

Let us define $\boldsymbol{U} = (B + C, D)$, $\boldsymbol{W} = (A + B + C, 0)$ and $\boldsymbol{T} = (E, G)$. Thus, $\Pr[\boldsymbol{U} \preceq_{\text{maj}} \boldsymbol{W}] = 1$, and for every $g \in \mathcal{C}_1$, we have that $\Pr[g(\boldsymbol{U}) \leq g(\boldsymbol{W})] = 1$.

Let $\boldsymbol{\Pi}$ be a 2×2 permutation matrix. Let $g \in \mathcal{C}_2$ be a symmetric convex function (and thus Schur increasing). Note that defining $\tilde{g}$ as $\tilde{g}(\boldsymbol{u}) = \sum_{\boldsymbol{\Pi}} g(\boldsymbol{u} + \boldsymbol{x}\boldsymbol{\Pi})$ over all permutation matrices $\mathbf{\Pi}$ defines another symmetric convex function (i.e. $\tilde{g} \in \mathcal{C}_2$). Then, denoting as F the joint distribution function of the pair (E, G),

$$
\begin{aligned}
& \mathbb{E}[g(X_1, X_2)] = \mathbb{E}[g(\boldsymbol{U} + \boldsymbol{T})] \\
= & \int \mathbb{E}[g(\boldsymbol{U} + \boldsymbol{t})] dF(\boldsymbol{t}) = \sum_{\mathbf{\Pi}} \int \mathbb{E}[g(\boldsymbol{U} + \boldsymbol{t}\mathbf{\Pi})] dF(\boldsymbol{t}) \\
= & \int \mathbb{E}\left[\sum_{\mathbf{\Pi}} \int \mathbb{E}[g(\boldsymbol{U} + \boldsymbol{t}\mathbf{\Pi})]\right] dF(\boldsymbol{t}) \leq \int \mathbb{E}\left[\sum_{\mathbf{\Pi}} \int \mathbb{E}[g(\boldsymbol{W} + \boldsymbol{t}\mathbf{\Pi})\right] dF(\boldsymbol{t}) \\
= & \mathbb{E}[g(\boldsymbol{W} + \boldsymbol{T}) = \mathbb{E}[g(Y_1, Y_2)],
\end{aligned}
$$

which ends the proof. □

Again, the result shows that an increase in the heterogeneity leads to an increase in risk even if, exactly as for the Poisson case, $\sum_{i=1}^n X_i =_d \sum_{i=1}^n Y_i$. Implications (2.4)-(2.5) still hold. It is interesting to compare Proposition 3.1 to Proposition 3.2. Even if both results hold true in the Poisson case, only the latter applies to Binomial and Negative Binomial distributions.

4 Heterogeneity in the claim severities

In this section, we allow for different distributions for the $C_{i,j}$'s, when i varies from 1 to n (but for fixed i, we keep identically distributed $C_{i,j}$'s), and we now consider $N_1, \ldots, N_n$ identically distributed. The heterogeneity is now at the severity level.

4.1 Claim sizes with the semi-group property

Assume that the N_i's are independent and identically distributed. The amount $C_{i,j}$ of claim j in business i is distributed as C_{θ_i}. We assume that the family $\{C_\theta,\ \theta \in \Theta\}$ possesses the semi-group property, that is, the sum $C_\alpha + C_\beta$ of two independent random variables C_α and C_β is distributed as $C_{\alpha+\beta}$.

Proposition 4.1. *Consider $\boldsymbol{X} = (X_1, \cdots, X_n)$ and $\boldsymbol{Y} = (Y_1, \cdots, Y_n)$, where $X_i = \sum_{j=1}^{N_i} C_{\gamma_i,j}$ and $Y_i = \sum_{j=1}^{N_i} C_{\eta_i,j}$. Assume that the N_i's are independent and identically distributed non-negative integer valued random variables. The random variables $C_{\gamma_i,j}$, $j = 1, 2, \ldots$ (resp. $C_{\eta_i,j}$, $j = 1, 2, \ldots$) are independent and distributed as C_{γ_i} (resp. C_{η_i}), and are independent of the N_i's, where the C_θ's possess the semi group property in θ. Then,*

(i) $\boldsymbol{\gamma} \preceq_{maj} \boldsymbol{\eta} \Rightarrow \boldsymbol{X} \preceq_{\mathcal{C}_3} \boldsymbol{Y}$*;*

(ii) $\boldsymbol{\gamma} \preceq_{maj} \boldsymbol{\eta} \Rightarrow \sum_{i=1}^n X_i \preceq_{cx} \sum_{i=1}^n Y_i$*.*

Proof. (i) Without loss of generality, we may assume that $n = 2$. Assume that $\gamma_1 \geq \gamma_2$ and $\eta_1 > \eta_2$. Note that $\boldsymbol{\gamma} \preceq_{\text{maj}} \boldsymbol{\eta}$ implies that there is $\alpha \in (0, 1)$, such that

$$
\begin{aligned}
\gamma_1 &= \alpha\eta_1 + (1-\alpha)\eta_2 &= \eta_2 + \alpha(\eta_1 - \eta_2) \\
\gamma_2 &= \alpha\eta_2 + (1-\alpha)\eta_1 &= \eta_2 + (1-\alpha)(\eta_1 - \eta_2).
\end{aligned}
$$

Consider $f \in \mathcal{C}_3$ so that there is a convex function g such that $f(\boldsymbol{x}) = g(x_1) + g(x_2)$. Then,

$$\begin{aligned}
\mathbb{E}[f(\boldsymbol{X})] &= \mathbb{E}\left[g(X_1) + g(X_2)\right] \\
&= \mathbb{E}\left[g\left(\sum_{j=1}^{N_1} C_{\gamma_1,j}\right) + g\left(\sum_{j=1}^{N_2} C_{\gamma_2,j}\right)\right] \\
&= \mathbb{E}\left[g\left(\sum_{j=1}^{N_1} C_{\eta_2,j} + \sum_{j=1}^{N_1} C_{\alpha(\eta_1-\eta_2),j}\right) + g\left(\sum_{j=1}^{N_2} C_{\eta_2,j} + \sum_{j=1}^{N_2} C_{(1-\alpha)(\eta_1-\eta_2),j}\right)\right].
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mathbb{E}[f(\boldsymbol{Y})] &= \mathbb{E}\left[g(Y_1) + g(Y_2)\right] \\
&= \mathbb{E}\left[g\left(\sum_{j=1}^{N_1} C_{\eta_1,j}\right) + g\left(\sum_{j=1}^{N_2} C_{\eta_2,j}\right)\right] \\
&= \mathbb{E}\left[g\left(\sum_{j=1}^{N_1} C_{\eta_2,j} + \sum_{j=1}^{N_1} C_{\alpha(\eta_1-\eta_2),j} + \sum_{j=1}^{N_1} C_{(1-\alpha)(\eta_1-\eta_2),j}\right) + g\left(\sum_{j=1}^{N_2} C_{\eta_2,j}\right)\right].
\end{aligned}$$

Recall that the random variables N_1 and N_2 are identically distributed. The convexity of g then implies that :

$$\begin{aligned}
&\mathbb{E}[f(\boldsymbol{Y})] - \mathbb{E}[f(\boldsymbol{X})] \\
= \; &\mathbb{E}\left[g\left(\sum_{j=1}^{N_1} C_{\eta_2,j} + \sum_{j=1}^{N_1} C_{\alpha(\eta_1-\eta_2),j} + \sum_{j=1}^{N_1} C_{(1-\alpha)(\eta_1-\eta_2),j}\right) - g\left(\sum_{j=1}^{N_1} C_{\eta_2,j} + \sum_{j=1}^{N_1} C_{\alpha(\eta_1-\eta_2),j}\right)\right] \\
&-\mathbb{E}\left[g\left(\sum_{j=1}^{N_2} C_{\eta_2,j} + \sum_{j=1}^{N_2} C_{(1-\alpha)(\eta_1-\eta_2),j}\right) - g\left(\sum_{j=1}^{N_2} C_{\eta_2,j}\right)\right] \geq 0
\end{aligned}$$

which ends the proof of (i).

Let us now turn to (ii). Let $\{C_{\eta_2,j},\;\; j = 1, 2, \ldots\}$, $\{C_{\alpha(\eta_1-\eta_2),j},\;\; j = 1, 2, \ldots\}$, $\{C'_{\eta_2,j},\;\; j = 1, 2, \ldots\}$, and $\{C_{(1-\alpha)(\eta_1-\eta_2),j},\;\; j = 1, 2, \ldots\}$ be independent sequences of independent and identically distributed nonnegative random variables. Let us define the following sums:

$$S_{\eta_2}(n) = \sum_{j=1}^{n} C_{\eta_2,j}, \;\; S_{\alpha(\eta_1-\eta_2)}(n) = \sum_{j=1}^{n} C_{\alpha(\eta_1-\eta_2,j)}$$

$$S_{(1-\alpha)(\eta_1-\eta_2)}(n) = \sum_{j=1}^{n} C_{(1-\alpha)(\eta_1-\eta_2,j)}, \;\; S'_{\eta_2}(n) = \sum_{j=1}^{n} C'_{\eta_2,j}$$

as well as, for some convex function g, the function $\Psi : \mathbb{N}^4 \to \mathbb{R}$ as

$$\Psi(n_1, n_2, n_3, n_4) = \mathbb{E}\Big[g\Big(S_{\eta_2}(n_1) + S_{\alpha(\eta_1-\eta_2)}(n_2) + S_{(1-\alpha)(\eta_1-\eta_2)}(n_3) + S'_{\eta_2}(n_4)\Big)\Big].$$

Then, we can write

$$\begin{aligned}
&\mathbb{E}[g(X_1 + X_2)] = \mathbb{E}[\Psi(N_1, N_1, N_2, N_2)] \\
= \;&\sum_{n_1 > n_2} \Pr[N_1 = n_1, N_2 = n_2]\Big(\Psi(n_1, n_1, n_2, n_2) + \Psi(n_2, n_2, n_1, n_1)\Big) \\
&+ \sum_{n} \Pr[N_1 = N_2 = n]\Psi(n, n, n, n)
\end{aligned}$$

and similarly

$$
\begin{aligned}
&\mathbb{E}[g(Y_1+Y_2)] = \mathbb{E}[\Psi(N_1,N_1,N_1,N_2)] \\
= &\sum_{n_1>n_2} \Pr[N_1=n_1,N_2=n_2]\Big(\Psi(n_1,n_1,n_1,n_2)+\Psi(n_2,n_2,n_2,n_1)\Big) \\
&+\sum_n \Pr[N_1=N_2=n]\Psi(n,n,n,n)
\end{aligned}
$$

Assume that $n_1>n_2$. Since g is convex the following inequality holds:

$$
\begin{aligned}
&\Psi(n_1,n_1,n_1,n_2)-\Psi(n_1,n_1,n_2,n_2)-\Psi(n_2,n_2,n_1,n_1)+\Psi(n_2,n_2,n_2,n_1) \\
= &\ \mathbb{E}\Big[g\Big(S_{\eta_2}(n_1)+S_{\alpha(\eta_1-\eta_2)}(n_1)+S_{(1-\alpha)(\eta_1-\eta_2)}(n_1)+S'_{\eta_2}(n_2)\Big) \\
&-g\Big(S_{\eta_2}(n_1)+S_{\alpha(\eta_1-\eta_2)}(n_1)+S_{(1-\alpha)(\eta_1-\eta_2)}(n_2)+S'_{\eta_2}(n_2)\Big)\Big] \\
&-\mathbb{E}\Big[g\Big(S_{\eta_2}(n_2)+S_{\alpha(\eta_1-\eta_2)}(n_2)+S_{(1-\alpha)(\eta_1-\eta_2)}(n_1)+S'_{\eta_2}(n_1)\Big) \\
&-g\Big(S_{\eta_2}(n_1)+S_{\alpha(\eta_1-\eta_2)}(n_1)+S_{(1-\alpha)(\eta_1-\eta_2)}(n_2)+S'_{\eta_2}(n_1)\Big)\Big] \geq 0,
\end{aligned}
$$

which concludes the proof. □

As it was the case for the heterogeneity at the frequency level, increasing the heterogeneity at the severity level leads to an increase in risk. Note that in this case, $\sum_{i=1}^n X_i$ and $\sum_{i=1}^n Y_i$ are no more identically distributed. The convex inequality in (ii) shows that the economic capital computed on the basis of any risk measure in agreement with the convex order will be higher for $\boldsymbol{Y}$ than for $\boldsymbol{X}$. In particular,

$$
\begin{aligned}
\boldsymbol{\gamma} \preceq_{\text{maj}} \boldsymbol{\eta} \quad &\Rightarrow \quad \text{TVaR}\left[\sum_{i=1}^n X_i;\alpha\right] \leq \text{TVaR}\left[\sum_{i=1}^n Y_i;\alpha\right] \ \text{ for all } \alpha\in[0,1] \\
&\Rightarrow \quad \rho_g\left[\sum_{i=1}^n X_i\right] \leq \rho_g\left[\sum_{i=1}^n Y_i\right] \ \text{ for any concave distortion } g.
\end{aligned}
$$

4.2 Claim severities in the same location-scale family

Let us now assume that the claim severities $C_{i,j}$'s belong to the same location-scale family, that is, $C_{ij}=a_iZ_{ij}+s_i$, $i=1,\ldots,n$, where the Z_{ij}'s are independent and identically distributed. In order to compare this portfolio with another one whose claim severities are of the form $D_{i,j}=b_iZ_{ij}+t_i$, we need a bivariate extension of majorization (in order to compare the $n\times 2$ arrays $(\boldsymbol{a},\boldsymbol{s})$ with $(\boldsymbol{b},\boldsymbol{t})$). To this end, let us recall that a linear transformation $\boldsymbol{T}:\ \mathbb{R}^n\to\mathbb{R}^n$ is said to be a T-transformation (or T-transform) if $\boldsymbol{T}$ has the form

$$
\boldsymbol{T}=\alpha\boldsymbol{I}+(1-\alpha)\boldsymbol{\Pi}, \tag{4.1}
$$

where $0\leq\alpha\leq 1$, $\boldsymbol{I}$ is the identity matrix, and $\boldsymbol{\Pi}$ is a permutation matrix that just interchanges two coordinates. Thus $\boldsymbol{Tx}$ has the form:

$$
\boldsymbol{Tx}=(x_1,\cdots,\alpha x_i+(1-\alpha)x_j,\cdots,x_{j-1},\alpha x_j+(1-\alpha)x_i,x_{j+1},\cdots,x_n).
$$

Such a T-transform is particularly appealing for actuaries in the context of heterogeneous portfolios. Starting with a vector $\boldsymbol{x}$, the T-transform applied with $\alpha = 1/2$ leads to another vector $\boldsymbol{Tx}$ where the heterogeneous x_i and x_j have been replaced with $(x_i+x_j)/2$. Considering the x_i's as heterogeneity parameters, switching from $\boldsymbol{x}$ to $\boldsymbol{Tx}$ homogeneizes policies i and j (replacing the original heterogeneous policies i and j with a couple of homogeneous ones). Arnold (1987) called the T-transform a Robin Hood transform (for obvious reasons), and the matrix $\boldsymbol{T}$ defined by (4.1) a Robin Hood matrix. We will adhere to this intuitive name in the remainder of this paper.

Let $\mathbb{R}^+_{n\times m}$ be the set of all $n \times m$ matrices with non-negative real elements. That is $\boldsymbol{M} \in \mathbb{R}^+_{n\times m}$ if $\boldsymbol{M} = (\boldsymbol{x}^{(1)}, \cdots, \boldsymbol{x}^{(m)})$, where $\boldsymbol{x}^{(j)}$ is an $n-$dimensional column vector with non-negative elements.

Definition 4.2. Let $\boldsymbol{M}_1, \boldsymbol{M}_2 \in \mathbb{R}^+_{n\times m}$. Then, $\boldsymbol{M}_1$ is said to be majorized in the Robin Hood sense by $\boldsymbol{M}_2$, which is denoted as $\boldsymbol{M}_1 \preceq_{\text{RH}} \boldsymbol{M}_2$, if there exists a finite set of Robin Hood matrices (of the form (4.1)) $\boldsymbol{T}_1, \cdots, \boldsymbol{T}_K$ such that $\boldsymbol{M}_1 = \boldsymbol{T}_1\boldsymbol{T}_2\cdots\boldsymbol{T}_K\boldsymbol{M}_2$.

Note that $\boldsymbol{x}^{(j)} \preceq_{\text{maj}} \boldsymbol{y}^{(j)}$ for $j = 1, \dots, m$, does not imply that $\boldsymbol{M}_1 \preceq_{\text{RH}} \boldsymbol{M}_2$. A comparison in the $\preceq_{\text{RH}}$-sense is thus stronger than the simple column-wise comparison in the $\preceq_{\text{maj}}$-sense.

We are now in a position to state the following result, which extends Proposition 4.1 to location-scale families of claim severities.

Proposition 4.3. *Consider* $\boldsymbol{X} = (X_1, \cdots, X_n)$ *and* $\boldsymbol{Y} = (Y_1, \cdots, Y_n)$, *where* $X_i = \sum_{j=1}^{N_i}(a_i Z_{ij} + s_i)$, *and* $Y_i = \sum_{j=1}^{N_i}(b_i Z_{ij} + t_i)$, $i = 1, \cdots, n$, *where the* Z_{ij}*'s are independent and identically distributed random variables, and the* N_i*'s are independent and identically distributed integer valued random variables, all the random variables being independent. Then,*

$$(\boldsymbol{a}, \boldsymbol{s}) \preceq_{RH} (\boldsymbol{b}, \boldsymbol{t}) \Rightarrow \boldsymbol{X} \preceq_{\mathcal{C}_3} \boldsymbol{Y}.$$

Proof. To simplify notations we consider $n = 2$. Assume that

$$\begin{array}{ccccccc} b_2 & < & a_2 & < & a_1 & < & b_1 \\ t_2 & < & s_2 & < & s_1 & < & t_1. \end{array}$$

Then, $(\boldsymbol{a}, \boldsymbol{s}) \preceq_{\text{RH}} (\boldsymbol{b}, \boldsymbol{t})$ ensures that there is a number β, $0 < \beta < 1$ such that

$$\begin{aligned} a_1 &= \beta b_1 + (1-\beta) b_2 \\ a_2 &= (1-\beta) b_1 + \beta b_2 \\ s_1 &= \beta b_1 + (1-\beta) b_2 \\ s_2 &= (1-\beta) t_1 + \beta t_2. \end{aligned}$$

Thus, considering $f \in \mathcal{C}_3$, there is a convex function g such that $f(\boldsymbol{x}) = g(x_1) + g(x_2)$ and we can represent $f(\boldsymbol{Y})$ and $f(\boldsymbol{X})$ as

$$\begin{aligned} f(\boldsymbol{Y}) &= g\left(\sum_{j=1}^{N_1}(b_2 Z_{1j}+t_2)+\beta\sum_{j=1}^{N_1}\Big((b_1-b_2)Z_{1j}+(t_1-t_2)\Big)\right.\\ &\quad \left.+(1-\beta)\sum_{j=1}^{N_1}\Big((b_1-b_2)Z_{1j}+(t_1-t_2)\Big)\right)+g\left(\sum_{j=1}^{N_2}(b_2 Z_{2j}+t_2)\right)\\ f(\boldsymbol{X}) &= g\left(\sum_{j=1}^{N_1}(b_2 Z_{1j}+t_2)+\beta\sum_{j=1}^{N_1}\Big((b_1-b_2)Z_{1j}+(t_1-t_2)\Big)\right)\\ &+ g\left(\sum_{j=1}^{N_2}(b_2 Z_{2j}+t_2)+(1-\beta)\sum_{j=1}^{N_2}\Big((b_1-b_2)Z_{2j}+(t_1-t_2)\Big)\right). \end{aligned}$$

Thus,

$$\begin{aligned} &\mathbb{E}[f(\boldsymbol{Y})-f(\boldsymbol{X})]\\ =\ &\mathbb{E}\left[g\left(\sum_{j=1}^{N_1}(b_2 Z_{1j}+t_2)+\beta\sum_{j=1}^{N_1}\Big((b_1-b_2)Z_{1j}+(t_1-t_2)\Big)\right.\right.\\ &\left.+(1-\beta)\sum_{j=1}^{N_1}\Big((b_1-b_2)Z_{1j}+(t_1-t_2)\Big)\right)\\ &\left.-g\left(\sum_{j=1}^{N_1}(b_2 Z_{1j}+t_2)+\beta\sum_{j=1}^{N_1}\Big((b_1-b_2)Z_{1j}+(t_1-t_2)\Big)\right)\right]\\ &-\mathbb{E}\left[g\left(\sum_{j=1}^{N_2}(b_2 Z_{2j}+t_2)+(1-\beta)\sum_{j=1}^{N_2}\Big((b_1-b_2)Z_{2j}+(t_1-t_2)\Big)\right)-g\left(\sum_{j=1}^{N_2}(b_2 Z_{2j}+t_2)\right)\right]\geq 0 \end{aligned}$$

where the last inequality follows from the convexity of g. □

5 Heterogeneity in both the arrival processes and claim severities

Next we consider heterogeneity in the claim arrival processes and in the claim amounts. To this end, let us consider the subclass $\mathcal{C}_3^{\uparrow}$ of $\mathcal{C}_3$ made of the functions $f:\mathbb{R}^n\to\mathbb{R}$ that can be represented as $f(\boldsymbol{x})=\sum_{i=1}^n g(x_i)$ with g non-decreasing and convex.

Proposition 5.1. *Let $\{N_\theta,\ \theta\in\Theta\}$ be a family of counting random variables, indexed by a parameter $\theta\in\Theta\subseteq\mathbb{R}$, with the semi-group property in θ. Let $\{C_\lambda,\ \lambda\in\Lambda\}$ be a family of positive random variables, indexed by a parameter $\lambda\in\Lambda\subseteq\mathbb{R}$, with the semi-group property in λ. Consider $\boldsymbol{X}=(X_1,\cdots,X_n)$ and $\boldsymbol{Y}=(Y_1,\cdots,Y_n)$, where $X_i=\sum_{j=1}^{N_{\lambda_i}}C_{\gamma_i,j}$ and $Y_i=\sum_{j=1}^{N_{\nu_i}}C_{\eta_i,j}$, $i=1,\cdots,n$, where the $C_{\gamma_i,j}$'s are independent and distributed as C_{γ_i}, and* *the $C_{\eta_i,j}$'s are independent and distributed as C_{η_i}. All the random variables are assumed to be independent. Then*

$$(\boldsymbol{\lambda}, \boldsymbol{\gamma}) \preceq_{RH} (\boldsymbol{\nu}, \boldsymbol{\eta}) \Rightarrow \boldsymbol{X} \preceq_{\mathcal{C}_3^{\uparrow}} \boldsymbol{Y}.$$

Proof. Consider $n = 2$. Assume that

$$\begin{array}{ccccccc} \nu_2 & < & \lambda_2 & < & \lambda_1 & < & \nu_1 \\ \eta_2 & < & \gamma_2 & < & \gamma_1 & < & \eta_1. \end{array}$$

The fact that $(\boldsymbol{\lambda}, \boldsymbol{\gamma}) \preceq_{\mathrm{RH}} (\boldsymbol{\nu}, \boldsymbol{\eta})$ then implies that there is β, $0 < \beta < 1$ such that

$$\begin{aligned} \lambda_1 &= \nu_2 + \beta(\nu_1 - \nu_2) \\ \gamma_1 &= \eta_2 + \beta(\eta_1 - \eta_2) \\ \lambda_2 &= \nu_2 + (1-\beta)(\nu_1 - \nu_2) \\ \gamma_2 &= \eta_2 + (1-\beta)(\eta_1 - \eta_2). \end{aligned}$$

We can now write

$$\begin{aligned} \mathbb{E}[f(\boldsymbol{Y})] &= \mathbb{E}\Bigg[g\Bigg(\sum_{j=1}^{N_{\nu_2}} C^{(1)}_{\eta_2,j} + \sum_{j=1}^{N_{\nu_2}} C^{(1)}_{\beta(\eta_1-\eta_2),j} + \sum_{j=1}^{N_{\nu_2}} C^{(1)}_{(1-\beta)(\eta_1-\eta_2),j} \\ &\quad + \sum_{j=1}^{N_{\beta(\nu_1-\nu_2)}} C^{(2)}_{\eta_2,j} + \sum_{j=1}^{N_{\beta(\nu_1-\nu_2)}} C^{(2)}_{\beta(\eta_1-\eta_2),j} + \sum_{j=1}^{N_{\beta(\nu_1-\nu_2)}} C^{(2)}_{(1-\beta)(\eta_1-\eta_2),j} \\ &\quad + \sum_{j=1}^{N_{(1-\beta)(\nu_1-\nu_2)}} C^{(3)}_{\eta_2,j} + \sum_{j=1}^{N_{(1-\beta)(\nu_1-\nu_2)}} C^{(3)}_{\beta(\eta_1-\eta_2),j} + \sum_{j=1}^{N_{(1-\beta)(\nu_1-\nu_2)}} C^{(3)}_{(1-\beta)(\eta_1-\eta_2),j} \Bigg) \Bigg] \\ &+ \mathbb{E}\left[g\left(\sum_{j=1}^{N_{\nu_2}} C_{\eta_2,j} \right) \right] \end{aligned}$$

where all the random variables N_{ν_2}, $N_{\beta(\nu_1-\nu_2)}$, $N_{(1-\beta)(\nu_1-\nu_2)}$, $C^{(i)}_{\eta_2,j}$, $C^{(i)}_{\beta(\eta_1-\eta_2),j}$, and $C^{(i)}_{(1-\beta)(\eta_1-\eta_2),j}$ are independent. Similarly, $\mathbb{E}[f(\boldsymbol{X})]$ can be written as

$$\begin{aligned} \mathbb{E}[f(\boldsymbol{X})] &= \mathbb{E}\Bigg[g\Bigg(\sum_{j=1}^{N_{\nu_2}} C^{(1)}_{\eta_2,j} + \sum_{j=1}^{N_{\nu_2}} C^{(1)}_{\beta(\eta_1-\eta_2),j} \\ &\quad + \sum_{j=1}^{N_{\beta(\nu_1-\nu_2)}} C^{(2)}_{\eta_2,j} + \sum_{j=1}^{N_{\beta(\nu_1-\nu_2)}} C^{(2)}_{\beta(\eta_1-\eta_2),j} \Bigg) \Bigg] \\ &+ \mathbb{E}\Bigg[g\Bigg(\sum_{j=1}^{N_{\nu_2}} C^{(1)}_{\eta_2,j} + \sum_{j=1}^{N_{\nu_2}} C^{(1)}_{(1-\beta)(\eta_1-\eta_2),j} \\ &\quad + \sum_{j=1}^{N_{(1-\beta)(\nu_1-\nu_2)}} C^{(3)}_{\eta_2,j} + \sum_{j=1}^{N_{(1-\beta)(\nu_1-\nu_2)}} C^{(3)}_{(1-\beta)(\eta_1-\eta_2),j} \Bigg) \Bigg]. \end{aligned}$$

Since g is convex non-decreasing, we obtain the following inequality:

$$
\begin{aligned}
& \mathbb{E}[f(\boldsymbol{Y})] - \mathbb{E}[f(\boldsymbol{X})] \\
\geq\ & \mathbb{E}\Bigg[g\Bigg(\sum_{j=1}^{N_{\nu_2}} C^{(1)}_{\eta_2,j} + \sum_{j=1}^{N_{\nu_2}} C^{(1)}_{\beta(\eta_1-\eta_2),j} + \sum_{j=1}^{N_{\nu_2}} C^{(1)}_{(1-\beta)(\eta_1-\eta_2),j} \\
& \qquad + \sum_{j=1}^{N_{\beta(\nu_1-\nu_2)}} C^{(2)}_{\eta_2,j} + \sum_{j=1}^{N_{\beta(\nu_1-\nu_2)}} C^{(2)}_{\beta(\eta_1-\eta_2),j} \\
& \qquad + \sum_{j=1}^{N_{(1-\beta)(\nu_1-\nu_2)}} C^{(3)}_{\eta_2,j} + \sum_{j=1}^{N_{(1-\beta)(\nu_1-\nu_2)}} C^{(3)}_{(1-\beta)(\eta_1-\eta_2),j} \Bigg) \\
& \qquad - g\Bigg(\sum_{j=1}^{N_{\nu_2}} C^{(1)}_{\eta_2,j} + \sum_{j=1}^{N_{\nu_2}} C^{(1)}_{\beta(\eta_1-\eta_2),j} + \sum_{j=1}^{N_{\beta(\nu_1-\nu_2)}} C^{(2)}_{\eta_2,j} + \sum_{j=1}^{N_{\beta(\nu_1-\nu_2)}} C^{(2)}_{\beta(\eta_1-\eta_2),j} \Bigg) \Bigg] \\
& - \mathbb{E}\Bigg[g\Bigg(\sum_{j=1}^{N_{\nu_2}} C^{(1)}_{\eta_2,j} + \sum_{j=1}^{N_{\nu_2}} C^{(1)}_{(1-\beta)(\eta_1-\eta_2),j} + \sum_{j=1}^{N_{(1-\beta)(\nu_1-\nu_2)}} C^{(3)}_{\eta_2,j} + \sum_{j=1}^{N_{(1-\beta)(\nu_1-\nu_2)}} C^{(3)}_{(1-\beta)(\eta_1-\eta_2),j} \Bigg) \\
& \qquad - g\Bigg(\sum_{j=1}^{N_{\nu_2}} C^{(1)}_{\eta_2,j} \Bigg) \Bigg] \geq 0,
\end{aligned}
$$

which ends the proof. □

This result shows that an increase in heterogeneity increases the risk, in the sense that the inequality (2.4) is valid. The price of an excess-of-loss reinsurance treaty is higher for the reinsurer facing $\boldsymbol{Y}$ than for the one facing $\boldsymbol{X}$.

6 Conclusion

Denuit & Frostig (2005) examined the effect of heterogeneity in the individual model of risk theory, where the claim amount for policy i writes $X_i = J_{q_i} C_i$ with J_{q_i} Bernoulli distributed with mean q_i ($J_{q_i} = 1$ if at least one claim has been reported by policyholder i, that is, if $X_i > 0$, and 0 otherwise). The random variable C_i is then the total cost of all these claims. All the random variables are assumed to be mutually independent as in the present paper.

Considering another portfolio, with individual claims $Y_i = K_{p_i} C_i$, with K_{p_i} Bernoulli distributed with mean p_i, the main finding was that switching from $\boldsymbol{q} = (q_1, \ldots, q_n)$ to $\boldsymbol{p} = (p_1, \ldots, p_n)$ larger in the sense of the majorization, that is increasing the heterogeneity of the portfolio, leads to a decrease of the dangerousness, in the sense that

$$\boldsymbol{q} \preceq_{\text{maj}} \boldsymbol{p} \Rightarrow \sum_{i=1}^{n} Y_i \preceq_{\text{cx}} \sum_{i=1}^{n} X_i,$$

provided the C_i's are identically distributed.

In general, the C_i's need not to to be identically distributed. If $X_i = J_{q_i} C_i$ and $Y_i = K_{p_i} C_i$, $i = 1, \ldots, n$, with

(i) $p_1 \geq \ldots \geq p_n$ and $q_1 \geq \ldots \geq q_n$,

(ii) $C_n \preceq_{\rm st} C_{n-1} \preceq_{\rm st} \ldots \preceq_{\rm st} C_1$

(iii) $\boldsymbol{q} \preceq_{\rm maj} \boldsymbol{p}$.

Then, the inequality

$$\mathbb{E}\left[g\left(\sum_{i=1}^{n} Y_i\right)\right] \leq \mathbb{E}\left[g\left(\sum_{i=1}^{n} X_i\right)\right]$$

holds true for any decreasing convex function, provided the expectations exist. This intuitively means that $\sum_{i=1}^{n} Y_i$ tends to be larger, but less variable than $\sum_{i=1}^{n} X_i$. Thus, increasing the degree of heterogeneity yields higher payments for the insurance company, but less variability. Since the danger comes from the variability, we can conclude that increasing the degree of heterogeneity decreases the risk.

Several other situations were investigated in that paper (severities in a location-scale family of distributions or with the semi-group property), and the conclusion was that in most cases, increasing the level of heterogeneity lead to a decrease in the dangerousness. The conclusion in the collective model of risk theory studied in this paper is different, in the sense that increasing the level of heterogeneity leads to an increase in the dangerousness of the portfolio. This shows that the effect of heterogeneity depends on the context.

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References

[1] Ammeter, H. (1964). Note concerning the distribution of the total loss excluding the largest individual claim. *ASTIN Bulletin* **3**, 132-143.

[2] Arnold, B. C. (1987). *Majorization and the Lorenz Order:A Brief Introduction.* Springer-Verlag, New York.

[3] Chang, C.-S. (1992), A new ordering for stochastic majorization: theory and applications, *Advances in Applied Probability* **24**, 604–634.

[4] Denuit, M., Dhaene, J., Goovaerts, M.J., & Kaas, R. (2005). *Actuarial Theory for Dependent Risks.* Wiley, New York.

[5] Denuit, M., & Frostig, E. (2005). Heterogeneity and the need for economic capital in the individual model. Working Paper 05-04, Institut des Sciences Actuarielles, Université Catholique de Louvain, Louvain-la-Neuve, Belgium.

[6] Marshall, A.W. (1991). Multivariate stochastic orderings and generating cones of functions. In *Stochastic Orders and Decision under Risk*, K.C. Mosler and M. Scarsini Eds. IMS Lecture Notes - Monograph Series **19**, 231-247.

[7] Marshall, A.W., & Olkin, I. (1979). *Inequalities : Theory of Majorization and its Applications*. Academic Press. New York.

[8] Nevius, S.E., Proschan, F., & Sethuraman, J. (1977). Schur Functions in Statistics II. Stochastic Majorization. *The Annals of Statistics* **5**, 263-273.

[9] Proschan, F., & Sethuraman, J. (1977). Schur Functions in Statistics I. The Preservation Theorem. *The Annals of Statistics* **5**, 256-262.

[10] Whitt, W. (1986). Stochastic comparisons for non-Markov processes. *Mathematics of Operations Research* **11**, 608-618.