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Abstract

It is known that the sums of the components of two random vectors $(X_1, X_2, \dots, X_n)$ and $(Y_1, Y_2, \dots, Y_n)$ ordered in the multivariate $(s_1, s_2, \dots, s_n)$-increasing convex order are ordered in the univariate $(s_1 + s_2 + \dots + s_n)$-increasing convex order. More generally, real-valued functions of $(X_1, X_2, \dots, X_n)$ and $(Y_1, Y_2, \dots, Y_n)$ are ordered in the same sense as long as these functions possess some specified non-negative cross derivatives. This note extends these results to multivariate functions. In particular, we consider vectors of partial sums $(S_1, S_2, \dots, S_n)$ and $(T_1, T_2, \dots, T_n)$ where $S_j = X_1 + \dots + X_j$ and $T_j = Y_1 + \dots + Y_j$ and we show that these random vectors are ordered in the multivariate $(s_1, s_1 + s_2, \dots, s_1 + \dots + s_n)$-increasing convex order. The consequences of these general results for the upper orthant order and the orthant convex order are discussed.

Keywords: Multivariate increasing convex order of higher degree, upper orthant (convex) order, stochastic recursive equations.

Subject classification: 60E15

1 Introduction and motivation

Let s be a positive integer. In this paper, we consider non-negative random variables and functions defined on the non-negative real line $\mathbb{R}^+ = [0, +\infty)$. Recall the definition of the univariate s-increasing convex orders introduced by Denuit, Lefevre and Shaked (1998). Let $\mathcal{U}_{s-\mathrm{icx}}$ be the class of all the regular s-increasing convex functions g, i.e. those functions g such that $\frac{d^k}{dx^k} g(x) \geq 0$ for all $x \geq 0$ and $k = 1, 2, \ldots, s$. Given two (non-negative) random variables X and Y, X is said to be smaller than Y in the s-increasing convex sense, denoted as $X \preceq_{s-\mathrm{icx}} Y$, if

$$\mathbb{E}[g(X)] \leq \mathbb{E}[g(Y)] \text{ for all } g \in \mathcal{U}_{s-\mathrm{icx}}, \tag{1.1}$$

provided that the expectations involved in (1.1) exist.

Boutsikas and Vaggelatou (2002) established that sums of components of n-dimensional random vectors ordered in the upper orthant order are ordered in the $\preceq_{n-\mathrm{icx}}$-sense. Recall that the random vector $(X_1, \ldots, X_n)$ is said to be smaller than $(Y_1, \ldots, Y_n)$ in the upper orthant order, which is denoted by $(X_1, \ldots, X_n) \preceq_{\mathrm{uo}} (Y_1, \ldots, Y_n)$, if the inequality

$$\Pr[X_1 > x_1, X_2 > x_2, \ldots, X_n > x_n] \leq \Pr[Y_1 > x_1, Y_2 > x_2, \ldots, Y_n > x_n] \tag{1.2}$$

is valid for all $x_1, x_2, \ldots, x_n$.

We refer the reader to Shaked and Shanthikumar (2007) for more information and for relevant references about $\preceq_{\mathrm{uo}}$. Now, Boutsikas and Vaggelatou (2002, Section 3.2) established that

$$(X_1, \ldots, X_n) \preceq_{\mathrm{uo}} (Y_1, \ldots, Y_n) \Rightarrow \sum_{i=1}^{n} X_i \preceq_{n-\mathrm{icx}} \sum_{i=1}^{n} Y_i. \tag{1.3}$$

Formula (1.3) indicates that an $\preceq_{\mathrm{uo}}$ ordering between random vectors translates into an $\preceq_{n-\mathrm{icx}}$ ordering between the sums of their respective components. Of course, the implication (1.3) can be strengthened as

$$(X_1, \ldots, X_n) \preceq_{\mathrm{uo}} (Y_1, \ldots, Y_n)$$

$$\Rightarrow \sum_{i=1}^{k} X_i \preceq_{k-\mathrm{icx}} \sum_{i=1}^{k} Y_i \text{ for } k = 1, \ldots, n$$

as $\preceq_{\mathrm{uo}}$ is closed under marginalization. In this paper, we show that in addition to these marginal comparisons, the vectors of partial sums are in fact ordered in the multivariate increasing convex order of higher degree (whose definition is recalled in Section 2).

The present paper is organized as follows. Section 2 extends (1.3) in two directions. First, more general functions of the components of the random vectors are considered, and not only partial sums. Then, more general orderings than $\preceq_{\mathrm{uo}}$ are used. Applications to stochastic recursive equations are briefly discussed. Section 3 examines several particular cases: vectors of partial sums as well as vectors ordered in the upper orthant order or in the orthant convex order.

2 Main result

Let $(s_1, \ldots, s_n)$ be a vector of positive integers. Let $\mathcal{U}_{(s_1,\ldots,s_n)-\mathrm{icx}}$ be the class of all the functions g such that

$$\frac{\partial^{k_1+k_2+\cdots+k_n}}{\partial x_1^{k_1}\partial x_2^{k_2}\ldots\partial x_n^{k_n}} g(x_1, x_2, \ldots, x_n) \geq 0 \text{ for all } x_1, x_2, \ldots, x_n \geq 0,$$

$$k_i = 0, 1, \ldots, s_i, \;\; i = 1, 2, \ldots, n, \;\; k_1 + k_2 + \cdots + k_n \geq 1. \tag{2.1}$$

Following Denuit and Mesfioui (2010), we say that the n-dimensional random vector $(X_1, \ldots, X_n)$ is smaller than the n-dimensional random vector $(Y_1, \ldots, Y_n)$ in the $(s_1, \ldots, s_n)$-increasing convex order, which is denoted by $(X_1, \ldots, X_n) \preceq_{(s_1,\ldots,s_n)-\mathrm{icx}} (Y_1, \ldots, Y_n)$, if

$$\mathbb{E}[g(X_1, \ldots, X_n)] \leq \mathbb{E}[g(Y_1, \ldots, Y_n)] \text{ for all } g \in \mathcal{U}_{(s_1,\ldots,s_n)-\mathrm{icx}}, \tag{2.2}$$

provided that the expectations exist. For $s_1 = s_2 = \ldots = s_n = 1$ we get $\preceq_{\mathrm{uo}}$.

The following technical result is useful to prove our main finding.

Lemma 2.1. *Let f and g be two non-negative functions.*

(i) If $f : \mathbb{R}^{n-1} \to \mathbb{R}^+$ belongs to $\mathcal{U}_{(s_1,\ldots,s_{n-1})-icx}$ and $g : \mathbb{R}^n \to \mathbb{R}^+$ belongs to $\mathcal{U}_{(s_1,\ldots,s_n)-icx}$, then the product fg belongs to $\mathcal{U}_{(s_1,\ldots,s_n)-icx}$.

(ii) If $f : \mathbb{R}^n \to \mathbb{R}^+$ belongs to $\mathcal{U}_{(s_1,\ldots,s_n)-icx}$ and and $g : \mathbb{R}^+ \to \mathbb{R}^+$ belongs to $\mathcal{U}_{(s_1+\ldots+s_n)-icx}$, then the composition $g \circ f$ belongs to $\mathcal{U}_{(s_1,\ldots,s_n)-icx}$.

Proof. Let us establish the validity of statement (i). As f is a function of $x_1, \ldots, x_{n-1}$ only and belongs to $\mathcal{U}_{(s_1,\ldots,s_{n-1})-\mathrm{icx}}$, it also belongs to $\mathcal{U}_{(s_1,\ldots,s_{n-1},s_n)-\mathrm{icx}}$. Now, the product of two non-negative functions in $\mathcal{U}_{(s_1,\ldots,s_n)-\mathrm{icx}}$ also belongs to $\mathcal{U}_{(s_1,\ldots,s_n)-\mathrm{icx}}$ which proves (i).

To get (ii), notice that the $(k_1, \ldots, k_n)$th cross derivative of $g \circ f$ involves lower order derivatives of f together with derivatives of g up to the $(k_1 + \ldots + k_{n-1})$th order of g. The conditions imposed on f and g ensure that these cross-derivatives are non-negative so that $g \circ f \in \mathcal{U}_{(s_1,\ldots,s_n)-icx}$, as announced.

□

Before stating our main result, recall from Denuit and Mesfioui (2010) that the stochastic inequality $(X_1, X_2, \ldots, X_n) \preceq_{(s_1,s_2,\ldots,s_n)-\mathrm{icx}} (Y_1, Y_2, \ldots, Y_n)$ holds if, and only if, the inequality

$$\mathbb{E}\left[\prod_{i=1}^{n} g_i(X_i)\right] \leq \mathbb{E}\left[\prod_{i=1}^{n} g_i(Y_i)\right]$$

is fulfilled for all the non-negative functions $g_1, \ldots, g_n$ such that $g_i \in \mathcal{U}_{s_i-\mathrm{icx}}$, $i = 1, \ldots, n$.

Proposition 2.2. *If $(X_1, X_2, \ldots, X_n) \preceq_{(s_1,\ldots,s_n)-icx} (Y_1, Y_2, \ldots, Y_n)$ then*

$$\big(h_1(X_1), h_2(X_1, X_2), \ldots, h_n(X_1, X_2, \ldots, X_n)\big)$$
$$\preceq_{(s_1,s_1+s_2,\ldots,s_1+s_2+\cdots+s_n)-icx} \big(h_1(Y_1), h_2(Y_1, Y_2), \ldots, h_n(Y_1, Y_2, \ldots, Y_n)\big)$$

provided $h_i : \mathbb{R}^i \to \mathbb{R}^+$ belongs to $\mathcal{U}_{(s_1,\ldots,s_i)-icx}$.

Proof. Consider $h_i : \mathbb{R}^i \to \mathbb{R}^+$ in $\mathcal{U}_{(s_1,\ldots,s_i)-\text{icx}}$ and $g_i : \mathbb{R}^+ \to \mathbb{R}^+$ in $\mathcal{U}_{(s_1+\ldots+s_i)-\text{icx}}$. From Lemma 2.1(ii), we know that $g_i \circ h_i$ belongs to $\mathcal{U}_{(s_1,\ldots,s_i)-\text{icx}}$, $i = 1,\ldots,n$. By applying Lemma 2.1(i) successively, we deduce that the function h defined by

$$h(x_1,\ldots,x_n) = \prod_{i=1}^n g_i(h_i(x_1,\ldots,x_i))$$

belongs to $\mathcal{U}_{(s_1,\ldots,s_n)-\text{icx}}$. Therefore $(X_1,X_2,\ldots,X_n) \preceq_{(s_1,s_2,\ldots,s_n)-\text{icx}} (Y_1,Y_2,\ldots,Y_n)$ implies

$$\mathbb{E}\left[\prod_{i=1}^n g_i\Big(h_i(X_1,\ldots,X_i)\Big)\right] \leq \mathbb{E}\left[\prod_{i=1}^n g_i\Big(h_i(Y_1,\ldots,Y_i)\Big)\right]$$

for all $h_i : \mathbb{R}^i \to \mathbb{R}^+$ in $\mathcal{U}_{(s_1,\ldots,s_i)-\text{icx}}$ and $g_i : \mathbb{R}^+ \to \mathbb{R}^+$ in $\mathcal{U}_{(s_1+\ldots+s_i)-\text{icx}}$ which proves the announced result. □

Proposition 2.2 is especially useful when stochastic recursive equations govern the dynamics of the random variables. Starting from a collection of non-negative random variables $X_1, X_2, X_3, \ldots$, we define the state $Z_X(t)$ of some stochastic system at time t by

$$Z_X(t) = h_t(X_1,\ldots,X_t) \text{ for } t = 1,2,\ldots.$$

Given another sequence of non negative random variables $Y_1, Y_2, Y_3, \ldots$, define $Z_Y(t)$ analogously. Then, provided

$$(X_1,\ldots,X_t) \preceq_{(s_1,\ldots,s_t)-\text{icx}} (Y_1,\ldots,Y_t)$$

and h_t belongs to $\mathcal{U}_{(s_1,\ldots,s_t)-\text{icx}}$ for all t, the trajectory of the two processes can be compared in the sense that the stochastic inequality

$$(Z_X(1),\ldots,Z_X(t)) \preceq_{(s_1,s_1+s_2,\ldots,s_1+s_2+\ldots+s_t)-\text{icx}} (Z_Y(1),\ldots,Z_Y(t))$$

holds true whatever t. For instance, if the X_i's and Y_i's represent multiplicative shocks to some initial random state Z, we have that

$$(ZX_1, ZX_1X_2, \ldots, ZX_1X_2\ldots X_t) \preceq_{(s_1,s_1+s_2,\ldots,s_1+s_2+\ldots+s_t)-\text{icx}} (ZY_1, ZY_1Y_2, \ldots, ZY_1Y_2\ldots Y_t).$$

The next section considers partial sums, or additive shocks to some initial state.

3 Particular cases

3.1 Partial sums

Proposition 2.2 applies in particular to partial sums. Define $S_j = X_1 + \ldots + X_j$ and $T_j = Y_1 + \ldots + Y_j$, $j = 1,\ldots,n$. If $(X_1,\ldots,X_n)$ and $(Y_1,\ldots,Y_n)$ are ordered in the $\preceq_{(s_1,\ldots,s_n)-\text{icx}}$-sense, we would like to know whether the partial sums S_j and T_j of their components can be compared. Marginal comparisons are available in the literature. More precisely, after Boutsikas and Vaggelatou (2002), Denuit and Mesfioui (2010) established that

$$
\begin{aligned}
(X_1, \ldots, X_n) \preceq_{(s_1,\ldots,s_n)-\text{icx}} (Y_1, \ldots, Y_n) \quad &\Rightarrow \quad (X_1, \ldots, X_k) \preceq_{(s_1,\ldots,s_k)-\text{icx}} (Y_1, \ldots, Y_k) \\
&\qquad \text{for } k = 1, 2, \ldots, n \\
&\Rightarrow \quad S_k \preceq_{(\sum_{i=1}^k s_i)-\text{icx}} T_k \text{ for } k = 1, 2, \ldots, n.
\end{aligned}
$$

for $k = 1, 2, \ldots, n$. The next result is a direct consequence of Proposition 2.2. It indicates that in addition to the aforementioned marginal comparisons, a stronger comparison for vectors of partial sums indeed holds.

Proposition 3.1. *If* $(X_1, X_2, \ldots, X_n) \preceq_{(s_1,s_2,\ldots,s_n)-icx} (Y_1, Y_2, \ldots, Y_n)$ *then*

$$(S_1, S_2, \ldots, S_n) \preceq_{(s_1,s_1+s_2,\ldots,s_1+s_2+\ldots+s_n)-icx} (T_1, T_2, \ldots, T_n).$$

3.2 Upper orthant order and orthant convex order

Considering $s_1 = s_2 = \ldots = s_n = 1$, we get the following corollary of Proposition 2.2 for the upper orthant order $\preceq_{\text{uo}}$. Consider the functions $h_i : \mathbb{R}^i \to \mathbb{R}^+$ in $\mathcal{U}_{(1,\ldots,1)-\text{icx}}$ for $i = 1, 2, \ldots, n$. Then, $(X_1, X_2, \ldots, X_n) \preceq_{\text{uo}} (Y_1, Y_2, \ldots, Y_n)$ implies

$$
\begin{aligned}
&\big(h_1(X_1), h_2(X_1, X_2), \ldots, h_n(X_1, X_2, \ldots, X_n)\big) \\
&\qquad \preceq_{(1,2,\ldots,n)-\text{icx}} \big(h_1(Y_1), h_2(Y_1, Y_2), \ldots, h_n(Y_1, Y_2, \ldots, Y_n)\big).
\end{aligned}
$$

In the case $s_1 = s_2 = \ldots = s_n = 1$, Proposition 3.1 gives

$$
\begin{aligned}
&(X_1, X_2, \ldots, X_n) \preceq_{\text{uo}} (Y_1, Y_2, \ldots, Y_n) \\
&\qquad \Rightarrow (S_1, S_2, \ldots, S_n) \preceq_{(1,2,\ldots,n)-\text{icx}} (T_1, T_2, \ldots, T_n).
\end{aligned}
$$

Thus, we get a multivariate extension of the implication (1.3) derived by Boutsikas and Vaggelatou (2002) who established that $S_k \preceq_{k-\text{icx}} T_k$ holds for $k = 1, 2, \ldots, n$.

The orthant convex order $\preceq_{\text{uo-cx}}$ corresponds to $s_1 = s_2 = \ldots = s_n = 2$. In this case, we get the following corollary of Proposition 2.2. Consider the functions $h_i : \mathbb{R}^i \to \mathbb{R}^+$ in $\mathcal{U}_{(2,\ldots,2)-\text{icx}}$ for $i = 1, 2, \ldots, n$. Then, $(X_1, X_2, \ldots, X_n) \preceq_{\text{uo-cx}} (Y_1, Y_2, \ldots, Y_n)$ implies

$$
\begin{aligned}
&\big(h_1(X_1), h_2(X_1, X_2), \ldots, h_n(X_1, X_2, \ldots, X_n)\big) \\
&\qquad \preceq_{(2,4,\ldots,2n)-\text{icx}} \big(h_1(Y_1), h_2(Y_1, Y_2), \ldots, h_n(Y_1, Y_2, \ldots, Y_n)\big).
\end{aligned}
$$

In the case $s_1 = s_2 = \ldots = s_n = 2$, Proposition 3.1 ensures that

$$
\begin{aligned}
&(X_1, X_2, \ldots, X_n) \preceq_{\text{uo-cx}} (Y_1, Y_2, \ldots, Y_n) \\
&\qquad \Rightarrow (S_1, S_2, \ldots, S_n) \preceq_{(2,4,\ldots,2n)-\text{icx}} (T_1, T_2, \ldots, T_n).
\end{aligned}
$$

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