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Unions and key players in network games with conflicts and spillovers

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Abstract

We study network games with social and private dissonance where each player in the network exerts some costly efforts. We allow for cooperative behavior in the sense that players may belong to unions and members of each union choose their efforts by maximizing the joint utility of the union. Each player not only benefits from the aggregate effort and efforts of network neighbors are strategic complements, but also suffers disutility when her effort differs from her neighbors' efforts or is inconsistent with her ideal effort. We characterize the unique Nash equilibrium of the network game with unions and we define a union intercentrality measure for finding the key player whose removal has the highest impact on the aggregate effort level. In addition, we explore the role of unions in fostering effort levels and we consider two alternative policies: the key addition to an existing union (the player who increases the most the aggregate effort by joining the union) and the key union that generates the highest total effort. Finally, we investigate the stability of unions.

Keywords: Social networks · peer effects · key players · unions · social and private dissonance

JEL Classifications: A14 · C72 · D85 · L14

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1 Introduction

Cooperation and collective acts are often a key factor in achieving a common goal or improving welfare in many social and economic contexts. Collaborative behavior is known to be present in human populations from ancient times.[1] Shared intentions are a basis for the formation of economic unions, climate treaties, military alliances, cooperatives, or labor unions. The interactions between people involved in the exchange of ideas, opinions, or beliefs, affect their behaviors that influence the well-being of the society. In addition, the decisions of agents, as individuals, households, firms, or countries, affect each other through the network of connections.[2] The composition and the structure of such networks, in turn, influence the evolution of cooperative behavior in the population.[3]

In this paper, we study a network game with social and private dissonance where players exert costly efforts and may belong to unions that allow for cooperative behavior.[4] We consider a global spillover effect between all players and an additional local spillover effect between neighbors in the network. Players get utility from the efforts exerted by all players with heterogeneous returns. Players suffer disutility when their efforts are different from those exerted by their neighbors. This disutility can be interpreted as social dissonance from non-compliance with the social norm.[5] In addition, players also incur disutility when they observe a discrepancy between their beliefs about the ideal effort and their actual efforts. This disutility can be described as private cognitive dissonance. The novelty of this paper is that we allow for cooperative behavior of the players. Players may belong to unions and members of each union choose their efforts by maximizing the joint utility of the union in the network.

We show that apart from individual efforts influencing the decisions of other players through spillovers, the efforts of all players in a connected network are additionally affected

[1] See Smith (2003) for a survey of the literature on evolution of human cooperation.

[2] Social and economic networks are not only the source of information transmission but also determine the structure of interdependence of individual outcomes on the actions of their neighbors (see Jackson, 2009). For instance, learning and information communication through social networks can result in the adoption of green products and innovative technologies, while the decisions about product or technology adoption are affected by potential spillovers from neighbors. Bramoullé, Djebbari and Fortin (2020) survey how behavioral spillovers and peer effects affect education, green behavior, technology adoption, and criminal behavior.

[3] Fowler and Christakis (2010) provide a survey about experiments showing that cooperative behavior spreads in social networks.

[4] See Galeotti, Goyal, Jackson, Vega-Redondo and Yariv (2010), Jackson and Zenou (2015), Bramoullé and Kranton (2016) for a comprehensive introduction to network games.

[5] That is, individuals are penalized if they deviate from the effort level of their neighbors in the social network. See e.g. Ballester, Calvó-Armengol and Zenou (2010), Patacchini and Zenou (2012), Liu, Patacchini and Zenou (2014), Boucher (2016), Lee, Liu, Patacchini and Zenou (2021), Atay, Mauleon, Schopohl and Vannetelbosch (2022) about peer effects and conformism in social networks.

by the union structure. That is, each player's contribution to the aggregate effort is affected not only by her idiosyncratic ideal effort, her return coefficient from global and local spillovers, her position in the network, but also by her membership to some union and the union structure. A union structure is simply a partition of the players into unions where members of a union do not need to be linked to each other in the network. Additionally, the efforts of the players in the network game are strategic complements: an increase of her neighbors' efforts fosters an increase of some given player's effort. We derive the unique Nash equilibrium of the network game. The equilibrium effort of each player is proportional to her Katz-Bonacich weighted centrality. Based on Ballester, Calvó-Armengol, and Zenou (2006) approach,[6] we define a union intercentrality measure for finding the key players in the network game with unions. The key player is the player in the network the removal of which entails the highest/lowest impact on the aggregate effort in the network. We look for both the positive key player, i.e. the player who once removed from the network has the highest impact on the total effort level, and the negative key player, i.e. the player who once removed from the network has the lowest impact.

We show that the key players in the network game with unions may differ from the key players in the same network game without unions as a result of an additional union effect in the player's union intercentrality. By means of examples, we highlight the role of social and private dissonance in the determination of the positive and negative key players. Social dissonance tends to decrease the effort of the positive key player while it has an ambiguous effect on the aggregate effort in the network. On the contrary, private dissonance tends to decrease the aggregate effort. Nevertheless, increasing the ideal effort of some player makes her more likely becoming the positive key player, and pushes up the effort exerted by others as well. Compared to the network game without unions, forming a union always increases the aggregate effort in the network.

Furthermore, we explore more in detail the role of unions in fostering effort levels across the network and we consider two alternative policies. First, we define the key addition to an existing union as the player who increases the most the aggregate effort in the network by joining the union. Second, we introduce the notion of key union. It is the union of a given size that generates the highest total effort compared to other unions of the same size. We find that the classic key player policy of removing the key player from the network may be dominated by the key addition and key union policy. For instance, instead of removing the key player, adding her to some union could increase the aggregate effort in the network even more. Finally, we briefly analyze the union structure that one might expect to emerge in the long run. We show that if the players are homogeneous and the

[6] Zenou (2016) gives an overview of the recent literature on key players in social and economic networks.

network structure is complete, then the grand union is stable and it is the union structure leading to the highest aggregate effort. However, once players become heterogeneous, the grand union is likely to be destabilized even though it may still be the best structure in terms of aggregate effort.

There are various examples of social and economic issues that can be described by games of spillovers and conflicts. Environmentally friendly behavior is an example where spillovers and miscoordination, together with private dissonance, affect individual choices. Studies show that social norms and private values are strong determinants of collective pro-environmental behaviors. Examples of collective pro-environmental behaviors are group activities, educational programs that aim to preserve the environment through recycling, collective clean-ups and beautification organized by communities. The collective action with a shared intention to promote green behavior is put through organizations and movements such as Greenpeace, Fridays for Future, or country-level treaties, such as the Paris agreement. Other factors behind the reasoning and pro-environmental decisions of individuals are the social influence and private values. The discomfort from acting differently from the neighbors, or the willingness to fit the social norms can drive the attitudes of individuals regardless of any monetary benefit.[7] In addition, private values or norms are another driving force of individual behavior.[8]

Another example of a network game with spillovers and conflicts is the adoption of technology when compatibility matters. For instance, the choice of technology such as messaging applications, software for video conferences, cloud computing services, or file transfer services is strongly affected by the choice of peers. A failure to coordinate creates incompatibility and one reason for such miscoordination can be personal preferences or ideal choices of individuals which can be based on trust or mistrust in the specific service provider, data protection issues, or simply personal habits.

The paper is organized as follows. In Section 2 we describe the network spillover game with social and private dissonance and we introduce unions. We characterize the unique Nash equilibrium efforts of the network game with unions. In Section 3 we define a union intercentrality measure for finding the key players in the network game with unions. In Section 4 we define the key addition to an existing union and we look for key unions. Finally, we investigate the stability of unions and we conclude.

[7]Ando, Ohnuma, Blöbaum, Matthies and Sugiura (2010) study the determinants of individual and collective environmentally conscious behaviors. They find that subjective norms from the network impact substantially the collective pro-environmental behaviors.

[8]For instance, Viscusi, Huber and Bell (2011) find that individuals who consider themselves environmentalists are more likely to be concerned about private and external norms experiencing green guilt from acting less environmentally friendly compared to their neighbors. See e.g. Gifford and Nilsson (2014) for an extensive survey of studies about personal and social determinants of pro-environmental behavior.

2 Unions in network games

2.1 The network spillover game

We study a network game with continuous action space and strategic complementaries between the actions of the neighbors. We have a set $N = \{1, 2, ..., n\}$ of players connected through a directed network $\boldsymbol{g}$ where g_{ij} is the positive weight on the directed link from player i to j for any neighbors $i, j \in N$ in the network. The weight is zero, $g_{ij} = 0$, when such directed link is absent. By convention, we assume there are no self-loops in the network. Thus, $g_{ii} = 0$ for all $i \in N$. We use the weighted adjacency matrix $\mathbf{G}^*$ associated with the network $\boldsymbol{g}$, with elements $g^*_{ij} \in [0, 1]$ corresponding to the original weights g_{ij} normalized by out-degrees $d^+_i(\boldsymbol{g})$ of player i, that is, her weighted number of outgoing links. For simplicity we assume that the network does not contain disconnected nodes, so that $d^+_i(\boldsymbol{g}) > 0$ for all players. Thus, the matrix $\mathbf{G}^*$ is row-normalized: $\sum_{j \in N} g^*_{ij} = 1$ for all $i \in N$.

The players are endowed with personal ideals regarding their own behavior. We denote the ideal efforts of the players with $y_i \in \mathbb{R}^+$.[9] They choose their efforts $x_i \in \mathbb{R}^+$ based on spillovers and conflicts from the neighbors in the network, while trying to remain consistent with their personal ideal effort levels to avoid disutility from private cognitive dissonance. We define this private dissonance as the discrepancy between the actual and ideal efforts. We distinguish two types of spillovers in the model: global and local spillovers. Similarly to public goods games, the players benefit from the efforts exerted by all players in the network. Yet, unlike public goods in networks studied by Bramoullé and Kranton (2007) or local-aggregate or -average models,[10] we assume that these spillovers are not restricted to direct network neighbors, thus we refer to those as global spillovers. Conversely, the local interactions between neighbors cause local spillovers. These are the spillovers occurring from complementarities in player's effort x_i and her descriptive social norm $\sum_{j \in N} g^*_{ij} x_j$,[11] which in network games is commonly referred to simply as social norm of player i. While the descriptive social norms is the representation of social

[9] The ideal efforts can be perceived as a personal norm of a player, personal standard, an attitude towards the specific issue, feeling of moral obligation, or self-expectations (disregarding the possibility that these expectations may derive from socially shared norms (Ajzen and Fishbein, 1970; Schwartz, 1973; Schwartz and Howard, 1980). While non-compliance with, or the violation of the personal norm may results in feeling of guilt, loss of self-esteem, self-depreciation (Schwartz, 1973), this discrepancy between the self-standard, or personal norm, and actual behavior causes aroused cognitive dissonance (Stone and Cooper, 2001).

[10] See e.g. Liu, Patacchini and Zenou (2014) and Ushchev and Zenou (2020).

[11] While players enjoy the collective effort of the whole population (the public good), their behavior is directly affected by those of their direct social contacts. Therefore, we assume no effort complementarities through global spillovers.

behavior, the injunctive norm is the perception of social approval. We assume that the disapproval is costly and players suffer disutility from social dissonance, or conflict, caused by the difference in their efforts and those of each of their neighbors. In other words, the deviation from injunctive norm, that is, the difference in behavior with each of direct neighbors, results in disutility. Minimization of this cost of conflict leads to conformism similarly to conventional local-average models with descriptive social norms. As also noted by Ushchev and Zenou (2020), this form of representation is still a conformist model and does not affect the choice of equilibrium efforts compared to the more common representation. However, it results in different welfare outcomes due to different utilities in equilibrium.

Let $\mathbf{x} = (x_1, ..., x_n)$ be the vector of efforts of players in the network $\boldsymbol{g}$, and $\mathbf{x}_{-i}$ the list of efforts of players $N \setminus \{i\}$. The utility of player i with effort x_i and idiosyncratic ideal y_i is then given by the linear-quadratic function:

$$u_i(x_i, \mathbf{x}_{-i}, y_i, \theta_i, \boldsymbol{g}) = \underbrace{\lambda_1 \theta_i \sum_{j \in N} x_j}_{\substack{\text{global spillover} \\ \text{effect}}} + \underbrace{\lambda_2 \theta_i x_i \sum_{j \in N} g^*_{ij} x_j}_{\substack{\text{local spillover} \\ \text{effect}}} - \underbrace{\frac{\omega_1}{2} \sum_{j \in N} g^*_{ij} (x_i - x_j)^2}_{\substack{\text{cost of} \\ \text{social dissonance}}} - \underbrace{\frac{\omega_2}{2} x_i^2}_{\text{effort cost}} - \underbrace{\frac{\omega_3}{2} (x_i - y_i)^2}_{\substack{\text{cost of} \\ \text{private dissonance}}}, \quad (1)$$

where $\lambda_1,\ \lambda_2,\ \omega_1,\ \omega_2$ and ω_3 are positive coefficients of the respective source of spillovers and costs. The players benefit from spillovers and suffer quadratic cost from their social dissonance, effort, and private dissonance. The coefficients λ_1 and λ_2 capture the effects of global and local spillovers, ω_1 is the sensitivity to conflict or taste for conformity, and ω_2 is the coefficient of effort cost. The sensitivity to cognitive dissonance or the taste for consistency with ideal behavior is represented by ω_3. Additionally, the players are heterogeneous in their return on spillovers captured by parameters $\theta_i \in \mathbb{R}^+$, for all $i \in N$. One can see that given the positive coefficients and parameters θ_i, the efforts of network neighbors are strategic complements:

$$\frac{\partial^2 u_i}{\partial x_i \partial x_j} = (\lambda_2 \theta_i + \omega_1) g^*_{ij} \geq 0. \quad (2)$$

Utility representation in (1) captures the benefits of players through positive spillovers from aggregate effort in a given network $\boldsymbol{g}$ and their direct neighbors, as well as the peer pressure to exert efforts close to those of the neighbors as an incentive to minimize the social dissonance and avoid miscoordination and conflict. In addition, the players have an incentive to be consistent with one's personal ideal y_i.[12] The efforts incur quadratic

[12] In network games literature, the notion of ideal efforts and consistency is present in works by Olcina, Panebianco and Zenou (2017) and Galeotti, Golub, Goyal and Rao (2021).

costs independently from the network structure and personal ideal effort.

Idiosyncratic ideal y_i together with parameter θ_i can be interpreted as the type of player i in the network. By setting either of the two parameters to be homogeneous across the network, we let the other indicate the player's type.

Utility representation in (1) can be rewritten as

$$u_i = (\lambda_1\theta_i + \omega_3 y_i)x_i + \lambda_1\theta_i \sum_{j\in N\setminus\{i\}} x_j + \sum_{j\in N\setminus\{i\}} g^*_{ij}\left((\lambda_2\theta_i + \omega_1)x_i - \frac{\omega_1}{2}x_j\right)x_j - \frac{1}{2}(\omega_1+\omega_2+\omega_3)x_i^2 - \frac{\omega_3}{2}y_i^2. \quad (3)$$

where we observe the direct effect of the player's effort on her utility, with type (y_i, θ_i) defining the return $(\lambda_1\theta_i + \omega_3 y_i)$ on effort, and $\frac{1}{2}(\omega_1+\omega_2+\omega_3)$ being the coefficient of its quadratic cost. We also observe a quadratic cost inflicted by player's personal ideal, $\frac{\omega_3}{2}y_i^2$. The component exhibiting the utility from interaction with neighbors shows the positive complementarities in their efforts with coefficient $(\lambda_2\theta_i + \omega_1)$, as already noted above, and a cost that player i bears from the effort x_j of her neighbors with coefficient $\frac{\omega_1}{2}$. It directly follows that the overall impact of direct interaction with neighbor j of player i on her utility is positive if [13]

$$x_j < 2\left(\frac{\lambda_2\theta_i + \omega_1}{\omega_1}\right)x_i. \quad (4)$$

2.2 Unions

Players may belong to unions or coalitions. Each player can be a member of at most one union and each union regroups players in the network $\boldsymbol{g}$ who aim at maximizing their joint utility. A union structure $P = \{p_1, p_2, \cdots, p_m\}$ is a partition of the set of players such that $p_i \cap p_j = \emptyset$ for $i \neq j$ and $\bigcup_{i=1}^m p_i = N$. Let m be the cardinality of P (i.e., the number of unions in P). We denote by $p(i) \in P$ the union to which player i belongs to. Player $i \in N$ is said to be a singleton player if she is the only player to belong to her union, i.e. $p(i) = \{i\}$. Let $\widehat{P} = \{N\}$ be the grand union, i.e. the union that regroups all players. Let $\widetilde{P}$ be the union structure such that all players are singleton players, i.e. $\widetilde{p}(i) = \{i\}$ for all $i \in N$. Given some union structure P, let $S_P \subseteq N$ be the set of players who are singleton players in P, i.e. $p(i) = \{i\}$ for all $i \in S_P$.[14] Player $i \in N$ is said to be a union player if she is not the only player to belong to her union, i.e. $p(i) \supset \{i\}$. So, $N \setminus S_P$ is the set of union players in P.

[13] We consider a fixed network structure. Similarly to Arifovic, Eaton and Walker (2015) one can study the dynamics of network structure and personal ideals. Yet, the study of network evolution is outside the scope of this paper.

[14] Throughout the paper we use the notation $\subseteq$ for weak inclusion and $\subset$ for strict inclusion. Finally, # will refer to the notion of cardinality.

For any union $p(i) \in P$ in the network $\boldsymbol{g}$, the utility of union $p(i)$ is simply the joint utility of all union members and is given by:

$$U_{p(i)} = \sum_{j \in p(i)} u_j. \tag{5}$$

Hence, given the utility of a singleton player defined in (1) we get:

$$U_{p(i)} = \sum_{j \in p(i)} \Big(\lambda_1 \theta_j \sum_{k \in N} x_k + \lambda_2 \theta_j x_j \sum_{k \in N} g^*_{jk} x_k - \frac{\omega_1}{2} \sum_{k \in N} g^*_{jk} (x_j - x_k)^2 - \frac{\omega_2}{2} x_j^2 - \frac{\omega_3}{2} (x_j - y_j)^2 \Big). \tag{6}$$

When $p(i) = \{i\}$ for $i \in S_P$, then $U_{\{i\}} = \sum_{j \in \{i\}} u_j = u_i$.

2.3 Nash equilibrium efforts with unions

Let α_i and ϕ_i be the coefficients of player i's utility from her own effort and of her neighbors' efforts weighted by the overall cost coefficient associated with player i's effort, respectively. That is,

$$\alpha_i = \frac{\lambda_1 \Theta_i + \omega_3 y_i}{\Omega_i}, \qquad \phi_i = \frac{\lambda_2 \theta_i + \omega_1}{\Omega_i}, \qquad \gamma_{ij} = \begin{cases} \dfrac{\lambda_2 \theta_j + \omega_1}{\Omega_i} & \text{if } p(i) = p(j) \\ 0 & \text{otherwise} \end{cases}, \tag{7}$$

$$\text{where} \qquad \Theta_i = \sum_{j \in p(i)} \theta_j \qquad \text{and} \qquad \Omega_i = \omega_1 \left(1 + \sum_{j \in p(i) \setminus \{i\}} g^*_{ji} \right) + \omega_2 + \omega_3.$$

Notice that for any singleton player $i \in S_P$, $\gamma_{ij} = 0$ for all $j \in N$. Let $\boldsymbol{\alpha}$ and $\boldsymbol{\phi}$ be n-dimensional vectors of α_i and ϕ_i respectively, and $\mathbf{I}$ be n-identity matrix. Further, let $[\boldsymbol{\phi} \otimes \mathbf{G}^*]$ be a matrix formed by coordinate-by-coordinate multiplication of the vector $\boldsymbol{\phi}$ and the matrix $\mathbf{G}^*$.[15] Let $\mathbf{H}$ be a matrix with elements h_{ij} that represent the union relation of any pair of players:

$$h_{ij} = \gamma_{ij} g^*_{ji} \tag{8}$$

for all $j \in N$. So we have $h_{ij} > 0$ for all $j \in p(i)$ with $g^*_{ji} > 0$.[16]

[15] Throughout the paper we use the symbol $\otimes$ to define matrix formed by coordinate-by-coordinate multiplication of a vector and a matrix, i.e. by each i-th row of the matrix multiplied by corresponding i-th element of the vector. Thus the matrix $[\boldsymbol{\phi} \otimes \mathbf{G}^*]$ is an $(n \times n)$ matrix resulting from multiplying each i-th element of the vector $\boldsymbol{\phi}$ with i-th row of the matrix $\mathbf{G}^*$ for each $i \in N$.

[16] In matrix terms $\mathbf{H} = \boldsymbol{\Gamma} \otimes \mathbf{G}^{*T}$ with $\boldsymbol{\Gamma}$ being a matrix of parameters γ_{ij} and $\otimes$ an element-by-element multiplication of the two matrices.

Using the notations above and given the definition of union utility in (6) we can find the best response for each player $i \in N$ in the network with unions. We obtain that the best response functions $BR_i(\mathbf{x}_{-i}; \mathbf{G}^*, \mathbf{H}, \phi_i, \alpha_i)$ are linear in players' efforts $\mathbf{x}_{-i}$:

$$\begin{aligned} BR_i(\mathbf{x}_{-i}; \mathbf{G}^*, \mathbf{H}, \phi_i, \alpha_i) &= \alpha_i + \phi_i \sum_{k \in N} g^*_{ik} x_k + \sum_{k \in N} \gamma_{ik} g^*_{ki} x_k \\ &= \alpha_i + \sum_{k \in N} (\phi_i g^*_{ik} + h_{ik})\, x_k. \end{aligned} \tag{9}$$

Compared to the best response without unions, the term $\sum_{k \in N} \gamma_{ik} g^*_{ki} x_k$ in the best response of the player is an additional term that captures the influence from the union peers of player i on her effort. In particular, this influence is non-zero when the union peers have directed links towards the player.

The following proposition provides the Nash equilibrium efforts of players in the network spillover game with unions.

Proposition 1. *Assume that the spectral radius of $[\boldsymbol{\phi} \otimes \mathbf{G}^* + \mathbf{H}]$ is smaller than 1. Then, the unique Nash equilibrium in pure strategies is given by*

$$\mathbf{x}^* = (\mathbf{I} - [\boldsymbol{\phi} \otimes \mathbf{G}^* + \mathbf{H}])^{-1} \boldsymbol{\alpha}. \tag{10}$$

All proofs can be found in the appendix. Given the strategic complementarities in efforts of players in the network game it is easy to see that an increase in efforts of players through formation of a union positively affects the effort of the rest in the network. The following proposition establishes the presence of strategic complementarity in efforts in the game with unions.

Proposition 2. *An increase in effort x_i of any player $i \in N$ in the network with union structure P weakly increases the efforts of all players in the network.*

While the efforts of players in the network game with unions are strategic complements, the effort of a singleton player does not always increase by joining a union. The following lemma provides a sufficient condition for the efforts of players and the aggregate effort in the network with unions being higher than those of the network with only singleton players.

Lemma 1. *Consider any network $\boldsymbol{g}$ and its row-normalized weighted adjacency matrix $\mathbf{G}^*$, with ideal efforts y_i and type coefficients θ_i for all players $i \in N$. Let $l \in p(i) \setminus \{i\}$ be such that $l = \arg\min_{k \in p(i) \setminus \{i\}} (\lambda_2 \theta_k + \omega_1) \frac{g^*_{ki}}{g^*_{ik}}$, $p(i) \in P$. If*

$$\frac{\omega_1 \sum\limits_{j \in p(i) \setminus \{i\}} g^*_{ji}}{\omega_1 + \omega_2 + \omega_3} \leq \min \left\{ \frac{(\lambda_2 \theta_l + \omega_1) g^*_{li}}{(\lambda_2 \theta_i + \omega_1) g^*_{il}}, \frac{\lambda_1 \sum\limits_{j \in p(i) \setminus \{i\}} \theta_j}{\lambda_1 \theta_i + \omega_3 y_i} \right\} \tag{11}$$

then the union $p(i)$ increases the equilibrium effort of player i with respect to the network game with $\widetilde{P}$. Moreover, if (11) holds for all $i \in N \setminus S_P$, then the union structure P increases the aggregate effort in the network with respect to the network game with $\widetilde{P}$.

Being in a union increases player i's utility through global spillovers. At the same time, the local spillovers and the social dissonance of union peers that are influenced by player i (all $j \in p(i) \setminus \{i\}$ such that $g^*_{ji} > 0$) now affect player i's choice of effort directly. Thus, the union $p(i)$ increases the effort of player i if the positive spillovers received from the union outweigh the cost of conflict created by her participation.

As a benchmark we consider now the special case without unions that reverts to the case where the union structure is $\widetilde{P}$ and consists of only singleton players. Remember that in $\widetilde{P}$, we have $\widetilde{p}(i) = \{i\}$ for all $i \in N$.

Lemma 2. *Suppose the union structure is $\widetilde{P}$. Let ρ be the spectral radius of the matrix $[\boldsymbol{\phi} \otimes \mathbf{G}^*]$. For any network $\boldsymbol{g}$ and types θ_i of players $i \in N$, if $\max_i \theta_i < \dfrac{\omega_2 + \omega_3}{\lambda_2}$ then the condition $\rho < 1$ is satisfied.*

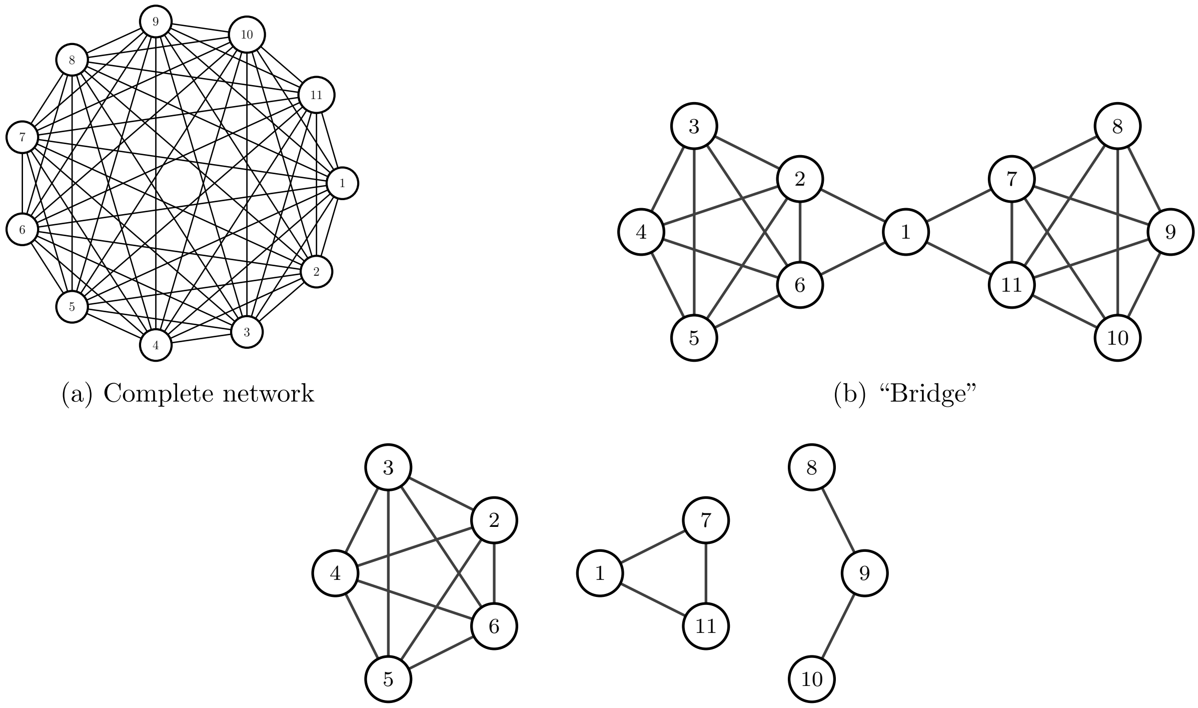
We now derive the equilibrium efforts for the class of homogeneous network games where all players are singletons and have the same ideal effort and return parameter: $y_i = y$, $\theta_i = \theta$ and $p(i) = \{i\}$ for all $i \in N$. Hence, $\alpha_i = \alpha$ and $\phi_i = \phi$ for all $i \in N$. The next corollary shows that if the network is such that each node has at least one outgoing link, then the equilibrium efforts will be homogeneous.

Corollary 1. *Suppose the union structure is $\widetilde{P}$. Take $\alpha_i = \alpha$ and $\phi_i = \phi$ for all $i \in N$. For any network $\boldsymbol{g}$ and its row-normalized weighted adjacency matrix $\mathbf{G}^*$, if $\theta < \dfrac{\omega_2 + \omega_3}{\lambda_2}$ then the unique Nash equilibrium in pure strategies is given by*

$$x_i^* = \frac{\alpha}{1 - \phi} \text{ for all } i \in N. \tag{12}$$

Using the original notations in (7) we find that $x^* = \dfrac{\lambda_1 \theta + \omega_3 y}{\omega_2 + \omega_3 - \lambda_2 \theta}$ for each player. As expected, in the homogeneous setup without unions of Corollary 1, the coefficient of social dissonance ω_1 does not affect the equilibrium outcomes due to absence of conflict when all neighbors are identical in their types. The most basic example satisfying Corollary 1 is the complete network. Given equal weights on each link, nodes are equivalent to each other. With homogeneous parameters of the model, the spillover game on the complete network is symmetric, and thus equilibrium efforts are equal for all players.

More interestingly, regardless of the network structure $\boldsymbol{g}$ and its size, the equilibrium efforts of all players in the network are still identical and depend only on the coefficients of the game, ideal efforts and type parameter values. This result is mainly driven by the fact that we associate any network with its row-normalized weighted adjacency matrix.



(a) Complete network

(b) “Bridge”

(c) Network with 3 components

Figure 1: Three network configurations.

Figure 1 illustrates three networks of the same size, yet with different structures. The three networks have players exerting the same effort in equilibrium. It also implies that the aggregate utilities and efforts are the same in three networks. This property of symmetry in players’ effort choices regardless of the network structure, and given that the type parameters are homogeneous, allows us to distinguish the effects of model coefficients for a given network when heterogeneity is introduced.

Assume there is a subset $C \subset N$ of players in the directed network $\boldsymbol{g}$ who forms a strongly connected component with no path connecting them to the rest of the network. In addition to strong connectivity, let $g^*_{ij} = 0$ for any $i \in C$ and any $j \in N \setminus C$.

Corollary 2. *Suppose the union structure is $\widetilde{P}$, $\boldsymbol{g}$ is such that a subset $C \subset N$ of players forms a strongly connected component with no outgoing path towards players in $N \setminus C$, and $y_i = y^c$, $\theta_i = \theta^c$ for all $i \in C$. If $\theta^c < \dfrac{\omega_2 + \omega_3}{\lambda_2}$ then the unique Nash equilibrium in pure strategies for each player in C is given by*

$$x^{c*} = \frac{\lambda_1 \theta^c + \omega_3 y^c}{\omega_2 + \omega_3 - \lambda_2 \theta^c}.$$

3 Key players

3.1 Finding the key player

We adopt the notion of key players introduced by Ballester, Calvó-Armengol and Zenou (2006) to find the most influential players in the network. To identify the key players we define the intercentrality (or key player centrality) measure for directed networks analogous to the one of Ballester, Calvó-Armengol and Zenou (2006) for our network game on the directed network $\boldsymbol{g}$, represented by row-normalized weighted adjacency matrix $\mathbf{G}^*$. We use the Katz-Bonacich centrality as an established measure of player's contribution in the aggregate outcomes of the network.

Introduced by Katz (1953) and redefined later by Bonacich (1987), the Katz-Bonacich centrality measures the centrality of player i as the total number of all possible paths that stem from i, with each path weighted inversely to its length. Thus, the higher number of shorter paths connecting the player to others in the network increases her Katz-Bonacich centrality.

Let

$$\mathbf{M}(\mathbf{G}^*, \boldsymbol{\phi}, \mathbf{H}) := (\mathbf{I} - [\boldsymbol{\phi} \otimes \mathbf{G}^* + \mathbf{H}])^{-1}$$

be well defined and non-negative with entries m_{ij}, $i, j \in N$. Then the vector of weighted Katz-Bonacich centralities of players is given by

$$\boldsymbol{b}(\mathbf{G}^*, \mathbf{H}, \boldsymbol{\phi}, \boldsymbol{\alpha}) := \boldsymbol{M}(\mathbf{G}^*, \boldsymbol{\phi}, \mathbf{H})\boldsymbol{\alpha}.$$

This measure corresponds to the equilibrium efforts $\mathbf{x}^*$ of the network game $(\mathbf{G}^*, \mathbf{H}, \boldsymbol{\phi}, \boldsymbol{\alpha})$ defined in Proposition 1. The effort of each player i is then simply given by

$$b_i(\mathbf{G}^*, \mathbf{H}, \boldsymbol{\phi}, \boldsymbol{\alpha}) = \sum_{k \in N} m_{ik}\alpha_k.$$

Let $\mathcal{B}(\mathbf{G}^*, \mathbf{H}, \boldsymbol{\phi}, \boldsymbol{\alpha})$ denote the total effort of all players,

$$\mathcal{B}(\mathbf{G}^*, \mathbf{H}, \boldsymbol{\phi}, \boldsymbol{\alpha}) := \sum_{j \in N} b_j(\mathbf{G}^*, \mathbf{H}, \boldsymbol{\phi}, \boldsymbol{\alpha}).$$

Next, we define the contribution of player i to the total effort in the network game with unions.

Definition 1. The contribution $\delta_{i,P}$ of player i to the total effort in the network with union structure P is the difference between the aggregate effort exerted in the initial game $(\mathbf{G}^*, \mathbf{H}, \boldsymbol{\phi}, \boldsymbol{\alpha})$ and the new game $(\mathbf{G}^{*-i}, \mathbf{H}^{-i}, \boldsymbol{\phi}^{-i}, \boldsymbol{\alpha}^{-i})$ after removing player i from the network:

$$\delta_{i,P}(\mathbf{G}^*, \mathbf{H}, \boldsymbol{\phi}, \boldsymbol{\alpha}) = \mathcal{B}(\mathbf{G}^*, \mathbf{H}, \boldsymbol{\phi}, \boldsymbol{\alpha}) - \mathcal{B}(\mathbf{G}^{*-i}, \mathbf{H}^{-i}, \boldsymbol{\phi}^{-i}, \boldsymbol{\alpha}^{-i}).$$

Notice that $\mathbf{G}^{*-i}$ is the row-normalized weighted adjacency matrix of the new network $\boldsymbol{g}^{-i}$. By removing player i from the network $\boldsymbol{g}$ we automatically assume her being removed from the union $p(i)$. Given that, the matrix $\mathbf{H}^{-i}$ and the vectors $\boldsymbol{\alpha}^{-i}$ and $\boldsymbol{\phi}^{-i}$ are constructed by reevaluating the respective entries of $\mathbf{H}$, $\boldsymbol{\alpha}$ and $\boldsymbol{\phi}$ through the change in parameters Θ_j and Ω_j in (7) and the weighted adjacency matrix $\mathbf{G}^{*-i}$ of the network $\boldsymbol{g}^{-i}$ for each $j \in N \setminus \{i\}$. Accordingly, we define $\mathbf{M}^{-i} = (\mathbf{I}_{n-1} - [\boldsymbol{\phi}^{-i} \otimes \mathbf{G}^{*-i}] - \mathbf{H}^{-i})^{-1}$ with new entries m_{ij}^{-i}. We can then find the relation of the player's contribution to the aggregate effort with her intercentrality and the additional effect resulting from her membership in a union. Player's union intercentrality is the combination of her intercentrality and a union effect.

Proposition 3. *The contribution of player i to the aggregate effort in the network with union structure P is given by her union intercentrality*

$$\delta_{i,P}(\mathbf{G}^*, \mathbf{H}, \boldsymbol{\phi}, \boldsymbol{\alpha}) = \underbrace{\sum_{k \in p(i)\setminus\{i\}} \sum_{j \in N} m_{jk}^{-i}(\alpha_k - \alpha_k^{-i})}_{\textit{union effect}} + \underbrace{\frac{b_i(\mathbf{G}^*, \mathbf{H}, \boldsymbol{\phi}, \boldsymbol{\alpha})}{m_{ii}} \sum_{j \in N} m_{ji}}_{\textit{intercentrality}} . \tag{13}$$

As part of her union intercentrality, the membership of player i to union $p(i)$ induces a union effect on her union peers through additional spillovers.[17] Given a non-negative matrix $\mathbf{M}^{-i}$, if $\alpha_k \geq \alpha_k^{-i}$ for all $k \in p(i)$, it is sufficient to say that the union effect in player's intercentrality is positive. Using the definition of α_k in (7) we can find a condition for positive union effect in player's intercentrality. The following lemma provides this condition.

Lemma 3. *For the union effect in union intercentrality of player i to be positive, it is sufficient to have either of the following conditions satisfied for all $i \in N \setminus S_P$ and $k \in p(i) \setminus \{i\}$:*

$$g_{ik}^* = 0 \qquad \textit{or} \qquad \frac{\lambda_1 \theta_k}{\omega_1 g_{ik}^*} \geq \frac{\lambda_1 \Theta_k + \omega_3 y_k - \lambda_1 \theta_i}{\Omega_k - \omega_1 g_{ik}^*}.$$

One may notice that the union effect in $\delta_{i,P}(\mathbf{G}^*, \mathbf{H}, \boldsymbol{\phi}, \boldsymbol{\alpha})$ disappears if player i becomes a singleton player, $i \in S_P$.

[17] Throughout the paper we assume the network $\boldsymbol{g}$ to be represented by weighted adjacency matrix such that the weights on outgoing links of each player i sum up to 1, i.e. $d_i^+(\boldsymbol{g}) = \sum_{j=1}^n g_{ij} = 1$. Changing this assumption affects the parameters $\alpha_i, \phi_i, \gamma_{ij}$ through the cost Ω_i which will then be as follows: $\Omega_i = \omega_1(d_i^+(\boldsymbol{g}) + \sum_{j \in p(i)\setminus\{i\}} g_{ji}) + \omega_2 + \omega_3$. The change in Ω_i entails a change in the definition of union intercentrality and with an additional neighborhood effect from the set of neighbors $N(i)$ of player i along with the union effect: $\delta_{i,P}(\mathbf{G}^*, \mathbf{H}, \boldsymbol{\phi}, \boldsymbol{\alpha}) = \sum_{\substack{k \in p(i) \cup N(i) \\ k \neq i}} \sum_{j \in N} m_{jk}^{-i}(\alpha_k - \alpha_k^{-i}) + \delta_i(\mathbf{G}^*, \mathbf{H}, \boldsymbol{\phi}, \boldsymbol{\alpha})$.

Corollary 3. *The contribution of a singleton player $j \in S_P$ to the aggregate effort in the network game with union structure P is given by her union intercentrality $\delta_{j,P}$,*

$$\delta_{j,P}(\mathbf{G}^*, \mathbf{H}, \boldsymbol{\phi}, \boldsymbol{\alpha}) = \frac{b_j(\mathbf{G}^*, \mathbf{H}, \boldsymbol{\phi}, \boldsymbol{\alpha})}{m_{jj}} \sum_{k \in N} m_{kj} \tag{14}$$

for all $j \in S_P$.

Using the definitions of the union intercentrality for players in the game with unions, we can now define the positive and negative key players in the network game with unions, i.e. the players who have the highest and the lowest impact on the aggregate effort in the game regardless of their affiliation to any union in the network.

Definition 2. The positive key player i^*_+ is the player with the highest union intercentrality:

$$i^*_+ = \arg\max_{j \in N} \delta_{j,P}(\mathbf{G}^*, \mathbf{H}, \boldsymbol{\phi}, \boldsymbol{\alpha}).$$

The negative key player i^*_- is the player with the lowest union intercentrality:

$$i^*_- = \arg\min_{j \in N} \delta_{j,P}(\mathbf{G}^*, \mathbf{H}, \boldsymbol{\phi}, \boldsymbol{\alpha}).$$

The problem of finding the key players defined above is equivalent to finding the $\arg\min_i \mathcal{B}(\mathbf{G}^{*-i}, \mathbf{H}^{*-i}, \boldsymbol{\phi}^{-i}, \boldsymbol{\alpha}^{-i})$ and $\arg\max_i \mathcal{B}(\mathbf{G}^{*-i}, \mathbf{H}^{*-i}, \boldsymbol{\phi}^{-i}, \boldsymbol{\alpha}^{-i})$ for positive and negative key players, respectively.

3.2 A benchmark: the key player without unions

As a benchmark we consider now the case without unions, i.e. $P = \widetilde{P}$. The contribution of a player to the aggregate effort in the network without unions is given by Expression (14) of her intercentrality. Looking at the benchmark without unions allows us to disentangle the effects of spillovers and social and private dissonance of players in network games by studying specific examples. We find the key players and discuss the resulting equilibrium outcomes given alternative parameter settings.

Example 1 (**Network spillovers**)**.** Consider the networks in Figure 1a and 1b, with players $N = \{1, \ldots, 11\}$. For simplicity, we assume there is no social and private dissonance, $\omega_1 = \omega_3 = 0$, and the effort cost coefficient $\omega_2 = 1$. The global spillovers have a coefficient of $\lambda_1 = 1$ and the local spillovers' coefficient is $\lambda_2 = 0.1$.

Given the initial parameter values we need the types θ_i to satisfy the condition in Lemma 2. As a reference point we use the game with homogeneous players. Recalling Corollary 1 we choose a unique $\theta < \frac{\omega_2}{\lambda_2}$ for all players. Setting $\theta = 1$ we find the unique Nash equilibrium effort equal to 1.11. The intercentrality δ_i for all players is the same and is equal to their efforts. Thus, in the homogeneous game there are no distinguished key players. We introduce heterogeneity by assigning higher type $\theta_i = 2$ to one of the players in given networks. In Table 1 we show how the total effort and key players in the network change with the introduction of a higher type player. As all players in the homogeneous game on the complete network (Figure 1a) are structurally equivalent, it does not matter which player is assigned a higher type. We see that the presence of a higher type player in the network increases the total effort in the network making the player herself the positive key player with higher intercentrality compared to that of the others'. Unlike complete networks, in the bridge network in Figure 1b the choice of the player matters due to structural differences. Exploiting the symmetries in this network we identify equivalent nodes which reduces the number of players we need to study down to players 1, 2 and 3. We see in Table 1 that while the positive key player is the one with higher type in each of the three settings, the highest aggregate effort in the network among the three is reached when the higher type is assigned to the player in position 2. When player 1 is the higher type player, the negative key players are those that are the furthest from 1 in the network. In case of player 2, when $\theta_2 = 2$, the negative key player is her direct neighbor, player 6, which can be explained by lack of spillovers from 6 given the absence of social links other than those of player 2. Player 3 when assigned $\theta_3 = 2$ has the highest intercentrality of all. Yet this setting makes player 1 a negative key player, resulting in a lower level of aggregate effort compared to the case of $\theta_2 = 2$.

Example 2 (Social dissonance). We change the setup in Example 1 by adding the social conflict. We set the conflict cost coefficient to $\omega_1 = 1$ in the games on networks 1a and 1b. So, $\omega_1 = \omega_2 = 1$, $\omega_3 = 0$, $\lambda_1 = 1$ and $\lambda_2 = 0.1$.

In the homogeneous game, where the players are identical in their types, the introduction of social conflict does not cause any change in outcomes. With heterogeneity in the player types the conflict with neighbors can affect equilibrium efforts. Assume all players but player i have their return parameter equal to 1, $\theta_j = 1$ for all $j \in N \setminus i$, and assign $\theta_i = 2$.

In the game on the complete network the conflict with neighbors, or the social dissonance, increases the aggregate effort in the game. As it is displayed in Table 1, when player 1 is the higher type player her effort in the game with social dissonance is lower than that of the game with $\omega_1 = 0$. While the higher type player has the highest intercentrality, thus is the positive key player, she has a lower effort due to peer pressure. The negative key players, thus the rest of the players in the complete network, increase their efforts when social dissonance is introduced. As a result, the presence of disutility from deviation from the social norm increases the efforts of the players.

Like in the previous example, the bridge network in 1b requires to study the effects of the social dissonance in three different cases by assuming players 1, 2 or 3 as the higher type player. While the efforts of negative key players i^*_- are increasing in all three cases, their intercentralities δ_i^- are decreasing. As in the game on the complete network, the efforts of the positive key players are decreasing in all three games on the bridge network when the conflict is introduced. While only the intercentrality of positive key player 2 increases with her having a higher type. The total effort is decreasing with the presence of conflict with neighbors when player 1 or 3 have higher type, while in case of $\theta_2 = 2$ the total effort in the game is increasing. The results are displayed in Table 1.

Conflict	**Type**	**Total effort**	**Positive key player**			**Negative key player**		
ω_1	$\theta_{j\neq i} = 1$	$\mathcal{B}$	$+$	b_+	δ_+	$-$	b_-	δ_-
	$\theta_i = 1$	12.2223	N	1.1112	1.1112	N	1.1112	1.1112
Complete Network								
$\omega_1 = 0$	$\theta_{i=1} = 2$	13.4582	$\{1\}$	2.2247	2.348	$N \setminus \{1\}$	1.1234	1.111
$\omega_1 = 1$	$\theta_{i=1} = 2$	13.464	$\{1\}$	1.7056	2.353	$N \setminus \{1\}$	1.1759	1.1104
Bridge Network								
$\omega_1 = 0$	$\theta_{i=1} = 2$	13.4365	$\{1\}$	2.2268	2.326	{3},{4},{5}, {8},{9},{10}	1.1124	1.0984
	$\theta_{i=2} = 2$	13.4838	$\{2\}$	2.228	2.373	$\{6\}^*$	1.1359	1.0948
	$\theta_{i=3} = 2$	13.4478	$\{3\}$	2.2277	2.337	$\{1\}^*$	1.1124	1.0988
$\omega_1 = 1$	$\theta_{i=1} = 2$	13.3791	$\{1\}$	1.7192	2.268	{3},{4},{5}, {8},{9},{10}	1.1443	1.0407
	$\theta_{i=2} = 2$	13.5847	$\{2\}$	1.7479	2.474	$\{6\}^*$	1.2413	1.0533
	$\theta_{i=3} = 2$	13.4104	$\{3\}$	1.7429	2.3	$\{1\}^*$	1.1444	1.0479

$*$ Despite being the negative key player, player 6 in the bridge network with $\theta_2 = 2$ and player 1 in the bridge network with $\theta_3 = 2$, they are not the players with lowest equilibrium effort. In both games the lowest effort is exerted by players 8, 9 and 10.

Table 1: Social dissonance and key players.

Example 3 (**Cognitive dissonance**). In addition to the conflict with neighbors in Example 2 we add the personal conflict, the private dissonance arising from difference in the ideal y_i and the actual efforts. To do so we set the coefficient $\omega_3 = 1$. So, $\omega_1 = \omega_2 = \omega_3 = 1$, $\lambda_1 = 1$ and $\lambda_2 = 0.1$.

To study the effect of the private dissonance we assume the type parameter θ to be homogeneous across the network and equal to 1. Endowing all players with same ideal efforts $y = 1$ satisfies Corollary 1 and results in identical equilibrium efforts of 1.0527. The aggregate effort is then equal to 11.579 for a network with $n = 11$ players. Note, that the ideal effort $y = 1$ and the equilibrium effort 1.0527 are lower than the equilibrium effort of 1.11 in the homogeneous game without private dissonance. This reflects the fact that the ideal effort incurs an additional cost on the utilities of players through the private dissonance.

Assuming one of the players in the network has higher ideal effort, $y_i = 2$ while $y_j = 1$ for all $j \in N \setminus \{i\}$, we find the key players and their contributions. Similarly to Examples 1 and 2 we study the complete and the bridge networks in Figure 1a and 1b, distinguishing three cases on the bridge network with players 1, 2 and 3 having a higher ideal effort. In presence of private dissonance, the introduction of a player with higher ideal effort increases the efforts of players in the network. The higher ideal effort increases the intercentrality of the player making her the key player on the network. Table 2 displays the results.

Type	**Total effort**	**Positive key player**			**Negative key player**		
$y_{j\neq i} = 1$	$\mathcal{B}$	$+$	b_+	δ_+	$-$	b_-	δ_-
$y_i = 1$	11.579	N	1.0527	1.0527	N	1.0527	1.0527
Complete Network							
$y_{i=1} = 2$	12.1053	$\{1\}$	1.3928	1.579	$N \setminus \{1\}$	1.0713	1.0527
Bridge Network							
$y_{i=1} = 2$	12.0766	$\{1\}$	1.3966	1.551	$\{3\},\{4\},\{5\},\{8\},\{9\},\{10\}$	1.0591	1.0332
$y_{i=2} = 2$	12.1389	$\{2\}$	1.4009	1.613	$\{6\}^*$	1.0904	1.0343
$y_{i=3} = 2$	12.0877	$\{3\}$	1.4001	1.562	$\{1\}^*$	1.0591	1.0351

$*$ Despite being the negative key player, player 6 in the bridge network with $\theta_2 = 2$ and player 1 in the bridge network with $\theta_3 = 2$, they are not the players with lowest equilibrium effort. In both games the lowest effort is exerted by players 8, 9 and 10.

Table 2: Cognitive dissonance and key players.

3.3 Comparing the benchmark to the case with unions

As a direct consequence of the condition in Lemma 1, the intercentrality of player i is increasing with union $p(i)$ with respect to being a singleton.

Example 4 (**No unions**)**.** Consider a game on the bridge network depicted in Figure 1b. Let the coefficients of the game be as follows: $\lambda_1 = 0.5$, $\lambda_2 = 0.125$, $\omega_1 = 0.125$, $\omega_2 = 0.5$,

$\omega_3 = 0.125$. We assume the ideal efforts y_i, $i \in N$ to be homogeneous and equal to 1. The return coefficients θ_i are randomly drawn from the range $[0.5; 1.5]$ and displayed on Figure 2.

The aggregate effort $\mathcal{B}$ of the game where all players are singletons is 14.8. The positive key player in the network is player 2 with type $\theta_2 = 1.5$, effort $b_2 = 1.78$ and intercentrality $\delta_2 = 2.07$. The negative key player of the game is player 7 with type $\theta_7 = 0.5$, effort $b_7 = 0.82$ and intercentrality $\delta_7 = 0.57$. The list of all efforts and contributions of each player is displayed on the Table 3.

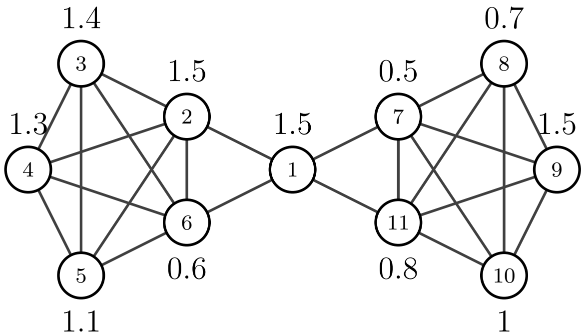


Figure 2: A bridge network.

Example 5 (**Union** $\{1, 2\}$)**.** Following the setting in Example 4 we introduce the union $p(1)$ formed by players 1 and 2. The other players in the network $\boldsymbol{g}$ remain singletons, $S_P = \{3, 4, \ldots, 11\}$. So, the union structure is $P = \{\{1, 2\}, \{3\}, \{4\}, \ldots, \{11\}\}$.

The total effort exerted in the network game with the union $\{1, 2\}$ increases to 18.41 compared to 14.8 in the no union case. The contribution of the union members to the total effort is much higher, with player 2 remaining the positive key player of the game with higher effort $b_2 = 3.13$ and union intercentrality $\delta_{2,P} = 5.68$. The negative key player of the game with union $\{1, 2\}$ is now player 6 with effort $b_6 = 1.17$ and union intercentrality $\delta_{6,P} = 0.3$. The list of efforts and contributions for each player is displayed on Table 3.

4 Discussion

4.1 Key unions

In social and economic problems seeking partnership, and looking for best options in forming unions in a given network is a commonly addressed issue. When addressing the issue of increasing the aggregate efforts in the population, finding the key player in the network helps to identify the most influential player affecting the total effort the most. Given the network $\boldsymbol{g}$, take some P such that $\emptyset \neq S_P \neq N$. Let $\overline{P}$ be such that

		No Unions		Union {1,2}		Union {1,2,3,4,5}	
Player	**Type**	**Effort**	**Contrib.**	**Effort**	**Contrib.**	**Effort**	**Contrib.**
i	θ_i	b_i	$\delta_{i,\widetilde{P}}$	b_i	$\delta_{i,P}$	b_i	$\delta_{i,P}$
1	1.5	1.65	1.76	2.99	5.37	6.81	15.51
2	1.5	1.78	2.07	3.13	5.68	9.44	20.43
3	1.4	1.68	1.78	1.87	1.72	9.09	18.24
4	1.3	1.59	1.65	1.78	1.57	9.02	17.44
5	1.1	1.43	1.38	1.6	1.27	8.89	15.77
6	0.6	1	0.59	1.17	0.3	2.87	-7.74
7	0.5	0.82	0.57	0.9	0.43	1.11	0.03
8	0.7	0.96	0.86	0.98	0.8	1.02	0.63
9	1.5	1.59	1.78	1.61	1.74	1.67	1.6
10	1	1.2	1.22	1.22	1.17	1.27	1.02
11	0.8	1.07	1.02	1.16	0.93	1.41	0.62
$\mathcal{B}$		**14.8**		**18.41**		**52.6**	

Table 3: Unions versus no unions.

$\overline{P} = P \setminus \{p, \{i\}\} \cup \{p \cup \{i\}\}$ with $p \in P$ and $i \in S_P$. That is, the new union structure $\overline{P}$ is obtained from the union structure P by simply adding a player i who is a singleton in P to some union in P.

Proposition 4. *Take $\overline{P}$ be such that $\overline{P} = P \setminus \{p, \{i\}\} \cup \{\overline{p}\}$ with $p \in P$, $i \in S_P$, $\emptyset \neq S_P \neq N$ and $\overline{p} = p \cup \{i\}$. The change in aggregate effort in the network $\boldsymbol{g}$ through the addition of a singleton player $i \in S_P$ to the union p is given by the difference of player i's union intercentralities for union structures $\overline{P}$ and P:*

$$\sigma_i^p = \underbrace{\delta_{i,\overline{P}}(\mathbf{G}^*, \mathbf{H}_{i\in\overline{p}}, \boldsymbol{\phi}_{i\in\overline{p}}, \boldsymbol{\alpha}_{i\in\overline{p}})}_{\substack{\textit{union intercentrality} \\ \textit{when } i \in \overline{p}}} - \underbrace{\delta_{i,P}(\mathbf{G}^*, \mathbf{H}_{i\in S_P}, \boldsymbol{\phi}_{i\in S_P}, \boldsymbol{\alpha}_{i\in S_P})}_{\substack{\textit{union intercentrality} \\ \textit{when } i \textit{ is a singleton player}}}$$

We call σ_i^p the union-induced intercentrality of player i from joining the union p. Using (13) and Corollary 3 we can rewrite σ_i^p in the following way.

$$\sigma_i^p = \underbrace{\sum_{k\in p(i)\setminus\{i\}} \sum_{j\in N} m_{jk}^{-i}(\alpha_k - \alpha_k^{-i})}_{\text{union effect}} + \underbrace{\frac{b_i(\mathbf{G}^*, \mathbf{H}_{i\in\overline{p}}, \boldsymbol{\phi}_{i\in\overline{p}}, \boldsymbol{\alpha}_{i\in\overline{p}})}{m_{ii}} \sum_{k\in N} m_{ki}}_{\text{intercentrality when } i \in \overline{p}}$$
$$- \underbrace{\frac{b_i(\mathbf{G}^*, \mathbf{H}_{i\in S_P}, \boldsymbol{\phi}_{i\in S_P}, \boldsymbol{\alpha}_{i\in S_P})}{m_{ii}} \sum_{k\in N} m_{ki}}_{\text{intercentrality when } i \in S_p} . \qquad (15)$$

If the condition (11) of Lemma 1 holds, then a player's intercentrality when she belongs to a union is greater than her intercentrality when she is a singleton. That is,

$$\frac{b_i(\mathbf{G}^*, \mathbf{H}_{i\in\overline{p}}, \boldsymbol{\phi}_{i\in\overline{p}}, \boldsymbol{\alpha}_{i\in\overline{p}})}{m_{ii}} \sum_{k\in N} m_{ki} \geq \frac{b_i(\mathbf{G}^*, \mathbf{H}_{i\in S_P}, \boldsymbol{\phi}_{i\in S_P}, \boldsymbol{\alpha}_{i\in S_P})}{m_{ii}} \sum_{k\in N} m_{ki} \text{ if (11) holds.}$$

In addition, Lemma 3 provides conditions for the union effect of player i to be positive. It is then straightforward to see that $\sigma_i^p \geq 0$ for all players that satisfy the conditions in Lemma 1 and Lemma 3. In other words, if the union $p \in P$ increases the intercentrality of player i, and the union effect of player i is positive, then the union-induced intercentrality σ_i^p of the player is positive.

Using the definition of the union-induced intercentrality we can find the key addition, i.e. the player that has the highest impact on the aggregate effort in the network when added to a given union.

Definition 3. Take any network $\boldsymbol{g}$ with union structure P such that $\emptyset \neq S_P \neq N$. The key addition to some union $p \in P$ is player $i \in S_P$ who has the highest union-induced intercentrality when added to the union,

$$i = \arg\max_{j\notin p} \sigma_j^p.$$

Given some network with union structure P, the key addition induces the highest increase in aggregate effort in the network game when added to some given union $p \in P$. The key addition to a singleton player $i \in S_P$ is player $j \in S_P$ with highest impact on the aggregate effort when the union $\{i, j\}$ is formed.

Given any network $\boldsymbol{g}$ with union structure P such that $2 \leq \#S_P \leq n$, we call a union formed by z initially singleton players a key union of size $z \leq \#S_P$ when it induces the highest total effort in the network game among all other possible unions of size z. Let $\mathcal{B}_P$ denote the total effort in network $\boldsymbol{g}$ with union structure P. The total effort exerted by all players once a union p of size z has been formed is given by

$$\mathcal{B}_P + \sigma_{i_2}^{\{i_1\}} + \sigma_{i_3}^{\{i_1,i_2\}} + \ldots + \sigma_{i_z}^{\{i_1,\ldots,i_{z-1}\}},$$

with $p = \{i_1, i_2, \ldots, i_z\} \subseteq S_P$.

Definition 4. Take any network $\boldsymbol{g}$ with union structure P such that $z \leq \#S_P \leq n$. The key union of z players $i_1, \ldots, i_z$ is the one that solves

$$\max_{i_1,\ldots,i_z\in S_P} \sigma_{i_2}^{\{i_1\}} + \sigma_{i_3}^{\{i_1,i_2\}} + \ldots + \sigma_{i_z}^{\{i_1,\ldots,i_{z-1}\}}.$$

The next example shows that the set of players belonging to the key union of size z may not be included in the set of players belonging to the key union of size $z+1$.

Example 6. (**Key addition and key union**) Consider again the game on the bridge network depicted in Figure 2 with $\lambda_1 = 0.5$, $\lambda_2 = 0.125$, $\omega_1 = 0.125$, $\omega_2 = 0.5$, $\omega_3 = 0.125$, $y_i = 1$ for all $i \in N$, and types θ_i randomly drawn from the range $[0.5; 1.5]$ and displayed on the figure. Suppose there is only one union $p \in P$. Given an opportunity to expand the union p by one member we find the best member to add, i.e. the key addition. Moreover, we can find the key union of a given size, that is, the union that provides the highest total effort compared to other same-size unions.

In Example 5 we have considered the union $p(1)$ consisting of players 1 and 2 in the network. Following Definition 3, we get that player 3 is the player with the highest union-induced intercentrality with $\sigma_3^{\{1,2\}} = 6.91 > \sigma_i^{\{1,2\}}$ for $i \in N \setminus \{1,2,3\}$. By definition, this player leads to the highest possible increase in the aggregate effort when joining the union $\{1,2\}$, which is then equal to 25.32. Table 4 provides the aggregate efforts and union-induced intercentralities of players in S_P from joining the union $\{1,2\}$.

It is worth noting that the union $p(1) = \{1,2\}$ is the key union of size 2, since $\max_{i,j \in S_{\widetilde{P}}} \sigma_i^{\{j\}} = \sigma_1^{\{2\}} = \sigma_2^{\{1\}} = 3.60$. When we gradually expand the union size up to three members by finding the key additions, players 1, 2 and 3 satisfy the following condition:

$$\max_{i_1,i_2} \sigma_{i_2}^{\{i_1\}} + \max_{i_3} \sigma_{i_3}^{\{i_1^*,i_2^*\}} = \sigma_2^{\{1\}} + \sigma_3^{\{1,2\}} = 10.51.$$

On the other hand, when we let the players forming a union of size 3 directly, from Definition 4, we get that players 2, 3 and 4 form the union of size 3 with the highest aggregate effort:

$$\max_{i_1,i_2,i_3} (\sigma_{i_2}^{\{i_1\}} + \sigma_{i_3}^{\{i_1,i_2\}}) = \sigma_2^{\{3\}} + \sigma_4^{\{2,3\}} = \sigma_2^{\{4\}} + \sigma_3^{\{2,4\}} = \sigma_3^{\{4\}} + \sigma_2^{\{3,4\}} = 11.19.$$

Indeed, the aggregate effort in the network game with union $\{2,3,4\}$ is $\mathcal{B} = 25.99$ which exceeds the aggregate effort $\mathcal{B} = 25.32$ of the game with union $\{1,2,3\}$. Thus, the key union of size 3 is the union $\{2,3,4\}$.

Example 6 shows an even more interesting result: the classic key player policy of removing the key player from the network may be dominated by the key addition and key union policy. Instead of removing the key player, adding her to some union could increase the aggregate effort in the network even more. For instance, consider now the union of players $p = \{1,2,3,4,5\} \in P$ with the rest being singleton players in the network, i.e. $S_P = \{6,7,8,9,10,11\}$. As shown in Table 3, the negative key player of the game is player 6. The positive externalities from the neighboring union p increase the incentives of player 6 to free-ride, increasing the conflict between the neighbors. Moreover, the union intercentrality of player 6 is negative. That is, removing player 6 from the network will increase the aggregate effort in the population. On the contrary, the union-induced intercentrality of player 6 is the highest among all singleton players (see Table 5). The latter

Player i	**Union**	$\mathcal{B}$	$\sigma_i^{\{1,2\}}$
	{1, 2}	18.41	
3	{1, 2, 3}	25.32	6.91
4	{1, 2, 4}	25.05	6.65
5	{1, 2, 5}	24.54	6.14
6	{1, 2, 6}	24.22	5.81
7	{1, 2, 7}	23.21	4.81
8	{1, 2, 8}	22.82	4.41
9	{1, 2, 9}	24.49	6.08
10	{1, 2, 10}	23.45	5.04
11	{1, 2, 11}	23.93	5.52

Table 4: Key addition to the union $\{1, 2\}$.

means that, while removing the player from the network would increase the aggregate effort to $\mathcal{B} = 60.34$, adding the player to the union $\{1, 2, 3, 4, 5\}$ will increase it even more.

Player i	**Union**	$\mathcal{B}$	$\sigma_i^{\{1,2,3,4,5\}}$
	{1, 2, 3, 4, 5}	52.6	
6	{1, 2, 3, 4, 5, 6}	72.95	20.35
7	{1, 2, 3, 4, 5, 7}	64.26	11.66
8	{1, 2, 3, 4, 5, 8}	63.69	11.09
9	{1, 2, 3, 4, 5, 9}	68.7	16.1
10	{1, 2, 3, 4, 5, 10}	65.57	12.97
11	{1, 2, 3, 4, 5, 11}	66.28	13.68

Table 5: Key player policy versus key addition policy.

4.2 Stable unions

A simple way to analyze the union structure that one might expect to emerge in the long run is to examine an equilibrium requirement that no group of players benefits from altering in some way the union structure.[18] Let $u_i(P)$ denote the Nash equilibrium utility

[18]Ray (2007) and Ray and Vohra (2015) provide surveys of models of coalition formation. Hart and Kurz (1983) have introduced the notions of γ-stability and δ-stability to study the stable coalition structures under the assumptions that the non-deviating players become singletons or remain together, re-

of player $i \in p(i)$ in the network game with union structure P.

Definition 5. A union structure P is μ-stable if for any $p_j, p_k \in P$

(i) $u_i(P) \geq u_i(P \setminus \{p_j, p_k\} \cup \{p_j \cup p_k\})$ for some $i \in p_j \cup p_k$, and

(ii) $u_i(P) \geq u_i(P \setminus \{p_j\} \cup \{k \mid k \in p_j\})$ for all $i \in p_j$.

Condition (i) in Definition 5 requires that members of two unions do not have incentives to merge both unions. Condition (ii) requires that there are no incentives for some members to dissolve one union. It is implicitly presumed that a singleton member can dissolve her union. Thus, a union structure is μ-stable if it is immune to deviations that consist of either merging two unions or dissolving one union.

Suppose now that the network structure is complete, $g^\star_{ij} = \frac{1}{n-1}$ for $i \neq j$, and players are homogeneous with $y_i = y$ and $\theta_i = \theta$ for all $i \in N$. In the case the union structure is $\widetilde{P}$ such that all players are singleton players, i.e. $\widetilde{p}(i) = \{i\}$ for all $i \in N$, the Nash equilibrium efforts and utilities are given by

$$\begin{aligned}
x_i^\star\left(\widetilde{P}, g^\star_{jk} = 1/(n-1), y, \theta\right) =& \frac{\lambda_1\theta + \omega_3 y}{\omega_2 + \omega_3 - \lambda_2\theta},\\
u_i\left(\widetilde{P}, g^\star_{jk} = 1/(n-1), y, \theta\right) =& \frac{\lambda_2\theta}{2}\left(\frac{\lambda_1\theta + \omega_3 y}{\omega_2 + \omega_3 - \lambda_2\theta}\right)^2 + \frac{1}{2}\frac{(\lambda_1\theta + \omega_3 y)^2}{(\omega_2 + \omega_3 - \lambda_2\theta)}\\
&+ (n-1)\lambda_1\theta\frac{\lambda_1\theta + \omega_3 y}{\omega_2 + \omega_3 - \lambda_2\theta} - \frac{\omega_3 y^2}{2}.
\end{aligned} \tag{16}$$

In the case the union structure is $\widehat{P}$ such that all players form the grand union, i.e. $\widehat{p}(i) = N$ for all $i \in N$, the Nash equilibrium efforts and utilities are given by

$$\begin{aligned}
x_i^\star\left(\widehat{P}, g^\star_{jk} = 1/(n-1), y, \theta\right) =& \frac{n\lambda_1\theta + \omega_3 y}{\omega_2 + \omega_3 - 2\lambda_2\theta},\\
u_i\left(\widehat{P}, g^\star_{jk} = 1/(n-1), y, \theta\right) =& \frac{1}{2}\frac{(n\lambda_1\theta + \omega_3 y)^2}{(\omega_2 + \omega_3 - 2\lambda_2\theta)} - \frac{\omega_3 y^2}{2}.
\end{aligned} \tag{17}$$

Comparing (16) with (17) leads to the next proposition that provides a sufficient condition for stabilizing the grand union.

Proposition 5. *Suppose that the network structure is complete and players are homogeneous with $y_i = y$ and $\theta_i = \theta$ for all $i \in N$. If*

$$n > \frac{\sqrt{\lambda_2}(\lambda_1\theta + \omega_3 y)}{\lambda_1\sqrt{\theta}\sqrt{\omega_2 + \omega_3 - \lambda_2\theta}} + 1$$

then the grand union $\widehat{P} = \{N\}$ is μ-stable.

spectively, after the deviation of some coalition members.

We illustrate the above proposition by means of a five-player example with $\lambda_1 = 0.25$, $\lambda_2 = 0.125$, $\omega_1 = 0.125$, $\omega_2 = 0.5$, $\omega_3 = 0$, $y_i = 0$ and $\theta_i = 1$ for all $i \in N$. Utilities are given in Table 6. We get that the grand union $\widehat{P} = \{\{1,2,3,4,5\}\}$ is μ-stable since from the grand union $\widehat{P}$ no singleton or group of players have incentives to dissolve the grand union to reach the union structure $\widetilde{P} = \{\{1\},\{2\},\{3\},\{4\},\{5\}\}$ where all players are singletons and worse off. The grand union is also the union structure leading to the highest aggregate effort. In addition, any union structure $\{p', p''\}$ such that $\#p' = 4$ and $\#p'' = 1$ is μ-stable since $i \in p''$ blocks the merger to form the grand union $\widehat{P}$ and no $j \in p'$ has an incentive to dissolve p' for reaching the union structure $\widetilde{P}$. Any other union structure is not μ-stable.

Union structure P	$\mathcal{B}$	u_1	u_2	u_3	u_4	u_5
{{1}, {2}, {3}, {4}, {5}}	3.33	0.78	0.78	0.78	0.78	0.78
{{1,2}, {3}, {4}, {5}}	4.87	0.97	0.97	1.15	1.15	1.15
{{1,2}, {3,4}, {5}}	6.46	1.36	1.36	1.36	1.36	1.54
{{1,2,3}, {4}, {5}}	8.35	1.43	1.43	1.43	1.99	1.99
{{1,2,3}, {4,5}}	10.06	1.83	1.83	1.83	2.21	2.21
{{1,2,3,4}, {5}}	14.55	2.15	2.15	2.15	2.15	3.46
{{1,2,3,4,5}}	25.00	3.13	3.13	3.13	3.13	3.13

Table 6: Total effort and utilities in the complete network with $n = 5$, $\lambda_1 = 0.25$, $\lambda_2 = 0.125$, $\omega_1 = 0.125$, $\omega_2 = 0.5$, $\omega_3 = 0$, $y_i = 0$ and $\theta_i = 1$ for all $i \in N$.

However, once players are no more homogeneous, the grand union is likely to be destabilized even when the network is complete. For instance, suppose that players 1 and 2 have higher returns on spillovers than the rest of players: $\lambda_1 = 0.25$, $\lambda_2 = 0.125$, $\omega_1 = 0.125$, $\omega_2 = 0.5$, $\omega_3 = 0$, $y_i = 0$ for all $i \in N$, $\theta_1 = \theta_2 = 2$ and $\theta_3 = \theta_4 = \theta_5 = 1$. Utilities are given in Table 7. We get that the grand union $\widehat{P} = \{\{1,2,3,4,5\}\}$ is no more μ-stable since players 3, 4 and 5 have incentives to dissolve $\widehat{P}$ to reach the union structure $\widetilde{P} = \{\{1\},\{2\},\{3\},\{4\},\{5\}\}$. Any union structure $\{p', p''\}$ such that $\#p' = 4$ and $\#p'' = 1$ is μ-stable since some $i \in p' \cup p''$ blocks the merger to form the grand union and no $j \in p'$ has an incentive to dissolve p' for reaching the union structure $\widetilde{P}$. Similarly, any union structure $\{p', p''\}$ such that $\#p' = 3$ and $\#p'' = 2$ is μ-stable. For completeness, any other union structure is not μ-stable.

The notion of μ-stability is a weak concept of stability since few deviations are considered. An alternative concept is the notion of γ-stability. A union structure P' is γ-obtainable from P via $p \subseteq N$, if (i) $p'_j = \{j\}$ for any $j \in \{p_i \setminus p \mid p_i \in P, p_i \setminus p \neq \emptyset\}$, (ii) $\{p'_i \in P' \mid p'_i \subseteq N \setminus p\} = \{p_i \in P \mid p_i \cap p = \emptyset\}$ and (iii) $p \in P'$. Condition (i) means that if

Union structure P	$\mathcal{B}$	u_1	u_2	u_3	u_4	u_5
{{1}, {2}, {3}, {4}, {5}}	5.32	2.51	2.51	1.26	1.26	1.26
{{1,2}, {3}, {4}, {5}}	9.37	3.68	3.68	2.20	2.20	2.20
{{1}, {2}, {3,4}, {5}}	7.16	3.47	3.47	1.52	1.52	1.71
{{1,3}, {2}, {4}, {5}}	8.25	3.70	4.04	1.38	1.96	1.96
{{1}, {2,3}, {4,5}}	10.22	5.11	4.76	1.84	2.25	2.25
{{1,2}, {3,4}, {5}}	11.40	4.74	4.74	2.49	2.49	2.70
{{1,3}, {2,4}, {5}}	11.40	5.39	5.39	2.09	2.09	2.72
{{1}, {2}, {3,4,5}}	11.36	5.75	5.75	2.14	2.14	2.14
{{1,3,4}, {2}, {5}}	14.08	6.24	7.22	1.99	1.99	3.34
{{1,2,3}, {4}, {5}}	17.17	6.79	6.79	1.35	3.99	3.99
{{1,2}, {3,4,5}}	16.05	7.25	7.25	3.17	3.17	3.17
{{1,3,4}, {2,5}}	17.69	8.27	8.91	2.75	2.75	3.48
{{1,2,3}, {4,5}}	19.57	8.11	8.11	1.79	4.34	4.34
{{1,3,4,5}, {2}}	24.50	11.12	13.37	2.84	2.84	2.84
{{1,2,3,4}, {5}}	31.39	12.95	12.95	1.49	1.49	7.17
{{1,2,3,4,5}}	59.25	26.68	26.68	-0.51	-0.51	-0.51

Table 7: Utilities in the complete network with $n = 5$, $\lambda_1 = 0.25$, $\lambda_2 = 0.125$, $\omega_1 = 0.125$, $\omega_2 = 0.5$, $\omega_3 = 0$, $y_i = 0$ for all $i \in N$, $\theta_1 = \theta_2 = 2$ and $\theta_3 = \theta_4 = \theta_5 = 1$.

the players in group p leave their respective union(s) in P, their previous union partners become singletons. Condition (ii) says that unions that are not affected by the deviation of group p remain together. Condition (iii) imposes the deviating players in p to form one union in the new union structure P'. A union structure P is γ-stable if for any $p \subseteq N$, P' γ-obtainable from P via p and $i \in p$ such that $u_i(P') > u_i(P)$, there exists $j \in p$ such that $u_j(P') \leq u_j(P)$. Characterizing γ-stable union structure is quite challenging and beyond the scope of the paper. Nevertheless, the notion of γ-stability leads exactly to the same outcomes as μ-stability in the above examples, except that $\{\{1,2,3,4\},\{5\}\}$ is not γ-stable in the example of Table 7 because the deviation of coalition $\{3,4\}$ to the union structure $\{\{1\},\{2\},\{3,4\},\{5\}\}$ is not blocked.

4.3 Conclusion

Social interactions and the structure of social networks have an important role in economic outcomes. We have studied network games with social and private dissonance where each player in the network exerts some costly efforts. The major novelty is that each player may belong to some union that allows for cooperative behavior. Members of each union maximize the joint utility of the union. Each player benefits from the aggregate effort and her neighbors' efforts. However, she suffers some disutility when her effort differs from her neighbors' efforts or is inconsistent with her ideal effort. We have characterized the unique Nash equilibrium of the network game with unions. The contribution of each player to the aggregate effort in the network with unions is not only proportional to her Katz-Bonacich weighted centrality but also complemented by the union effect. Accordingly, we have introduced a union intercentrality measure for finding the key player whose removal has the highest impact on the aggregate effort level. We have investigated policies to increase the aggregate effort in the network by forming unions of a fixed size, and by adding players to some given union. Hence, we have defined the key addition to an existing union as being the player who increases the most the aggregate effort by joining the union, and we have looked for the key union that provides the highest aggregate effort. It turns out that the classic key player policy of removing the key player from the network may be dominated by the key addition and key union policy. Instead of removing the key player, adding her to some union could increase the aggregate effort even more. Finally, we have considered the stability of unions. A welfare analysis with consideration of a fair utility allocation rule or endogenizing the network formation is left for future research.[19]

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Appendix

The next lemma is an adaptation of Lemma 1 in Ballester, Calvó-Armengol and Zenou (2006).

Lemma 4. *Let* $\boldsymbol{M}(\boldsymbol{\Phi}(\boldsymbol{g})) = [\mathbf{I} - \Phi(\boldsymbol{g})]^{-1}$ *be well defined and nonnegative. Then we have* $m_{ji}(\boldsymbol{\Phi}(\boldsymbol{g}))m_{ik}(\boldsymbol{\Phi}(\boldsymbol{g})) = m_{ii}(\boldsymbol{\Phi}(\boldsymbol{g}))[m_{jk}(\boldsymbol{\Phi}(\boldsymbol{g})) - m_{jk}(\boldsymbol{\Phi}(\boldsymbol{g}^{-i}))]$ *for all* $k \neq i \neq j$.

Proof. Let f_{ij} with $i, j \in N$, be the elements of the matrix $\boldsymbol{\Phi}(\boldsymbol{g})$ defined on the weighted adjacency matrix of the network $\boldsymbol{g}$, and similarly f_{kj}^{-i} the elements of the matrix $\boldsymbol{\Phi}(\boldsymbol{g}^{-i})$

[19] Mauleon and Vannetelbosch (2016) provide a comprehensive introduction to network formation games.

defined on the weighted adjacency matrix of the network $\boldsymbol{g}^{-i}$ formed by removing player i from $\boldsymbol{g}$. We get

$$\begin{aligned} m_{ii}(\boldsymbol{\Phi}(\boldsymbol{g}))[m_{jk}(\boldsymbol{\Phi}(\boldsymbol{g})) - m_{jk}(\boldsymbol{\Phi}(\boldsymbol{g}^{-i}))] &= \sum_{s=1}^{\infty} f_{ii}^{[s]} \sum_{r=0}^{\infty} \left(f_{jk}^{[r]} - f_{jk}^{-i[r]} \right) = \\ &= \sum_{s=1}^{\infty} f_{ii}^{[s]} \sum_{r=0}^{\infty} f_{j(i)k}^{[r]} = \sum_{s=1}^{\infty} \sum_{r=0}^{\infty} f_{ii}^{[s]} f_{j(i)k}^{[r]} \\ &= \sum_{\substack{s=1 \\ s' \geq 1}}^{\infty} \sum_{\substack{r=0 \\ r' \geq 1}}^{\infty} f_{ji}^{[r-r']} f_{ii}^{[s-s']} f_{ii}^{[s']} f_{ik}^{[r']} \\ &= \sum_{p=1}^{\infty} \sum_{p' \geq 1} f_{ji}^{[p-p']} f_{ik}^{[p']} = m_{ji} m_{ik} \end{aligned}$$

where $f_{j(i)k}^{[r]}$ is the weighted number of the r-step paths from j to k passing through the node i, it is the (i,j)th element of the r-power of matrix $\boldsymbol{\Phi}(\boldsymbol{g})$ associated to the Bonacich centrality of player j. □

Proof of Proposition 1. We first find the best response of player $i \in p(i)$ for any $p(i) \in P$ from the first order conditions on the joint utility in (6).

$$\begin{aligned} \frac{\partial U_{p(i)}(x)}{\partial x_i} =& \lambda_1 \theta_i + \omega_3 y_i + (\lambda_2 \theta_i + \omega_1) \sum_{k \in N} g_{ik}^* x_k - (\omega_1 + \omega_2 + \omega_3) x_i + \\ & + \sum_{j \in p(i) \setminus \{i\}} \left(\lambda_1 \theta_j + (\lambda_2 \theta_j + \omega_1) g_{ji}^* x_j + \omega_1 g_{ji}^* x_i \right) = \\ =& \lambda_1 \sum_{j \in p(i)} \theta_j + \omega_3 y_i + (\lambda_2 \theta_i + \omega_1) \sum_{k \in N} g_{ik}^* x_k + \sum_{j \in p(i) \setminus \{i\}} (\lambda_2 \theta_j + \omega_1) g_{ji}^* x_j + \\ & - (\omega_1 (1 + \sum_{j \in p(i) \setminus \{i\}} g_{ji}^*) + \omega_2 + \omega_3) x_i \end{aligned}$$

Then,

$$\frac{\partial U_{p(i)}(x)}{\partial x_i} \equiv 0 \Leftrightarrow$$

$$(\omega_1(1+\sum_{j \in p(i) \setminus \{i\}} g_{ji}^*)+\omega_2+\omega_3)x_i^* = \lambda_1 \sum_{j \in p(i)} \theta_j+\omega_3 y_i+(\lambda_2\theta_i+\omega_1)\sum_{k \in N} g_{ik}^* x_k+\sum_{j \in p(i) \setminus \{i\}} (\lambda_2\theta_j+\omega_1) g_{ji}^* x_j$$

Using the notations introduced in equations (7) and (8), x_i^* becomes:

$$x_i^* = \frac{\lambda_1 \Theta_i + \omega_3 y_i}{\Omega_i} + \frac{(\lambda_2 \theta_i + \omega_1)}{\Omega_i} \sum_{k \in N} g_{ik}^* x_k + \sum_{j \in p(i) \setminus \{i\}} \frac{\lambda_2 \theta_j + \omega_1}{\Omega_i} g_{ji}^* x_j$$

$$x_i^* = \alpha_i + \phi_i \sum_{k \in N} g_{ik}^* x_k + \sum_{j \in p(i) \setminus \{i\}} \gamma_{ij} g_{ji}^* x_j$$

$$x_i^* = \alpha_i + \phi_i \sum_{k\in N} g_{ik}^* x_k + \sum_{k\in N} h_{ik} x_k \tag{18}$$

With the following matrix form representation:

$$\mathbf{x} = \boldsymbol{\alpha} + [\boldsymbol{\phi} \otimes \mathbf{G}^*]\mathbf{x} + \mathbf{H}\mathbf{x} \Leftrightarrow (\mathbf{I} - [\boldsymbol{\phi} \otimes \mathbf{G}^*] - \mathbf{H})\mathbf{x} = \boldsymbol{\alpha} \Leftrightarrow \mathbf{x} = (\mathbf{I} - [\boldsymbol{\phi} \otimes \mathbf{G}^*] - \mathbf{H})^{-1}\boldsymbol{\alpha}.$$

□

Proof of Proposition 2. Let $v = \left(\frac{\partial x_1}{\partial x_i}, \ldots, \frac{\partial x_n}{\partial x_i}\right)$ denote the vector of derivatives of the effort of the players with respect to x_i. From $\mathbf{x} = \boldsymbol{\alpha} + [\boldsymbol{\phi} \otimes \mathbf{G}^* + \mathbf{H}]\mathbf{x}$ and the fact that α and ϕ are independent from x_i, it follows $\mathbf{v} = [\boldsymbol{\phi} \otimes \mathbf{G}^* + \mathbf{H}]\mathbf{v}$. Solving for v yields: $\mathbf{v} = (\mathbf{I} - [\boldsymbol{\phi} \otimes \mathbf{G}^*] - \mathbf{H})^{-1}\mathbf{1}$ where $\mathbf{1}$ is $n \times 1$ vector of 1's.
Under our assumption that the spectral radius of $[\boldsymbol{\phi} \otimes \mathbf{G}^*] + \mathbf{H}$ is smaller than 1, the matrix $(\mathbf{I} - [\boldsymbol{\phi} \otimes \mathbf{G}^*] - \mathbf{H})^{-1}$ is non-negative. It then follows that the entries of v are also non-negative.

□

Proof of Lemma 1. The best response of a singleton player $i \in S_P$ in the network with union structure P is given by:

$$x_i^{S_P} = \alpha_i^{S_P} + \phi_i^{S_P} \sum_{j\in N} g_{ij}^* x_j.$$

with

$$\alpha_i^{S_P} = \frac{\lambda_1\theta_i + \omega_3 y_i}{(\omega_1 + \omega_2 + \omega_3)} \qquad \text{and} \qquad \phi_i^{S_P} = \frac{\lambda_2\theta_i + \omega_1}{(\omega_1 + \omega_2 + \omega_3)}.$$

The best response of player $i \in p(i)$ in the network with union structure P is given by:

$$x_i = \alpha_i + \phi_i \sum_{k\in N} g_{ik}^* x_k + \sum_{k\in N} \gamma_{ik} g_{ki}^* x_k.$$

We can derive sufficient conditions to ensure $x_i \geq x_i^{S_P}$ using the two steps below.

Step 1: Find the conditions for $\phi_i^{S_P} g_{ik}^* \leq \phi_i g_{ik}^* + \gamma_{ik} g_{ki}^*$ for all $k \in p(i) \setminus \{i\}$.

$$\frac{\lambda_2\theta_i + \omega_1}{\omega_1 + \omega_2 + \omega_3} g_{ik}^* \leq \frac{\lambda_2\theta_i + \omega_1}{\omega_1(1 + \sum\limits_{j\in p(i)\setminus\{i\}} g_{ji}^*) + \omega_2 + \omega_3} g_{ik}^* + \frac{\lambda_2\theta_k + \omega_1}{\omega_1(1 + \sum\limits_{j\in p(i)\setminus\{i\}} g_{ji}^*) + \omega_2 + \omega_3} g_{ki}^*,$$

$$\frac{\lambda_2\theta_i + \omega_1}{\omega_1 + \omega_2 + \omega_3} g_{ik}^* - \frac{\lambda_2\theta_i + \omega_1}{\omega_1(1 + \sum\limits_{j\in p(i)\setminus\{i\}} g_{ji}^*) + \omega_2 + \omega_3} g_{ik}^* \leq \frac{\lambda_2\theta_k + \omega_1}{\omega_1(1 + \sum\limits_{j\in p(i)\setminus\{i\}} g_{ji}^*) + \omega_2 + \omega_3} g_{ki}^*,$$

$$\frac{(\lambda_2\theta_i+\omega_1)g^*_{ik}}{(\omega_1+\omega_2+\omega_3)}\frac{\omega_1\sum\limits_{j\in p(i)\setminus\{i\}} g^*_{ji}}{(\omega_1(1+\sum\limits_{j\in p(i)\setminus\{i\}} g^*_{ji})+\omega_2+\omega_3)} \leq \frac{\lambda_2\theta_k+\omega_1}{\omega_1(1+\sum\limits_{j\in p(i)\setminus\{i\}} g^*_{ji})+\omega_2+\omega_3} g^*_{ki},$$

$$\frac{\lambda_2\theta_i+\omega_1}{\omega_1+\omega_2+\omega_3} g^*_{ik}\omega_1 \sum_{j\in p(i)\setminus\{i\}} g^*_{ji} \leq (\lambda_2\theta_k+\omega_1)g^*_{ki},$$

$$\frac{\lambda_2\theta_i+\omega_1}{\omega_1+\omega_2+\omega_3} \leq \frac{\lambda_2\theta_k+\omega_1}{\omega_1\sum\limits_{j\in p(i)\setminus\{i\}} g^*_{ji}}\frac{g^*_{ki}}{g^*_{ik}},$$

$$\phi_i^{S_P} \leq \frac{\lambda_2\theta_j+\omega_1}{\omega_1\sum\limits_{j\in p(i)\setminus\{i\}} g^*_{ji}}\frac{g^*_{ki}}{g^*_{ik}}.$$

Restructuring the condition above we get:

$$\frac{\omega_1\sum\limits_{j\in p(i)\setminus\{i\}} g^*_{ji}}{\omega_1+\omega_2+\omega_3} \leq \frac{(\lambda_2\theta_k+\omega_1)g^*_{ki}}{(\lambda_2\theta_i+\omega_1)g^*_{ik}}.$$

Step 2: Find the condition for $\alpha_i^{S_P} \leq \alpha_i$ for all $i \in N \setminus S_P$

$$\alpha_i^{S_P} \leq \alpha_i,$$

$$\frac{\lambda_1\theta_i+\omega_3 y_i}{\omega_1+\omega_2+\omega_3} \leq \frac{\lambda_1\sum\limits_{j\in p(i)}\theta_j+\omega_3 y_i}{\omega_1(1+\sum\limits_{j\in p(i)\setminus\{i\}} g^*_{ji})+\omega_2+\omega_3},$$

$$\frac{\lambda_1\theta_i+\omega_3 y_i}{\omega_1+\omega_2+\omega_3} \leq \frac{\lambda_1\theta_i+\omega_3 y_i+\lambda_1\sum\limits_{j\in p(i)\setminus\{i\}}\theta_j}{\omega_1(1+\sum\limits_{j\in p(i)\setminus\{i\}} g^*_{ji})+\omega_2+\omega_3},$$

$$\frac{\lambda_1\theta_i+\omega_3 y_i}{\omega_1+\omega_2+\omega_3} - \frac{\lambda_1\theta_i+\omega_3 y_i}{\omega_1(1+\sum\limits_{j\in p(i)\setminus\{i\}} g^*_{ji})+\omega_2+\omega_3} \leq \frac{\lambda_1\sum\limits_{j\in p(i)\setminus\{i\}}\theta_j}{\omega_1(1+\sum\limits_{j\in p(i)\setminus\{i\}} g^*_{ji})+\omega_2+\omega_3},$$

$$\frac{(\lambda_1\theta_i+\omega_3 y_i)}{(\omega_1+\omega_2+\omega_3)}\frac{\omega_1\sum\limits_{j\in p(i)\setminus\{i\}} g^*_{ji}}{(\omega_1(1+\sum\limits_{j\in p(i)\setminus\{i\}} g^*_{ji})+\omega_2+\omega_3)} \leq \frac{\lambda_1\sum\limits_{j\in p(i)\setminus\{i\}}\theta_j}{\omega_1(1+\sum\limits_{j\in p(i)\setminus\{i\}} g^*_{ji})+\omega_2+\omega_3},$$

$$\frac{\lambda_1\theta_i+\omega_3 y_i}{\omega_1+\omega_2+\omega_3} \leq \frac{\lambda_1\sum\limits_{j\in p(i)\setminus\{i\}}\theta_j}{\omega_1\sum\limits_{j\in p(i)\setminus\{i\}} g^*_{ji}},$$

$$\alpha_i^{S_P} \leq \frac{\lambda_1\sum\limits_{j\in p(i)\setminus\{i\}}\theta_j}{\omega_1\sum\limits_{j\in p(i)\setminus\{i\}} g^*_{ji}}.$$

Restructuring this condition we have the following:

$$\frac{\omega_1\sum\limits_{j\in p(i)\setminus\{i\}} g^*_{ji}}{\omega_1+\omega_2+\omega_3} \leq \frac{\lambda_1\sum\limits_{j\in p(i)\setminus\{i\}}\theta_j}{\lambda_1\theta_i+\omega_3 y_i}.$$

Using the conditions derived in the steps above we can conclude that the participation in the union $p(i)$ increases the effort of player i with respect to her effort as a singleton player if:

$$\frac{\omega_1 \sum\limits_{j\in p(i)\setminus\{i\}} g^*_{ji}}{\omega_1+\omega_2+\omega_3} \leq \frac{(\lambda_2\theta_k+\omega_1)g^*_{ki}}{(\lambda_2\theta_i+\omega_1)g^*_{ik}}, \text{ for all } k\in p(i)\setminus\{i\}, \text{ and } \frac{\omega_1 \sum\limits_{j\in p(i)\setminus\{i\}} g^*_{ji}}{\omega_1+\omega_2+\omega_3} \leq \frac{\lambda_1 \sum\limits_{j\in p(i)\setminus\{i\}} \theta_j}{\lambda_1\theta_i+\omega_3 y_i}.$$

To simplify, assume player $l \in p(i)\setminus\{i\}$ is the player that solves the following problem:

$$l = \underset{k\in p(i)\setminus\{i\}}{\arg\min}(\lambda_2\theta_k+\omega_1)\frac{g^*_{ki}}{g^*_{ik}}.$$

Combining the two conditions we have that, for a given union structure P and a player $i \in p(i)$, for $x_i \geq x_i^{S_P}$ it is sufficient to have the following condition hold:

$$\frac{\omega_1 \sum\limits_{j\in p(i)\setminus\{i\}} g^*_{ji}}{\omega_1+\omega_2+\omega_3} \leq \min\left\{\frac{(\lambda_2\theta_l+\omega_1)g^*_{li}}{(\lambda_2\theta_i+\omega_1)g^*_{il}}, \frac{\lambda_1 \sum\limits_{j\in p(i)\setminus\{i\}} \theta_j}{\lambda_1\theta_i+\omega_3 y_i}\right\}.$$

The aggregate effort in the network increases with the formation of unions in P if the sufficient condition above is satisfied for all $i \in N\setminus S_P$. □

Proof of Lemma 2. Since $p(i)=\{i\}$ for all $i\in N$, we have that $h_{ij}=0$ for all $i,j\in N$. The proof of Lemma 2 follows directly from the fact that the weighted adjacency matrix $\mathbf{G}^*$ is row-normalized, and the application of Gershgorin circle theorem. Indeed the spectral radius ρ lies within the unit circle if the maximum row-sum norm of the matrix $[\boldsymbol{\phi}\otimes\mathbf{G}^*]$ is smaller than one. This is satisfied if the maximum of ϕ_i for all $i\in N$ is smaller than one. Using the notations in (7) it follows that the condition $\rho<1$ in Proposition 1 is satisfied when $\max\limits_i \theta_i < \dfrac{\omega_2+\omega_3}{\lambda_2}$. □

Proof of Corollary 1. Take $p(i)=\{i\}$, $\alpha_i=\alpha$ and $\phi_i=\phi$ for all $i\in N$. Hence, $h_{ij}=0$ for all $i,j\in N$. From Proposition 1 the spectral radius of the matrix $\phi\mathbf{G}^*$ has to be smaller than 1 to ensure the existence of equilibrium. Similarly to the proof of Lemma 2, it is easy to show that this condition is satisfied when $\theta < \dfrac{\omega_2+\omega_3}{\lambda_2}$. Then, the equilibrium efforts are equal to

$$\mathbf{x}^* = \alpha(\mathbf{I}-\phi\mathbf{G}^*)^{-1}\mathbf{1},$$

with $\mathbf{1}$ being a unit vector of size n. Given G^* is a row-normalized adjacency matrix, we have that $(\mathbf{I}-\phi\mathbf{G}^*)\mathbf{1}=(1-\phi)\mathbf{1}$. Using the properties of matrix inverse we get

$$\mathbf{1} = (\mathbf{I}-\phi\mathbf{G}^*)^{-1}(\mathbf{I}-\phi\mathbf{G}^*)\mathbf{1} = (\mathbf{I}-\phi\mathbf{G}^*)^{-1}(1-\phi)\mathbf{1}.$$

Thus $(\mathbf{I} - \phi\mathbf{G}^*)^{-1}\mathbf{1} = \dfrac{1}{1-\phi}\mathbf{1}$. It follows that

$$\alpha(\mathbf{I} - \phi\mathbf{G}^*)^{-1}\mathbf{1} = \frac{\alpha}{1-\phi}\mathbf{1}$$

and the equilibrium efforts are given by

$$\mathbf{x}^* = \frac{\alpha}{1-\phi}\mathbf{1}.$$

□

Proof of Corollary 2. Suppose $p(i) = \{i\}$ for all $i \in N$. Take $\boldsymbol{g}$ such that a subset $C \subset N$ of players forms a strongly connected component with no outgoing path towards players in $N \setminus C$, and $y_i = y^c$, $\theta_i = \theta^c$ for all $i \in C$. Hence, the subset of players C is disconnected from the rest of the network due to absence of outgoing links from the nodes in the component to those in $N \setminus C$. It follows that that the weighted adjacency matrix associated with any strongly connected component without outgoing paths on $\boldsymbol{g}$ is the sub-matrix $\mathbf{G}^*_c$ of $\mathbf{G}^*$ formed by the subset of rows and columns corresponding to players in C. Using the properties of the best response function (9) the results in Corollary 1 can be applied. Thus the equilibrium effort choice of any player in C with homogeneous types (y^c, θ^c) is equal to

$$x = \frac{\alpha^c}{1-\phi^c},$$

where $\alpha^c = \dfrac{\lambda_1\theta^c + \omega_3 y^c}{(\omega_1+\omega_2+\omega_3)}$ and $\phi^c = \dfrac{\lambda_2\theta^c + \omega_1}{(\omega_1+\omega_2+\omega_3)}$. □

Proof of Proposition 3.

$$\begin{aligned}
&\delta_{i,P}(\mathbf{G}^*, \mathbf{H}, \boldsymbol{\phi}, \boldsymbol{\alpha}) = \mathcal{B}(\mathbf{G}^*, \mathbf{H}, \boldsymbol{\phi}, \boldsymbol{\alpha}) - \mathcal{B}(\mathbf{G}^{*-i}, \mathbf{H}^{-i}, \boldsymbol{\phi}^{-i}, \boldsymbol{\alpha}^{-i}) \\
&= \sum_{j\in N}\sum_{k\in N}(m_{jk}\alpha_k - m_{jk}^{-i}\alpha_k^{-i}) \\
&\overset{(*1)}{=} \sum_{j\in N}\sum_{k\in N\setminus p(i)}(m_{jk} - m_{jk}^{-i})\alpha_k + \sum_{j\in N}\sum_{\substack{k\in p(i)\\ k\neq i}}(m_{jk}\alpha_k - m_{jk}^{-i}\alpha_k + m_{jk}^{-i}\alpha_k - m_{jk}^{-i}\alpha_k^{-i}) + \sum_{j\in N} m_{ji}\alpha_i \\
&= \sum_{j\in N}\sum_{k\in N\setminus\{i\}}(m_{jk} - m_{jk}^{-i})\alpha_k + \sum_{j\in N} m_{ji}\alpha_i + \sum_{j\in N}\sum_{\substack{k\in p(i)\\ k\neq i}} m_{jk}^{-i}(\alpha_k - \alpha_k^{-i}) \\
&\overset{(*2)}{=} \frac{b_i(\mathbf{G}^*, \mathbf{H}, \boldsymbol{\phi}, \boldsymbol{\alpha})}{m_{ii}}\sum_{j\in N} m_{ji} + \sum_{j\in N}\sum_{\substack{k\in p(i)\\ k\neq i}} m_{jk}^{-i}(\alpha_k - \alpha_k^{-i})
\end{aligned}$$

With:

$$(*1) : \alpha_k = \alpha_k^{-i} \; \forall \; k \notin p(i)$$

$$\begin{aligned}
(*2) : &\sum_{j \in N} \sum_{k \in N \setminus \{i\}} (m_{jk} - m_{jk}^{-i}) \alpha_k + \sum_{j \in N} m_{ji} \alpha_i \\
&= \sum_{j \in N} b_j(\mathbf{G}^*, \mathbf{H}, \boldsymbol{\phi}, \boldsymbol{\alpha}) - \sum_{j \in N \setminus \{i\}} b_j(\mathbf{G}^{*-i}, \mathbf{H}^{-i}, \boldsymbol{\phi}^{-i}, \boldsymbol{\alpha}) \\
&= b_i(\mathbf{G}^*, \mathbf{H}, \boldsymbol{\phi}, \boldsymbol{\alpha}) + \sum_{j \in N \setminus \{i\}} b_j(\mathbf{G}^*, \mathbf{H}, \boldsymbol{\phi}, \boldsymbol{\alpha}) - \sum_{j \in N \setminus \{i\}} b_j(\mathbf{G}^{*-i}, \mathbf{H}^{-i}, \boldsymbol{\phi}^{-i}, \boldsymbol{\alpha}) \\
&= b_i(\mathbf{G}^*, \mathbf{H}, \boldsymbol{\phi}, \boldsymbol{\alpha}) + \sum_{j \in N \setminus \{i\}} \sum_{k \in N} (m_{jk} \alpha_k - m_{jk}^{-i} \alpha_k) \\
&\overset{(*2A)}{=} b_i(\mathbf{G}^*, \mathbf{H}, \boldsymbol{\phi}, \boldsymbol{\alpha}) + \sum_{j \in N \setminus \{i\}} \sum_{k \in N} \frac{m_{ji} m_{ik}}{m_{ii}} \alpha_k \\
&\overset{(*2B)}{=} b_i(\mathbf{G}^*, \mathbf{H}, \boldsymbol{\phi}, \boldsymbol{\alpha}) + b_i(\mathbf{G}^*, \mathbf{H}, \boldsymbol{\phi}, \boldsymbol{\alpha}) \sum_{j \in N \setminus \{i\}} \frac{m_{ji}}{m_{ii}} \\
&= b_i(\mathbf{G}^*, \mathbf{H}, \boldsymbol{\phi}, \boldsymbol{\alpha})(1 + \sum_{j \in N \setminus \{i\}} \frac{m_{ji}}{m_{ii}}) \\
&= b_i(\mathbf{G}^*, \mathbf{H}, \boldsymbol{\phi}, \boldsymbol{\alpha})(\frac{m_{ii}}{m_{ii}} + \sum_{j \in N \setminus \{i\}} \frac{m_{ji}}{m_{ii}}) \\
&= b_i(\mathbf{G}^*, \mathbf{H}, \boldsymbol{\phi}, \boldsymbol{\alpha}) \sum_{j \in N} \frac{m_{ji}}{m_{ii}}
\end{aligned}$$

$$(*2A) : \text{Lemma 4 with} \boldsymbol{\Phi}(\boldsymbol{g}) = [\boldsymbol{\phi} \otimes \mathbf{G}^*] + \mathbf{H}$$

$$\begin{aligned}
(*2B) : &\sum_{j \in N \setminus \{i\}} \sum_{k \in N} \frac{m_{ji} m_{ik}}{m_{ii}} \alpha_k \\
&= \sum_{j \in N \setminus \{i\}} \left(\frac{m_{ji} m_{ii}}{m_{ii}} \alpha_i + \sum_{k \in N \setminus \{i\}} \frac{m_{ji} m_{ik}}{m_{ii}} \alpha_k \right) \\
&= \sum_{j \in N \setminus \{i\}} \frac{m_{ji}}{m_{ii}} \left(m_{ii} \alpha_i + \sum_{k \in N \setminus \{i\}} m_{ik} \alpha_k \right) \\
&= \sum_{j \in N \setminus \{i\}} \frac{m_{ji}}{m_{ii}} b_i(\mathbf{G}^*, \mathbf{H}, \boldsymbol{\phi}, \boldsymbol{\alpha})
\end{aligned}$$

□

Proof of Proposition 4.

$$
\begin{aligned}
&\mathcal{B}(\mathbf{G}^*, \mathbf{H}_{i\in\overline{p}}, \boldsymbol{\phi}_{i\in\overline{p}}, \boldsymbol{\alpha}_{i\in\overline{p}}) - \mathcal{B}(\mathbf{G}^*, \mathbf{H}_{i\in S_P}, \boldsymbol{\phi}_{i\in S_P}, \boldsymbol{\alpha}_{i\in S_P})\\
&= \mathcal{B}(\mathbf{G}^*, \mathbf{H}_{i\in\overline{p}}, \boldsymbol{\phi}_{i\in\overline{p}}, \boldsymbol{\alpha}_{i\in\overline{p}}) - \mathcal{B}(\mathbf{G}^{*-i}, \mathbf{H}^{-i}_{i\in\overline{p}}, \boldsymbol{\phi}^{-i}_{i\in\overline{p}}, \boldsymbol{\alpha}^{-i}_{i\in\overline{p}})\\
&\ + \mathcal{B}(\mathbf{G}^{*-i}, \mathbf{H}^{-i}_{i\in\overline{p}}, \boldsymbol{\phi}^{-i}_{i\in\overline{p}}, \boldsymbol{\alpha}^{-i}_{i\in\overline{p}}) - \mathcal{B}(\mathbf{G}^*, \mathbf{H}_{i\in S_P}, \boldsymbol{\phi}_{i\in S_P}, \boldsymbol{\alpha}_{i\in S_P})\\
&= \mathcal{B}(\mathbf{G}^*, \mathbf{H}_{i\in\overline{p}}, \boldsymbol{\phi}_{i\in\overline{p}}, \boldsymbol{\alpha}_{i\in\overline{p}}) - \mathcal{B}(\mathbf{G}^{*-i}, \mathbf{H}^{-i}_{i\in\overline{p}}, \boldsymbol{\phi}^{-i}_{i\in\overline{p}}, \boldsymbol{\alpha}^{-i}_{i\in\overline{p}})\\
&\ - \big(\mathcal{B}(\mathbf{G}^*, \mathbf{H}_{i\in S_P}, \boldsymbol{\phi}_{i\in S_P}, \boldsymbol{\alpha}_{i\in S_P}) - \mathcal{B}(\mathbf{G}^{*-i}, \mathbf{H}^{-i}_{i\in\overline{p}}, \boldsymbol{\phi}^{-i}_{i\in\overline{p}}, \boldsymbol{\alpha}^{-i}_{i\in\overline{p}})\big)\\
&= \delta_{i,\overline{P}}(\mathbf{G}^*, \mathbf{H}_{i\in\overline{p}}, \boldsymbol{\phi}_{i\in\overline{p}}, \boldsymbol{\alpha}_{i\in\overline{p}}) - \delta_{i,P}(\mathbf{G}^*, \mathbf{H}_{i\in S_P}, \boldsymbol{\phi}_{i\in S_P}, \boldsymbol{\alpha}_{i\in S_P}).
\end{aligned}
$$

□

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