PORTFOLIO INSURANCE UNDER ROUGH VOLATILITY AND VOLTERRA PROCESSES

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Abstract

Affine Volterra processes have gained more and more interest in recent years. In particular, this class of processes generalizes the classical Heston model for which widely-used calibration techniques have long been known, as well as the rough Heston model which has garnered lots of attention from academicians and practitioners since 2014. The aim of this work is therefore to revisit and generalize the constant proportion portfolio insurance (CPPI) under the class of affine Volterra processes. Indeed, existing simulation-based methods for CPPI do not apply easily to affine Volterra processes, in particular when the variance process of the underlying risky asset is non-Markovian in the current variance state (as in the rough Heston model). We instead propose an approach based on the characteristic function of the log-cushion which appears to be more consistent, stable and particularly efficient in the case of affine Volterra processes compared with classical simulation techniques. Using such approach, we describe in this paper several properties of CPPI (moments, density and risk measures), which naturally result from the form of the log-cushion's characteristic function under affine Volterra processes. This allows to consider different behaviors and more complex dynamics for the underlying risky asset in the context of CPPI and hence build portfolio strategies that are extremely tractable and consistent with financial data.

Keywords: Finance, Portfolio insurance, CPPI, Volterra processes, Rough volatility

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1. Introduction

The constant proportion portfolio insurance (CPPI) is a dynamic strategy of investment that yields a super-linear participation in future asset returns while retaining a security guarantee on a part of the invested capital (called the floor). This strategy therefore protects a portfolio of stocks against the downside risk but still allows for an upside potential. Basically, the CPPI method consists in maintaining the risk exposure equal to a constant multiple of the excess of wealth over a floor. The CPPI was introduced by Perold & Sharpe (1988) for a portfolio of bonds and revisited by Black & Jones (1987) for equities. A first stream of literature extensively studies the properties of continuous-time CPPI strategies. A comparison of OBPI (option-based portfolio insurance) and CPPI is provided in Bertrand & Prigent (2001) under the classical Black and Scholes (B&S) framework and under the assumption of frictionless trading. This first set of research also deals with the effects of jump processes, stochastic volatility models, regime switching models and extreme value approaches on the CPPI method, cfr Cont & Tankov (2009); Prigent & Bertrand (2003); Hainaut (2011); Bertrand & Prigent (2016) and references therein. The second stream of literature focuses on solving an optimisation problem that arises when we assume investors maximize their end-of-period expected utility. In this setting, the optimality of the CPPI strategy depends on the risk profile of the investor, which has been thoroughly studied in Kingston (1989); Grossman & Vila (1992); Branger et al. (2010); Bertrand & Prigent (2019). Finally, the third stream tries to identify problems with the design of CPPI strategies and then proposes modifications to mitigate them. We refer to Pain & Rand (2008) for a summary of the different variations of CPPI proposed in the literature. Among them, we highlight the *cushion insurance* of Prigent & Tahar (2005) and the *time varying floor* of Ameur & Prigent (2018).

Our contribution lies within the framework of the first stream of literature. In this paper, we derive a methodology based on the characteristic function of the log-cushion, which enables us to derive and study the properties of continuous-time CPPI with frictionless trading under the very general class of affine Volterra processes, as defined in Abi Jaber et al. (2019). In fact, the characteristic function of the log-cushion for affine Volterra

processes is obtained in quasi-closed form, which then allows to easily and efficiently compute the moments, the density as well as several risk-measures for the CPPI value at maturity. Our approach is therefore not restricted to the study of CPPI properties under a particular model as already done in the literature but extends directly and easily to a wide class of models, allowing to take into account different behaviors and more complex dynamics for the underlying risky asset in the context of CPPI. Moreover, using the log-cushion's characteristic function instead of the existing simulation-based techniques allows to derive far more rapidly and more accurately numerous portfolio properties.

In particular, we will focus our analysis on CPPI properties under the rough Heston model, which is an important member among the class of affine Volterra processes (based on a power-law kernel). Indeed, the rough Heston model of El Euch & Rosenbaum (2019) is an highly tractable implementation of rough volatility models, which are remarkably consistent with financial time series data and with the observed implied volatility surface as initially explained in Gatheral et al. (2018). The rough Heston model is therefore an excellent candidate for reproducing historical and risk-neutral data while enjoying only three parameters to calibrate. This explains the growing popularity of this model among practitioners and academicians. However, the non-Markovian nature of the variance process under the rough Heston model requires to deviate from the classical stochastic volatility approach for CPPI described in Prigent & Bertrand (2003) and to work with the characteristic function of the log-cushion instead. Moreover, the analysis of CPPI properties (performance, density and risk measures) will be carried out for the classical Heston model as well, which also belongs to the class of affine Volterra processes and for which well-established calibration methods exist. Finally, we will compare the obtained results with the B&S model, which has already been extensively studied in the context of CPPI. In the last part of the paper, we will discuss the possible ways to introduce gap risk, borrowing constraints and transaction costs in CPPI strategy and hence go beyond the assumption of frictionless/continuous trading.

2. Model setup under rough volatility

As mentioned above, the CPPI strategy consists in maintaining the risk exposure equal to a constant multiple m of the excess of wealth over a floor F . This way, the value of the portfolio V is above the floor F at any time t in the considered period of time $[0, T]$. The value of the floor thus gives the dynamical insured amount and is assumed to evolve at a constant interest rate r following

$$dF_t = F_t r dt.$$

Obviously, the initial floor F_0 is less than the initial portfolio value V_0 and is equal to $V_0 = e^{-rT}G$, where G is the guaranteed amount at maturity. The difference $V_0 - F_0$ is called the cushion, denoted by C_0 . Its value C_t at any time t in $[0, T]$ is given by

$$C_t = V_t - F_t.$$

Now denote the exposure E_t , which is the total amount invested in the risky asset S_t . The standard CPPI method consists of letting $E_t = mC_t$ where m is the constant multiple. In order to have a convex payoff in the underlying, we impose $m > 1$. From Bertrand & Prigent (2016), we have that m is a exogenous parameter chosen by the portfolio manager. A first alternative is to choose m as the inverse of the maximal negative daily return anticipated by the fund manager. For example, if the manager expects that the worst daily return will be -20% over $[0, T]$, then the upper bound on the multiple m such that the value of the fund V_t remains above the floor F_t for all time $t \in [0, T]$ is given by 5. Another alternative is to choose m according to the occurrence probabilities of extreme events in the risky asset.

Moreover, we consider in this paper a continuous-time CPPI strategy and we assume in a first step continuous/frictionless trading over the whole investment horizon $[0, T]$. In the last section, we will then briefly discuss the possible ways of taking into account gap risk, borrowing constraints and transaction costs by restricting trading on a discrete time grid.

We also assume in the sequel that the risky asset S_t is a diffusion process such that

$$dS_t = S_t \left(\mu dt + \sqrt{v_t} dW_t^{\mathbb{P}} \right), \quad (2.1)$$

with $W_t^{\mathbb{P}}$ a standard Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a right-continuous filtration $(\mathcal{F}_t)_{t \geq 0}$. The cornerstone of this section is to consider that the variance process v_t exhibits a rough behavior. More precisely, we consider the generalized rough Heston model of El Euch et al. (2018) where they naturally introduce the following power-law kernel $(t-u)^{H-1/2}$ in the variance process v_t of the classical Heston model in order to allow for a rough behavior of v_t in a Heston-type model. This model is directly specified in a risk-neutral probability measure $\mathbb{Q}$ under which the stock price process now follows the dynamic

$$dS_t = S_t \left(r dt + \sqrt{v_t} dW_t^{\mathbb{Q}} \right), \quad (2.2)$$

where $W_t^{\mathbb{Q}}$ is a Brownian motion defined on $(\Omega, \mathcal{F}, \mathbb{Q})$. The variance process under the generalized rough Heston model is given as in El Euch et al. (2018) by

$$v_t = \frac{1}{\Gamma(H+1/2)} \int_s^t (t-u)^{H-1/2} \lambda (\theta_s(u) - v_u) du + \frac{\nu}{\Gamma(H+1/2)} \int_s^t (t-u)^{H-1/2} \sqrt{v_u} d\widehat{W}_u^{\mathbb{Q}}, \quad (2.3)$$

where $\theta_s(\cdot)$ is assumed to be deterministic, continuous and represents a time-dependent mean reversion level observed at time s . $\widehat{W}_t^{\mathbb{Q}}$ is again a standard Brownian motion defined on $(\Omega, \mathcal{F}, \mathbb{Q})$ such that $\langle W^{\mathbb{Q}}, \widehat{W}^{\mathbb{Q}} \rangle_t = \rho t$ with a constant correlation ρ . When $H = 1/2$, we can verify that we indeed recover the classical Heston model. It can also be shown that the trajectories of the variance itself under the rough Heston are almost surely Hölder-continuous of order $H - \varepsilon$, for any $\varepsilon > 0$, leading to a rough behavior of the variance process with short-range memory when $H < 1/2$ and smoother trajectories with long-range dependence when $H > 1/2$. These authors also provide a microstructure foundation of such model as shown in El Euch & Rosenbaum (2019).

We now define the forward variance curve $\xi_s(t) := \mathbb{E}^{\mathbb{Q}}[v_t | \mathcal{F}_s]$, $t \geq s$. We can easily recover v_s from $\xi_s(t)$ as $v_s = \lim_{t \rightarrow s} \xi_s(t)$. Following Bergomi & Guyon (2012), forward variance models are models that can be written as a function of this curve $\xi_s(t)$. It is

shown in El Euch et al. (2018) that the mean-reversion function $\theta_0(t)$ at time $s = 0$ and the speed of mean-reversion λ can be chosen so that the model is consistent with the forward variance curve observed in the market (thanks to the fundamental relation (5.11) studied in the sequel). Hence, since the forward variance curve can be directly retrieved from the financial market, El Euch et al. (2019) are able to rewrite the risk-neutral variance process (2.3) at time $s = 0$ in the asymptotic setting $\lambda \rightarrow 0$:

$$v_t = \xi_0(t) + \frac{\nu}{\Gamma(H + 1/2)} \int_0^t (t - u)^{H-1/2} \sqrt{v_u} d\widehat{W}_u^{\mathbb{Q}}, \quad (2.4)$$

and this model can be generalized at a later time s with

$$v_t = \xi_s(t) + \frac{\nu}{\Gamma(H + 1/2)} \int_s^t (t - u)^{H-1/2} \sqrt{v_u} d\widehat{W}_u^{\mathbb{Q}}. \quad (2.5)$$

Finally, these authors also show that under the rough Heston model, we have the following forward variance curve dynamic in this asymptotic setting $\lambda \rightarrow 0$ with $\alpha = H + 1/2$:

$$d\xi_s(t) = \nu \frac{(t - s)^{\alpha-1}}{\Gamma(\alpha)} \sqrt{v_s} d\widehat{W}_s^{\mathbb{Q}}. \quad (2.6)$$

Let $\eta_s(t) = \sqrt{v_s} \kappa(t - s)$ where $\kappa(t - s) = \nu(t - s)^{\alpha-1} / \Gamma(\alpha)$, we can then rewrite (2.6) as

$$d\xi_s(t) = \eta_s(t) d\widehat{W}_s^{\mathbb{Q}}. \quad (2.7)$$

This writing will be useful in the sequel to ease notations. Note that $\eta_s(t)$ satisfies the Assumption 2.1. of Gatheral & Keller-Ressel (2019) as shown in the proof of Appendix 11.2 and that $\kappa(t - s)$ is a decreasing L_2 -kernel, which is necessary in the following of the paper to define the characteristic function of the log-cushion.

3. CPPI in the rough Heston model

Under the assumption of frictionless trading for CPPI, we can consider the stochastic volatility case of Prigent & Bertrand (2003) where the volatility σ_t of the underlying risky asset S_t is stochastic. They first show that the cushion value at any time t (here under a risk-neutral measure $\mathbb{Q}$) is

$$C_t = C_0 \exp \left\{ rt - \frac{1}{2} m^2 \int_0^t v_s ds + m \int_0^t \sqrt{v_s} dW_s^{\mathbb{Q}} \right\}. \quad (3.1)$$

Then, from

$$S_t^m = S_0^m \exp \left\{ m \left(r t - \frac{1}{2} \int_0^t v_s ds \right) + m \int_0^t \sqrt{v_s} dW_s^{\mathbb{Q}} \right\}, \quad (3.2)$$

Prigent & Bertrand (2003) find that :

$$C_t = S_t^m \alpha_t, \quad (3.3)$$

with $\alpha_t = \left(\frac{C_0}{S_0^m} \right) \exp\{\beta_t t\}$ and $\beta_t = -(m-1)r - \frac{1}{2} (m^2 - m) \frac{1}{t} \int_0^t v_s ds$.

Finally, since the CPPI fund is the sum of the cushion and the floor, we get that $V_t = F_t + S_t^m \alpha_t$. On the contrary to the B&S framework, we no longer have that the parameter α_t is a constant but a random variable instead. The portfolio value V_t now depends on the whole path of the variance process $\int_0^t v_s ds$ and hence, the short-memory of the rough-Heston variance process with $H < 1/2$ will have an impact on V_t . However, expressions (3.2) and (3.3) will not be useful in our case, *i.e.* when the variance process exhibits a rough behavior as in equation (2.3). Indeed, as explained in El Euch et al. (2018) and Dupret et al. (2021), the rough Heston model is non-Markovian with respect to the current variance state v_s , which makes the quantity $\int_0^t v_s ds$ hard to compute and to obtain from Monte-Carlo simulations and even from more advanced numerical schemes such as the Hybrid scheme of Bennedsen et al. (2017). However, it is directly visible from equation (2.5) that the rough Heston is Markovian in the forward variance curve $\xi_s(t)$. Indeed, given the state vector $\xi_s(t)$, the dynamics of the model are well-determined. Hence, we will now capitalize on this Markov property with respect to $\xi_s(t)$ in order to compute more precisely and efficiently the value of the CPPI portfolio at any time t . Denoting $X_t = \log C_t$, it is useful to rewrite the risk-neutral equation (3.1) as

$$dX_t = r dt - \frac{1}{2} m^2 v_t dt + m \sqrt{v_t} dW_t^{\mathbb{Q}} = r dt + dY_t, \quad (3.4)$$

where $dY_t = -\frac{1}{2} m^2 v_t dt + m \sqrt{v_t} dW_t^{\mathbb{Q}}$.

We now want to obtain the characteristic function of the forward variance model (Y, ξ) , *i.e.* of the process Y together with the family of processes $(\xi_s(t))_{t \geq s}$. From *Theorem 2.6.* of Gatheral & Keller-Ressel (2019) and from the dynamic of $\xi_s(t)$ given in equation (2.7),

we have that that our forward variance model (Y, ξ) has an affine cumulant-generating function (CGF) in the sense that its CGF is of the form

$$\log \mathbb{E} \left[e^{u(Y_T - Y_s)} | \mathcal{F}_s \right] = \int_s^T g(T - r, u) \xi_s(r) dr, \quad (3.5)$$

for all $u \in [0, 1]$, $0 \leq s \leq T$ and where $g(\cdot, u)$ is $\mathbb{R}_-$ -valued and continuous on $[0, T]$ for all $T > 0$ and $u \in [0, 1]$. This leads to the following theorem :

Theorem 3.1. *The function $g(\cdot, u) : \mathbb{R}_+ \rightarrow \mathbb{R}_-$ is the unique global continuous solution of the fractional Riccati equation¹ :*

$$g(t, u) = R_v \left(m, u, \int_0^t \kappa(t - r) g(r, u) dr \right) = R_v(m, u, (\kappa \star g)(t, u)), \quad (3.6)$$

where

$$R_v(m, u, w) = \frac{1}{2} m^2 (u^2 - u) + m \rho u w + \frac{1}{2} w^2. \quad (3.7)$$

Proof : Cfr Appendix 11.2.

We now show how to determine the function $g(\cdot, u)$ in the case of the CPPI under the rough Heston model. First let denote

$$\psi(t, u) = \frac{1}{\nu} (\kappa \star g)(t, u) = \int_0^t \frac{(t - s)^{\alpha-1}}{\Gamma(\alpha)} g(s, u) ds.$$

Thanks to Appendix 11.1 on fractional calculus, it directly leads to

$$D^\alpha \psi(t, u) = g(t, u).$$

Then, using relations (3.6) and (3.7) with the fact that $(\kappa \star g)(t, u) = \nu \psi(t, u)$, we obtain

$$D^\alpha \psi(t, u) = \frac{1}{2} m^2 (u^2 - u) + m \rho u \nu \psi(t, u) + \frac{1}{2} \nu^2 \psi(t, u)^2. \quad (3.8)$$

Finally, we can transform the affine CGF (3.5) into a characteristic function by imposing $u = iz$. Therefore, we obtain the following risk-neutral characteristic function in forward-variance form of (Y, ξ) for the rough Heston, conditionally on the initial state (Y_s, ξ_s) :

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¹The convolution operator $\star$ between two functions f, g is defined as : $(f \star g)(t) := \int_0^t f(t - r) g(r) dr$.

$$\Phi_s^Y(T, z) = \mathbb{E}^{\mathbb{Q}} \left[e^{izY_T} \middle| Y_s, \xi_s \right] = \exp \left\{ izY_s + \int_s^T D^\alpha \psi(T-r, iz) \xi_s(r) dr \right\}. \quad (3.9)$$

The characteristic function of the log-cushion in forward variance form (X, ξ) for the rough Heston model under $\mathbb{Q}$ directly follows from (3.9) :

$$\Phi_s^X(T, z) = \mathbb{E}^{\mathbb{Q}} \left[e^{izX_T} \middle| X_s, \xi_s \right] = \exp \left\{ iz(X_s + r(T-s)) + \int_s^T D^\alpha \psi(T-r, iz) \xi_s(r) dr \right\}. \quad (3.10)$$

As explained in El Euch et al. (2019) and in Dupret et al. (2021), the characteristic function of the log-stock price, $\log(S_t)$, is also available in quasi-closed form, which enables an efficient and tractable calibration of the rough Heston model based on the observed prices of European options. The main results of this calibration on CAC40 option data are summarized in the next section. Moreover, having a characteristic function in quasi-closed form for the log-cushion value X_t thanks to equation (3.10), we can obtain several risk measures for C_t such as the VaR and TVaR, as well as its moments of all orders. However, the rough Heston model is given and calibrated directly under the risk-neutral measure and hence the characteristic function of the log-cushion is computed under $\mathbb{Q}$. Therefore, since we are interested in the dynamic of the portfolio under the real probability measure $\mathbb{P}$, we need to introduce a change of measure in order to obtain the characteristic function of the log-cushion under $\mathbb{P}$. In Section 5, two choices for the change of measure $d\mathbb{Q}/d\mathbb{P}$ will be considered and discussed.

4. Calibration of the rough Heston model

In this paper, we will focus our analysis on the CAC40 index and we summarize here the main steps and results of the calibration of the rough Heston model based on observed vanilla options on this index. A detailed methodology of this calibration is presented in Dupret et al. (2021). We first choose as starting date $t = 0$, the 6th of July 2020. The maturity of vanilla options as well as the risk-free term structure will be defined with respect to this date. The risk-free rates used for pricing purposes are then derived from the observed swap term structure. Table 11.1 in Appendix provides a summary of the risk-free rates used in the calibration between 0 and 10 years. In particular, we assume in this paper that the interest rate is constant for each maturity. Moreover, we decide to

consider a fixed investment horizon of $T = 2$ years for the CPPI strategy and hence we will work with a constant interest rate equal to $r = -0.38755\%$. Finally, after filtering the raw data retrieved from vanilla options on the CAC40 index, we obtain a database of 140 OTM options for which we observe the strike, the maturity, the bid, ask and mid implied volatility.

Once we have gathered our option database, the next step is to derive the forward variance curve $\xi_0(t)$ from the financial markets. We refer to Gatheral (2011) for a detailed approach of the derivation of the forward variance curve based on the observed values of log-contracts. Moreover, a classical parametrization of the forward variance curve is given by the Gompertz function

$$\xi_0(t) = z_1 e^{-z_2 e^{-z_3 t}}, \quad (4.1)$$

where z_1 is the asymptote, z_2 sets the displacement along the x-axis and z_3 sets the growth rate. This Gompertz function can be fitted on the obtained values of log-contracts and we find for the CAC40 at time $t = 0$: $z_1 = \mathbf{0.03954}$, $z_2 = \mathbf{-0.7992}$ and $z_3 = \mathbf{0.5672}$. Therefore, the asymptotic level of volatility is given by $\sqrt{0.03954} = 19.88\%$. Then, using the well-known Carr-Madan formula based on inverse Fourier transform and the characteristic function of the log-stock price under the rough Heston model, we can derive its optimal parameters (H, ν, ρ) . From our dataset of 140 OTM options on the CAC40 index, we find the following parameter values for the rough Heston : $H = \mathbf{0.1207}$, $\nu = \mathbf{0.4132}$ and $\rho = \mathbf{-0.8016}$. The detailed methodology for obtaining these optimal parameters is again explained in Dupret et al. (2021).

5. Changes of measure

Since the volatility is stochastic, there are two sources of risk but only one risky asset traded (the index) and therefore, the market is incomplete and the risk-neutral measure is not unique. As we are interested in the real-world dynamic of the CPPI portfolio, we need to use a Girsanov change of measure in order to find an equivalent martingale measure $\mathbb{P}$ such that we recover equation (2.1), which was given by $dS_t = S_t (\mu dt + \sqrt{v_t} dW_t^{\mathbb{P}})$.

Hence, the change of measure is

$$dW_t^{\mathbb{P}} = dW_t^{\mathbb{Q}} + \frac{r - \mu}{\sqrt{v_t}} dt. \quad (5.1)$$

Moreover, we have that

$$\widehat{W}_t^{\mathbb{Q}} = \rho W_t^{\mathbb{Q}} + \sqrt{1 - \rho^2} \bar{W}_t^{\mathbb{Q}},$$

where $\bar{W}_t^{\mathbb{Q}}$ is an independent $\mathbb{Q}$ -Brownian motion from $W_t^{\mathbb{Q}}$. A general change of measure for the second component $\bar{W}_t^{\mathbb{Q}}$ is of the form

$$d\bar{W}_t^{\mathbb{P}} = d\bar{W}_t^{\mathbb{Q}} - \gamma_t dt,$$

where $\gamma = \gamma(t, \omega)$, for $t \in [0, T]$, is a suitable adapted process called the market price of volatility risk. Let $\rho' = \sqrt{1 - \rho^2}$ and $\widehat{W}_t^{\mathbb{P}} = \rho W_t^{\mathbb{P}} + \rho' \bar{W}_t^{\mathbb{P}}$. As a result,

$$d\widehat{W}_t^{\mathbb{P}} = d\widehat{W}_t^{\mathbb{Q}} + \left(\rho \frac{r - \mu}{\sqrt{v_t}} - \rho' \gamma_t \right) dt, \quad (5.2)$$

which we can rewrite

$$d\widehat{W}_t^{\mathbb{P}} = d\widehat{W}_t^{\mathbb{Q}} + \vartheta_t dt, \quad (5.3)$$

where ϑ_t is an adapted process. We now consider the two following cases :

1. In the first case, we consider the minimal measure (MM) introduced by Föllmer & Schweizer (1991) (as assumed for example in Prigent & Bertrand (2003) or Hull & White (1987)), which corresponds to setting ϑ_t to zero for any time t .
2. The second case assumes a stochastic change of measure (SM). More precisely, we take ϑ_t of the form of equation (5.2) where $\gamma_t = \pi_t \sqrt{v_t}$ and where π_t is the volatility risk premium. For a detailed methodology on the estimation of π_t , we refer to Bollerslev et al. (2011). For this paper, we consider w.l.o.g. that $\pi_t \equiv 3$ in order to have a significant effect of the market price of volatility risk. For the rest of the paper, we will denote $\bar{\rho} = 3\rho'$.

5.1. Minimal measure (MM)

In the first case, the dynamic of the variance process v_t under $\mathbb{P}$ remains the same as under the risk-neutral measure $\mathbb{Q}$ and is then given by the same equation as (2.4) :

$$v_t = \xi_0(t) + \frac{\nu}{\Gamma(H + 1/2)} \int_0^t (t - u)^{H-1/2} \sqrt{v_u} d\widehat{W}_u^{\mathbb{P}}.$$

However, due to the change of drift in the stock price process from (5.1), the cushion value C_t at any time t now becomes under $\mathbb{P}$:

$$C_t = C_0 \exp \left\{ (r + m(\mu - r))t - \frac{1}{2}m^2 \int_0^t v_s ds + m \int_0^t \sqrt{v_s} dW_s^{\mathbb{P}} \right\}, \quad (5.4)$$

and hence, the real-world dynamic of the log-cushion X_t is now

$$dX_t = (r + m(\mu - r))dt - \frac{1}{2}m^2 v_t dt + m\sqrt{v_t} dW_t^{\mathbb{P}} = (r + m(\mu - r))dt + dY_t. \quad (5.5)$$

Finally, the characteristic function of the log-cushion (X, ξ) in forward variance form under $\mathbb{P}$ for the MM rough Heston at time $s = 0$ can be built in the exact same way as for (3.10) :

$$\begin{aligned} \Phi_X(T, z) = \mathbb{E}^{\mathbb{P}} \left[e^{izX_T} \middle| X_0, \xi_0 \right] &= \exp \left\{ iz \left(X_0 + (r + m(\mu - r))T \right) \right. \\ &\quad \left. + \int_0^T D^\alpha \psi(T - r, iz) \xi_0(r) dr \right\}, \end{aligned} \quad (5.6)$$

where $D^\alpha \psi(\cdot, iz)$ is again given by (3.8) and where the parameters (H, ν, ρ) are the same as in Section 4. Therefore, it only remains to estimate the drift of the stock price process μ based on real-world data. From Bloomberg, we have that the average market return for the CAC40 in 2020 is around 6.63% and is stable since 2015. Hence, we simply consider in this paper that the drift is constant and we estimate it by $\hat{\mu} = 6.63\%$. This minimal change of measure imposes that the market forward variance curve under $\mathbb{Q}$ is the same as the forward variance curve under $\mathbb{P}$ and hence that

$$\mathbb{E}^{\mathbb{Q}} [v_t | \mathcal{F}_0] = \mathbb{E}^{\mathbb{P}} [v_t | \mathcal{F}_0]$$

Bayer et al. (2016) show from data on the S&P500 that, in fact, $\mathbb{Q}$ is close to $\mathbb{P}$ in the sense of the forward variance curve.

5.2. Stochastic change of measure (SM)

However, the minimal change of measure is too simplistic and we thus consider a more realistic stochastic change of measure based on (5.2) with $\gamma_t = 3\sqrt{v_t}$:

$$d\widehat{W}_t^{\mathbb{P}} = d\widehat{W}_t^{\mathbb{Q}} + \left(\rho \frac{r - \mu}{\sqrt{v_t}} - \bar{\rho} \sqrt{v_t} \right) dt,$$

where typically $\rho < 0$, $\bar{\rho} = 3\rho' = 3\sqrt{1-\rho^2} > 0$ and $r - \mu < 0$. We can verify that the market price of volatility risk is well positive since $\gamma_t = 3\sqrt{v_t}$. In this SM case, we now focus on equation (2.3) based on the parameters λ and $\theta_0(t)$ for the variance process instead of the asymptotic setting (2.4). This way, we keep the dynamic of a generalized rough Heston model under the real-world measure $\mathbb{P}$ and hence a characteristic function in quasi-closed form. Therefore, the dynamic of the variance process v_t under $\mathbb{P}$ now becomes

$$v_t = v_0 + \frac{1}{\Gamma(H+1/2)} \int_0^t (t-u)^{H-1/2} (\lambda(\theta_0(u) - v_u) - \rho\nu(r-\mu) + \nu\bar{\rho}v_u) du \\ + \frac{\nu}{\Gamma(H+1/2)} \int_0^t (t-u)^{H-1/2} \sqrt{v_u} d\widehat{W}_u^{\mathbb{P}},$$

which can be rewritten

$$v_t = v_0 + \frac{1}{\Gamma(H+1/2)} \int_0^t (t-u)^{H-1/2} (\lambda - \bar{\rho}\nu) \left(\frac{\lambda\theta_0(u) - \rho\nu(r-\mu)}{\lambda - \bar{\rho}\nu} - v_u \right) du \\ + \frac{\nu}{\Gamma(H+1/2)} \int_0^t (t-u)^{H-1/2} \sqrt{v_u} d\widehat{W}_u^{\mathbb{P}}. \quad (5.7)$$

In order to satisfy the definition of the generalized rough Heston model as given in *Definition 1* of El Euch et al. (2018), we impose the assumption that $\lambda > \bar{\rho}\nu$. Moreover, $\theta_0(u)$ must be a deterministic function, positive and continuous on $\mathbb{R}_+$ in order to be consistent with this definition. We then denote

$$\tilde{\lambda} = \lambda - \bar{\rho}\nu \quad \text{and} \quad \tilde{\theta}_0(t) = \frac{\lambda\theta_0(t) - \rho\nu(r-\mu)}{\lambda - \bar{\rho}\nu}. \quad (5.8)$$

We can now rewrite the dynamic (5.7) of v_t under $\mathbb{P}$ as

$$v_t = v_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1} \tilde{\lambda} (\tilde{\theta}_0(u) - v_u) du + \frac{\nu}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1} \sqrt{v_u} d\widehat{W}_u^{\mathbb{P}}, \quad (5.9)$$

with $\alpha = H + 1/2$. Therefore, we again have a generalized rough Heston model under $\mathbb{P}$. We can then rewrite the dynamic of the variance process as a functional of the forward variance curve under $\mathbb{P}$ in the asymptotic setting $\tilde{\lambda} = 0$, as already done in Section 1 (cfr El Euch et al. (2019)) :

$$v_t = \xi_0^{\mathbb{P}}(t) + \frac{\nu}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1} \sqrt{v_u} d\widehat{W}_u^{\mathbb{P}},$$

where $\xi_0^{\mathbb{P}}(t) = \mathbb{E}^{\mathbb{P}}[v_t | \mathcal{F}_0]$ is the forward variance curve under $\mathbb{P}$. The characteristic function of the log-cushion in forward variance form of the SM rough Heston is then obtained in the exact same way as for (3.10) and (5.6) with

$$\Phi_X(T, z) = \mathbb{E}^{\mathbb{P}} \left[e^{izX_T} \middle| X_0, \xi_0 \right] = \exp \left\{ iz \left(X_0 + (r + m(\mu - r))T \right) + \int_0^T D^\alpha \psi(T - r, iz) \xi_0^{\mathbb{P}}(r) dr \right\}, \quad (5.10)$$

where $\psi(\cdot, iz)$ is again the unique solution of the fractional Riccati equation (3.8) :

$$D^\alpha \psi(t, iz) = -\frac{1}{2}(z^2 + iz) + iz\rho\nu\psi(t, iz) + \frac{\nu^2}{2}\psi^2.$$

Hence, this form of the characteristic function again involves the forward variance curve as in the MM setting with formula (5.6), but this time the forward variance curve is computed under the real probability measure $\mathbb{P}$ based on the parameters $\tilde{\lambda}$ and $\tilde{\theta}_0(t)$. However, this forward variance curve under $\mathbb{P}$ cannot be directly retrieved from the market anymore and we now explain how to derive $\xi_0^{\mathbb{P}}(t)$.

The first step is to obtain the values of the risk-neutral parameter λ and initial function $\theta_0(t)$, which are not directly obtained from the calibration. Indeed, these parameters are written as a functional of the market forward variance curve which allows initially to simplify the rough Heston model by reducing the number of parameters to calibrate (cfr Section 1). Hence, we first need to retrieve λ and $\theta_0(u)$ so that the model is consistent with the observed market forward variance curve $\xi_0(t) = \mathbb{E}^{\mathbb{Q}}[v_t]$. As explained in El Euch et al. (2019), this can easily be done using the fundamental relation of El Euch et al. (2018) for the risk-neutral forward variance curve under the generalized rough Heston model

$$\lambda\theta_0(t) + v_0 \frac{t^{-\alpha}}{\Gamma(1-\alpha)} = D^\alpha \left(\mathbb{E}^{\mathbb{Q}}[v_t] \right) + \lambda \mathbb{E}^{\mathbb{Q}}[v_t], \quad (5.11)$$

and minimizing the absolute value between the left and right-hand side. Since no restriction is imposed on the form of $\theta_0(t)$, we tried several growth functions and polynomials and the one that best satisfies the constraint (5.11) is the Weibull function

$$\theta_0(t) = \omega - \beta \exp(-\kappa t^p),$$

with the following values :

$$\lambda = 8.30895, \quad \omega = 0.03882, \quad \beta = -0.04841, \quad \kappa = 0.85658, \quad p = 0.87982$$

We can verify that $\lambda > \bar{\rho}\nu = 0.247$ and that $\theta_0(t)$ is continuous and positive on $\mathbb{R}_+$. We confirm with Figure 5.1 the perfect fit of the forward variance curve computed using the parameters $(\lambda, \omega, \beta, \kappa, p)$ in relation (5.11) to the original market forward variance curve derived from (4.1) with $z_1 = 0.0395$, $z_2 = -0.7992$ and $z_3 = 0.5672$.

Fig. 5.1. Initial risk-neutral market forward variance curve (black line) compared to the risk-neutral forward variance curve obtained from relation (5.11)(red dots).

Now that we have all the parameters of the rough Heston under $\mathbb{Q}$, we can find the following values for the SM rough Heston under $\mathbb{P}$ using equations (5.8):

$$H = 0.1207, \quad \nu = 0.4132, \quad \rho = -0.8016, \quad \tilde{\lambda} = 7.567842, \quad \mu = 0.063$$

and where the mean-reverting level $\tilde{\theta}_0(t)$ of v_t under $\mathbb{P}$ is obtained using the above parameters of the Weibull function. Note that the chosen change of measure does not affect ν , ρ nor H . Empirically, from Gatheral et al. (2018) and Dupret et al. (2021), the parameters H and ρ are the same under the real and the risk-neutral probability measure, but this is not the case for ν . Hence, more advanced changes of measure could be proposed in order to take into account the effect of the volatility risk premium on ν .

Finally, based on the real-world parameters above of the SM rough Heston, we infer the initial forward variance curve $\xi_0^{\mathbb{P}}$ under the physical measure thanks to the the fundamental relation (5.11), which also holds under $\mathbb{P}$:

$$\tilde{\lambda}\tilde{\theta}_0(t) + v_0 \frac{t^{-\alpha}}{\Gamma(1-\alpha)} = D^\alpha \left(\mathbb{E}^{\mathbb{P}}[v_t] \right) + \tilde{\lambda} \mathbb{E}^{\mathbb{P}}[v_t]. \quad (5.12)$$

The next Figure 5.2 compares the risk-neutral forward variance curve to the real-world forward variance curve based on the stochastic change of measure (with parameters $\tilde{\lambda}$ and $\tilde{\theta}_0(t)$). We see that the real-world forward variance curve is slightly above the risk-

Fig. 5.2. Initial risk-neutral forward variance curve (red) vs. real-world forward variance curve (black).

neutral one due to the market price of volatility risk. However, both curves are very close to each other, which confirms that $\mathbb{P}$ and $\mathbb{Q}$ are close as supposed in the minimal measure hypothesis, even for significant market price of risk π_t .

Remark : Note that an alternative representation of the characteristic function of the log-cushion (5.10) under the SM rough Heston is given by :

$$\begin{aligned} \mathbb{E}^{\mathbb{P}} \left[e^{izX_T} \middle| \mathcal{F}_0 \right] &= \exp \left\{ iz \left(X_0 + (r + m(\mu - r)) T \right) \right. \\ &\quad \left. + \int_0^T \tilde{\psi}(T - r, iz) \left(\tilde{\lambda}\tilde{\theta}_0(r) + \frac{V_0 r^{-\alpha}}{\Gamma(1-\alpha)} \right) dr \right\}, \end{aligned} \quad (5.13)$$

and where $\tilde{\psi}(\cdot, iz)$ is now the solution of

$$D^\alpha \tilde{\psi}(t, iz) = -\frac{1}{2}m^2(z^2 + iz) + (miz\rho\nu - \tilde{\lambda})\tilde{\psi}(t, iz) + \frac{\nu^2}{2}\tilde{\psi}(t, iz)^2.$$

Clearly, this form of the characteristic function does not involve the initial forward variance curve $\xi_0(\cdot)$ anymore but can be derived from the characteristic function (5.10) as shown and explained in El Euch et al. (2018) (in the case $m = 1$). We see that this alternative version of the characteristic function also involves the real-world parameters $\tilde{\lambda}$ and $\tilde{\theta}_0(\cdot)$.

6. Other models

In order to compare the rough Heston model with other models such as the Heston or B&S models, we need the characteristic function of the log-cushion in these models so that we can apply the same method as above. We also show in this section that such log-cushion's characteristic function can be well defined analytically under affine Volterra processes and we can hence generalize the above results for this larger class of processes.

6.1. Heston model

First, the Heston model under a risk neutral measure $\mathbb{Q}$ is given by

$$dS_t = S_t r dt + S_t \sqrt{v_t} dW_t^{\mathbb{Q}}, \quad (6.1)$$

$$dv_t = \lambda(\theta - v_t) dt + \nu \sqrt{v_t} d\widehat{W}_t^{\mathbb{Q}}. \quad (6.2)$$

The two Brownian motions are defined as above under $\mathbb{Q}$. We have that $v_0 = \sigma_0^2$ is the initial variance, θ is the (constant) long-run variance of v_t , λ denotes the speed of mean-reversion and ν is the volatility of variance parameter. Following Cui et al. (2017), the calibration of the Heston model based on our database of OTM vanilla options on the CAC40 index leads to the following optimal parameters : $\sigma_0 = \mathbf{0.2867814}$, $\lambda = \mathbf{2.010226}$, $\theta = \mathbf{0.08454243}$, $\nu = \mathbf{0.81320}$ and $\rho = \mathbf{-0.9193485}$. Now, using the same change of measure (5.2) as in the SM case with $\pi_t \equiv 3$ and $\bar{\rho} = 3\rho' = 3\sqrt{1 - \rho^2}$, we again find that

$$d\widehat{W}_t^{\mathbb{P}} = d\widehat{W}_t^{\mathbb{Q}} + \left(\rho \frac{r - \mu}{\sqrt{v_t}} - \bar{\rho} \sqrt{v_t} \right) dt.$$

We can then rewrite the Heston model under the physical probability measure by

$$dS_t = S_t \mu dt + S_t \sqrt{v_t} dW_t^{\mathbb{P}}, \quad (6.3)$$

$$dv_t = \tilde{\lambda}(\tilde{\theta} - v_t) dt + \nu \sqrt{v_t} d\widehat{W}_t^{\mathbb{P}}, \quad (6.4)$$

where as in 5.8, we denote

$$\tilde{\lambda} = \lambda - \bar{\rho} \nu \quad \text{and} \quad \tilde{\theta} = \frac{\lambda \theta - \rho \nu (r - \mu)}{\lambda - \bar{\rho} \nu}.$$

Hence, we find $\tilde{\lambda} = 1.05038$ and $\tilde{\theta} = 0.2093971$ (again for an investment horizon $T = 2$ years and thus with $r = -0.38755\%$).

From Bergomi & Guyon (2012), we know that the Heston model can be rewritten as a forward variance model with an affine CGF of the form of equation (3.5). Furthermore, based on *Example 2.10.* of Gatheral & Keller-Ressel (2019) and from Theorem 3.1, we obtain directly that the characteristic function of the log-cushion in the Heston model under $\mathbb{P}$ is

$$\begin{aligned} \Phi_X(T, z) = \mathbb{E}^{\mathbb{P}} \left[e^{izX_T} \middle| \mathcal{F}_0 \right] = \exp \left\{ iz (X_0 + (r + m(\mu - r)) T) \right. \\ \left. + v_0 \psi(T, iz) + \tilde{\lambda} \tilde{\theta} \int_0^T \tilde{\psi}(r, iz) dr \right\}, \end{aligned} \quad (6.5)$$

and where $\tilde{\psi}(\cdot, iz)$ solves a (non-fractional) Riccati equation

$$\frac{\partial}{\partial t} \tilde{\psi}(t, iz) = -\frac{1}{2} m^2 (z^2 + iz) + (miz\rho\nu - \tilde{\lambda}) \tilde{\psi}(t, iz) + \frac{\nu^2}{2} \tilde{\psi}(t, iz)^2. \quad (6.6)$$

Note that this Riccati equation is consistent with the classical ODE's initially derived for the Heston model, such as in Gatheral (2011) (with $m = 1$). We also see that the fractional derivative of the rough Heston (which conveys the short memory to the model with $H < 1/2$) is replaced by a classical time derivative in the Riccati equation. Finally, recall that $\mu = 6.63\%$ and that m is chosen exogenously.

6.2. Affine Volterra processes

The methodology described above for deriving the characteristic function of the rough Heston and Heston models can be extended to the more general class of affine Volterra

processes. This class of processes is defined in Abi Jaber et al. (2019) by the following d -dimensional stochastic convolution equation of the form

$$Z_t = Z_0 + \int_0^t K_Z(t-s)b(Z_s)ds + \int_0^t K_Z(t-s)\sigma(Z_s)dW_s, \quad (6.7)$$

where W is a d -dimensional $\mathbb{P}$ -Brownian motion, where K_Z is a convolution kernel such that $K_Z \in L_{loc}^2(\mathbb{R}_+, \mathbb{R}^{d \times d})$ and where the coefficients b and σ satisfy the regularity and integrability conditions studied in Abi Jaber et al. (2019). This class of processes is called *affine* since the coefficient $a(z) = \sigma(z)\sigma(z)^T$ (with $\sigma(z)$ positive semidefinite) and the coefficient $b(z)$ are both affine of the form

$$\begin{aligned} a(z) &= A^0 + z_1 A^1 + \cdots + z_d A^d, \\ b(z) &= b^0 + z_1 b^1 + \cdots + z_d b^d, \end{aligned}$$

for d -dimensional symmetric matrices A^i and vectors b^i . We next introduce the $d \times d$ matrix

$$B = \begin{pmatrix} b^1 & \cdots & b^d \end{pmatrix}, \quad (6.8)$$

and for any row vector $u \in \mathbb{C}^d$, we define the row vector

$$A(u) = \left(uA^1 u^T, \dots, uA^d u^T \right). \quad (6.9)$$

We then recall from the log-cushion dynamic (5.5) that

$$Y_t = -\frac{1}{2}m^2 \int_0^t v_s ds + m \int_0^t \sqrt{v_s} dW_s^{\mathbb{P}}. \quad (6.10)$$

Moreover, we now consider that the variance process v_t follows the general (one-dimensional) convolution equation given by

$$v_t = v_0 + \int_0^t K(t-s)(\beta + bv_s)ds + \int_0^t K(t-s)(\alpha + a\sqrt{v_s})d\widehat{W}_s^{\mathbb{P}}, \quad (6.11)$$

again with a constant correlation ρ such that $\langle W^{\mathbb{P}}, \widehat{W}^{\mathbb{P}} \rangle_t = \rho t$, a convolution kernel $K \in L_{loc}^2(\mathbb{R}_+, \mathbb{R})$ and $\beta, b, \alpha, a \in \mathbb{R}$. We thus take $d = 2$ and consider the two-dimensional process $Z = (Y, v)$. In order to have an affine Volterra process Z compatible with the

given equation (6.10) of Y , we need to impose the coefficient $a(z)$ of Z to be of the form

$$A^0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad A^1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad A^2 = \begin{pmatrix} m^2 & m\rho a \\ m\rho a & a^2 \end{pmatrix}. \quad (6.12)$$

This amounts to set $\alpha = 0$ in the variance process 6.11. Otherwise, there does not exist a positive semidefinite matrix $\sigma(z)$ which is such that $a(z) = \sigma(z)\sigma(z)^T$ is affine. The coefficients $b(z)$ of Z are given by

$$b^0 = \begin{pmatrix} 0 \\ \beta \end{pmatrix} \quad B = \begin{pmatrix} 0 & -\frac{1}{2} \\ 0 & b \end{pmatrix}, \quad (6.13)$$

and the kernel of the two-dimensional affine Volterra process Z is given by

$$K_Z = \begin{pmatrix} 1 & 0 \\ 0 & K \end{pmatrix}. \quad (6.14)$$

From coefficients (6.12) and (6.13), we can rewrite more precisely the variance process (6.11) by the following equation such that $Z = (Y, v)$ verifies the conditions of an affine Volterra process

$$v_t = v_0 + \int_0^t K(t-s) (\beta + bv_s) ds + \int_0^t K(t-s) (0 + a\sqrt{v_s}) d\widehat{W}_s^{\mathbb{P}}. \quad (6.15)$$

We clearly see that the real-world Heston model described in Section 6.1 can be retrieved by setting the kernel $K(t) \equiv 1$ and the coefficients $a = \nu$, $\beta = \tilde{\lambda}\tilde{\theta}$ and $b = -\tilde{\lambda}$ in the variance process. For the rough Heston model, we need to impose a power law kernel $K(t) = t^{\alpha-1}/\Gamma(\alpha)$ for the variance process instead. We now derive from Abi Jaber et al. (2019) the characteristic function of the log-cushion under the two-dimensional affine Volterra process Z for general coefficients a, β, b and for general convolution kernel K .

Theorem 6.1. *Let $Z = (Y \ v)^T$ be a solution of (6.7) with $K_Z \in L^2(\mathbb{R}_+, \mathbb{R}^{2 \times 2})$ and $A(\cdot)$ given by (6.9). Let $u = (u_1 \ u_2) \in \mathbb{C}^2$ and assume that $\psi = (\psi_1 \ \psi_2)^T \in L^2([0, T], \mathbb{C}^2)$ solves the Riccati-Volterra equation*

$$\psi = uK_Z + \left(\psi B + \frac{1}{2}A(\psi) \right) \star K_Z. \quad (6.16)$$

Using the form of the coefficients $b(z)$ and $a(z)$ given by (6.12) and (6.13) and the form of K_Z (6.14), we obtain

$$\psi_1 = u_1, \quad (6.17)$$

$$\psi_2 = u_2 K + K \star \mathcal{R}_\Psi(\psi), \quad (6.18)$$

where we define

$$\mathcal{R}_\Psi(\psi) = \frac{1}{2}m^2(\psi_1^2 - \psi_1) + (b + m\psi_1\rho a)\psi_2 + \frac{a^2}{2}\psi_2^2 \quad \text{and} \quad \mathcal{R}_\Phi(\psi_2) = \beta\psi_2.$$

Then, the Laplace transform of Z_T is given by

$$\mathbb{E}^\mathbb{P}[e^{uZ_T}] = e^{\phi(T) + \chi(T)V_0 + u_1Y_0}, \quad (6.19)$$

where

$$\phi' = \frac{d}{dt}\phi = \mathcal{R}_\Phi(\psi_2), \quad \chi' = \frac{d}{dt}\chi = \mathcal{R}_\Psi(\psi),$$

and

$$\phi(0) = 0, \quad \chi(0) = u_2.$$

Proof : Cfr Appendix 11.3

However, we are especially interested in the characteristic function of the log-cushion X_t which can be obtained from the dynamic of Y_t using equation (5.5). Therefore, we now study the following affine Volterra process $\tilde{Z} = (X, v)$ and obtain the next Corollary 6.2.

Corollary 6.2. *The Laplace transform of the the log-cushion X_T at horizon T associated with the affine Volterra process $\tilde{Z}$ is given by*

$$\mathbb{E}^\mathbb{P}[e^{u_1X_T}] = e^{\phi(T) + \chi(T)V_0 + u_1(X_0 + (r + m(\mu - r))t)}, \quad (6.20)$$

with ϕ and χ defined by

$$\phi' = \frac{d}{dt}\phi = \mathcal{R}_\Phi(\psi_2), \quad \chi' = \frac{d}{dt}\chi = \mathcal{R}_\Psi(\psi),$$

where $\mathcal{R}_\Phi(\psi_2)$ and $\mathcal{R}_\Psi(\psi)$ are defined in Theorem 6.1 and with initial conditions defined by

$$\phi(0) = 0, \quad \chi(0) = 0.$$

The process $\psi = (\psi_1 \ \psi_2)^T \in L^2([0, T], \mathbb{C}^2)$ is now the solution of the following Riccati-Volterra equation

$$\psi_1 = u_1, \quad (6.21)$$

$$\psi_2 = K \star \mathcal{R}_\Psi(\psi) = K \star \left(\frac{1}{2} m^2 (u_1^2 - u_1) + (b + m u_1 \rho a) \psi_2 + \frac{a^2}{2} \psi_2^2 \right). \quad (6.22)$$

Proof : Cfr Appendix 11.4

It is important to note that we only consider here the unconditional characteristic function of affine Volterra processes at time $s = 0$. Indeed, Theorem 6.1 and Corollary 6.2 do not hold anymore at a later time s since we cannot write the conditional characteristic function $\mathbb{E}^\mathbb{P}[e^{u_1 X_T} | \mathcal{F}_s]$ as a linear function of $\phi(\cdot)$ and $\chi(\cdot)$.

Based on this Corollary 6.2, we can easily recover the characteristic function of the log-cushion under the real-world Heston model, which was given by equation (6.5). It suffices to take $K \equiv 1$ as convolution kernel with $u_1 = iz, a = \nu, \beta = \tilde{\lambda} \tilde{\theta}$ and $b = -\tilde{\lambda}$ and we hence obtain that

$$\chi(T) = \int_0^T \mathcal{R}_\Psi(\psi(T-s)) ds = \psi_2(T),$$

and that

$$\phi(T) = \int_0^T \mathcal{R}_\Phi(\psi_2(T-s)) ds = \tilde{\lambda} \tilde{\theta} \int_0^T \psi_2(T-s) ds.$$

For the SM rough Heston model under the real probability measure $\mathbb{P}$, we need to take the power-law kernel $K(t) = t^{\alpha-1}/\Gamma(\alpha)$. Using fractional calculus (cfr Appendix 11.2) with equation (6.22), we first obtain

$$\psi_2(T) = I^\alpha (\mathcal{R}_\Psi(\psi(T))) \iff \mathcal{R}_\Psi(\psi(T)) = I^{-\alpha} (\psi_2(T)).$$

Using a change of variable, we then find

$$\chi(T) = \int_0^T \mathcal{R}_\Psi(\psi(T-s)) ds = I^{1-\alpha} (\psi_2(T)) = \frac{1}{\Gamma(1-\alpha)} \int_0^T r^{-\alpha} \psi_2(T-r) dr,$$

and

$$\phi(T) = \int_0^T \mathcal{R}_\Phi(\psi_2(T-r)) dr = \int_0^T \tilde{\lambda} \tilde{\theta}_0(r) \psi_2(T-r) dr.$$

Therefore, this confirms that the characteristic function of the log-cushion under the SM rough Heston model is given by the representation (5.13). It is important to note that under our generalized rough Heston, the coefficient b of $\tilde{Z}$ now depends on time t through the parameter β . More precisely, we have $\beta(t) = \tilde{\lambda}\tilde{\theta}_0(t)$. Theorem 6.1 and Corollary 6.2 can be generalized under this inhomogeneous case provided that $b(t, z)$ verifies the regularity and integrability conditions of Ackermann et al. (2020) (*Assumption A.2*). In the inhomogeneous case (when the coefficients σ and b also depend on t), these authors find similar results as in the homogenous case analyzed in Abi Jaber et al. (2019). These results can therefore be used to easily extend Theorem 6.1 and Corollary 6.2 to the generalized rough Heston model with time-dependent mean reversion $\tilde{\theta}_0(t)$.

We finally conclude that the class of Volterra processes tends to be suitable for building CPPI strategies that are analytically tractable through their characteristic function.

6.3. Black and Scholes model

Finally, for the sake of comparison we analyze the characteristic function of the log-cushion in the B&S model. In this B&S model, we have that the log-cushion dynamic can be rewritten, in a similar way as in equation (5.5), by

$$dX_t = (r + m(\mu - r))dt - \frac{1}{2}m^2\sigma^2 dt + m\sigma dW_t^{\mathbb{P}}. \quad (6.23)$$

Hence, it directly comes that the characteristic function of the log-cushion is given by

$$\begin{aligned} \Phi_s^X(T, z) = \mathbb{E}^{\mathbb{P}} \left[e^{izX_T} \middle| \mathcal{F}_s \right] &= \exp \left\{ iz \left(X_s + (T - s)(r + m(\mu - r)) \right) \right. \\ &\quad \left. - \frac{1}{2}m^2(z^2 + iz)\sigma^2(T - s) \right\}. \end{aligned} \quad (6.24)$$

The calibration of the implied volatility parameter based on option data on the CAC40 gives us a value $\sigma = 0.2563$. We again consider a drift μ equal to 6.63%.

6.4. (Fractional) Riccati equations

In order to obtain the characteristic function of the log-cushion in the rough Heston model, we need to solve the fractional Riccati equation (3.8). Based on Momani &

Shawagfeh (2006), we use an Adomian decomposition method combined with Padé approximants to obtain numerically the solution of such fractional Riccati equations. This method is highly effective and stable in order to capture the essential behavior of the solution $\psi(t, u)$ in the fractional Riccati case and provides solutions that are consistent with the numerical scheme described in the classical paper of Diethelm et al. (2002) for finding solutions of fractional differential equations. For the Heston model, we need to solve the ordinary Riccati equation (6.6). No analytical solution can be found to this first-order nonlinear ODE and we hence also resort to the Adomian method of Momani & Shawagfeh (2006) with $\alpha = 1$ to derive numerically the solution $\psi(t, u)$.

We now show how to compute the moments of different orders for the portfolio value V_T at maturity. We especially consider the Heston and the (MM and SM) rough Heston models, which are two important members of our class of affine Volterra processes $\tilde{Z}$. We also compare these results under the B&S framework for the log-cushion dynamic.

7. Moments

From the characteristic function of the log-cushion, the moments of order k of the cushion C_t can be directly obtained by imposing $iz = k$. Indeed, we have

$$\Phi_X(T, -ki) = \mathbb{E} \left[e^{kX_T} \middle| \mathcal{F}_0 \right] = \mathbb{E} \left[e^{k \log C_T} \middle| \mathcal{F}_0 \right] = \mathbb{E} \left[C_T^k \right]. \quad (7.1)$$

We consider here the situation at time $s = 0$ where the initial amount in the fund V_0 is 100 and where the guaranteed amount $G(= F_T)$ at maturity $T = 2$ is 90. We find that $F_0 = 90e^{-2r} = 90.7$ and hence that $C_0 = V_0 - F_0 = 9.3$. For the multiple, we choose exogenously a value $m = 4.5$ such that the worst daily return anticipated by the fund manager over the investment horizon is 22%. Using the models introduced above, we are able to find the k^{th} order moment of the cushion value C_T and we hence obtain the following Table (7.1) describing the expectation, standard deviation/volatility, skewness and kurtosis of the portfolio value V_T at maturity, using the relation $\mathbb{E}[V_T^k] = \mathbb{E}[(C_T + F_T)^k]$:

	Expectation	Volatility	Relative Skewness	Relative Kurtosis
B&S	106.8467	61.4443	59.4593	48 307.2101
Rough Heston MM	106.8467	15.4499	1.3670	12.9791
Rough Heston SM	106.8467	15.8033	1.4332	13.6312
Heston	106.8467	30.4749	3.1143	16.4971

Table 7.1. Comparison of the first four moments of V_T based on the characteristic function of the log-cushion under each of the four models.

First, we have that the B&S model provides moments that are much higher than in the other models. This is due to the strong negative correlation ρ of the Heston and rough Heston models which decreases their moment values. Therefore, since the B&S model does not take into account the leverage effect, it tends to highly overestimate the moments of the fund V_T . We also see that the two rough Heston models (MM and SM) have moments which are below the ones of the Heston model. For the volatility of the fund $\sigma[V_T]$, this is due to the fact that the forward variance curve is decreasing in the rough Heston model (see Figure 5.2) while the long-term mean $\tilde{\theta}$ of the variance process under the Heston model is 0.2094 and hence lies well above this forward variance curve. The stronger leverage effect (lower correlation ρ between the price process and the variance process) in the rough Heston explains the lower values of skewness and kurtosis compared to the Heston case. Moreover, we naturally expect the moments of the rough Heston with the stochastic change of measure (SM) to be higher than the ones of the minimal measure rough Heston (MM) due to the market price of volatility risk. However, as mentioned above, the minimal measure and the real probability measure are close to each other in the sense that the forward variances under each measure are relatively close and hence, we do not observe large differences in the moment values between the two models.

In the next Table 7.2, we again present the values of the first four moments of V_T but this time based on simulations². This way, we can confirm the consistency and

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²Simulation and discretization schemes for the studied models are presented in Dupret et al. (2021)

robustness of the method proposed in this paper based on the characteristic function of the log-cushion compared with the classical stochastic volatility method for CPPI (which requires simulation) of Prigent & Bertrand (2003) :

	Expectation	Volatility	Relative Skewness	Relative Kurtosis
B&S	106.8481	61.5675	63.0853	44 116.7211
Rough Heston MM	106.8333	15.5751	1.2772	8.9482
Rough Heston SM	106.8515	15.9621	1.3554	9.5214
Heston	106.8408	30.5386	3.1113	16.3770

Table 7.2. Comparison of the first four moments of V_T for each model based on simulations.

We first point out that the higher the order of the moments, the higher the number of simulations we need to generate so as to have stable results. Here, we used 1 million simulations and the skewness and kurtosis for each model are still highly unstable. This explains why we observe larger differences between the two Tables 7.1 and 7.2 for the skewness and kurtosis values (especially for B&S and rough Heston models). Nevertheless, we can conclude that the method based on the characteristic functions of the log-cushion provides results that are consistent with simulations while being much more effective and accurate. Hence, the methodology derived in this paper overcomes the instability and slowness of simulations and allows to easily and efficiently generate the moments of the fund value V_T .

In a similar way as for the calculation of the moments of the portfolio value V_T , we now show that we can derive efficiently and consistently several risk measures as well as the density of V_T based on the characteristic function of the log-cushion.

8. Risk Measure and probability density

Based on Artzner et al. (1999), we define the Value-at-Risk (VaR) as the ε -quantile of the portfolio value at the term of a chosen time horizon T . This risk measure is used to appraise the shortfall risk and despite that it is not a coherent risk measure (cfr Artzner et al. (1999)), the Value-at-Risk still remains an essential tool for asset allocation. If

$\varepsilon \in [0, 1]$ is the maximum shortfall probability accepted by the investor, then the VaR is defined as the shortfall level, q , such that

$$P(V_T \leq q) = \varepsilon \iff P(C_T \leq q - F_T) = \varepsilon \iff P(X_T \leq \log(q - F_T)) = \varepsilon \quad (8.1)$$

where $X_T = \log C_T$. If we then denote the probability density function of X_T by $f_T(\cdot)$, this relation becomes

$$\int_{-\infty}^{\log(q - F_0 e^{rT})} f_T(x) dx = \varepsilon. \quad (8.2)$$

Another common measure of risk is the Tail Value-at-Risk (TVaR), denoted tv , which is the expected shortfall level if the portfolio value breaches the VaR q ,

$$\begin{aligned} tv = \mathbb{E}[V_T | V_T \leq q] &\iff tv = \mathbb{E}\left[C_T + F_0 e^{rT} \mid C_T \leq q - F_0 e^{rT}\right] \\ &\iff tv = \mathbb{E}\left[e^{X_T} \mid X_T \leq \log(q - F_0 e^{rT})\right] + F_0 e^{rT} \end{aligned} \quad (8.3)$$

Equation (8.3) can be rewritten in terms of the probability density function as follows

$$tv = \frac{1}{\varepsilon} \int_{-\infty}^{\log(q - F_0 e^{rT})} e^x f_T(x) dx + F_0 e^{rT}. \quad (8.4)$$

The TVaR is a coherent risk measure, also called expected shortfall and can identically be defined as the conditional tail expectation when applied to continuous random variables.

The probability distribution of the log-cushion X_T (conditionally to $\mathcal{F}_0$) under the above models is not analytically determined (except for the B&S model). However, its characteristic function $\Phi_X(T, z)$, which is also the Fourier Transform of its density, has a quasi-closed form under these models. By inverting numerically the characteristic function of X_T , it is hence possible to determine its density of probability and to compute numerically the VaR and TVaR by equations (8.2) and (8.4). This numerical scheme, adapted from Hainaut (2011), relies on the Fast Fourier Transform (FFT) algorithm and is detailed in Appendix 11.5.

8.1. Application

The log-cushion density obtained numerically from the FFT algorithm (cfr Appendix 11.5) is plotted in the following Figure 8.1 for each of the studied models. The number

of steps involved in the FFT algorithm is set to $N = 2048$ and the other parameters of the CPPI are kept as in Table 7.1. We also compare the log-cushion density of each model for two values of the multiplier : $m = 4.5$ (as in Section 7, in black) and $m = 6$ (in red).

Fig. 8.1. Density of the log-cushion for each model with $m = 4.5$ in black and $m = 6$ in red. Figures (a) and (b) have y-axis limits of $[0, 0.25]$ and Figures (c) and (d) of $[0, 0.5]$.

The log-cushion is normally distributed under the B&S model, which explains the shape of the curves in (a). Compared to the other models, we see that the probability density function of the B&S log-cushion is shifted to the right with significant probability attributed to high values of the log-cushion and extremely low probability attributed to really low values of the log-cushion. This explains the high values of skewness and

kurtosis observed in Table 7.1 for the portfolio value V_T when taking the exponential of the log-cushion plus the floor. We also see that the log-cushion values are more dispersed around the mean in the B&S model, especially compared to the rough Heston models. This explains the higher volatility $\sigma[V_T]$ of the portfolio value. Moreover, we observe that when the value of the multiplier m increases, the density of the log-cushion widens and is shifted to the left.

The density function of the log-cushion under the Heston model is negatively-skewed with the fattest left tail, making this model the most risky. This negative skewness of the log-cushion explains why the skewness of the portfolio value V_T is low when taking the exponential of the distribution (but still positive since the effect of the exponential function on the skewness overcomes the negative skewness of the log-cushion). Compared with rough Heston models, we clearly see that we have a broader dispersion of the log-cushion values which are less peaked. This leads to a higher volatility of V_T . We again have that the distribution of the log-cushion widens when the multiplier increases.

Both rough Heston models are highly peaked and negatively skewed, which explains the low volatility of V_T and the low skewness and kurtosis. We can however observe that the minimal measure yields a slightly higher peak but is less dispersed compared with the stochastic measure. This will be confirmed below with Table 8.1 of quantile values for the fund V_T . It explains why the volatility of V_T in the MM rough Heston is lower compared with the SM rough Heston. Finally, when m increases, the distribution slightly broadens and is shifted this time to the right.

Based on the probability density function of the log-cushion, we can derive its quantiles and hence use formula (11.12) to compute the VaR q of the portfolio value V_T . We then obtain the following Table (8.1) describing 7 percentiles of interest for V_T with $m = 4.5$:

This Table 8.1 confirms and details Figure 8.1 of the log-cushion density. We indeed have under the B&S model that the distribution of the portfolio value is shifted to the right and is more dispersed than in the other models. In fact, the lowest quantiles (1%,

	1%	5%	25%	50%	75%	95%	99%
B&S	90.1001140	90.304417	91.4824	94.4516	103.3813	155.1296	287.8409
RH MM	90.0000892	90.009376	93.0089	104.0985	116.6913	134.3294	153.0880
RH SM	90.0000797	90.008239	92.8993	103.7776	116.7701	135.1751	155.4732
Heston	90.0000025	90.000107	90.1564	93.3854	110.5970	174.8740	244.6398

Table 8.1. Comparison of different quantiles of the portfolio value V_T between models.

5%) and largest quantiles (99%) tend to have higher values in the B&S model than in the other models, while the middle quantiles (25%, 50%, 75%) are the highest for the rough Heston models (which are the most peaked). The Heston model tends to have the lowest values for small percentiles since it has the fattest left tail and it also exhibits high values for large percentiles, which explains the high dispersion of the portfolio value in this model. Since this model is the less peaked (maximal density of 0.157), the middle quantiles have low values. Finally, as mentioned above, the rough Heston models tend to be the most peaked and are the less dispersed models. We confirm that the SM rough Heston tends to have lower values V_T for small quantiles and hence a fatter left tail than the MM rough Heston, while for large percentiles we observe larger portfolio values and hence a fatter right tail for the SM rough Heston. This explains the higher volatility of the fund $\sigma[V_T]$ in the SM case.

We now turn to the TVaR of the portfolio value which is defined by formula (8.4). We consider the TVaR at levels 1%, 5% and 25% since we are interested in the left tail of the distribution and we then obtain the following Table 8.2 for all our models with $m = 4.5$:

	1%	5%	25%
B&S	90.063982	90.177641	90.71245
Rough Heston MM	90.000029	90.003298	90.81467
Rough Heston SM	90.000026	90.002789	90.73518
Heston	90.0000012	90.000048	90.03154

Table 8.2. Comparison of the TVaR at levels 1%, 5% and 25% of V_T between models.

By definition, we have that the TVaR is below the VaR at corresponding levels. Again, we confirm that the Heston model has the fattest left tail with significant probability attributed to extremely small values of V_T while the B&S portfolio value exhibits the highest TVaR for small percentile levels. At level 25%, the rough Heston models have the highest TVaR again due to the shape of their log-cushion density. Since the SM rough Heston has a fatter left tail than the MM rough Heston, it has lower TVaR for the three considered levels and is hence more risky.

Finally, it can again be confirmed that the values of VaR and TVaR derived in Table 8.1 and Table 8.2 are consistent and close to the values obtained from simulations. However, simulation-based methods (Monte-Carlo, Hybrid scheme, ...) are way more slow and less stable than working with the characteristic function of the log-cushion, especially for rough Heston models due to the non-Markovian nature of their variance process.

9. Discrete-time CPPI

Since we have assumed continuous trading on the whole investment horizon $[0, T]$, we cannot take into account the gap risk, *i.e.* the risk of violating the floor protection. We can indeed observe in Table 8.1 that the final value of the portfolio V_T is systematically above the guarantee $G = 90$, even for very small quantiles. In practice, this gap risk needs to be managed and is caused by liquidity constraints and steep drops in the risky asset occurring before the fund manager can rebalance the portfolio. Gap risk is discussed thoroughly in Cont & Tankov (2009) and in Ameur & Prigent (2014). Nevertheless, both liquidity constraints and gap risk can be modeled in the setup of Balder et al. (2009) where the price dynamic of the risky asset is again described by a continuous-time stochastic process but where trading is now restricted to discrete time. In their framework, Balder et al. (2009) consider that the price dynamic simply follows a geometric Brownian motion. From the results derived above with Theorem 6.1 and Corollary 6.2, their discrete-time CPPI could be easily extended to the class of affine Volterra processes in order to handle and measure gap risk associated to such models.

Moreover, transaction costs could also be taken into consideration with this discrete setting.

10. Conclusion

This paper proposes to derive and study the properties of continuous-time CPPI strategies based on the characteristic function of the log-cushion. The class of affine Volterra processes appears to be particularly adapted for building such CPPI strategies in frictionless markets since their log-cushion's characteristic function is given in quasi-closed form and can be evaluated rapidly using numerical schemes. This method allows to generalize results previously established in the literature for a large class of models, especially when the variance process associated with the risky asset is non-Markovian in the current variance state. Through the choice of the coefficients a and b as well as the convolution kernel K of affine Volterra processes, various and more complex dynamics can be envisaged for the underlying risky asset on which the CPPI strategy is defined. In particular, we investigate the CPPI strategy under rough volatility with the rough Heston model (using a power law kernel K). We then show how to obtain effectively and accurately very consistent results from the characteristic function of the log-cushion, such as the moments, the density and several risk measures for the portfolio value V_T . Moreover, the Heston model, which belongs to the class of affine Volterra processes as well (with an identity kernel), is also studied in this paper and we again get results in a more effective and accurate way using the characteristic function of the log-cushion compared with classical simulation-based methods. For the sake of comparison, this characteristic function is also given under the simple B&S framework.

The Heston and rough Heston models being calibrated under a risk-neutral measure as explained in Dupret et al. (2021), the introduction of some changes of measure is required since we are interested in the dynamic of the CPPI portfolio under the real probability measure $\mathbb{P}$. Two different changes of measure are considered in this paper : one based on the minimal measure (MM) from Föllmer & Schweizer (1991), which corresponds to setting the volatility risk premium to zero, and the other based on a stochastic change of

measure (SM) with a positive risk premium. From the calibration and the study realized on the CAC40 index, we find that the B&S model has the highest values of the portfolio V_T at maturity for very small and very large percentiles, leading to a high volatility, skewness and kurtosis of the fund. The Heston model also leads to a high dispersion of the fund values and takes the lowest values of VaR and TVaR at small levels (1% and 5%), making this model the most risky. Finally, the log-cushion densities under rough Heston models are highly peaked and concentrated around their mean, leading to small values of volatility, skewness and kurtosis for the fund V_T . We can also observe that the density of the log-cushion (and hence of the fund value) in each model tends to widen when the multiplier m increases.

As future work, CPPI strategy with trading restricted to discrete time should be further studied and convergence results such as in Balder et al. (2009) should be demonstrated. In this way, we will be able to manage the gap risk and include borrowing constraints as well as transaction costs under our CPPI strategy. Other changes of measure could also be envisaged, especially for analyzing the effect of the volatility risk premium on the volatility of variance parameter ν . Finally, an extension of our current framework to more advanced versions of CPPI strategies (cushion insurance, varying floor, ...) may be desirable for a more thorough discussion on portfolio insurance.

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Availability of data and material

The datasets and code generated during and/or analysed during the current study are available from the corresponding author on reasonable request.

Declaration of interest

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11. Appendix

11.1. Fractional Calculus

We define the fractional integral of order $r \in (0, 1)$ of a function f as

$$I^r f(t) = \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} f(s) ds,$$

whenever the integral exists, and its fractional derivative of order $r \in (0, 1)$ as

$$D^r f(t) = \frac{1}{\Gamma(1-r)} \frac{d}{dt} \int_0^t (t-s)^{-r} f(s) ds,$$

whenever it exists. If we take the power-law kernel $K(t) = t^{r-1}/\Gamma(r)$, the fractional integral is given by the following convolution operator

$$I^r f(t) = (K \star f)(t),$$

and the fractional derivative by

$$D^r f(t) = \frac{d}{dt} \left(I^{1-r} f(t) \right).$$

11.2. Proof of Theorem 2.1

The proof closely follows the one of Gatheral & Keller-Ressel (2019). Let

$$G_s = (g \star \xi)_s(T, u) = \int_s^T g(T - r, u) \xi_s(r) dr, \quad (11.1)$$

$$M_s = \exp(uY_s + G_s). \quad (11.2)$$

Hence, it directly comes from (3.5) that M is a martingale. It is also shown in Gatheral & Keller-Ressel (2019) that G can be rewritten as an Itô process of the form

$$G_s = \int_0^T g(T - r, u) \xi_0(r) dr - \int_0^s g(T - r, u) v_r dr + \int_0^s h_r(T, u) dW_r^{\mathbb{Q}}, \quad (11.3)$$

where

$$h_s(T, u) = (g \star \eta)_s(T, u) = \int_s^T g(T - r, u) \eta_s(r) dr.$$

This equation 11.3 for G is obtained using the stochastic Fubini theorem which requires the *Assumption 2.1* of Gatheral & Keller-Ressel (2019) imposing the two following conditions on $\eta_s(\cdot)$:

- (a) For $dt \otimes d\mathbb{Q}$ -almost all (t, ω) , it hold that $t \mapsto \eta_s(s + t, \omega)$ is non-negative and decreasing on $(0, T)$
- (b) For any $T > 0$, the following integrability condition holds almost surely :

$$\int_0^T \left(\int_0^T \eta_s(r)^2 ds \right)^{1/2} dr < \infty.$$

Now recall that the dynamic of Y is given by

$$dY_s = -\frac{1}{2}m^2 v_s ds + m\sqrt{v_s} dW_s^{\mathbb{Q}}.$$

Then, applying Itô's lemma to $M_s = \exp(uY_s + G_s)$, we have that

$$\begin{aligned} \frac{dM_s}{M_s} &= (dW_s^{\mathbb{Q}} \text{ terms}) \\ &+ \left(\frac{1}{2}m^2(u^2 - u)v_s - g(T - s, u)v_s + mu\rho\sqrt{v_s}h_s(T, u) + \frac{1}{2}h_s(T, u)^2 \right) ds. \end{aligned}$$

Since we know from Section 1 that $\eta_s(T) = \sqrt{v_s} \kappa(T - s)$, we find the following simplification for $h_s(T, u)$:

$$h_s(T, u) = (g \star \eta)_s(T, u) = \sqrt{v_s} \int_0^{T-s} g(T - s - r, u) \kappa(r) dr = \sqrt{v_s} (g \star \kappa)(T - s).$$

Therefore, if we define

$$R_v(m, u, w) = \frac{1}{2}m^2(u^2 - u) + m\rho uw + \frac{1}{2}w^2, \quad (11.4)$$

we find that

$$\frac{dM_s}{M_s} = (dW_s^{\mathbb{Q}} \text{ terms}) + (R_v(m, u, (g \star \kappa)(T - s)) - g(T - s, u)) v_s ds.$$

In order to have that M is a martingale, we need to cancel the terms in ds and we hence find the following expression for $g(\cdot, u)$,

$$g(t, u) = R_v(m, u, (g \star \kappa)(t, u)) = R_v\left(m, u, \int_0^t \kappa(t - r)g(r, u)dr\right), \quad (11.5)$$

Finally, expression 11.5 enables to compute the CGF of the forward variance model (Y, ξ) using

$$\log \mathbb{E}^{\mathbb{Q}} \left[e^{u(Y_T - Y_s)} | \mathcal{F}_s \right] = \int_s^T g(T - r, u) \xi_s(r) dr. \quad (11.6)$$

□

11.3. Proof of Theorem 6.1

For an investment horizon T , we define $M_t = e^{U_t}$, with U_t given by

$$\begin{aligned} U_t &= \phi(T - t) + \chi(T)V_0 - \int_0^t \chi'(T - s)v_s ds + \int_0^t \psi_2(T - s)dL_s + \psi_1(T - s)Y_t \\ &= \phi(T - t) + \chi(T)V_0 - \int_0^t \chi'(T - s)v_s ds + \int_0^t \psi_2(T - s)dL_s + u_1 Y_t, \end{aligned} \quad (11.7)$$

since $\psi_1 = u_1$ due to the form (6.17) and where $dL_t = (\beta + bv_s)ds + (0 + a\sqrt{v_s})d\widehat{W}_s^{\mathbb{P}}$ as in (6.15) to ensure that the process $Z = (Y, v)$ is an affine Volterra process. Using Itô's lemma on $M_t = e^{U_t}$, we have

$$\begin{aligned} \frac{dM_t}{M_t} &= -\phi'(T - t) - \chi'(T - t)v_t dt + \psi_2(T - t)(\beta + bv_t)dt + \frac{1}{2}\psi_2(T - t)^2 a^2 v_t dt \\ &\quad + \frac{1}{2}m^2(u_1^2 - u_1)v_t dt + \rho\psi_2(T - t)(0 + a\sqrt{v_t})u_1 m\sqrt{v_t} dt + (dW_t \text{ terms}), \end{aligned}$$

which gives

$$\frac{dM_t}{M_t} = (-\phi'(T - t) + \mathcal{R}_{\Phi}(\psi_2(T - t))) dt + (-\chi'(T - t) + \mathcal{R}_{\Psi}(\psi(T - t))) v_t dt + (dW_t \text{ terms}),$$

where we define

$$\mathcal{R}_{\Phi}(\psi_2) = \beta\psi_2 + 0,$$

and

$$\begin{aligned}\mathcal{R}_\Psi(\psi) &= \frac{1}{2}m^2(\psi_1^2 - \psi_1) + (b + m\psi_1\rho a)\psi_2 + \frac{a^2}{2}\psi_2^2 \\ &= \frac{1}{2}m^2(u_1^2 - u_1) + (b + mu_1\rho a)\psi_2 + \frac{a^2}{2}\psi_2^2.\end{aligned}$$

Hence for M_t to be a martingale, we find that $\phi' = \mathcal{R}_\Phi(\psi_2)$ and that $\chi' = \mathcal{R}_\Psi(\psi)$.

Imposing that $\phi(0) = 0$ and $\chi(0) = u_2$ and using the definition of U_t as well as the form (6.18) of ψ_2 , it is straightforward to show that

$$U_T = \phi(0) + \chi(0)V_T + u_1Y_T = u_2V_T + u_1Y_T, \quad (11.8)$$

Hence, since M is martingale, we have that

$$\mathbb{E}^\mathbb{P}[e^{uZ_T}] = \mathbb{E}^\mathbb{P}[e^{u_2V_T + u_1Y_T}] = \mathbb{E}^\mathbb{P}[M_T] = M_0 = e^{\phi(T) + \chi(T)V_0 + u_1Y_0},$$

with $u = (u_1 \ u_2)$ and $Z = \begin{pmatrix} Y \\ v \end{pmatrix}$. □

11.4. Proof of corollary 6.2

We build the exact same proof as in Theorem 6.1. except that we now impose $u_2 = 0$ such that $\chi(0) = 0$. Hence, the terminal value U_T of the process U defined by (11.7) is now

$$U_T = \phi(0) + \chi(0)V_T + u_1Y_T = u_1Y_T.$$

Since M is still martingale when $u_2 = 0$, we have that

$$\mathbb{E}^\mathbb{P}[e^{uZ_T}] = \mathbb{E}^\mathbb{P}[e^{u_1Y_T}] = \mathbb{E}^\mathbb{P}[e^{U_T}] = \mathbb{E}^\mathbb{P}[M_T] = M_0 = e^{\phi(T) + \chi(T)V_0 + u_1Y_0},$$

and the solution ψ now becomes

$$\begin{aligned}\psi_1 &= u_1, \\ \psi_2 &= K \star \mathcal{R}_\Psi(\psi).\end{aligned} \quad (11.9)$$

Finally, recall from (5.5) that the dynamic of the log-cushion is denoted X_t and is given by

$$dX_t = (r + m(\mu - r))dt + dY_t.$$

Hence, the Laplace transform of the affine Volterra process $\tilde{Z} = (X, v)$ with $u_2 = 0$ becomes

$$\mathbb{E}^{\mathbb{P}}[e^{u\tilde{Z}_T}] = \mathbb{E}^{\mathbb{P}}[e^{u_1 X_T}] = e^{\phi(T) + \chi(T)V_0 + u_1(X_0 + (r+m(\mu-r))t)}.$$

And finally the characteristic function of the log-cushion under the general affine Volterra process $\tilde{Z}$ is obtained by replacing u_1 by iz . $\square$

11.5. Fast Fourier Transform

We present here the approach used to determine numerically the density of probability of the log-cushion X_T , which then allows to obtain the VaR and TVaR of the portfolio value V_T for each model.

First, the characteristic function of X_T , denoted $\Phi_X(T, z)$, is by definition the Fourier Transform of the density $f_T(x)$ of X_T :

$$\Phi_X(T, z) = \mathbb{E}^{\mathbb{P}}[e^{izX_T}] = \int_{-\infty}^{+\infty} e^{izx} f_T(x) dx.$$

Since we know the characteristic function of X_T for the B&S, Heston and rough Heston models, we can derive the density function $f_T(x)$ using the Fourier inversion theorem which states that

$$f_T(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \Phi_X(T, z) e^{-izx} dz = \frac{1}{\pi} \int_0^{+\infty} \Phi_X(T, z) e^{-izx} dz, \quad (11.10)$$

where the second equality results from the symmetry of the integrand. Next, we illustrate how to derive the density function $f_T(x)$ from equation (11.10) via the Fast Fourier Transform (FFT). FFT is an efficient algorithm for computing the following transformation of a vector $(\alpha_n, n = 1, \dots, N-1)$ into a vector $(\beta_n, n = 1, \dots, N-1)$, where N is typically a power of 2 :

$$\beta_n = \sum_{j=0}^{N-1} \exp\left(-\frac{2\pi i}{N} jn\right) \alpha_j.$$

This method is quicker than computing the discretization of the integral (11.10) since the FFT algorithm computes in only $O(n \log n)$ operations (rather than $O(n^2)$). The first step of the FFT consists in discretizing the integral (11.10). We denote Δz the step of discretization and N the number of steps. The discretization mesh is defined by

$$\{z_j\} = \{j\Delta z \in \mathbb{R}^+ \mid 0 \leq j \leq N-1\}.$$

We then need to define a mesh of discretization for the density values $f_T(x)$, spaced by Δx and counting the same number N of elements as $\{z_j\}$. The mesh of the density is then

$$\{x_n\} = \{-x_{min} + n\Delta x \in \mathbb{R}^+ \mid 0 \leq n \leq N-1\},$$

where $x_{min} = \frac{N\Delta x}{2}$ and with the following condition on the steps of discretizations $\Delta z \Delta x = \frac{2\pi}{N}$. Therefore, we can reformulate (11.10) into the following form

$$\begin{aligned} f_T(x_n) &= \frac{1}{\pi} \int_0^{+\infty} \Phi_X(T, z) e^{-izx_n} dz \\ \iff f_T(x_n) &= \frac{1}{\pi} \sum_{j=0}^{N-1} \Phi_X(T, z_j) e^{-iz_j x_n} \Delta z \\ \iff \underbrace{\pi f_T(x_n)}_{\beta_n} &= \sum_{j=0}^{N-1} e^{-i\frac{2\pi}{N}jn} \underbrace{e^{iz_j x_{min}} \Phi_X(T, z_j) \Delta z}_{\alpha_j}. \end{aligned} \quad (11.11)$$

This way we can apply the FFT with $\alpha_j = e^{iz_j x_{min}} \Phi_X(T, z_j) \Delta z$ in order to obtain the output vector $\beta_n = \pi f_T(x_n)$ for $n \in 0, \dots, N-1$. Finally, once we estimated the density via the FFT, we can easily derive the ε -quantile of X_T denoted x_ε . Using equation (8.1), we find that the VaR level q of the portfolio value V_T at maturity is equal to

$$q = e^{x_\varepsilon} + F_T. \quad (11.12)$$

The TVaR tv at level ε of V_T can then be computed by discretizing the integral (8.4) :

$$tv = \frac{1}{\varepsilon} \sum_{x_n: x_n \leq \log(q - F_0 e^{rT})} e^{x_n} f_T(x_n) \Delta x + F_0 e^{rT}. \quad (11.13)$$

11.6. Interest rates

Maturity	Rate	Maturity	Rate
Overnight	-0.55486 %	6 months	-0.35386 %
7 days	-0.52686 %	1 year	-0.36767 %
1 month	-0.494 %	2 years	-0.38775 %
2 months	-0.44157 %	5 years	-0.35563 %
3 months	-0.43514 %	10 years	-0.19118 %

Table 11.1: Risk-free interest rates between 0 and 10 years