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Local Farsightedness in Network Formation

Pierre de Callataÿ* Ana Mauleon† Vincent Vannetelbosch‡

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Abstract

We propose the concept of local-k farsighted consistent network for analysing network formation games where players only consider a limited number of feasible networks. A network g is said to be local-k farsightedly consistent if, for any network g' within the distance-k neighbourhood of g, either g is not defeated by g', or g defeats g'. We show that if the utility function is (componentwise) egalitarian or satisfies reversibility or excludes externalities across components, then local-k farsightedness is more likely to be a good proxy for what would happen when players have full knowledge of all feasible networks.

Key words: networks; local farsightedness; stability.
JEL Classification: A14, C70, D20.

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1 Introduction

Many social and economic networks do not emerge randomly but are rather the outcome of decisions or choices made by individuals or agents (e.g. R&D networks, trade networks, criminal networks, ...). Alternative solution concepts for predicting which network structures are likely to emerge in the long run have been proposed in the literature depending on whether and how far agents anticipate that their action may also induce others to change the network relations they maintain.[1] On the one hand, pairwise stability (Jackson and Wolinsky, 1996) involves fully myopic players in the sense that they do not anticipate that others might react to their actions. Players are able to modify the network one link at a time, and choose to change the network if the resulting network implies higher payoffs for the deviating agents. On the other hand, farsighted pairwise stability (Jackson, 2008) involves perfectly farsighted players. Players fully anticipate the complete sequence of reactions that results from their own actions in the network.[2]

But, there are many network formation games where a farsightedly pairwise stable network fails to exist. In fact, requiring a network to be immune to any profitable farsighted deviation is very demanding. One solution is to adopt set-valued concepts like the vNM farsighted stable set (Herings, Mauleon and Vannetelbosch, 2009; Mauleon, Vannetelbosch and Vergote, 2011; Ray and Vohra, 2015), the largest consistent set (Chwe, 1994; Page, Wooders and Kamat, 2005), the pairwise farsightedly stable set (Herings, Mauleon and Vannetelbosch, 2009) or the farsighted absorbing set (de Callataÿ, Mauleon and Vannetelbosch, 2023). Another solution is to compare networks by comparing their sets of farsighted defeating networks (de Callataÿ, Mauleon and Vannetelbosch, 2021). A network is minimally farsighted unstable if there is no other network which is more farsightedly stable.[3] However, both approaches are quite demanding in terms of computational complexity since they require that players have full knowledge of all feasible networks and their associated payoffs.

In this paper, we consider network formation games where players only consider a limited number of feasible networks. A network g is said to be local-k farsightedly stable if there is no network within the distance-k neighbourhood of g that farsightedly defeats g. A network farsightedly defeats another network if there is a farsighted outgoing improving path from the latter to the former network. A farsighted outgoing improving path from g to g' is a sequence of networks within the distance-k neighbourhood of g that can emerge

[1] See Goyal (2007) and Jackson (2008) for an introduction to social and economic networks. Mauleon and Vannetelbosch (2016) provide a comprehensive overview of the solution concepts for solving network formation games.

[2] Kirchsteiger, Mantovani, Mauleon and Vannetelbosch (2016) provide experimental evidence suggesting that subjects are consistent with (limited) farsightedness. Most of the players anticipate a number of reactions by the other players to the actions they take themselves. See also Teteryatnikova and Tremewan (2020).

[3] A network is more farsightedly stable than another network if the set of farsighted defeating networks in the former is a proper subset of the set of farsighted defeating networks in the latter network.

when players form or cut links based on the improvement the end network offers relative to the current network. The distance-k neighbourhood of g consists of all networks that are at most at a distance k of g. The distance between g and g' is simply the number of links that g does have while g' does not, plus the number of links that g does not have while g' does. Similarly to farsighted pairwise stability (Jackson, 2008), a local-k farsightedly stable network often fails to exist.[4]

Hence, we propose the notion of local-k farsightedly consistent network. A network g is local-k farsightedly consistent if any farsighted deviation is deterred: either because the deviating players are worse off at the end network g' or because there is a farsighted ingoing improving path to g from the end network g'. In other words, the threat of a farsighted deviation to g' is consistently deterred by a subsequent farsighted deviation leading back to g. Local-k farsighted consistency is a coarsening of local-k farsighted stability.

Our next objective is to question the robustness of the predictions made under local farsightedness. In general, some networks that could emerge in the long run when all players are farsighted may not be locally farsighted stable or consistent, and vice versa. Under which circumstances is looking at local farsightedness a good proxy for what would happen when players have full knowledge of all feasible networks? On the contrary, when restricting the analysis to local farsightedness could lead to false predictions?

First, we show that, under the egalitarian utility function, the set of local-k farsightedly consistent networks is equal to the set of local-k farsightedly stable networks. In addition, strongly efficient networks are always local-k farsightedly consistent. A public good investment model illustrates this property. Second, in the case the utility function is componentwise egalitarian, we characterize a set of networks that are local-k farsightedly consistent independently of k. Under a componentwise egalitarian utility function, players belonging to the same component get the same utility and there are no externalities across components. Third, if any profitable farsighted deviation from a network g within its distance-k neighbourhood is reversible, then g is local-k farsightedly consistent. We show that, the distance-based utility model of Bloch and Jackson (2007) satisfies this property, and a star network $g^\star$ is local-k farsightedly consistent if k is greater than the number of links in $g^\star$, even tough $g^\star$ is not local-k farsightedly stable. Fourth, we show that, if the utility function allows for externalities across components, then the size of the neighbourhood matters much more for determining the farsighted stability of a network. To sum up, if the utility function is (componentwise) egalitarian, or satisfies reversibility, or excludes externalities across components, then local-k farsighted stability or consistency is more likely to be a good predictor for farsightedness.

Finally, we investigate the relationships between our concepts of local farsightedness and set-valued concepts of farsightedness. Beside situations where players have limited

[4]Another approach for overcoming the lack of farsighted stability is to require the consent of component members for adding or deleting links. See Diamantoudi and Xue (2003), Caulier, Mauleon and Vannetelbosch (2013) and Caulier, Mauleon, Sempere-Monerris and Vannetelbosch (2013).

knowledge about feasible networks, there are also situations where players typically have incomplete information about others' connections. Expectations about these connections might then influence players' decisions with whom to connect. Recently, Foerster, Mauleon and Vannetelbosch (2021) propose a solution concept for network formation games where individuals can form two types of links: public links observed by everyone and shadow links generally not observed by others.[5]

The paper is organized as follows. In Section 2 we introduce some notation and basic properties of networks. In Section 3 we define the concepts of local farsightedness. In Section 4 we look at general properties for identifying locally farsighted consistent networks. In Section 5 we determine the relationships between the concepts of local farsightedness and set-valued concepts. In Section 6 we consider farsighted group deviations and externalities. Finally, in Section 7 we conclude.

2 Networks

The set of players is denoted by $N = \{1, 2, ..., n\}$, where n is the total number of players. A network g is a list of which pairs of players are linked to each other and $ij \in g$ indicates that i and j are linked under g. The complete network on the set of players $S \subseteq N$ is denoted by g^S and is equal to the set of all subsets of S of size 2. It follows in particular that the empty network is denoted by $g^{\emptyset}$. The set of all possible networks on N is denoted by $\mathcal{G}$ and consists of all subsets of g^N. The network obtained by adding link ij to an existing network g is denoted $g + ij$ and the network that results from deleting link ij from an existing network g is denoted $g - ij$. Let $N(g) = \{i \mid \text{there is } j \text{ such that } ij \in g\}$ be the set of players who have at least one link in the network g. Let $N_i(g) = \{j \in N \mid ij \in g\}$ be the set of neighbours of player i in g. A star network, denoted by $g_i^{\star}$, is a network such that there exists some player i (the center) who is linked to every other player $j \neq i$ (the peripherals) and that contains no other links (i.e. g is such that $N_i(g) = N \setminus \{i\}$ and $N_j(g) = \{i\}$ for all $j \in N \setminus \{i\}$). A path in a network g between i and j is a sequence of players $i_1, \ldots, i_K$ such that $i_k i_{k+1} \in g$ for each $k \in \{1, \ldots, K-1\}$ with $i_1 = i$ and $i_K = j$. A network g is connected if for all $i \in N$ and $j \in N \setminus \{i\}$, there exists a path in g connecting i and j. A non-empty network $h \subseteq g$ is a component of g, if for all $i \in N(h)$ and $j \in N(h) \setminus \{i\}$, there exists a path in h connecting i and j, and for any $i \in N(h)$ and $j \in N(g)$, $ij \in g$ implies $ij \in h$. The set of components of g is denoted by $H(g)$. A component h of g is minimally connected if h has $\#N(h) - 1$ links (i.e. every pair of players in the component are connected by exactly one path). The partition of N induced by g is denoted by $\Pi(g)$, where $S \in \Pi(g)$ if and only if either there exists $h \in H(g)$ such

[5]Once players can have shadow links, some players may overestimate others' connections and hence under-connect (relative to stable networks under correct beliefs), while others may underestimate connections and hence over-connect.

that $S = N(h)$ or there exists $i \notin N(g)$ such that $S = \{i\}$.[6] We denote by $S(i)$ the coalition $S \in \Pi(g)$ such that $i \in S$.

A network utility function is a mapping $u : \mathcal{G} \to \mathbb{R}^N$ that assigns to each network g a utility $u_i(g)$ for each player $i \in N$. A network $g \in \mathcal{G}$ is strongly efficient relative to u if it maximizes $\sum_{i \in N} u_i(g)$; i.e. if $\sum_{i \in N} u_i(g) \geq \sum_{i \in N} u_i(g')$ for all $g' \in \mathcal{G}$. Let E be the set of strongly efficient networks. A network $g \in \mathcal{G}$ Pareto dominates a network $g' \in \mathcal{G}$ relative to u if $u_i(g) \geq u_i(g')$ for all $i \in N$, with strict inequality for at least one $i \in N$. A network $g \in \mathcal{G}$ is Pareto efficient relative to u if it is not Pareto dominated, and a network $g \in \mathcal{G}$ is Pareto dominant if it Pareto dominates any other network. A simple way to determine which networks are stable is Jackson and Wolinsky (1996) myopic notion of pairwise stability: a network g is pairwise stable with respect to u if and only if (i) for all $ij \in g$, $u_i(g) \geq u_i(g-ij)$ and $u_j(g) \geq u_j(g-ij)$, and (ii) for all $ij \notin g$, if $u_i(g) < u_i(g+ij)$ then $u_j(g) > u_j(g+ij)$.[7]

Let us define a notion of distance between two networks which will be crucial for defining local farsightedness. For $g, g' \subseteq g^N$, $g \neq g'$ we denote by

$$d(g, g') \equiv \#\{ij \in g^N \mid (ij \in g \wedge ij \notin g') \vee (ij \notin g \wedge ij \in g')\},$$

the distance between g and g'. That is, $d(g, g')$ is the number of links that g does have while g' does not, plus the number of links that g does not have while g' does. Hence,

$$1 \leq d(g, g') \leq \frac{n(n-1)}{2}.$$

For a given network g, let $\Delta(g, k) \subseteq \mathcal{G}$ be the distance-k neighbourhood of g. It is the set of networks which are at a distance of at most k from g. Formally,

$$\Delta(g, k) \equiv \{g' \in \mathcal{G} \mid d(g, g') \leq k, g' \neq g\},$$

and

$$\#\Delta(g, k) = \sum_{l=1}^{k} \begin{pmatrix} n(n-1)/2 \\ l \end{pmatrix} = \sum_{l=1}^{k} \left(\frac{(n(n-1)/2)!}{l!(n(n-1)/2 - l)!} \right)$$

is simply the number of networks in the distance-k neighbourhood of g.

3 Local Farsightedness

3.1 Locally farsighted stable networks

We first define the notion of local-k farsightedly stable networks. A local-k farsightedly stable network is a network g such that farsighted players do not have incentives to deviate

[6] Throughout the paper we use the notation $\subseteq$ for weak inclusion and $\subsetneq$ for strict inclusion. Finally, $\#$ will refer to the notion of cardinality.

[7] Tercieux and Vannetelbosch (2006) provide a refinement of pairwise stability, p-pairwise stability, which allows them to characterize the stochastically stable networks.

from it looking forward towards some network within the neighbourhood of g. Precisely, a network is a local-k farsightedly stable network if there is no farsighted outgoing improving path emanating out of it within its distance-k neighbourhood. A farsighted outgoing improving path from g to g' is a sequence of networks that can emerge when farsighted players form or delete links based on the improvement the end network offers relative to the current network.[8] All networks along the sequence belong to the distance-k neighbourhood of g. Since we only allow for pairwise deviations, each network in the sequence differs from the previous one in that either a new link is formed between two players or an existing link is deleted. If a link ij is deleted, then it must be that either player i or player j prefers the end network to the current network. If a link is added between player i and player j, then both player i and player j must prefer the end network to the current network.

Definition 1. A farsighted outgoing improving path $(g_1, \ldots, g_L)$ from a network g to a network $g' \in \Delta(g,k)$ within the distance-k neighbourhood of g is a finite sequence of networks $g_1, \ldots, g_L$ with $g_1 = g$ and $g_L = g'$ such that for any $l \in \{1, \ldots, L-1\}$, $g_{l+1} \in \Delta(g,k)$ and either

(i) $g_{l+1} = g_l - ij$ for some ij such that $u_i(g_L) > u_i(g_l)$ or $u_j(g_L) > u_j(g_l)$; or

(ii) $g_{l+1} = g_l + ij$ for some ij such that $u_i(g_L) > u_i(g_l)$ and $u_j(g_L) \geq u_j(g_l)$.

If there exists a farsighted outgoing improving path from a network g to a network g' within the distance-k neighbourhood of g, then we write $g \to_k g'$. For a given network $g \in \mathcal{G}$, let $\phi_k^{\text{out}}(g) \subseteq \Delta(g,k)$ be the set of all networks within the distance-k neighbourhood of g that can be reached from g by a farsighted improving path. That is,

$$\phi_k^{\text{out}}(g) = \{g' \in \Delta(g,k) \mid g \to_k g'\}.$$

In other words, $\phi_k^{\text{out}}(g)$ is the set of networks within the distance-k neighbourhood of g that farsightedly defeat g.

Definition 2. A network $g \in \mathcal{G}$ is local-k farsightedly stable if $\phi_k^{\text{out}}(g) = \emptyset$.

Notice that $\phi_{k-1}^{\text{out}}(g) \subseteq \phi_k^{\text{out}}(g)$ for $k > 1$ and if $\phi_k^{\text{out}}(g) = \emptyset$ then $\phi_{k-1}^{\text{out}}(g) = \emptyset$. For $k \geq n(n-1)/2$, $\Delta(g,k)$ is simply $\mathcal{G} \setminus \{g\}$ and $\phi_k^{\text{out}}(g) = \phi_{k+1}^{\text{out}}(g) = \phi_\infty^{\text{out}}(g)$ where $\phi_\infty^{\text{out}}(g)$ gives us the set of all networks that farsightedly defeat g.

Let $P(k)$ be the set of local-k farsightedly stable networks. Notice that $P(k) \subseteq P(k-1)$. Unfortunately, $P(k)$ might turn to be empty. Since $\mathcal{G}$ is finite, there exists k' such that $P(k) = P(k+1) = P(\infty)$ for $k > k'$. Obviously, $P(1)$ is the set of pairwise stable networks (Jackson and Wolinsky, 1996) and $P(\infty)$ is the set of farsightedly pairwise stable networks (Jackson, 2008). There is no guarantee that $P(1)$ is non-empty.

[8] See Jackson (2008) or Herings, Mauleon and Vannetelbosch (2009) for the original definition of a farsighted improving path. Jackson and Watts (2002) define the notion of improving path in the case that all players are myopic. Herings, Mauleon and Vannetelbosch (2020) and Luo, Mauleon and Vannetelbosch (2021) extend this notion to a mixed population composed of both myopic and farsighted players.

Remark 1. $P(1) \supseteq P(k) \supseteq P(k+1) \supseteq P(\infty)$.

Hence, emptiness of $P(k)$ (or instability) is more likely to become a problem when k increases. A network $g \in \mathcal{G}$ is said to be distance-k farsightedly unstable if $\phi_k^{\text{out}}(g) \neq \emptyset$ and $\phi_{k-1}^{\text{out}}(g) = \emptyset$. A network $g \in \mathcal{G}$ is more locally farsighted stable than a network $g' \in \mathcal{G}$ if g is distance-k farsightedly unstable and g' is distance-k' farsightedly unstable with $k > k'$.

Example 1. Consider a situation where three players can form links. The payoffs they obtain from the different network configurations are given in Figure 1. We have $\phi_1^{\text{out}}(g_0) = \{g_1, g_2, g_3\}$, $\phi_k^{\text{out}}(g_0) = \{g_1, g_2, g_3, g_4, g_5, g_6\}$ for $k \geq 2$, $\phi_1^{\text{out}}(g_1) = \{g_4, g_5\}$, $\phi_k^{\text{out}}(g_1) = \{g_4, g_5, g_6\}$ for $k \geq 2$, $\phi_1^{\text{out}}(g_2) = \{g_4, g_6\}$, $\phi_k^{\text{out}}(g_2) = \{g_4, g_5, g_6\}$ for $k \geq 2$, $\phi_1^{\text{out}}(g_3) = \{g_5, g_6\}$, $\phi_k^{\text{out}}(g_3) = \{g_4, g_5, g_6\}$ for $k \geq 2$, $\phi_1^{\text{out}}(g_4) = \emptyset$, $\phi_k^{\text{out}}(g_4) = \{g_5, g_6\}$ for $k \geq 2$, $\phi_1^{\text{out}}(g_5) = \emptyset$, $\phi_k^{\text{out}}(g_5) = \{g_4, g_6\}$ for $k \geq 2$, $\phi_1^{\text{out}}(g_6) = \emptyset$, $\phi_k^{\text{out}}(g_6) = \{g_4, g_5\}$ for $k \geq 2$, and $\phi_k^{\text{out}}(g_7) = \{g_4, g_5, g_6\}$ for $k \geq 1$.

Hence, $P(1) = \{g_4, g_5, g_6\}$ while $P(k) = \emptyset$ for $k \geq 2$. But, are some networks more locally farsighted stable than others? The networks g_4, g_5, g_6 are more locally farsighted stable than any other network.

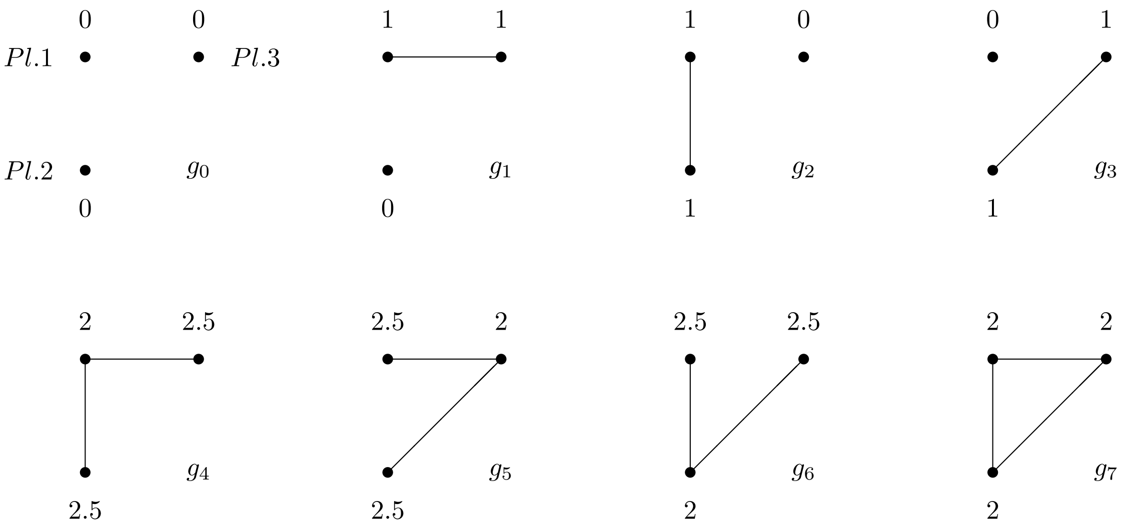

Figure 1: An example with three players.

3.2 Locally farsighted absorbing networks

We now define the notion of local-k farsightedly absorbing networks. A network g is a local-k farsightedly absorbing network if from any network within its distance-k neighbourhood there is a farsighted ingoing improving path leading to g.

Definition 3. A farsighted ingoing improving path $(g_1, \ldots, g_L)$ to a network g from a network $g' \in \Delta(g, k)$ within the distance-k neighbourhood of g is a finite sequence of networks $g_1, \ldots, g_L$ with $g_1 = g'$ and $g_L = g$ such that for any $l \in \{1, \ldots, L-1\}$, $g_{l+1} \in \Delta(g,k)$ and either

(i) $g_{l+1} = g_l - ij$ for some ij such that $u_i(g_L) > u_i(g_l)$ or $u_j(g_L) > u_j(g_l)$; or

(ii) $g_{l+1} = g_l + ij$ for some ij such that $u_i(g_L) > u_i(g_l)$ and $u_j(g_L) \geq u_j(g_l)$.

If there exists a farsighted ingoing improving path from a network g' to a network g within the distance-k neighbourhood of g, then we write $g \leftarrow_k g'$. For a given network $g \in \mathcal{G}$, let $\phi_k^{\text{in}}(g) \subseteq \Delta(g,k)$ be the set of all networks within the distance-k neighbourhood of g from which there is a farsighted ingoing improving path leading to g. That is,

$$\phi_k^{\text{in}}(g) = \{g' \in \Delta(g,k) \mid g \leftarrow_k g'\}.$$

In other words, $\phi_k^{\text{in}}(g)$ is the set of networks within the distance-k neighbourhood of g that are farsightedly defeated by g.

Definition 4. A network $g \in \mathcal{G}$ is local-k farsightedly absorbing if $\phi_k^{\text{in}}(g) = \Delta(g,k)$.

For $k \geq n(n-1)/2$, $\Delta(g,k)$ is simply $\mathcal{G} \setminus \{g\}$ and $\phi_k^{\text{in}}(g) = \phi_{k+1}^{\text{in}}(g) = \phi_\infty^{\text{in}}(g)$ where $\phi_\infty^{\text{in}}(g)$ gives us the set of all networks that are farsightedly defeated by g.

Let $A(k)$ be the set of local-k farsightedly absorbing networks.

Example 1 (Continued). We reconsider the situation where three players can form links of Figure 1. We have $\phi_k^{\text{in}}(g_0) = \emptyset$ for $k \geq 1$, $\phi_k^{\text{in}}(g_1) = \{g_0\}$ for $k \geq 1$, $\phi_k^{\text{in}}(g_2) = \{g_0\}$ for $k \geq 1$, $\phi_k^{\text{in}}(g_3) = \{g_0\}$ for $k \geq 1$, $\phi_1^{\text{in}}(g_4) = \{g_1, g_2, g_7\} = \Delta(g_4, 1)$, $\phi_2^{\text{in}}(g_4) = \{g_0, g_1, g_2, g_5, g_6, g_7\} = \Delta(g_4, 2)$, $\phi_k^{\text{in}}(g_4) = \{g_0, g_1, g_2, g_3, g_5, g_6, g_7\} = \Delta(g_4, k) = \mathcal{G} \setminus \{g_4\}$ for $k \geq 3$, $\phi_1^{\text{in}}(g_5) = \{g_1, g_3, g_7\} = \Delta(g_5, 1)$, $\phi_2^{\text{in}}(g_5) = \{g_0, g_1, g_3, g_4, g_6, g_7\} = \Delta(g_5, 2)$, $\phi_k^{\text{in}}(g_5) = \{g_0, g_1, g_2, g_3, g_4, g_6, g_7\} = \Delta(g_5, k) = \mathcal{G} \setminus \{g_5\}$ for $k \geq 3$, $\phi_1^{\text{in}}(g_6) = \{g_2, g_3, g_7\} = \Delta(g_6, 1)$, $\phi_2^{\text{in}}(g_6) = \{g_0, g_2, g_3, g_4, g_5, g_7\} = \Delta(g_6, 2)$, $\phi_k^{\text{in}}(g_6) = \{g_0, g_1, g_2, g_3, g_4, g_5, g_7\} = \Delta(g_6, k) = \mathcal{G} \setminus \{g_6\}$ for $k \geq 3$, and $\phi_k^{\text{in}}(g_7) = \emptyset$ for $k \geq 1$.

Hence, we obtain that $A(k) = \{g_4, g_5, g_6\}$ for $k \geq 1$.

Example 2. Consider another situation where three players can form links. The payoffs they obtain from the different network configurations are given in Figure 2. We have $\phi_k^{\text{in}}(g_0) = \emptyset$ for $k \geq 1$, $\phi_k^{\text{in}}(g_1) = \{g_0\}$ for $k \geq 1$, $\phi_k^{\text{in}}(g_2) = \{g_0\}$ for $k \geq 1$, $\phi_k^{\text{in}}(g_3) = \{g_0\}$ for $k \geq 1$, $\phi_1^{\text{in}}(g_4) = \{g_1, g_2, g_7\} = \Delta(g_4, 1)$, $\phi_2^{\text{in}}(g_4) = \{g_0, g_1, g_2, g_6, g_7\} \subsetneq \Delta(g_4, 2)$, $\phi_k^{\text{in}}(g_4) = \{g_0, g_1, g_2, g_3, g_5, g_6, g_7\} = \Delta(g_4, k) = \mathcal{G} \setminus \{g_4\}$ for $k \geq 3$, $\phi_1^{\text{in}}(g_5) = \{g_1, g_3, g_7\} = \Delta(g_5, 1)$, $\phi_2^{\text{in}}(g_5) = \{g_0, g_1, g_3, g_4, g_6, g_7\} = \Delta(g_5, 2)$, $\phi_k^{\text{in}}(g_5) = \{g_0, g_1, g_2, g_3, g_4, g_6, g_7\} = \Delta(g_5, k) = \mathcal{G} \setminus \{g_5\}$ for $k \geq 3$, $\phi_1^{\text{in}}(g_6) = \{g_2, g_3\} \subsetneq \Delta(g_6, 1)$, $\phi_2^{\text{in}}(g_6) = \{g_0, g_2, g_3\} \subsetneq \Delta(g_6, 2)$, $\phi_k^{\text{in}}(g_6) = \{g_0, g_1, g_2, g_3, g_4, g_5, g_7\} = \Delta(g_6, k) = \mathcal{G} \setminus \{g_6\}$ for $k \geq 3$, and $\phi_1^{\text{in}}(g_7) = \{g_6\} \subsetneq \Delta(g_7, 1)$, $\phi_2^{\text{in}}(g_7) = \{g_1, g_2, g_3, g_4, g_5, g_6\} = \Delta(g_7, 2)$, $\phi_k^{\text{in}}(g_7) = \{g_0, g_1, g_2, g_3, g_4, g_5, g_6\} = \Delta(g_7, k) = \mathcal{G} \setminus \{g_7\}$ for $k \geq 3$.

Hence, $A(1) = \{g_4, g_5\}$ while $A(2) = \{g_5, g_7\}$ and $A(k) = \{g_4, g_5, g_6, g_7\}$ for $k \geq 3$. Thus, $A(1) \nsubseteq A(2) \subseteq A(k)$ for $k \geq 3$. So, the set of local-k farsightedly absorbing networks may depend in a non-monotonic way on the distance k.

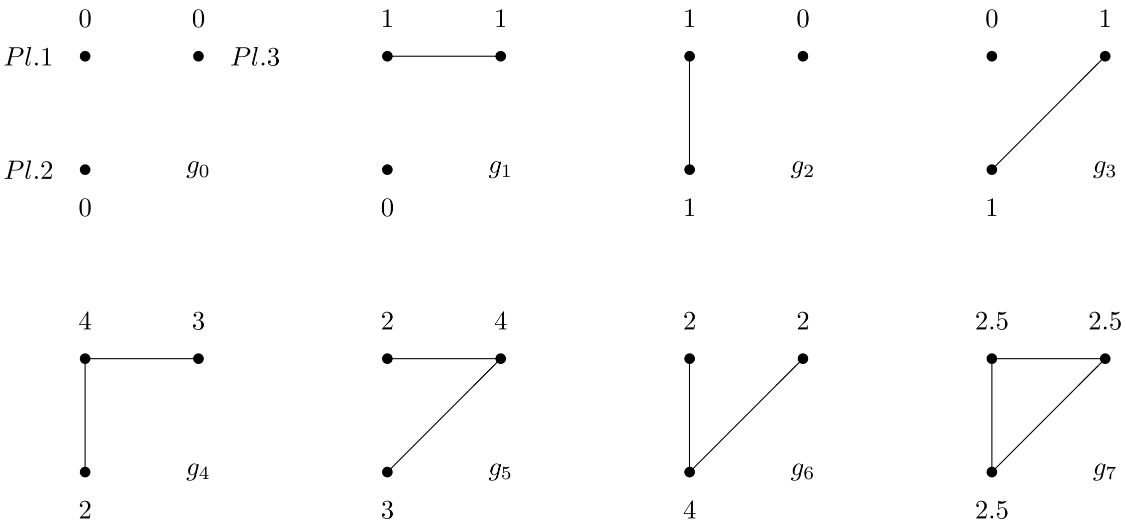

Figure 2: Another example with three players.

3.3 Locally farsighted consistent networks

We now define the notion of local-k farsightedly consistent networks. A local-k farsightedly consistent network is a network g such that any farsighted deviation from g within its distance-k neighbourhood is deterred. Either because the deviating players are worse off at the end network g' or because there is a farsighted ingoing improving path to g from the end network g'. In other words, the threat of a farsighted deviation to g' is consistently deterred by a subsequent farsighted deviation leading back to g.

Definition 5. A network $g \in \mathcal{G}$ is local-k farsightedly consistent if for any $g' \in \Delta(g,k)$ either $g' \notin \phi_k^{\text{out}}(g)$ or $g' \in \phi_k^{\text{in}}(g)$.

Let $F(k)$ be the set of local-k farsightedly consistent networks.

Remark 2. $P(k) \subseteq F(k)$ and $A(k) \subseteq F(k)$ for all $k \geq 1$.

Example 2 (Continued). We reconsider the situation where three players can form links of Figure 2. We have $\phi_1^{\text{out}}(g_0) = \{g_1, g_2, g_3\}$, $\phi_2^{\text{out}}(g_0) = \{g_1, g_2, g_3, g_4, g_5, g_6\}$, $\phi_k^{\text{out}}(g_0) = \{g_1, g_2, g_3, g_4, g_5, g_6, g_7\}$ for $k \geq 3$, $\phi_k^{\text{out}}(g_1) = \{g_4, g_5\}$ for $k \leq 2$, $\phi_k^{\text{out}}(g_1) = \{g_4, g_5, g_6, g_7\}$ for $k \geq 3$, $\phi_1^{\text{out}}(g_2) = \{g_4, g_6\}$, $\phi_2^{\text{out}}(g_2) = \{g_4, g_6, g_7\}$, $\phi_k^{\text{out}}(g_2) = \{g_4, g_5, g_6, g_7\}$ for $k \geq 3$, $\phi_1^{\text{out}}(g_3) = \{g_5, g_6\}$, $\phi_2^{\text{out}}(g_3) = \{g_5, g_6, g_7\}$, $\phi_k^{\text{out}}(g_3) = \{g_4, g_5, g_6, g_7\}$ for $k \geq 3$, $\phi_1^{\text{out}}(g_4) = \emptyset$, $\phi_k^{\text{out}}(g_4) = \{g_5, g_6, g_7\}$ for $k \geq 2$, $\phi_1^{\text{out}}(g_5) = \emptyset$, $\phi_k^{\text{out}}(g_5) = \{g_4, g_6, g_7\}$ for $k \geq 2$, $\phi_1^{\text{out}}(g_6) = \{g_7\}$, $\phi_k^{\text{out}}(g_6) = \{g_4, g_5, g_7\}$ for $k \geq 2$, $\phi_k^{\text{out}}(g_7) = \{g_4, g_5\}$ for $k \leq 2$, and $\phi_k^{\text{out}}(g_7) = \{g_4, g_5, g_6\}$ for $k \geq 3$.

Hence, $P(1) = \{g_4, g_5\}$ and $P(k) = \emptyset$ for $k \geq 2$. We get $F(1) = \{g_4, g_5\}$ while $F(2) = \{g_5, g_7\}$ and $F(k) = \{g_4, g_5, g_6, g_7\}$ for $k \geq 3$. Hence, as for the locally farsighted absorbing networks, the set of local-k farsightedly consistent networks may depend in a non-monotonic way on the distance k.

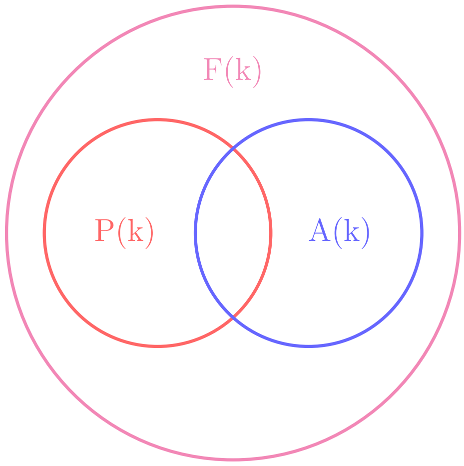

Figure 3: Relationship between the alternative concepts of local farsightedness.

Example 3. Consider the situation where three players can form links and where the payoffs are given in Figure 4. For $k \geq 3$ we have $\phi_k^{\text{out}}(g_0) = \{g_4, g_7\} \subseteq \phi_k^{\text{in}}(g_0) = \{g_1, g_2, g_4, g_5, g_6, g_7\}$, $\phi_k^{\text{out}}(g_1) = \{g_0, g_3, g_4, g_7\} \nsubseteq \phi_k^{\text{in}}(g_1) = \emptyset$, $\phi_k^{\text{out}}(g_2) = \{g_0, g_3, g_4, g_7\} \nsubseteq \phi_k^{\text{in}}(g_2) = \emptyset$, $\phi_k^{\text{out}}(g_3) = \{g_4, g_7\} \subseteq \phi_k^{\text{in}}(g_3) = \{g_1, g_2, g_4, g_5, g_6, g_7\}$, $\phi_k^{\text{out}}(g_4) = \{g_0, g_3\} \subseteq \phi_k^{\text{in}}(g_4) = \{g_0, g_1, g_2, g_3, g_5, g_6\}$, $\phi_k^{\text{out}}(g_5) = \{g_0, g_3, g_4, g_7\} \nsubseteq \phi_k^{\text{in}}(g_5) = \emptyset$, $\phi_1^{\text{out}}(g_6) = \{g_0, g_3, g_4, g_7\} \nsubseteq \phi_k^{\text{in}}(g_6) = \emptyset$, and $\phi_k^{\text{out}}(g_7) = \{g_0, g_3\} \subseteq \phi_k^{\text{in}}(g_7) = \{g_0, g_1, g_2, g_3, g_5, g_6\}$.

Hence, $P(k) = \emptyset$ and $A(k) = \emptyset$ for $k \geq 3$. However, $F(k) = \{g_0, g_3, g_4, g_7\}$ for $k \geq 3$. Thus, $F(k)$ is a strict coarsening of $P(k)$ and $A(k)$ in this example. In Figure 3 we depict the relationship between the three notions of local farsightedness.

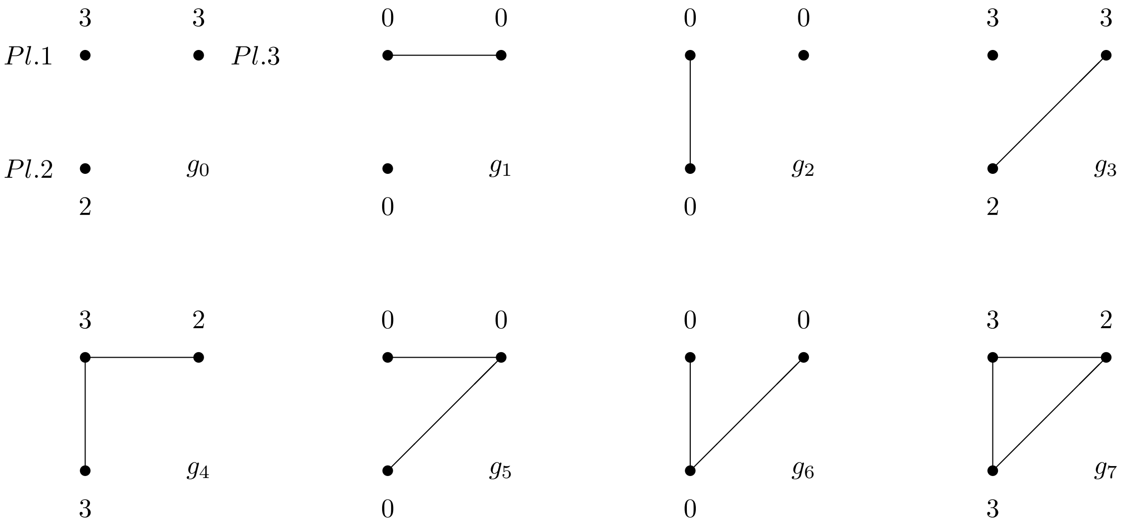

Figure 4: $F(k)$ is a strict coarsening of $P(k)$ and $A(k)$.

Proposition 1. *A network $g \in \mathcal{G}$ is local-k farsightedly consistent if and only if $\phi_k^{out}(g) \subseteq \phi_k^{in}(g)$.*

Proof. (⇐) Suppose that $\phi_k^{\text{out}}(g) \nsubseteq \phi_k^{\text{in}}(g)$. Then, there exists $g' \in \Delta(g,k)$ such that $g' \in \phi_k^{\text{out}}(g)$ and $g' \notin \phi_k^{\text{in}}(g)$, and so it follows that g is not a local-k farsightedly consistent network, a contradiction. (⇒) Suppose that g is not a local-k farsightedly consistent network. Then, there exists $g' \in \Delta(g,k)$ such that $g' \in \phi_k^{\text{out}}(g)$ and $g' \notin \phi_k^{\text{in}}(g)$, and so it follows that $\phi_k^{\text{out}}(g) \nsubseteq \phi_k^{\text{in}}(g)$, a contradiction. □

Example 4. Consider the situation where three players can form links and where the payoffs are given in Figure 5. We have $\phi_1^{\text{out}}(g_0) = \{g_3\}$, $\phi_1^{\text{in}}(g_0) = \{g_1, g_2\}$, $\phi_2^{\text{out}}(g_0) = \{g_3\}$, $\phi_2^{\text{in}}(g_0) = \{g_1, g_2, g_3, g_4, g_5, g_6\}$, $\phi_k^{\text{out}}(g_0) = \{g_3\}$, $\phi_k^{\text{in}}(g_0) = \{g_1, g_2, g_3, g_4, g_5, g_6, g_7\}$ for $k \geq 3$, $\phi_1^{\text{out}}(g_3) = \emptyset$, $\phi_1^{\text{in}}(g_3) = \{g_0, g_5, g_6\}$, $\phi_k^{\text{out}}(g_3) = \{g_0, g_1\}$, $\phi_k^{\text{in}}(g_3) = \{g_0, g_2, g_5, g_6, g_7\}$ for $k \geq 2$. From Proposition 1, since $\phi_1^{\text{out}}(g_0) = \{g_3\} \nsubseteq \{g_1, g_2\} = \phi_1^{\text{in}}(g_0)$ and $\phi_1^{\text{out}}(g_3) = \emptyset$, we have that g_3 is a local-1 farsightedly consistent network while g_0 is not. Similarly, since $\phi_k^{\text{out}}(g_0) \subseteq \phi_k^{\text{in}}(g_0)$ and $\phi_k^{\text{out}}(g_3) \nsubseteq \phi_k^{\text{in}}(g_3)$ for $k \geq 2$, we have that g_0 is a local-k farsightedly consistent network while g_3 is not. Hence, $F(1) = \{g_3\}$ while $F(k) = \{g_0\}$ for $k \geq 2$.

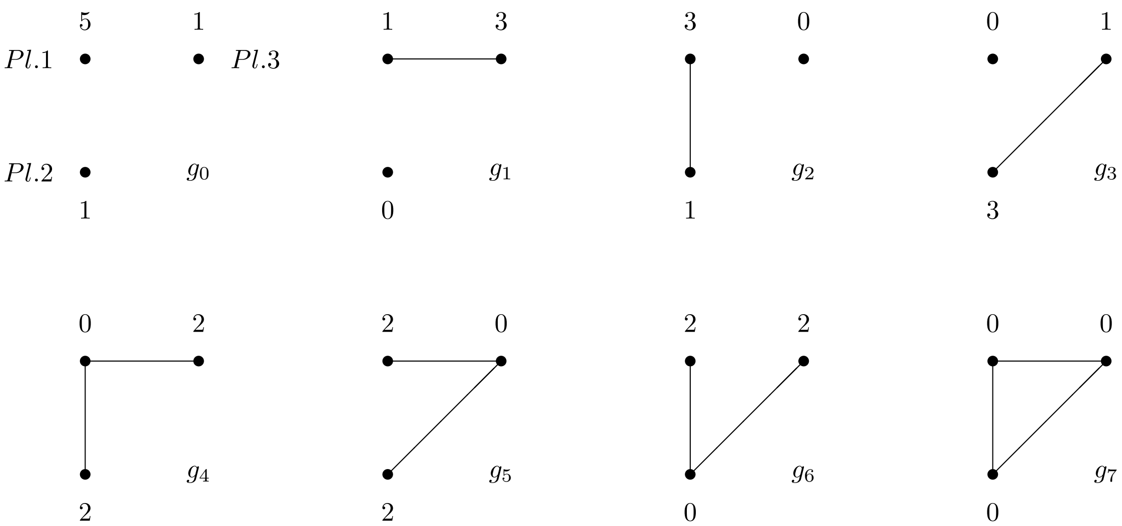

Figure 5: Another example with three players.

4 General Properties

4.1 Reversibility

Definition 6. A farsighted outgoing improving path $(g_1, g_2, \ldots, g_{L-1}, g_L)$ from a network $g_1 = g$ to a network $g_L = g' \in \Delta(g,k)$ within the distance-k neighbourhood of g is reversible if the finite sequence of networks $(g_L, g_{L-1}, \ldots, g_2, g_1)$ is a farsighted ingoing improving path to g from g'.

Let $\Phi_k(g)$ be the set of all networks within the distance-k neighbourhood of g that can be reached from g by a farsighted outgoing improving path that is reversible.

Proposition 2. *If for any $g' \in \Delta(g,k)$, either $g' \notin \phi_k^{out}(g)$ or $g' \in \Phi_k(g)$, then g is a local-k farsightedly consistent network.*

Proof. Suppose that g is not a local-k farsightedly consistent network. Then, there exists $g' \in \Delta(g,k)$ such that $g' \in \phi_k^{\text{out}}(g)$ and $g' \notin \phi_k^{\text{in}}(g)$ and so $g' \notin \Phi_k(g)$, a contradiction. □

Corollary 1. *If $\phi_k^{out}(g) = \Phi_k(g)$ then g is a local-k farsightedly consistent network.*

Example 1 (Continued). We reconsider the situation where three players can form links of Figure 1. Take for instance the network g_4. We had $\phi_1^{\text{in}}(g_4) = \{g_1, g_2, g_7\} = \Delta(g_4, 1)$, $\phi_2^{\text{in}}(g_4) = \{g_0, g_1, g_2, g_5, g_6, g_7\} = \Delta(g_4, 2)$, $\phi_k^{\text{in}}(g_4) = \{g_0, g_1, g_2, g_3, g_5, g_6, g_7\} = \Delta(g_4, k) = \mathcal{G} \setminus \{g_4\}$ for $k \geq 3$. In addition, $\phi_1^{\text{out}}(g_4) = \emptyset$, $\phi_k^{\text{out}}(g_4) = \{g_5, g_6\}$ for $k \geq 2$. There is a unique farsighted outgoing improving path from g_4 to g_5 consisting of the sequence (g_4, g_1, g_5). There is also a unique farsighted outgoing improving path from g_4 to g_6 consisting of the sequence (g_4, g_2, g_6). Both farsighted outgoing improving paths are reversible. Indeed, the sequence (g_5, g_1, g_4) is a farsighted ingoing improving path to g_4 from g_5 within $\Delta(g_4, k)$ and the sequence (g_6, g_2, g_4) is a farsighted ingoing improving path to g_4 from g_6 within $\Delta(g_4, k)$. Hence, $\phi_k^{\text{out}}(g_4) = \Phi_k(g_4)$ for $k \geq 2$. Similarly for g_5 and g_6. So, $\phi_k^{\text{out}}(g_5) = \Phi_k(g_5)$ and $\phi_k^{\text{out}}(g_6) = \Phi_k(g_6)$ for $k \geq 2$.

Example 5. Distance-based utility model (Bloch and Jackson, 2007). Example 1 is a three-player version of the distance-based model. In general, if player i is connected to player j by a path of t links, then player i receives a benefit of $b(t)$ from her indirect connection with player j. It is assumed that $b(t) \geq b(t+1) > 0$ for any t. Each direct link $ij \in g$ results in a benefit $b(1)$ and a cost c to both i and j. This cost can be interpreted as the time a player must spend with another player in order to maintain a direct link. Player i's distance-based payoff from a network g is given by

$$u_i(g) = \sum_{j \neq i} b\left(t(ij)\right) - \#N_i(g) \cdot c,$$

where $t(ij)$ is the number of links in the shortest path between i and j (setting $t(ij) = \infty$ if there is no path between i and j), $c \geq 0$ is a cost per link, and b is a non-increasing function. The symmetric connections model ($b(t) = \delta^t$) of Jackson and Wolinsky (1996) is a special case of distance-based payoffs. Remember that a star network $g_i^\star$ is a network such that there exists some player i (the center) who is linked to every other player $j \neq i$ (the peripherals) and that contains no other links (i.e. g is such that $N_i(g) = N \setminus \{i\}$ and $N_j(g) = \{i\}$ for all $j \in N \setminus \{i\}$). For intermediate costs, $b(1) - b(2) < c < b(1)$, all star networks are the strongly efficient networks. In Figure 1, the three-player case for $b(1) = 3$, $b(2) = 1.5$, and $c = 2$ is depicted.

Proposition 3. *Consider the distance-based utility model in the case $b(1) - b(2) < c < b(1)$. If g is a star network then g is local-k farsightedly consistent for $k \geq n - 1$.*

Proof. Take any network $g \neq g_i^\star$ such that $g \in \Delta(g_i^\star, k)$ and $k \geq n-1$. We show that $\phi_k^{\text{in}}(g_i^\star) \ni g$. (i) Suppose $g \neq g_j^\star$ $(j \neq i)$. From g, looking forward to $g_i^\star$ (where they obtain their highest possible payoff), players $(\neq i)$ delete all their links successively to reach the empty network $g^\emptyset$. Since $k \geq n-1$ and $g_i^\star$ has exactly $n-1$ links, $g^\emptyset \in \Delta(g_i^\star, k)$. Hence, along any sequence $(g_1 = g, g_2, \ldots, g_L = g^\emptyset)$ with $g_l = g_{l-1} - jk$ such that $jk \in g_{l-1}$, we have that $g_l \in \Delta(g_i^\star, k)$ for $l = 1, \ldots, L$. From $g^\emptyset$, players have incentives (since $b(1) > c$) to add links successively to build the star network $g_i^\star$ with player i in the center. (ii) Suppose $g = g_j^\star$ $(j \neq i)$). From g, looking forward to $g_i^\star$, player j deletes all her links successively to reach the empty network. From $g^\emptyset$, players have incentives (since $b(1) > c$) to add links successively to build the star network $g_i^\star$ with player i in the center. Hence, $g_i^\star \in A(k) \subseteq F(k)$. □

The following corollary follows from the proof of Proposition 3. It shows that farsighted outgoing improving paths from a star network to another star network are reversible.

Corollary 2. *Consider the distance-based utility model in the case $b(1) - b(2) < c < b(1)$. If $k \geq 2(n-2)$ then $g_i^\star \in \Phi_k(g_j^\star)$ for $j \neq i$, $i, j \in N$.*

4.2 Egalitarian utility function

Suppose that the utility function u is such that, for any given network, all players get the same payoff: $u_i(g) = u_j(g)$ for all $i, j \in N$. With the egalitarian utility function u, each player's payoff depends on the network but not on the specific role she plays within the network. The next proposition shows that, under the egalitarian utility function, (i) the set of local-k farsightedly consistent networks is equal to the set of local-k farsightedly stable networks, (ii) increasing the size of the neighbourhood refines the set of locally farsighted consistent networks, and (iii) only the strongly efficient networks remain locally farsighted consistent for k large enough.

Proposition 4. *Take any u such that $u_i(g) = u_j(g)$ for all $i, j \in N$. We have*

(i) $F(k) = \{g \in \mathcal{G} \mid \phi_k^{out}(g) = \emptyset\} \equiv P(k)$.

(ii) $F(k) \supseteq F(k+1)$.

(iii) $F(k) = E$ *for* $k \geq n(n-1)/2$.

Proof. **(i)** Given the egalitarian utility function u, we have that $u_i(g_l) < u_i(g_L)$ for all $i \in N$, $l = 1, \ldots, L-1$, along any farsighted (outgoing or ingoing) improving path $(g_1, \ldots, g_L)$. Hence, if $g \in \phi_k^{\text{out}}(g')$ then $u_i(g') < u_i(g)$ for all $i \in N$ and $g \notin \phi_k^{\text{in}}(g')$. So, any profitable farsighted deviation from g' to g cannot be deterred by the threat of going back to the original network g'. Thus, a network g is a local-k farsightedly consistent network if and only if $\phi_k^{\text{out}}(g) = \emptyset$. So, $F(k) = P(k)$.

(ii) Since $P(k+1) \subseteq P(k)$ and $F(k) = P(k)$ for $k \geq 1$, it follows that $F(k) \supseteq F(k+1)$.

(iii) Take $k \geq n(n-1)/2$. Hence, $\Delta(g,k) = \mathcal{G} \setminus \{g\}$ for every $g \in \mathcal{G}$. Given the egalitarian utility function u, any strongly efficient network $g \in E$ Pareto dominates any network $g' \notin E$. That is, $u_i(g) > u_i(g')$ for all $g \in E, g' \notin E$, for all $i \in N$. Hence, we have that $\phi_k^{\text{out}}(g) = \emptyset$ for all $g \in E$ and $\phi_k^{\text{out}}(g') \cap E \neq \emptyset$ for all $g' \in \mathcal{G} \setminus E$. So, $F(k) = E$ for $k \geq n(n-1)/2$. □

Example 6. Public good investment model. We consider a public good investment model to illustrate the above proposition. Each player or city decides about the links she wants to form with other players. Every player shares equally the total cost for the provision of the public good and every player can only benefit from the public good if the whole community is connected. Let c be the cost of forming a link between two players; $0 < c < 1$. Thus, player i's payoff is simply given by

$$u_i(g) = \begin{cases} a \cdot (n-1) - c \cdot \#g & \text{if } \Pi(g) = \{N\} \\ -c \cdot \#g & \text{if } \Pi(g) \neq \{N\} \end{cases},$$

where $\#g$ denotes the number of links in the network g and $a \geq 1$. In Figure 6, we have depicted the 3-player case.

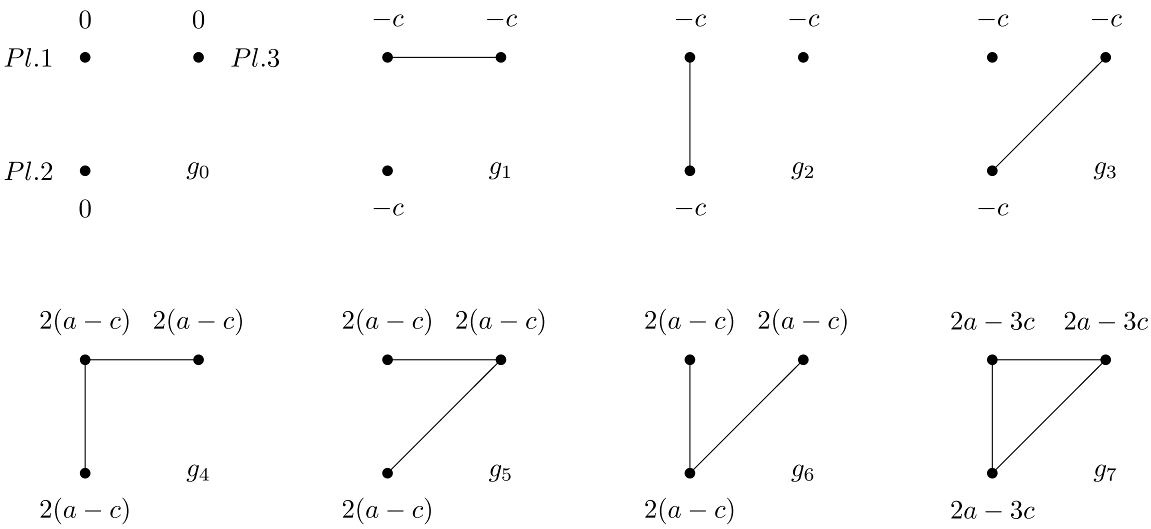

Figure 6: The public good investment game with three players.

Proposition 5. *The set of local-k farsightedly consistent networks in the public good investment model is*

(i) $F(k) = \{g^{\emptyset}\} \cup \{g \in \mathcal{G} \mid \#g = n-1 \text{ and } \Pi(g) = \{N\}\}$ *if* $k < n-1$, *and*

(ii) $F(k) = \{g \in \mathcal{G} \mid \#g = n-1 \text{ and } \Pi(g) = \{N\}\}$ *if* $k \geq n-1$.

Proof. In the public good investment model, the utility function is an egalitarian utility function. Thus, any strongly efficient network $g \in E$ Pareto dominates any network

$g' \notin E$. That is, $u_i(g) > u_i(g')$ for all $g \in E, g' \notin E$, for all $i \in N$. We get that g is strongly efficient if and only g is a minimally connected network; i.e. $E = \{g \in \mathcal{G} \mid \#g = n-1 \text{ and } \Pi(g) = \{N\}\}$. For any $g \notin E \cup \{g^{\emptyset}\}$ we have $u_i(g - kl) > u_i(g)$ for all $i \in N$, for any $kl \in g$. From Proposition 4 we have that $\phi_k^{\text{out}}(g) = \emptyset$ for all $g \in E$ and $g \notin F(k)$ if $\phi_k^{\text{out}}(g) \neq \emptyset$. Hence, $g \notin F(k)$ if $g \notin E \cup \{g^{\emptyset}\}$.

(i) Take $k < n-1$. We have that $\phi_k^{\text{out}}(g^{\emptyset}) = \emptyset$ and $\phi_k^{\text{in}}(g^{\emptyset}) = \Delta(g^{\emptyset}, k)$ since $\Delta(g^{\emptyset}, k) \cap E = \emptyset$. Hence, $F(k) = \{g^{\emptyset}\} \cup \{g \in \mathcal{G} \mid \#g = n-1 \text{ and } \Pi(g) = \{N\}\} = \{g^{\emptyset}\} \cup E$.

(ii) Take $k \geq n-1$. Since $u_i(g^{\emptyset}) < u_i(g)$ for al $i \in N$, for all $g \in E$, and $\Delta(g^{\emptyset}, k) \cap E \neq \emptyset$, we have that $\phi_k^{\text{out}}(g^{\emptyset}) \cap E \neq \emptyset$ and $\phi_k^{\text{in}}(g^{\emptyset}) = \Delta(g^{\emptyset}, k) \setminus E$. Hence, $g^{\emptyset} \notin F(k)$, and so $F(k) = \{g \in \mathcal{G} \mid \#g = n-1 \text{ and } \Pi(g) = \{N\}\} = E$. □

5 Relationship with Set-Valued Concepts

In this section we study the relationship between local farsightedness and set-valued concepts such as the largest pairwise consistent set, the vNM pairwise farsightedly stable set, the pairwise farsightedly stable and the farsightedly absorbing set.[9]

5.1 The largest pairwise consistent set

The largest pairwise consistent set has been defined in Chwe (1994) for general social environments. By considering a network as a social environment and by allowing only pairwise deviations, Herings, Mauleon and Vannetelbosch (2009) define the largest pairwise consistent set.

Definition 7. G is a pairwise consistent set if for all $g \in G$,

(ia) for all $ij \notin g$, there exists $g' \in G$, where $g' = g + ij$ or $g' \in \phi_\infty^{\text{out}}(g + ij) \cap G$, such that $u_i(g') < u_i(g)$ or $u_j(g') < u_j(g)$ or $(u_i(g'), u_j(g')) = (u_i(g), u_j(g))$,

(ib) for all $ij \in g$, there exists $g', g'' \in G$, where $g' = g - ij$ or $g' \in \phi_\infty^{\text{out}}(g - ij) \cap G$, and $g'' = g - ij$ or $g'' \in \phi_\infty^{\text{out}}(g - ij) \cap G$, such that $u_i(g') \leq u_i(g)$ and $u_j(g'') \leq u_j(g)$.

The largest pairwise consistent set is the pairwise consistent set that contains any pairwise consistent set.

The set G is a pairwise consistent set if both external and internal deviations are deterred. The largest pairwise consistent set is the set that contains any pairwise consistent set. It follows from the results in Chwe (1994) that the largest pairwise consistent set exists, is non-empty, and satisfies external stability.

[9]Other approaches to farsightedness can be found in Xue (1998), Herings, Mauleon and Vannetelbosch (2004, 2019, 2023), Mauleon and Vannetelbosch (2004), Dutta, Ghosal and Ray (2005), Page and Wooders (2009), Dutta and Vohra (2017), Ray and Vohra (2019), Kimya (2020), Bloch and van den Nouweland (2020).

Two networks g and g' are adjacent if they differ by one link. The utility function u exhibits no indifference if for any g and g' that are adjacent either g defeats g' or g' defeats g.

Proposition 6. *Suppose that u exhibits no indifference. If g is local-k farsightedly consistent then $\{g\}$ is a pairwise consistent set and g belongs to the largest pairwise consistent set.*

Proof. Since u exhibits no indifference, we have that a local-k farsightedly consistent network g defeats (i) $g+ij$ for all $ij \notin g$ and (ii) $g-ij$ for all $ij \in g$. Thus, $g \in \phi_{\infty}^{\text{out}}(g+ij)$, $g+ij \in \phi_1^{\text{in}}(g)$, $g \in \phi_{\infty}^{\text{out}}(g-ij)$, and $g-ij \in \phi_1^{\text{in}}(g)$. So, $\{g\}$ is a pairwise consistent set and g belongs to the largest pairwise consistent set. □

5.2 The vNM pairwise farsightedly stable set

Incorporating the notion of farsighted improving paths into the original definition of the von Neumann-Morgenstern stable set, Herings, Mauleon and Vannetelbosch (2009) obtain the vNM pairwise farsightedly stable set. A set of networks is a vNM pairwise farsightedly stable set if (i) there is no farsighted improving path between networks within the set (i.e. the internal stability condition); (ii) there is a farsighted improving path from any network outside the set to some network within the set (i.e. the external stability condition).

Definition 8. The set G is a vNM pairwise farsightedly stable set if **(i)** for all $g \in G$, $\phi_{\infty}^{\text{out}}(g) \cap G = \emptyset$ and **(ii)** for all $g' \in \mathcal{G} \setminus G$, $\phi_{\infty}^{\text{out}}(g') \cap G \neq \emptyset$.

Notice that vNM pairwise farsightedly stable sets do not always exist. In Example 4 we have that $\{g_0\}$ is the unique vNM farsightedly stable set while g_0 is the unique local-k farsightedly consistent for $k \geq 2$ but is not local-k farsightedly consistent for $k=1$ (g_3 is the unique local-k farsightedly consistent for $k=1$).

Proposition 7. *If $\{g\}$ is a vNM pairwise farsightedly stable set then there exists $\bar{k}$ such that g is local-k farsightedly consistent for $k \geq \bar{k}$.*

Proof. Take $\bar{k} = n(n-1)/2$. Suppose that g is not a local-k farsightedly consistent network. Hence, there exists g' such that $g' \in \phi_k^{\text{out}}(g)$ and $g' \notin \phi_k^{\text{in}}(g)$. Since $\Delta(g,k) = \mathcal{G} \setminus \{g\}$ and $\Delta(g',k) = \mathcal{G} \setminus \{g'\}$, we have that $g \notin \phi_k^{\text{out}}(g') = \phi_{\infty}^{\text{out}}(g')$, and so $\{g\}$ is not a vNM pairwise farsightedly stable set, a contradiction.

□

5.3 The pairwise farsightedly stable set

Herings, Mauleon and Vannetelbosch (2009) introduce the pairwise farsightedly stable set. It is obtained by requiring the deterrence of external deviations, external stability, and minimality. More precisely, a set of networks G is pairwise farsightedly stable if (i) all possible pairwise deviations from any network $g \in G$ to a network outside G are deterred by a credible threat of ending worse off or equally well off, (ii) there exists a farsighted improving path from any network outside the set leading to some network in the set, and (iii) there is no proper subset of G satisfying (i) and (ii).

Definition 9. A set of networks $G \subseteq \mathcal{G}$ is pairwise farsightedly stable if

(i) for all $g \in G$,

- **(ia)** for all $ij \notin g$ such that $g + ij \notin G$, there exists $g' \in \phi_\infty^{\text{out}}(g + ij) \cap G$ such that $(u_i(g'), u_j(g')) = (u_i(g), u_j(g))$ or $u_i(g') < u_i(g)$ or $u_j(g') < u_j(g)$,
- **(ib)** for all $ij \in g$ such that $g - ij \notin G$, there exists $g', g'' \in \phi_\infty^{\text{out}}(g - ij) \cap G$ such that $u_i(g') \leq u_i(g)$ and $u_j(g'') \leq u_j(g)$,

(ii) for all $g' \in \mathcal{G} \setminus G$, $\phi_\infty^{\text{out}}(g') \cap G \neq \emptyset$.

(iii) there does not exist $G' \subsetneq G$ such that G' satisfies **(ia)**, **(ib)**, and **(ii)**.

Pairwise farsightedly stable sets need not to be consistent in the sense of Chwe (1994) since internal deviations need not to be deterred. However, a pairwise consistent set does not necessarily satisfies the external stability condition. Only the largest pairwise consistent set is guaranteed to satisfy external stability.

Corollary 4 in Herings, Mauleon and Vannetelbosch (2009) tells us that the set $\{g\}$ is a pairwise farsightedly stable set if and only if it is a vNM pairwise farsightedly stable set. Hence, together with Proposition 7 we obtain the following corollary.

Corollary 3. *If $\{g\}$ is a pairwise farsightedly stable set then there exists $\bar{k}$ such that g is local-k farsightedly consistent for $k \geq \bar{k}$.*

5.4 The farsightedly absorbing set

de Callataÿ, Mauleon and Vannetelbosch (2023) propose the concept of farsightedly absorbing set. A set of networks is said to be farsightedly absorbing if it satisfies the following three conditions: no external deviation, external stability and minimality. That is, (i) there is no farsighted improving path from networks within the set to some networks outside the set; (ii) there is a farsighted improving path from any network outside the set to some network within the set; and (iii) there is no proper subset satisfying (i) and (ii).

Definition 10. A set of networks $G \subseteq \mathcal{G}$ is a farsightedly absorbing set if

(i) for every $g \in G$, it holds that $\phi_\infty^{\text{out}}(g) \cap (\mathcal{G} \setminus G) = \emptyset$,

(ii) for every $g \in \mathcal{G} \setminus G$, it holds that $\phi_\infty^{\text{out}}(g) \cap G \neq \emptyset$,

(iii) there does not exist $G' \subsetneq G$ such that G' satisfies **(i)** and **(ii)**.

Proposition 8. *If $\{g\}$ is a farsightedly absorbing set then g is a local-k farsightedly consistent network.*

Proof. Since $\{g\}$ is a farsightedly absorbing set, we have that $\phi_{\infty}^{\text{out}}(g) = \emptyset$. Given that, in general, $\phi_{k-1}^{\text{out}}(g) \subseteq \phi_{k}^{\text{out}}(g)$ for $k > 1$ and $\phi_{k-1}^{\text{out}}(g) = \emptyset$ if $\phi_{k}^{\text{out}}(g) = \emptyset$, it follows that g is a local-k farsightedly consistent network for $k \geq 1$. □

6 Group Deviations and Externalities

6.1 Local farsightedness with group deviations

The notion of local-k farsightedly consistent network only considers deviations by at most a pair of players at a time. It might be that some group of players could all be made better off by some complicated reorganization of their links, which is not accounted for under local-k farsighted consistency. A network $g' \in \mathcal{G}$ is obtainable from $g \in \mathcal{G}$ via deviations by group $S \subseteq N$ if (i) $ij \in g'$ and $ij \notin g$ implies $\{i, j\} \subseteq S$, and (ii) $ij \in g$ and $ij \notin g'$ implies $\{i, j\} \cap S \neq \emptyset$.[10]

Definition 11. A groupwise farsighted outgoing improving path $(g_1, \ldots, g_L)$ from a network g to a network $g' \in \Delta(g, k)$ within the distance-k neighbourhood of g is a finite sequence of networks $g_1, \ldots, g_L$ with $g_1 = g$ and $g_L = g'$ such that for any $l \in \{1, \ldots, L-1\}$, $g_{l+1} \in \Delta(g, k)$ is obtainable from g_l via deviations by $S_l \subseteq N$, $u_i(g_L) \geq u_i(g_l)$ for all $i \in S_l$ and $u_i(g_L) > u_i(g_l)$ for some $i \in S_l$.

For a given network $g \in \mathcal{G}$, let $\widehat{\phi}_{k}^{\text{out}}(g)$ be the set of networks within the distance-k neighbourhood of g that can be reached from g by a groupwise farsighted outgoing improving path.

Definition 12. A groupwise farsighted ingoing improving path $(g_1, \ldots, g_L)$ to a network g from a network $g' \in \Delta(g, k)$ within the distance-k neighbourhood of g is a finite sequence of networks $g_1, \ldots, g_L$ with $g_1 = g'$ and $g_L = g$ such that for any $l \in \{1, \ldots, L-1\}$, $g_{l+1} \in \Delta(g, k)$ is obtainable from g_l via deviations by $S_l \subseteq N$, $u_i(g_L) \geq u_i(g_l)$ for all $i \in S_l$ and $u_i(g_L) > u_i(g_l)$ for some $i \in S_l$.

For a given network $g \in \mathcal{G}$, let $\widehat{\phi}_{k}^{\text{in}}(g)$ be the set of networks within the distance-k neighbourhood of g from which there is a groupwise farsighted outgoing improving path leading to g.

Definition 13. A network $g \in \mathcal{G}$ is local-k groupwise farsightedly consistent if for any $g' \in \Delta(g, k)$ either $g' \notin \widehat{\phi}_{k}^{\text{out}}(g)$ or $g' \in \widehat{\phi}_{k}^{\text{in}}(g)$.

Let $\widehat{F}(k)$ be the set of local-k groupwise farsightedly consistent networks.

[10]Dutta and Mutuswami (1997), Jackson and van den Nouweland (2005) extend the notion of pairwise stability to group deviations, while Luo, Mauleon and Vannetelbosch (2022) propose the notion of coalition-proof stability for predicting the networks that could emerge when group deviations are allowed.

6.2 Componentwise egalitarian utility function

Let

$$g(S) = \left\{ g \subseteq g^S \,\middle|\, \frac{\sum_{i \in N(g)} u_i(g)}{\#N(g)} \geq \frac{\sum_{i \in N(g')} u_i(g')}{\#N(g')} \; \forall g' \subseteq g^S,\, g' \neq \emptyset \right\}$$

be the set of networks with the highest average payoff out of those that can be formed by players in $S \subseteq N$.

Suppose that the utility function u is componentwise egalitarian. That is, u is such that (i) $u_i(g) = u_j(g)$ for all $i, j \in S \in \Pi(g)$ and (ii) $u_i(g) = u_i(h)$ with $h \in H(g)$ and $i \in N(h)$. With a componentwise egalitarian utility function u, (i) players belonging to the same component get the same utility and (ii) there are no externalities across components (i.e. payoffs of players belonging to a component in a given network do not depend on the structure of other components). Given the componentwise egalitarian utility function u, we find a network $\widehat{g}$ through the following algorithm due to Banerjee (1999). Pick some $h_1 \in g(N)$. Next, pick some $h_2 \in g(N \setminus N(h_1))$. At stage l pick some $h_l \in g(N \setminus \cup_{m \leq l-1} N(h_m))$. Since N is finite this process stops after a finite number L of stages. The union of the components picked in this way defines a network $\widehat{g}$. We denote by $\widehat{G}$ the set of all networks that can be found through this algorithm.[11]

The next proposition shows that if (i) players belonging to the same component get the same utility, (ii) there are no externalities across components, (iii) players prefer to be more connected inside their component, and (iv) players are not indifferent between components of different sizes, then every network obtained through the above algorithm is locally groupwise farsightedly consistent whatever the size of the neighbourhood.

Proposition 9. *Take any u such that*

(i) $u_i(g) = u_j(g)$ *for all* $i, j \in S \in \Pi(g)$,

(ii) $u_i(g) = u_i(h)$ *with* $h \in H(g)$ *and* $i \in N(h)$,

(iii) $u_i(h) > u_i(h')$ *if* $\#h > \#h'$ *and* $N(h) = N(h')$ *with* $h \in H(g)$, $h' \in H(g')$ *and* $i \in N(h)$,

(iv) $u_i(g) \neq u_i(g')$ *if* $\#S(i) \neq \#S'(i)$ *with* $S(i) \in \Pi(g), S'(i) \in \Pi(g')$.

Every $g \in \widehat{G}$ is a local-k groupwise farsightedly consistent network for $k \geq 1$.

Proof. Take any $g = \bigcup_{l=1}^{L} h_l \in \widehat{G}$ and any $g' \in \Delta(g,k)$ such that $u_i(g') \neq u_i(g)$ for some $i \in N$. Otherwise, $g' \notin \phi_k^{\text{out}}(g)$. We show in a constructive way that $g' \in \phi_k^{\text{in}}(g)$. Notice that each component h_l is fully connected, i.e. $\#h_l = \#N(h_l)(\#N(h_l) - 1)/2$, for $l = 1, \ldots, L$.

[11] More than one network may be picked up through this algorithm since players may be permuted or even be indifferent between components of different sizes.

Step 1.1**:** Let $T_1 \subseteq N$ be the set of players i such that $u_i(g') \neq u_i(g)$, $i \in N$. Pick some $j \in T_1$ and let $\hat{i}^1 = j$. If $h_1 \in H(g')$ then go to Step 1.2 with $g_1 = g'$ and $\hat{i}^2 = j$. Otherwise, looking forward towards g, the coalition $S_1 = \{\hat{i}^1\} \cup N(h_1)$ deviates jointly from g' to:

(i) either $g' - \{ij \mid i \in N(h_1),\ ij \in g' \text{ and } ij \notin h_1\} + \{ij \mid i \in N(h_1),\ ij \in h_1 \text{ and } ij \notin g'\} - k_1^1 k_2^1$ with $k_1^1, k_2^1 \in N(h_1)$ if $\hat{i}^1$ is already linked to some $i \in N(h_1)$ in g' and so all players belonging to S_1 are worse off in g' than in g;

(ii) or $g' + \hat{i}^1 l_1 - \{ij \mid i \in N(h_1),\ ij \in g' \text{ and } ij \notin h_1\} + \{ij \mid i \in N(h_1),\ ij \in h_1 \text{ and } ij \notin g'\} - k_1^1 k_2^1$ with $l_1, k_1^1, k_2^1 \in N(h_1)$ if $\hat{i}^1$ is not already linked to some $i \in N(h_1)$ in g'.

That is, $\hat{i}^1$ links to some $l_1 \in N(h_1)$ (if not yet linked) and all players belonging to $N(h_1)$ delete their links that are in g' but not in g and build the missing links that are not in g' but are in g, and two players $k_1^1, k_2^1 \in N(h_1)$ delete (or do not build) the link between them. Player $\hat{i}^1$ is strictly better off in g compared to g' while players in $N(h_1)$ are at least as well off.

Next, if player $\hat{i}^1 \notin N(h_1)$ then player l_1 cuts the link to player $\hat{i}^1$. We reach the network $g_1 = g' - \{ij \mid i \in N(h_1),\ ij \in g' \text{ and } ij \notin h_1\} + \{ij \mid i \in N(h_1),\ ij \in h_1 \text{ and } ij \notin g'\} - k_1^1 k_2^1$ with $k_1^1, k_2^1 \in N(h_1)$ where $h_1 - k_1^1 k_2^1 \in H(g_1)$. Along the sequence from g' to g_1 all networks belong to $\Delta(g, k)$. Notice that at g_1 all players in $N(h_1)$ are worse off than at g since the link $k_1^1 k_2^1$ is missing. Let $\hat{i}^2 = k_1^1$ and go to Step 1.2.

Step 1.2**:** If $h_2 \in H(g_1)$ then go to Step 1.3 with $g_2 = g_1$ and $\hat{i}^3 = \hat{i}^2$. Otherwise, looking forward towards g, the coalition $S_2 = \{\hat{i}^2\} \cup N(h_2)$ deviates jointly from g_1 to:

(i) either $g_1 - \{ij \mid i \in N(h_2),\ ij \in g_1 \text{ and } ij \notin h_2\} + \{ij \mid i \in N(h_2),\ ij \in h_2 \text{ and } ij \notin g_1\} - k_1^2 k_2^2$ with $k_1^2, k_2^2 \in N(h_2)$ if $\hat{i}^2$ is already linked to some $i \in N(h_2)$ in g_1 and so all players belonging to S_2 are worse off in g_1 than in g;

(ii) or $g_1 + \hat{i}^2 l_2 - \{ij \mid i \in N(h_2),\ ij \in g_1 \text{ and } ij \notin h_2\} + \{ij \mid i \in N(h_2),\ ij \in h_2 \text{ and } ij \notin g_1\} - k_1^2 k_2^2$ with $l_2, k_1^2, k_2^2 \in N(h_2)$ if $\hat{i}^2$ is not already linked to some $i \in N(h_2)$ in g_1.

That is, $\hat{i}^2$ links to some $l_2 \in N(h_2)$ (if not yet linked) and all players belonging to $N(h_2)$ delete their links that are in g_1 but not in g and build the missing links that are not in g_1 but are in g, and two players $k_1^2, k_2^2 \in N(h_2)$ delete (or do not build) the link between them. Player $\hat{i}^2$ is strictly better off in g compared to g_1 while players in $N(h_2)$ are at least as well off.

Next, if player $\hat{i}^2 \notin N(h_2)$ then player l_2 cuts the link to player $\hat{i}^2$. We reach the network $g_2 = g_1 - \{ij \mid i \in N(h_2),\ ij \in g_1 \text{ and } ij \notin h_2\} + \{ij \mid i \in N(h_2),\ ij \in h_2 \text{ and } ij \notin g_1\} - k_1^2 k_2^2$ with $k_1^2, k_2^2 \in N(h_2)$ where $h_2 - k_1^2 k_2^2 \in H(g_2)$. Along the sequence from g_1 to g_2 all networks belong to $\Delta(g, k)$. Notice that at g_2 all players in $N(h_2)$ are worse off than at g since the link $k_1^2 k_2^2$ is missing. Let $\hat{i}^3 = k_1^2$ and go to Step 1.3.

Step 1.l**:** If $h_l \in H(g_{l-1})$ then go to Step 1.$l+1$ with $g_l = g_{1-1}$ and $\hat{i}^{l+1} = \hat{i}^l$. Otherwise, looking forward towards g, the coalition $S_l = \{\hat{i}^l\} \cup N(h_l)$ deviates jointly from g_{l-1} to:

(i) either $g_{l-1} - \{ij \mid i \in N(h_l), ij \in g_{l-1}$ and $ij \notin h_l\} + \{ij \mid i \in N(h_l), ij \in h_l$ and $ij \notin g_{l-1}\} - k_1^l k_2^l$ with $k_1^l, k_2^l \in N(h_l)$ if $\hat{i}^l$ is already linked to some $i \in N(h_l)$ in g_{l-1} and so all players belonging to S_l are worse off in g_{l-1} than in g;

(ii) or $g_{l-1} + \hat{i}^l l_l - \{ij \mid i \in N(h_l), ij \in g_{l-1}$ and $ij \notin h_l\} + \{ij \mid i \in N(h_l), ij \in h_l$ and $ij \notin g_{l-1}\} - k_1^l k_2^l$ with $l_l, k_1^l, k_2^l \in N(h_l)$ if $\hat{i}^l$ is not already linked to some $i \in N(h_l)$ in g_{l-1}.

That is, $\hat{i}^l$ links to some $l_l \in N(h_l)$ (if not yet linked) and all players belonging to $N(h_l)$ delete their links that are in g_{l-1} but not in g and build the missing links that are not in g_{l-1} but are in g, and two players $k_1^l, k_2^l \in N(h_l)$ delete (or do not build) the link between them. Player $\hat{i}^l$ is strictly better off in g compared to g_{l-1} while players in $N(h_l)$ are at least as well off.

Next, if player $\hat{i}^l \notin N(h_l)$ then player l_l cuts the link to player $\hat{i}^l$. We reach the network $g_l = g_{l-1} - \{ij \mid i \in N(h_l), ij \in g_{l-1}$ and $ij \notin h_l\} + \{ij \mid i \in N(h_l), ij \in h_l$ and $ij \notin g_{l-1}\} - k_1^l k_2^l$ with $k_1^l, k_2^l \in N(h_l)$ where $h_l - k_1^l k_2^l \in H(g_l)$. Along the sequence from g_{l-1} to g_l all networks belong to $\Delta(g, k)$. Notice that at g_l all players in $N(h_l)$ are worse off than at g since the link $k_1^l k_2^l$ is missing. Let $\hat{i}^{l+1} = k_1^l$ and go to Step 1.$l+1$.

We proceed so until we reach g_L at Step L where $h_l - k_1^l k_2^l \in H(g_L)$ for $l = 1, \ldots, L$. Thus, $g_L = \bigcup_{l=1}^L (h_l - k_1^l k_2^l)$, where $h_l \in H(g)$ and $k_1^l, k_2^l \in N(h_l)$ for $l = 1, \ldots, L$.

Step 2.1**:** From g_L, looking forward towards g, players $k_1^1, k_2^1 \in N(h_l)$ add the missing link between them to form $g_{L+1} = g_L + k_1^1 k_2^1$ and so we have $h_1 \in H(g_{L+1})$.

Step 2.l**:** From g_{L+l-1}, looking forward towards g, players $k_1^l, k_2^l \in N(h_l)$ add the missing link between them to form $g_{L+l} = g_{L+l-1} + k_1^l k_2^l$ and so we have $h_l \in H(g_{L+l})$; and so on until we reach the network $g = \bigcup_{l=1}^L h_l$.

Step 2.L**:** From g_{L+L-1}, looking forward towards g, players $k_1^L, k_2^L \in N(h_L)$ add the missing link between them to form $g_{L+L} = g_{L+L-1} + k_1^L k_2^L$. So we have $h_L \in H(g_{L+L}) = H(g)$ and we have reached the network $g = \bigcup_{l=1}^L h_l$.

Thus, we have build a farsightedly ingoing improving path to $g \in \widehat{G}$ from any $g' \in \Delta(g, k)$ such that $u_i(g') \neq u_i(g)$ for some $i \in N$; so $g' \in \phi_k^{\text{in}}(g)$. If $u_i(g') = u_i(g)$ for all $i \in N$ then $g' \notin \phi_k^{\text{out}}(g)$. Hence, every $g \in \widehat{G}$ is a local-k farsightedly consistent network for $k \geq 1$.[12]

[12]Notice that if a coalition only deviates when all members of the coalition are strictly better off at the

□

6.3 Local public good networks with externalities

We consider now a local public good network model with externalities across components. Each municipality or player decides about the links she wants to form with other players. Every player shares common facilities with other players they are connected to. There is a fixed amount A for the whole provision of the public good financed by the government. Costs for the provision of the public good are shared equally among the connected players. Let c be the cost of forming a link between two players; $0 < c < 1$ and $c \cdot n(n+1) < A$. Thus, player i's payoff is simply given by

$$u_i(g) = \frac{A}{[\#S(i)] \cdot [\#\Pi(g)+1]^2} - c \cdot \frac{\#g+1}{\#S(i)}.$$

where $S(i)$ is the coalition $S \in \Pi(g)$ such that $i \in S$.[13] In Figure 7 we have depicted the 3-player case for $A = 144$. This utility function satisfies some general properties (**P1-P5**) that are useful for characterizing the networks that are locally groupwise farsighted consistent. **P1** (*Positive spillovers*) states that, in any network, linking two components increases the payoffs of the players that do not belong to those components. **P2** (*Negative association*) states that, in any network, players belonging to smaller components obtain greater payoffs than players belonging to larger components. **P3** (*No redundant links*) states that, in any network, players belonging to any component increase their payoffs by deleting links without modifying the groups of connected players. **P4** (*Free-riding*) states that if a player leaves her component to become isolated, then she is better off. **P5** (*Efficiency*) states that every minimally connected network is strongly efficient. Formally, the utility function satisfies the following five properties.

P1 In any network g, we have that $u_i(g+jk) > u_i(g)$ if $S(i) \neq S(j) \neq S(k)$ and $S(i), S(j), S(k) \in \Pi(g)$.

P2 In any network g, we have that $u_i(g) < u_j(g)$ if and only if $\#S(i) > \#S(j)$.

P3 In any network g, we have that $u_i(g-ij) > u_i(g)$ if $\Pi(g-ij) = \Pi(g)$.

P4 In any network g, we have that $u_i(g') > u_i(g)$ if $\Pi(g') = \Pi(g) \setminus \{S_l\} \cup \{S_l \setminus \{i\}, \{i\}\}$ for all $i \in S_l, S_l \in \Pi(g)$.

P5 The set of strongly efficient networks is $E = \{g \in \mathcal{G} \mid \#g = n-1 \text{ and } \Pi(g) = \{N\}\}$.

end network, then every network $g \in \widehat{G}$ would already be locally groupwise farsightedly consistent under only conditions (i) and (ii) of Proposition 9.

[13]The utility function reflects the fact that having disconnected municipalities may lead to managing inefficiencies like the duplication of tasks or facilities. The formulation of the cost is chosen so that each player in a minimally connected component simply bears a cost equal to c.

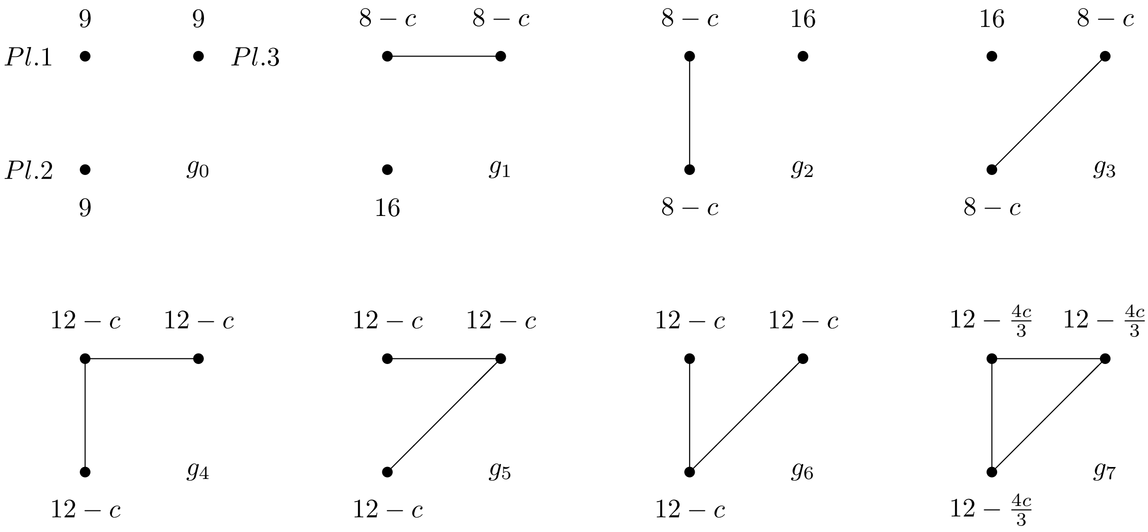

Figure 7: The local public good model with three players.

The next proposition shows that, in the local public good network model, every minimally connected network connecting all players is locally farsightedly consistent when k is large enough.

Proposition 10. *Consider the local public good network model. If* $k \geq (n-1)(n-2)/2$ *then every* $g \in \{g \in \mathcal{G} \mid \#g = n-1$ *and* $\Pi(g) = \{N\}\}$ *is a local-*k *groupwise farsightedly consistent network.*

Proof. Take any g such that $g \in \{g \in \mathcal{G} \mid \#g = n-1 \text{ and } \Pi(g) = \{N\}\}$. Since $k \geq (n-1)(n-2)/2$, we have that $g^{\emptyset} \in \Delta(g,k)$. In addition, $g' \in \Delta(g,k)$ for g' such that $\#g' = (n-1)(n-2)/2$ and $\Pi(g') = \{N \setminus \{j\}, \{j\}\}$. From **P4** we have that $\widehat{\phi}_k^{\text{out}}(g) \neq \emptyset$. From **P3**, if $g' \notin E$ and $\Pi(g') = \{N\}$, then $g' \notin \widehat{\phi}_k^{\text{out}}(g)$. Thus, it is sufficient to show that, for any $g' \notin E$ such that $\Pi(g') \neq \{N\}$ and $g' \in \Delta(g,k)$, then $g' \in \widehat{\phi}_k^{\text{in}}(g)$. We show it in two steps.

[Step A.] Given **P2** and **P4**, the players belonging to the largest component of g' are worse off than in g. Moreover, all players prefer g to $g^{\emptyset}$ (i.e. when all players are isolated).

[Step B.] From g' we build a groupwise farsighted ingoing improving path to g as follows. The improving path consists of a sequence of moves where at each move one player belonging to the largest component in the current network deviates to become an isolated player, until the empty network $g^{\emptyset}$ is reached. Since $k \geq (n-1)(n-2)/2$, it is guaranteed that along the sequence every network belongs to $\Delta(g,k)$. From the empty network $g^{\emptyset}$, all the players deviate jointly to g with $g \in \{g \in \mathcal{G} \mid \#g = n-1 \text{ and } \Pi(g) = \{N\}\}$. By (A) and (B) we have that, for any $g' \notin E$ such that $\Pi(g') \neq \{N\}$ and $g' \in \Delta(g,k)$, then $g' \in \widehat{\phi}_k^{\text{in}}(g)$. So, g is a local-k groupwise farsightedly consistent network. □

7 Conclusion

We have proposed the concept of local-k farsightedly consistent network for analysing network formation games where players do not have full knowledge of the set of feasible networks. Precisely, players only consider a limited number of feasible networks when they decide about the links they want to form or keep with other players. A network g is said to be local-k farsightedly consistent if, for any network g' within the distance-k neighbourhood of g, either g is not defeated by g', or g defeats g'. That is, either there is no farsighted deviation to g' or the farsighted deviation to g' is deterred by the threat of a subsequent farsighted deviation that leads back to g. We have shown that if the utility function is (componentwise) egalitarian or satisfies reversibility or excludes externalities across components, then local-k farsightedness is more likely to be a good proxy for what would happen when players have full knowledge of all feasible networks. Finally, we have studied the relationship with set-valued concepts of farsightedness.

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