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Abstract

A time-consistent evaluation is a dynamic pricing method according to which a risk that will be almost surely cheaper than another one at a future date should already be cheaper today. Common actuarial pricing approaches are usually not time-consistent. Pelsser and Ghalehjooghi (2016) derived time-consistent valuation principles from time-inconsistent ones. The aim of this paper is twofold. Firstly, we propose a model for credit insurance portfolios taking into account the contagion risk via self-exciting jump processes. Secondly, we extend the approach of Pelsser and Ghalehjooghi to credit insurance in this framework. Starting from classical time-inconsistent actuarial pricing methods, we derive partial integro-differential equations (PIDE) for their time-consistent counterparts. We discuss numerical methods for solving these PIDE and their results. We draw two conclusions from these results. On the one hand, we show that working with time-consistent evaluations in the absence of a risk of contagion does not make a significant difference compared to time-inconsistent evaluations. On the other hand, our results show that the time-consistency of evaluations allows to better take into account the risk of contagion in credit insurance, if such a risk exists.

Keywords : Credit risk, Self-exciting processes, Time-consistency.

Introduction

Insurance markets are incomplete by nature since most of actuarial risks are not hedgeable with financial instruments. Choosing an appropriate valuation method is therefore a challenging task. This problem was initially tackled in a static way with the concepts of evaluations or risk measures. We refer the reader to e.g. Artzner et. al. (1999) for an introduction about coherent risk measures and Kaas et al. (2008) for examples. Unlike insurance pricing, financial pricing is treated in a dynamic way, focused on hedging and replicating portfolios, see for example Black and Scholes (1973). Delbaen and Schachermayer (1994) show that this intrinsically dynamic approach can be applied with various securities, as long

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as the asset prices are bounded real-valued semimartingales. Dynamic evaluation methods extend this dynamic nature of financial pricing to insurance pricing. This type of evaluations, as well as the concept of time-consistency (or dynamic-consistency), were introduced by Riedel (2004) in discrete time. Rosazza Gianin (2006) uses g -expectations to extend the concept of time-consistency to continuous time. Pelsser and Stadje (2014) introduce the concept of two step market evaluations, that allows to apply time-consistent evaluations on products that incorporate both a financial risk and an actuarial risk. Pelsser and Ghalehjooghi (2016) use the definition of time-consistency in order to turn time-inconsistent classical evaluation principles into time-consistent ones. They derive partial differential equations (PDE) that characterize time-consistent evaluations.

Credit risk for transactions involving large quoted corporations can be hedged with derivatives like credit default swaps. Nevertheless financial markets do not propose similar covers for small or medium enterprises (SME). For this category of companies, the only solution to limit credit exposure is provided by trade credit insurances. Insurers proposing these contracts rely on diversification to hedge themselves but are exposed to contagion between defaults. This motivates us to propose a time-consistent valuation framework managing both default and contagion risks.

The two main paradigms for credit risk valuation are the structural and reduced approaches. In the first one, the balance sheet of the firm value is explicitly modelled and the default occurs when the asset value falls below liabilities. For details, we refer the reader e.g. to Merton (1974), Black and Cox (1976), Geske (1977) and Longstaff and Schwartz (1995). The second approach is called the reduced-form or intensity-based credit model. The default is in this case triggered by the first jump of a point process. The reduced-form was initiated by the work of Duffie and Singleton (1999). Textbooks concerning this second kind of models are Bielecki and Rutkowski (2002), Duffie and Singleton (2003), Lando (2004), Jeanblanc et al. (2009) and Schoutens and Cariboni (2009). In this article, we opt for a reduced-form approach.

An endogenous way to model the contagion between defaults of companies consists to use self-exciting point processes. In this approach, any jump of this process triggers an increase of the instantaneous expected number of future jumps. Hawkes (1971a,b) and Hawkes and Oakes (1974) were among the first to introduce such point processes. Since then, Hawkes processes have been used to model earthquake occurrences, see e.g. Musmeci and Vere-Jones (1992) or Ogata (1998), or terrorist activity, see Porter and White (2012). They are also used for modelling financial transactions, see e.g. Bauwens and Hautsch (2009), Bacry et. al. (2015), Hainaut (2017), Hainaut and Moraux (2019), Hainaut and Goutte (2019). For a survey of properties and other possible applications of Hawkes processes, the reader is referred to Reinhart (2018).

The contributions of this article are organized as follows. Firstly, we introduce a new model based on self-exciting processes in which the default intensity of each company is an affine function of a common market intensity. We provide closed-form expressions for the moments of this intensity process. Secondly we find semi-closed form expressions for the Laplace transform of the intensity process. We use next this Laplace transform to compute probabilities of default. Thirdly, we extend the approach of Pelsser and Ghalehjooghi

(2016) to our setting in two directions. On the one hand, we evaluate in a time-consistent manner credit insurance portfolio. On the other hand, we take into account the risk of contagion. We obtain partial integro-differential equations (PIDE) for the time-consistent values and solve them numerically. Fourthly, we provide a numerical method for solving the PIDE. This method solves PIDE's for which we miss some of the boundary conditions. To conclude, we present a numerical illustration and analyze the impacts of contagion risk and time-consistency on the value of a credit insurance product.

1 The Model

In the intensity approach to credit risk, defaults of companies are modelled by jumps of point processes. A jump of a point process corresponds to the default of a company. In this paper, we are interested in the defaults observed in a credit insurance portfolio. The defaults correspond to the jumps of a point process $(P_t)_{t \geq 0}$. The losses induced by the defaults are modelled by the sizes of the jumps of $(P_t)_{t \geq 0}$. We start by describing precisely how the process $(P_t)_{t \geq 0}$ is built.

Let $(N_t)_{t \geq 0}$ be a point process on a filtered probability space $(\Omega, \mathbb{F}, P)$ where $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ is the natural filtration of all the processes to be described in this section. We denote the intensity of $(N_t)_{t \geq 0}$ by $(\lambda_t)_{t \geq 0}$, another stochastic process. The intensity represents the instantaneous probability to observe a jump in the process $(N_t)_{t \geq 0}$. Let $\xi_1, \xi_2, \dots$ be independent identically distributed exponential random variables with parameter $\rho > 0$. Consequently, if the distribution of $\xi_1, \xi_2, \dots$ is given by the measure ν on $(\mathbb{R}^+, \mathcal{B}(\mathbb{R}^+))$, we have

$$d\nu(x) = \rho e^{-\rho x} dx, \quad \rho > 0 \quad (1.1)$$

for all $x \in \mathbb{R}^+$, and $\mathbb{E}[\xi_i] = \rho^{-1}$ for all i . We also define the process $(P_t)_{t \geq 0}$ as

$$P_t = \sum_{k=1}^{N_t} \xi_k. \quad (1.2)$$

It is a pure positive jump process whose jump intensity is the stochastic process $(\lambda_t)_{t \geq 0}$. The dynamics of this intensity process is assumed to satisfy the stochastic differential equation (SDE)

$$d\lambda_t = \kappa(\theta - \lambda_t)dt + \eta dP_t + \sqrt{\lambda_t} \sigma dW_t \quad (1.3)$$

where $(W_t)_{t \geq 0}$ is a standard Brownian motion, $\kappa, \theta, \eta, \sigma > 0$ and $\lambda_0 \in [\theta, +\infty[$. As a consequence of SDE (1.3), the intensity depends upon the history of the process $(P_t)_{t \geq 0}$. This allows the process $(P_t)_{t \geq 0}$ to exhibit a self-exciting behaviour : A jump of the process $(P_t)_{t \geq 0}$ induces a jump in the intensity $(\lambda_t)_{t \geq 0}$, which in turn implies an increase of the probability to observe another jump in $(P_t)_{t \geq 0}$. The presence of the term $\sqrt{\lambda_t} \sigma dW_t$ in SDE (1.3) ensures that $\lambda_t \geq 0$ almost surely for any $t > 0$. As early as λ_t gets close to 0, the noise term $\sqrt{\lambda_t} \sigma dW_t$ gets close to zero, so that λ_t cannot fall below 0. The parameter θ represents a level towards which the intensity tends to revert. The parameter κ in equation (1.3) represents the speed of the reversion towards θ and σ is a volatility parameter.

Since the jumps of $(P_t)_{t \geq 0}$ correspond to defaults, our model introduces a phenomenon of contagion between defaults : A default of a company in the portfolio triggers an increase of the intensity of defaults, thereby increasing the chances to observe other defaults in the portfolio. The self exciting behavior of $(\lambda_t, P_t)_{t \geq 0}$ thus aims to model the clustering of defaults observed in practice, see e.g. Ait-Sahalia et al. (2015).

2 Expectation and Variance of the λ Process

This section is concerned with the computation of the expectation, variance and autocovariance of the intensity process described in equation (1.3). These computations will allow us to derive stability conditions for our model, that is $\lim_{t \rightarrow +\infty} (\mathbb{E}[\lambda_t] \vee \mathbb{V}\text{ar}(\lambda_t)) < +\infty$.

Proposition 2.1. *Assume that $\kappa > \frac{\eta}{\rho}$. For any $t \geq 0$, we have*

$$\mathbb{E}[\lambda_t] = e^{a_3 t} \lambda_0 + \frac{\kappa \theta}{a_3} (e^{a_3 t} - 1) \quad (2.1)$$

and

$$\mathbb{V}\text{ar}(\lambda_t) = \frac{(e^{a_3 t} - 1)(2a_1 \lambda_0 e^{a_3 t} + a_2(e^{a_3 t} - 1))}{2a_3} \quad (2.2)$$

where $a_1 = \sigma^2 + 2\left(\frac{\eta}{\rho}\right)^2$, $a_2 = \frac{\kappa \theta \left(\sigma^2 + 2\left(\frac{\eta}{\rho}\right)^2\right)}{\frac{\eta}{\rho} - \kappa}$ and $a_3 = \frac{\eta}{\rho} - \kappa$.

Proof. First note that SDE 1.3 implies that

$$\lambda_t = \theta + e^{-\kappa t}(\lambda_0 - \theta) + \sigma \int_0^t e^{-\kappa(t-s)} \sqrt{\lambda_s} dW_s + \eta \int_0^t e^{-\kappa(t-s)} dP_s. \quad (2.3)$$

By taking the expectation and deriving with respect to t , we obtain the ordinary differential equation (ODE)

$$\frac{\partial \mathbb{E}[\lambda_t]}{\partial t} = \kappa(\theta - \mathbb{E}[\lambda_t]) + \frac{\eta}{\rho} \mathbb{E}[\lambda_t] \quad (2.4)$$

whose solution is well given by equation (2.1). For the variance, noting that $\mathbb{E}[\xi^2] = \rho^{-2}$ and using Ito's lemma yields

$$\frac{\partial \mathbb{E}[\lambda_t^2]}{\partial t} = \mathbb{E}[\lambda_t] \left(2\kappa\theta + \sigma^2 + 2\left(\frac{\eta}{\rho}\right)^2 \right) + 2\mathbb{E}[\lambda_t^2] \left(\frac{\eta}{\rho} - \kappa \right). \quad (2.5)$$

By the chain rule, we find

$$\frac{\partial}{\partial t} (\mathbb{E}[\lambda_t])^2 = 2\kappa\theta \mathbb{E}[\lambda_t] + 2(\mathbb{E}[\lambda_t])^2 \left(-\kappa + \frac{\eta}{\rho} \right). \quad (2.6)$$

Putting together equations (2.5) and (2.6), we obtain $\frac{\partial}{\partial t} \mathbb{V}\text{ar}(\lambda_t)$. Then, replacing $\mathbb{E}[\lambda_t]$ by its expression (2.1) and performing some simplifications leads to the following ODE

$$\frac{\partial \mathbb{V}\text{ar}(\lambda_t)}{\partial t} = e^{a_3 t} (a_1 \lambda_0 + a_2) - a_2 + 2a_3 \mathbb{V}\text{ar}(\lambda_t) \quad (2.7)$$

whose solution is given by equation (2.2). $\square$

From this proposition, we infer that the stability condition $\lim_{t \rightarrow +\infty} (\mathbb{E}[\lambda_t] \vee \mathbb{V}\text{ar}(\lambda_t)) < +\infty$ is equivalent to $a_3 < 0$, that is $\eta\rho^{-1} < \kappa$. This condition means that the speed of the mean reversion κ must be sufficiently large.

In the next section, we derive semi-closed form expressions for the Laplace transform of the intensity.

3 Laplace Transform

3.1 Laplace Transform of the Jump Processes

Since $(N_t)_{t \geq 0}$ is an increasing process, it is a submartingale. It follows from the Doob-Meyer decomposition that there exists a unique decomposition of $(N_t)_{t \geq 0}$ of the form $(M_t + \Lambda_t)_{t \geq 0}$ where $(\Lambda_t)_{t \geq 0}$ is an increasing right continuous predictable process called the compensator of $(N_t)_{t \geq 0}$ and $(M_t)_{t \geq 0}$ is a local martingale. From our model specification, we have that

$$\Lambda_t = \int_0^t \lambda_s ds$$

for all $t \geq 0$. Furthermore, since the time of default is a totally inaccessible stopping time, $(\Lambda_t)_{t \geq 0}$ has continuous paths.

Define the function $\psi : \mathbb{R} \rightarrow \mathbb{R}$ as $\psi(u) = 1 - e^{-u}$ and for each $u \in \mathbb{R}$, let $(Z_t(u))_{t \geq 0}$ be the unique stochastic process satisfying the stochastic differential equation

$$dZ_t(u) = -\psi(u)Z_{t-}(u)dM_t \quad (3.1)$$

with initial condition $Z_0(u) = 1$. From Lemma 9.1.2.3 in Jeanblanc et. al. (2009), we find that $(Z_t(u))_{t \geq 0}$ is given by

$$Z_t(u) = \exp \{ \psi(u)\Lambda_t - uN_t \}. \quad (3.2)$$

This process is called the Doléans-Dade exponential of $(-\psi(u)M_t)_{t \geq 0}$. Since $(M_t)_{t \geq 0}$ is a local martingale, equation (3.1) implies that $(Z_t(u))_{t \geq 0}$ is also a local martingale. Note that if $\mathbb{E}[\exp\{\Lambda_t\}] < +\infty$, then $(Z_t(u))_{t \geq 0}$ is a martingale. In the following, the process $(Z_t(u))_{t \geq 0}$ will be used to change the measure.

Since $Z_0(u) = 1$ and $(Z_t(u))_{t \geq 0}$ is a martingale, we use $Z_T^i(u)$ to define a measure $\mathbb{P}^u$ that is equivalent to $\mathbb{P}$ via

$$\mathbb{P}^u(B) := \mathbb{E} [\mathbb{I}_B Z_T(u)] \quad (3.3)$$

for any $B \in \mathcal{F}_T$. More generally,

$$\mathbb{P}^u(B|\mathcal{F}_t) = \mathbb{E} \left[\mathbb{I}_B \frac{Z_T(u)}{Z_t(u)} | \mathcal{F}_t \right]. \quad (3.4)$$

For each $u \geq 0$, we have a Laplace transform for the compensator under the measure $\mathbb{P}^u$, denoted as

$$\mathcal{L}^u(v, t, T) = \mathbb{E}^u [\exp\{-v(\Lambda_T - \Lambda_t)\} | \mathcal{F}_t]. \quad (3.5)$$

This Laplace transform is linked to the $\mathbb{P}$ -Laplace transform of $(N_t)_{t \geq 0}$ by the following result.

Proposition 3.1. *For any $u \in \mathbb{R}$, we have*

$$\mathbb{E} [\exp\{-u(N_T - N_t)\} | \mathcal{F}_t] = \mathcal{L}^u(\psi(u), t, T). \quad (3.6)$$

Proof. This result is a direct consequence of the following development

$$\begin{aligned} \mathbb{E} [\exp\{-u(N_T - N_t)\} | \mathcal{F}_t] &= \mathbb{E} [\exp\{-u(N_T - N_t) - \psi(u)(\Lambda_T - \Lambda_t) \\ &\quad + \psi(u)(\Lambda_T - \Lambda_t)\} | \mathcal{F}_t] \\ &= \mathbb{E} \left[\exp\{-\psi(u)(\Lambda_T - \Lambda_t)\} \frac{Z_T(u)}{Z_t(u)} | \mathcal{F}_t \right] \\ &= \mathbb{E}^u [\exp\{-\psi(u)(\Lambda_T - \Lambda_t)\} | \mathcal{F}_t] = \mathcal{L}^u(\psi(u), t, T), \end{aligned} \quad (3.7)$$

where the second equality comes from the definition of the measure $\mathbb{P}^u$. $\square$

In order to be able to exploit this relation, we have to determine the dynamics of $(\lambda_t)_{t \geq 0}$ under the measure $\mathbb{P}^u$. In order to do so, we will need the following Lemma, which is a Corollary to Girsanov's Theorem.

Lemma 3.2. *Let ζ be a $\mathbb{P}$ -local martingale and define the equivalent probability measure $\mathbb{Q}$ by $\frac{d\mathbb{Q}}{d\mathbb{P}} = \mathcal{E}(\zeta)$, where $\mathcal{E}(\zeta)$ denotes the Doléans-Dade exponential of ζ . If Y is a $\mathbb{P}$ -local martingale such that $\langle Y, \zeta \rangle$ exists, then $\tilde{Y} := Y - \langle Y, \zeta \rangle$ is a $\mathbb{Q}$ -local martingale.*

Proof. See Jeanblanc et. al. (2009), Corollary 9.4.4.5. $\square$

This result implies the following Proposition.

Proposition 3.3. *Fix $u \in \mathbb{R}$. Under the measure $\mathbb{P}^u$, the process $(\lambda_t)_{t \geq 0}$ obeys the stochastic differential equation*

$$d\lambda_t = \kappa(\theta - \lambda_t)dt + \sigma\sqrt{\lambda_t}dW_t + \eta dP_t \quad (3.8)$$

where, under $\mathbb{P}^u$,

- $(W_t)_{t \geq 0}$ is a standard Brownian motion.
- $(N_t)_{t \geq 0}$ is a pure jump process with jumps of size 1 and intensity $(e^{-u}\lambda_t)_{t \geq 0}$.
- $(P_t)_{t \geq 0}$ satisfies

$$P_t = \sum_{k=1}^{N_t} \xi_k$$

where $\xi_1, \xi_2, \dots$ are iid exponential random variables of mean ρ^{-1} that are independent from $(N_t)_{t \geq 0}$ and $(W_t)_{t \geq 0}$.

Proof. The proof contains two steps. Firstly, we show that the $\mathbb{P}$ -Brownian motion $(W_t)_{t \geq 0}$ remains a Brownian motion under $\mathbb{P}^u$. Secondly, we determine the intensities of the jump processes $(N_t)_{t \geq 0}$ and $(P_t)_{t \geq 0}$ under $\mathbb{P}^u$. The main tool for the proof is Lemma 3.2 with the $\mathbb{P}$ -local martingale ζ chosen to be $(-\psi(u)M_t)_{t \geq 0}$.

Since the $\mathbb{P}$ -Brownian motion $(W_t)_{t \geq 0}$ is a $\mathbb{P}$ -martingale, we have that the process $(W_t + \langle W, \psi(u)M \rangle_t)_{t \geq 0}$ is a local martingale under $\mathbb{P}^u$. Then, notice that $\langle W, \psi(u)M^N \rangle_t = 0$ for all t . The process $(\langle W, M \rangle_t)_{t \geq 0}$ is indeed the compensator $([W, M]_t)_{t \geq 0}$ which is constant and equal to zero. To see this, note that for any $s > 0$ and $0 = s_0 < s_1 < \dots < s_n = s$ where $n \in \mathbb{N}$, we have

$$\left| \sum_{i=0}^{n-1} (W_{s_{i+1}} - W_{s_i})(M_{s_{i+1}} - M_{s_i}) \right| \leq \left(\sup_i |W_{s_{i+1}} - W_{s_i}| \right) \text{Var}_s(M) \quad (3.9)$$

where $\text{Var}_s(M)$ is the total variation of $(M_t)_{t \geq 0}$ up to time s . Since $(W_t)_{t \geq 0}$ has continuous paths and $(M_t)_{t \geq 0}$ has bounded variation, this proves that $[W, M] \equiv 0$. This finishes the first part of the proof. For the second part, we apply the same Lemma on the $\mathbb{P}$ -local martingale $(M_t)_{t \geq 0}$. We obtain that $(M + \psi(u)\langle M, M \rangle)_{t \geq 0}$ is a local martingale under $\mathbb{P}^u$. Furthermore,

$$\begin{aligned} [M, M]_t &= [N, N]_t - [N, \Lambda]_t - [\Lambda, N]_t + [\Lambda, \Lambda]_t \\ &= [N, N]_t = \sum_{s \leq t} \Delta N_s \Delta N_s = N_t \end{aligned} \quad (3.10)$$

which is obtained because $[N, \Lambda]_t = [\Lambda, N]_t = [\Lambda, \Lambda]_t = 0$ for all t . Now, considering that $(\langle M, M \rangle_t)_{t \geq 0}$ is the compensator of $([M, M]_t)_{t \geq 0}$, it follows that

$$\langle M, M \rangle_t = \Lambda_t. \quad (3.11)$$

Therefore, with the help of Lemma 3.2, we obtain that $(M_t + \psi(u)N_t)_{t \geq 0} = (N_t - e^{-u}\Lambda_t)_{t \geq 0}$ is a $\mathbb{P}^u$ -local martingale, so that $(N_t)_{t \geq 0}$ has intensity $(e^{-u}\lambda_t)_{t \geq 0}$. Finally, we apply Lemma 3.2 on the local martingale $(M_t^P)_{t \geq 0} := (P_t - \rho^{-1}\Lambda_t)_{t \geq 0}$. It follows that $(M_t^P + \psi(u)\langle M^P, M \rangle_t)_{t \geq 0}$ is a local martingale under $\mathbb{P}^u$. Similarly to equation (3.10), we have

$$\begin{aligned} [M^P, M]_t &= [P, N]_t - [P, \Lambda]_t - \rho^{-1}[\Lambda, N]_t + \rho^{-1}[\Lambda, \Lambda]_t \\ &= [P, N]_t = \sum_{s \leq t} \Delta P_s \Delta N_s = P_t, \end{aligned} \quad (3.12)$$

so that $(M_t^P - \rho^{-1}\psi(u)\Lambda_t)_{t \geq 0} = (P_t - e^{-u}\rho^{-1}\Lambda_t)_{t \geq 0}$ is a $\mathbb{P}^u$ -local martingale. $\square$

From now, we know the dynamics of our processes under the new measure $\mathbb{P}^u$. The next tool we need will allow us to be able to compute the Laplace transform of the intensity. This tool is given in the following Proposition.

Proposition 3.4. *Let $v \in \mathbb{R}$ be fixed. There exist functions $A^v, B^v : \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq y\} \rightarrow \mathbb{R}$ such that for any $T \geq t \geq 0$, we have*

$$\mathbb{E}[\exp\{-v(\Lambda_T - \Lambda_t)ds\} | \mathcal{F}_t] = \exp\{A^v(t, T) + B^v(t, T)\lambda_t\}. \quad (3.13)$$

Moreover, these functions satisfy the ordinary differential equations

$$\frac{\partial B^v}{\partial t}(t, T) = \kappa B^v(t, T) + v - \frac{\sigma^2}{2}(B^v)^2(t, T) - \left[\left(1 - \frac{\eta B^v(t, T)}{\rho} \right)^{-1} - 1 \right] \quad (3.14)$$

$$\frac{\partial A^v}{\partial t}(t, T) = -\kappa \theta B^v(t, T) \quad (3.15)$$

with terminal conditions $A^v(T, T) = B^v(T, T) = 0$.

Proof. First note that the terminal conditions are clearly implied by equation (3.13). It suffices to show that there exist functions A, B such that the process

$$(S_t)_{t \in [0, T]} := (\exp \{A^v(t, T) + B^v(t, T)\lambda_t - v\Lambda_t\})_{t \in [0, T]} \quad (3.16)$$

is a martingale. Indeed if the process $(S_t)_{t \in [0, T]}$ is a martingale, we can write using the terminal conditions on A and B

$$\mathbb{E}[\exp \{-v\Lambda_T\} | \mathcal{F}_t] = \exp \{A^v(t, T) + B^v(t, T)\lambda_t - v\Lambda_t\} \quad (3.17)$$

and multiplying both sides by $\exp \{v\Lambda_t\}$ yields the desired result. The next step is to apply Ito's Lemma on the stochastic process $(S_t)_{t \in [0, T]} = (h(t, \lambda_t, \Lambda_t))_{t \in [0, T]}$, where $h : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \rightarrow \mathbb{R}^+$ is defined as $h(t, x, y) = \exp \{A^v(t, T) + B^v(t, T)x - vy\}$. By doing so we obtain

$$\begin{aligned} dS_t &= \left(\frac{\partial A^v}{\partial t}(t, T) + \lambda_t \frac{\partial B^v}{\partial t}(t, T) \right) h(t, \lambda_t, J_t) dt + \frac{\sigma^2}{2}(B^v)^2(t, T) \lambda_t h(t, \lambda_t, J_t) dt \\ &\quad + B^v(t, T) h(t, \lambda_t, J_t) \left(\kappa(\theta - \lambda_t) dt + \sigma \sqrt{\lambda_t} dW_t \right) - v h(t, \lambda_t, \Lambda_t) \lambda_t dt \\ &\quad + h(t, \lambda_t, J_t) (e^{\eta \xi B^v(t, T)} - 1) dN_t \\ &= h(t, \lambda_t, J_t) \left[\frac{\partial A^v}{\partial t}(t, T) + \lambda_t \frac{\partial B^v}{\partial t}(t, T) + B^v(t, T) \kappa(\theta - \lambda_t) - v \lambda_t \right. \\ &\quad \left. + \frac{\sigma^2}{2} \lambda_t (B^v)^2(t, T) + (\exp \{ \eta \xi B^v(t, T) \} - 1) \lambda_t \right] dt + B^v(t, T) h(t, \lambda_t, J_t) \sigma \sqrt{\lambda_t} dW_t \\ &\quad + h(t, \lambda_t, J_t) (\exp \{ \eta \xi B^v(t, T) \} - 1) (dN_t - \lambda_t dt). \end{aligned} \quad (3.18)$$

Using the fact that $\xi \sim \text{Exp}(\rho)$, $\mathbb{E}[\exp \{ \eta \xi B^v(t, T) \}] = \left(1 - \frac{\eta B^v(t, T)}{\rho} \right)^{-1}$ provided that $\rho > \eta B^v(t, T)$. Since $(S_t)_{t \in [0, T]}$ is required to be a martingale, the drift term should average zero. Therefore, by rearranging the terms in the drift and dividing by $h(t, \lambda_t, J_t)$, we obtain

$$\begin{aligned} 0 = \lambda_t &\left[\frac{\partial B^v}{\partial t}(t, T) - \kappa B^v(t, T) - v + \frac{\sigma^2}{2}(B^v)^2(t, T) + \left(1 - \frac{\eta B^v(t, T)}{\rho} \right)^{-1} - 1 \right] \\ &+ \frac{\partial A^v}{\partial t}(t, T) + \kappa \theta B^v(t, T). \end{aligned} \quad (3.19)$$

This equation must be valid for any value of the random variable λ_t , which permits to obtain the announced result. $\square$

By combining Propositions 3.3 and 3.4, we find that

$$\begin{aligned}\mathcal{L}^u(v, t, T) &= \mathbb{E}^u [\exp \{-v(\Lambda_T - \Lambda_t)\} | \mathcal{F}_t] \\ &= \exp \{A^v(t, T) + B^v(t, T)\lambda_t\}\end{aligned}\tag{3.20}$$

where A^v, B^v satisfy

$$\begin{aligned}\frac{\partial B^v}{\partial t}(t, T) &= \kappa B^v(t, T) + \psi(u) - \frac{\sigma^2}{2}(B^v)^2(t, T) \\ &\quad - e^{-u} \left[\left(1 - \frac{\eta B^v(t, T)}{\rho}\right)^{-1} - 1 \right]\end{aligned}\tag{3.21}$$

$$\frac{\partial A^v}{\partial t}(t, T) = -\kappa \theta B^v(t, T)\tag{3.22}$$

with terminal conditions $A^v(T, T) = B^v(T, T) = 0$. Note that equations (3.21) and (3.14) differ by the fact that the last term of equation (3.21) is multiplied by e^{-u} . This is the consequence of changing the measure from $\mathbb{P}$ to $\mathbb{P}^u$: The process $(P_t)_{t \geq 0}$ has jump intensity $(e^{-u\lambda_t})_{t \geq 0}$ instead of $(\lambda_t)_{t \geq 0}$, as shown in Proposition 3.3. Those two equations (3.21) and (3.22) can easily be solved numerically. From Proposition 3.1, it follows that

$$\begin{aligned}\mathbb{E}[\exp\{-u(N_T - N_t)\} | \mathcal{F}_t] &= \mathcal{L}^u(\psi(u), t, T) \\ &= \exp\{A^{\psi(u)}(t, T) + B^{\psi(u)}(t, T)\lambda_t\}.\end{aligned}\tag{3.23}$$

where $A^{\psi(u)}, B^{\psi(u)}$ satisfy equations (3.21) and (3.22) with $v = \psi(u)$. This shows that the Laplace transforms of the process $(N_t)_{t \geq 0}$ can be found by numerically solving a system of two ordinary differential equations.

4 Distributions of the Jump Processes

In this section, we use the results of the previous section to find the distribution of the random variables N_t , with $t \geq 0$, i.e. $\mathbb{P}(N_t = k)$ for $k = 0, 1, \dots$. We use the probability generating function of N_t , which is defined as the mapping $g_t : x \mapsto \mathbb{E}[x^{N_t}]$. Recall the following property of probability generating functions

$$\left. \frac{d^k g_t(x)}{dx^k} \right|_{x=0} = \mathbb{P}(N_t = k)\tag{4.1}$$

and observe the following obvious link between the probability generating function and the Laplace transform

$$g_t(x) = \mathbb{E}[\exp\{-(-\ln(x))N_t\}].\tag{4.2}$$

Thanks to this relation, we immediately conclude from equation (3.23) that

$$\mathbb{P}(N_t = k) = \lim_{x \rightarrow 0} \frac{\partial^k \exp\{A^{1-x}(0, t) + B^{1-x}(0, t)\lambda_0\}}{\partial x^k}.\tag{4.3}$$

The right-hand side of this equation can be approximated numerically. It is also possible to compute the moments of N_t numerically by using

$$\begin{aligned}
\mathbb{E}[(N_T - N_t)^k | \mathcal{F}_t] &= (-1)^k \frac{\partial^k}{\partial u^k} \mathbb{E}[\exp\{-u(N_T - N_t)\} | \mathcal{F}_t] \Big|_{u=0} \\
&= (-1)^k \frac{\partial^k}{\partial u^k} \exp\{A^{\psi(u)}(t, T) + B^{\psi(u)}(t, T)\lambda_t\} \Big|_{u=0},
\end{aligned} \tag{4.4}$$

which can easily be approximated by finite differences. This will be useful in the numerical illustration section.

5 Time-Consistent Evaluations

This section extends the approach of Pelsser and Ghalehjooghi to the framework we introduced. A time-consistent evaluation principle is an evaluation principle that has the following property : if it is known today that, at a future date, a risk X will be more expensive than another risk Y , then the risk X should already be more expensive today. An equivalent characterisation of time-consistency is the following: let $(\Pi_t)_{t \geq 0}$ a dynamic evaluation. Such an evaluation is time-consistent if and only if

$$\Pi_t(X) = \Pi_t(\Pi_{t+\Delta}(X)) \tag{5.1}$$

for any risk X and any $t, \Delta \geq 0$. This equivalent characterisation is the one we will use in this section.

In the following, we assume that the companies we work are small or medium companies (SME) for which no credit derivatives are available. These risks being not hedgeable by financial instruments, the market is incomplete and a unique risk-neutral measure does not exist. To manage this uncertainty about the risk neutral measure, we use actuarial valuation methods and work under the real probability measure $\mathbb{P}$.

5.1 Variance Principle

For a risk X expressed in a certain monetary unit, the static variance principle is computed as

$$\mathbb{E}[X] + \alpha \text{Var}(X). \tag{5.2}$$

Note that in equation (5.2), the expectation $\mathbb{E}[X]$ is expressed in the monetary unit, whereas the term $\text{Var}(X)$ is expressed in squared monetary units. Therefore the parameter α should be replaced by α/L , where $L > 0$ is a reference amount expressed in the monetary unit. Equivalently, this means that α is a quantity expressed $1/m_u$, where m_u is the monetary unit.

Let Π^v be a function that is the time-consistent valuation function derived from the variance principle of the trade credit risk insurance portfolio. This function depends on (t, λ_t) . In the case of the variance principle, the time-consistency requirement (5.1) writes

$$\Pi^v(t, \lambda_t) = \mathbb{E} \left[e^{-r\varepsilon} \Pi^v(t + \varepsilon, \lambda_{t+\varepsilon}) | \mathcal{F}_t \right] + \alpha \text{Var} \left(e^{-r\varepsilon} \Pi^v(t + \varepsilon, \lambda_{t+\varepsilon}) | \mathcal{F}_t \right). \tag{5.3}$$

This means that we should have

$$\lim_{\varepsilon \downarrow 0} \frac{\mathbb{E}[e^{-r\varepsilon} \Pi^v(t + \varepsilon, \lambda_{t+\varepsilon}) | \mathcal{F}_t] + \alpha \text{Var}(e^{-r\varepsilon} \Pi^v(t + \varepsilon, \lambda_{t+\varepsilon}) | \mathcal{F}_t) - \Pi^v(t, \lambda_t)}{\varepsilon} = 0, \quad (5.4)$$

for any $t, \varepsilon \geq 0, t + \varepsilon \in [0, T]$, where $T > 0$ is the maturity of the contract. We will separately look at the expectation term and the variance term at the right end side of equation (5.3). For the expectation part, using Ito's lemma for semimartingales (see e.g. chapter 2 of Protter (2003) or chapter 6 section 7 of Rogers and Williams (2000)) yields

$$\begin{aligned} \lim_{\varepsilon \downarrow 0} \mathbb{E} \left[\frac{\Pi^v(t + \varepsilon, \lambda_{t+\varepsilon})}{\varepsilon} | \mathcal{F}_t \right] &= \Pi^v(t, \lambda_t) + \Pi_t^v(t, \lambda_t) + \kappa(\theta - \lambda_t) \Pi_\lambda^v(t, \lambda_t) \\ &+ \frac{\sigma^2 \lambda_t}{2} \Pi_{\lambda\lambda}^v(t, \lambda_t) + \mathbb{E}[\xi + \Pi^v(t, \lambda_t + \eta\xi) - \Pi^v(t, \lambda_t)] \lambda_t \end{aligned} \quad (5.5)$$

where Π_t^v, Π_λ^v and $\Pi_{\lambda\lambda}^v$ are shorthands for $\frac{\partial \Pi^v}{\partial t}, \frac{\partial \Pi^v}{\partial \lambda}$ and $\frac{\partial^2 \Pi^v}{\partial \lambda^2}$ respectively and $\xi \sim \text{Exp}(\rho)$. Note that the result of equation (5.5) is the infinitesimal generator of Π^v , that we will denote by $\mathcal{A}\Pi^v$. For the variance part, we will use infinitesimal generators. First, let us recall the following elementary identity for variance

$$\text{Var}(Y_{t+\varepsilon} | \mathcal{F}_t) = \mathbb{E}[Y_{t+\varepsilon}^2 | \mathcal{F}_t] - Y_t^2 - (\mathbb{E}[Y_{t+\varepsilon} | \mathcal{F}_t] - Y_t)^2 - 2Y_t(\mathbb{E}[Y_{t+\varepsilon} | \mathcal{F}_t] - Y_t) \quad (5.6)$$

where $(Y_t)_{t \geq 0}$ is used instead of $(\Pi^v(t, \lambda_t))_{t \geq 0}$ for shorter notations. Dividing equation (5.6) by ε on both sides and letting $\varepsilon \downarrow 0$ leads to

$$\lim_{\varepsilon \rightarrow 0} \frac{\text{Var}(\Pi^v(t + \varepsilon, \lambda_{t+\varepsilon}) | \mathcal{F}_t)}{\varepsilon} = \mathcal{A}(\Pi^{v2}) - 2\Pi^v(\mathcal{A}\Pi^v) \quad (5.7)$$

where Π^{v2} means $(\Pi^v)^2$ and is used to avoid too many parentheses. We also remind that Π^v is of course a function of (t, λ_t) . The use of Ito's lemma for semimartingales and the chain rule allows us to get the following expression for the infinitesimal generator $\mathcal{A}(\Pi^{v2})$:

$$\begin{aligned} \mathcal{A}(\Pi^{v2}) &= 2\Pi^v \Pi_t^v + 2\Pi^v \Pi_\lambda^v \kappa(\theta - \lambda_t) + \frac{1}{2} \sigma^2 \lambda_t (2(\Pi_\lambda^v)^2 + 2\Pi^v \Pi_{\lambda\lambda}^v) \\ &+ (\mathbb{E}[(\xi + \Pi^v(t, \lambda_t + \eta\xi))^2] - \Pi^{v2}) \lambda_t \end{aligned} \quad (5.8)$$

where, again, $\xi \sim \text{Exp}(\rho)$. Note that in equation (5.8), $\Pi^v, \Pi_t^v, \Pi_\lambda^v$ and $\Pi_{\lambda\lambda}^v$ are functions of (t, λ_t) . Then, starting from equation (5.4), using that $e^{-r\varepsilon} = 1 - r\varepsilon + o(\varepsilon)$ and using our computations regarding the expectation and the variance, we obtain a PIDE for the time-consistent variance price:

$$\begin{aligned} r\Pi^v &= \Pi_t^v + \kappa(\theta - \lambda_t) \Pi_\lambda^v + \frac{1}{2} \sigma^2 \lambda_t [\Pi_{\lambda\lambda}^v + 2\alpha(\Pi_\lambda^v)^2] \\ &+ \mathbb{E}[\xi + \Pi^v(t, \lambda_t + \eta\xi) - \Pi^v(t, \lambda_t)] \lambda_t + \alpha \mathbb{E}[(\xi + \Pi^v(t, \lambda_t + \eta\xi) - \Pi^v(t, \lambda_t))^2] \lambda_t \end{aligned} \quad (5.9)$$

Given the computations we already performed, deriving a PIDE for the time-consistent standard-deviation principle is an easy matter.

5.2 Standard-Deviation Principle

For a risk X expressed in a certain monetary unit, the static variance principle is computed as

$$\mathbb{E}[X] + \beta \sqrt{\mathbb{V}\text{ar}(X)}. \quad (5.10)$$

Let Π^s be a function that is the time-consistent valuation derived from the standard deviation principle. Again, this function depends on (t, λ_t) . For any $t + \varepsilon \in [0, T]$, we should have

$$\Pi^s(t, \lambda_t) = \mathbb{E} \left[e^{-r\varepsilon} \Pi^s(t + \varepsilon, \lambda_{t+\varepsilon}) \mid \mathcal{F}_t \right] + \beta \sqrt{\mathbb{V}\text{ar} \left(e^{-r\varepsilon} \Pi^s(t + \varepsilon, \lambda_{t+\varepsilon}) \mid \mathcal{F}_t \right)} \quad (5.11)$$

for a fixed constant $\beta > 0$. As noted in Pelsser and Ghalehjooghi (2016), the terms $\mathbb{E}[X]$ and $\sqrt{\mathbb{V}\text{ar}(X)}$ have different time scales. That is, the expectation term scales linearly with the time whereas the standard deviation term scales with the square root of the time. It follows that the β parameter must be replaced by $\sqrt{\varepsilon}\beta$ in equation (5.11), which yields

$$\Pi^s(t, \lambda_t) = \mathbb{E} \left[e^{-r\varepsilon} \Pi^s(t + \varepsilon, \lambda_{t+\varepsilon}) \mid \mathcal{F}_t \right] + \beta \sqrt{\varepsilon} \sqrt{\mathbb{V}\text{ar} \left(e^{-r\varepsilon} \Pi^s(t + \varepsilon, \lambda_{t+\varepsilon}) \mid \mathcal{F}_t \right)}. \quad (5.12)$$

As in the previous section, we divide by ε and let ε tends to 0. Using the results we obtained for the variance principle, we have

$$\begin{aligned} r\Pi^s(t, \lambda_t) &= \Pi_t^s(t, \lambda_t) + \kappa(\theta - \lambda_t)\Pi_\lambda^s(t, \lambda_t) + \frac{\sigma^2 \lambda_t}{2} \Pi_{\lambda\lambda}^s(t, \lambda_t) \\ &+ \mathbb{E} [\xi + \Pi^s(t, \lambda_t + \eta\xi) - \Pi^s(t, \lambda_t)] \lambda_t + \beta \lim_{\varepsilon \downarrow 0} \sqrt{\frac{\mathbb{V}\text{ar}(\Pi^s(t + \varepsilon, \lambda_{t+\varepsilon}) \mid \mathcal{F}_t)}{\varepsilon}}. \end{aligned} \quad (5.13)$$

With the help of equations (5.7) and (5.8) we can compute the limit of the standard deviation term and arrive to the following PIDE

$$\begin{aligned} r\Pi^s(t, \lambda_t) &= \Pi_t^s(t, \lambda_t) + \kappa(\theta - \lambda_t)\Pi_\lambda^s(t, \lambda_t) + \frac{\sigma^2 \lambda_t}{2} \Pi_{\lambda\lambda}^s(t, \lambda_t) \\ &+ \mathbb{E} [\xi + \Pi^s(t, \lambda_{t-} + \eta\xi) - \Pi^s(t, \lambda_{t-})] \lambda_t \\ &+ \beta \left[\mathbb{E} [(\xi + \Pi^s(t, \lambda_t + \eta\xi) - \Pi^s)^2] \lambda_t + \sigma^2 \lambda_t (\Pi_\lambda)^2 \right]^{1/2}. \end{aligned} \quad (5.14)$$

In the next subsection, we derive the time-consistent counterpart of an evaluation known as the mean value principle.

5.3 Mean Value Principle

The static mean value principle is given by

$$\Pi(X) = v^{-1} \left(\mathbb{E}[v(X)] \right) \quad (5.15)$$

where v can be any convex increasing function. The time-consistent evaluation operator Π^m derived from the static Mean Value principle should satisfy

$$\Pi^m(t, \lambda_t) = v^{-1}(\mathbb{E}[v(e^{-r\varepsilon}\Pi^m(t+\varepsilon, \lambda_{t+\varepsilon}))|\mathcal{F}_t]) \quad (5.16)$$

for any $0 \leq t + \varepsilon \leq T$, with $t, \varepsilon \geq 0$. It implies that

$$\lim_{\varepsilon \downarrow 0} \frac{\mathbb{E} \left[v \left(\frac{\Pi^m(t+\varepsilon, \lambda_{t+\varepsilon})}{e^{r\varepsilon}} \right) | \mathcal{F}_t \right] - v(\Pi^m(t, \lambda_t))}{\varepsilon} = 0 \quad (5.17)$$

for all $t \geq 0$. Using this condition and Ito's Lemma on the process $\left(v \left(\frac{\Pi^m(t+\varepsilon, \lambda_{t+\varepsilon})}{e^{r\varepsilon}} \right) \right)_{\varepsilon \geq 0}$, we will derive a partial differential equation for the time-consistent price. In order to apply Ito's Lemma, we compute the partial derivatives of the function $(\varepsilon, \lambda) \mapsto f(\varepsilon, \lambda) := v \left(\frac{\Pi^m(t+\varepsilon, \lambda)}{e^{r\varepsilon}} \right)$ by the use of the chain rule. This leads to

$$\begin{aligned} \frac{\partial f}{\partial \varepsilon}(s, \lambda_{t+s}) &= \\ v' \left(\frac{\Pi^m(t+s, \lambda_{t+s})}{e^{rs}} \right) \frac{\Pi_t^m(t+s, \lambda_{t+s}) - r\Pi^m(t+s, \lambda_{t+s})}{e^{rs}} \end{aligned} \quad (5.18)$$

$$\frac{\partial f}{\partial \lambda}(s, \lambda_{t+s}) = v' \left(\frac{\Pi^m(t+s, \lambda_{t+s})}{e^{rs}} \right) \frac{\Pi_\lambda^m(t+s, \lambda_{t+s})}{e^{rs}} \quad (5.19)$$

$$\begin{aligned} \frac{\partial^2 f}{\partial \lambda^2}(s, \lambda_{t+s}) &= \left(\frac{\Pi_\lambda^m(t+s, \lambda_{t+s})}{e^{rs}} \right)^2 v'' \left(\frac{\Pi^m(t+s, \lambda_{t+s})}{e^{rs}} \right) \\ &\quad + v \left(\frac{\Pi^m(t+s, \lambda_{t+s})}{e^{rs}} \right) \frac{\Pi_{\lambda\lambda}^m(t+s, \lambda_{t+s})}{e^{rs}}. \end{aligned} \quad (5.20)$$

As before, from Ito's Lemma, we find

$$\begin{aligned} 0 &= \frac{\sigma^2}{2} \lambda_t (\Pi_\lambda^m(t, \lambda_t))^2 v''(\Pi^m(t, \lambda_t)) + \mathbb{E}[v(\xi + \Pi^m(t, \lambda_t + \eta\xi)) - v(\Pi^m(t, \lambda_t))] \lambda_t \\ &\quad + v'(\Pi^m(t, \lambda_t)) \left[\Pi_t^m(t, \lambda_t) - r\Pi^m(t, \lambda_t) + \Pi_\lambda^m(t, \lambda_t)\kappa(\theta - \lambda_t) + \frac{\sigma^2}{2} \lambda_t \Pi_{\lambda\lambda}^m(t, \lambda_t) \right]. \end{aligned} \quad (5.21)$$

which can be used as a PIDE for the time-consistent price.

5.4 The Exponential Principle

The exponential principle is a particular case of the mean value principle. It is obtained by defining the function v by $v(x) = e^{\gamma^{-1}x}$, $\gamma > 0$. A PIDE for the time-consistent exponential principle is obtained by replacing v , $v'(x) = \gamma^{-1}e^{\gamma^{-1}x}$ and $v''(x) = \gamma^{-2}e^{\gamma^{-1}x}$ in equation (5.21). We obtain

$$\begin{aligned} 0 &= \frac{\sigma^2}{2} \lambda_t (\Pi_\lambda^m(t, \lambda_t))^2 \gamma^{-2} e^{\gamma^{-1}\Pi^m(t, \lambda_t)} \\ &\quad + \frac{\Pi_t^m(t, \lambda_t) - r\Pi^m(t, \lambda_t) + \Pi_\lambda^m(t, \lambda_t)\kappa(\theta - \lambda_t) + \frac{\sigma^2}{2} \lambda_t \Pi_{\lambda\lambda}^m(t, \lambda_t)}{\gamma e^{-\gamma^{-1}\Pi^m(t, \lambda_t)}} \\ &\quad + \mathbb{E} \left[e^{\gamma^{-1}(\xi + \Pi^m(t, \lambda_t + \eta\xi))} - e^{\gamma^{-1}\Pi^m(t, \lambda_t)} \right] \lambda_t. \end{aligned} \quad (5.22)$$

Since $e^{\gamma^{-1}\Pi^m(t, \lambda_t)} > 0$, we can divide both sides of equation (5.22) by $e^{\gamma^{-1}\Pi^m(t, \lambda_t)}$, so that we get

$$0 = \frac{\Pi_t^m(t, \lambda_t) - r\Pi^m(t, \lambda_t) + \Pi_\lambda^m(t, \lambda_t)\kappa(\theta - \lambda_t) + \frac{\sigma^2}{2}\lambda_t\Pi_{\lambda\lambda}^m(t, \lambda_t)}{\gamma} + \mathbb{E} \left[e^{\gamma^{-1}(\xi + \Pi^m(t, \lambda_t + \eta\xi) - \Pi^m(t, \lambda_t))} - 1 \right] \lambda_t + \frac{\sigma^2}{2}\lambda_t (\Pi_\lambda^m(t, \lambda_t))^2 \gamma^{-2}. \quad (5.23)$$

This plays a role in the numerical methods part. Indeed, the exponential of the evaluation $e^{\gamma^{-1}\Pi^m(t, \lambda_t)}$ can lead to huge numbers, causing problems in the numerical solving of the PDE. It is therefore important to get rid of it.

6 Numerical Solutions of PIDE

In this section, we explain the numerical method that allows us to solve the 3 PIDE derived above. To this end, we start by explaining the general setting of the method. Afterwards we address the problem of solving of each PIDE specifically.

We work with a finite difference method. There are two variables to discretize: the intensity λ and the time t . The finite difference method is performed on a finite grid of values for the two variables. The limits for the time will be taken to be 0 and T for the time, T being the maturity of the contract whereas the limits for λ are taken to be 0 and $\lambda_{\max} > 0$. $\lambda_{\max}$ will be fixed so that it is large enough to be attained by $(\lambda_t)_{t \geq 0}$ with a very low probability. We define n_λ and n_t , strictly positive integers, to be the number of steps in the grid for λ and t respectively. These integers imply mesh sizes for both discretized variables, we will denote those mesh sizes by $\Delta\lambda := \lambda_{\max}/n_\lambda$ and $\Delta t := T/n_t$. The grid for the time is $(t_0, t_1, \dots, t_{n_t})$, where $t_i = i\Delta t$ for each i . The grid for the intensity λ_t is $(\lambda_0, \lambda_1, \dots, \lambda_{n_\lambda})$ where $\lambda_j = j\Delta\lambda$ for each j .

The approximated values of the function Π on the grid will be denoted by $p_{i,j} \approx \Pi(t_i, \lambda_j)$. The matrix that contains all approximated values $p_{i,j}$ will be denoted by $P = (p_{i,j})_{\substack{i=0, \dots, n_t \\ j=0, \dots, n_\lambda}}$. The approximations of partial derivatives will be denoted as

$$\partial_t p_{i,j} \approx \frac{\partial \Pi}{\partial t}(t_i, \lambda_j) \quad \partial_\lambda p_{i,j} \approx \frac{\partial \Pi}{\partial \lambda}(t_i, \lambda_j) \quad \partial_{\lambda\lambda} p_{i,j} \approx \frac{\partial^2 \Pi}{\partial \lambda^2}(t_i, \lambda_j) \quad (6.1)$$

so that $\partial_t P = (\partial_t p_{i,j})_{\substack{i=1, \dots, n_t+1 \\ j=1, \dots, n_\lambda+1}}$, $\partial_\lambda P = (\partial_\lambda p_{i,j})_{\substack{i=1, \dots, n_t+1 \\ j=1, \dots, n_\lambda+1}}$ and $\partial_{\lambda\lambda} P = (\partial_{\lambda\lambda} p_{i,j})_{\substack{i=1, \dots, n_t+1 \\ j=1, \dots, n_\lambda+1}}$ are matrices that approximate the needed partial derivatives of Π .

The only boundary condition in this problem is $\Pi(T, \lambda) = 0$ for any possible intensity λ . Indeed, at time T the contract is over. Consequently, we set $p_{n_t, j} = 0$ for all $j \in \{0, 1, \dots, n_\lambda\}$. All the other $p_{i,j}$ will be found with a finite difference method based on the PIDE's. The approximations of the partial derivatives with respect to time are, for all $i \in \{0, 1, \dots, n_t - 1\}$ and $j \in \{0, 1, \dots, n_\lambda\}$, given by

$$\partial_t p_{i,j} = \frac{p_{i+1,j} - p_{i,j}}{\Delta t}. \quad (6.2)$$

Concerning the approximations of the partial derivatives with respect to λ , we use the following formulas

$$\partial_\lambda p_{i,j} = \begin{cases} \theta_{\text{or}} \frac{p_{i,j+2}-p_{i,j}}{2\Delta t} + (1-\theta_{\text{or}}) \frac{p_{i+1,j+2}-p_{i+1,j}}{2\Delta t} & \text{if } j = 0 \\ \theta_{\text{or}} \frac{p_{i,j}-p_{i,j-2}}{2\Delta\lambda} + (1-\theta_{\text{or}}) \frac{p_{i+1,j}-p_{i+1,j-2}}{2\Delta\lambda} & \text{if } j = n_\lambda \\ \theta_{\text{or}} \frac{p_{i,j+1}-p_{i,j-1}}{2\Delta\lambda} + (1-\theta_{\text{or}}) \frac{p_{i+1,j+1}-p_{i+1,j-1}}{2\Delta\lambda} & \text{if } j \notin \{0, n_\lambda\} \end{cases} \quad (6.3)$$

$$\partial_{\lambda\lambda} p_{i,j} = \begin{cases} \theta_{\text{or}} \frac{p_{i,j+2}-2p_{i,j+1}+p_{i,j}}{(\Delta\lambda)^2} + (1-\theta_{\text{or}}) \frac{p_{i+1,j+2}-2p_{i+1,j+1}+p_{i+1,j}}{(\Delta\lambda)^2} & \text{if } j = 0 \\ \theta_{\text{or}} \frac{p_{i,j}-2p_{i,j-1}+p_{i,j-2}}{(\Delta\lambda)^2} + (1-\theta_{\text{or}}) \frac{p_{i+1,j}-2p_{i+1,j-1}+p_{i+1,j-2}}{(\Delta\lambda)^2} & \text{if } j = n_\lambda \\ \theta_{\text{or}} \frac{p_{i,j+1}-2p_{i,j}+p_{i,j-1}}{(\Delta\lambda)^2} + (1-\theta_{\text{or}}) \frac{p_{i+1,j+1}-2p_{i+1,j}+p_{i+1,j-1}}{(\Delta\lambda)^2} & \text{if } j \notin \{0, n_\lambda\} \end{cases} \quad (6.4)$$

where $\theta_{\text{or}} \in [0, 1]$ is an over relaxation parameter (the subscript "or" stands for over relaxation). This parameter was fixed to 0.5 in our computations. Our approximations of partial derivatives with respect to λ present a special feature. At equations (6.3) and (6.4) in the cases $j \in \{0, n_\lambda\}$, the approximations are modified so that we do not fall outside of the grid. This trick allows us to solve the PIDE's in spite of the lack of boundary conditions.

6.1 Variance Principle

In this subsection, we focus on solving PIDE (5.9) numerically. The partial derivatives will be approached according to equations (6.2), (6.3) and (6.4). In addition to partial derivatives with respect to t and λ , this equation also contains two terms that correspond to the expected impact of a default on the price, that is $\mathbb{E}[\Pi^v(t, \lambda_t + \eta\xi) - \Pi^v(t, \lambda_t)]$ and $\mathbb{E}[(\xi + \Pi^v(t, \lambda_t + \eta\xi) - \Pi^v(t, \lambda_t))^2]$. These terms will be approximated by discretization of the random variable ξ , which is equivalent to Riemann sum approximations. More precisely the first term is approximated by

$$\begin{aligned} & \mathbb{E}[\xi + \Pi^v(t_i, \lambda_j + \eta\xi) - \Pi^v(t_i, \lambda_j)] \\ & \approx \rho^{-1} + \sum_{k=j+1}^{n_\lambda} p_{i,k}^v P\left(\frac{(k-j-1)}{\eta}\Delta\lambda \leq \xi < \frac{(k-j)}{\eta}\Delta\lambda\right) \\ & + p_{i,n_\lambda}^v P\left(\xi \geq \frac{n_\lambda-j}{\eta}\Delta\lambda\right) - p_{i,j}^v. \end{aligned} \quad (6.5)$$

Before approaching the second term, it is rewritten as

$$\begin{aligned} & \mathbb{E}[(\xi + \Pi^v(t_i, t_j + \eta\xi) - \Pi^v(t_i, \lambda_j))^2] \\ & \approx \mathbb{E}[\xi^2] + \mathbb{E}[\Pi^{v2}(t_i, \lambda_j + \eta\xi)] + 2\mathbb{E}[\xi\Pi^v(t_i, \lambda_j + \eta\xi)] \\ & + \Pi^{v2}(t_i, \lambda_j) - 2\Pi^v(t_i, \lambda_j)\mathbb{E}[\xi] - 2\Pi^v(t_i, \lambda_j)\mathbb{E}[\Pi^v(t_i, t_j + \eta\xi)]. \end{aligned} \quad (6.6)$$

In equation (6.6), the term $\mathbb{E}[\Pi^{v2}(t_i, t_j + \eta\xi)]$ is approximated by

$$\begin{aligned} & \mathbb{E}[\Pi^{v2}(t_i, t_j + \eta\xi)] \\ & \approx \sum_{k=j+1}^{n_\lambda+1} (p_{i,k})^2 P\left(\frac{(k-j-1)}{\eta}\Delta\lambda \leq \xi < \frac{(k-j)}{\eta}\Delta\lambda\right) \\ & + (p_{i,n_\lambda+1})^2 P\left(\xi \geq \frac{n_\lambda+1-j}{\eta}\Delta\lambda\right), \end{aligned} \quad (6.7)$$

whereas $\mathbb{E}[\xi\Pi^v(t_i, t_j + \eta\xi)]$ is approximated by

$$\begin{aligned}
& \mathbb{E}[\xi \Pi^v(t_i, t_j + \eta \xi)] \\
& \approx \sum_{k=j+1}^{n_\lambda+1} \frac{(k-j)}{\eta} \Delta_\lambda p_{i,k} P \left(\frac{(k-j-1)}{\eta} \Delta_\lambda \leq \xi < \frac{(k-j)}{\eta} \Delta_\lambda \right) \\
& + \rho^{-1} p_{i,n_\lambda+1} P \left(\xi \geq \frac{n_\lambda+1-j}{\eta} \Delta_\lambda \right).
\end{aligned} \tag{6.8}$$

Since ξ is exponentially distributed, we have $\mathbb{E}[\xi] = \rho^{-1}$ and $\mathbb{E}[\xi^2] = 2\rho^{-2}$. Let $p_{i,j}^v \approx \Pi^v(t_i, \lambda_j)$ and P^v the matrix that contains all the $p_{i,j}^v$. As already pointed out, the last column of P^v , denoted by¹ $P_{\star n_t}^v$ is filled with zeros. We can now replace the terms of PIDE (5.9) by their approximations described above. For each $k \in \{0, 1, \dots, n_t - 1\}$, we can find $P_{\star k}^v$ from $P_{\star(k+1)}^v$ by solving numerically a system of equations.

Solving numerically PIDE (5.14) is done in the same way. That is, the method and the approximations are the same, but the system of equations that allows to find $P_{\star k}^s$ from $P_{\star(k+1)}^s$ is modified so that it correspond to PIDE (5.14).

6.2 Exponential Principle

This subsection focus on the specificity of the numerical resolution of PIDE (5.23). Appart from the partial derivatives that are again approached by the formulas (6.2), (6.3) and (6.4), we need to approximate $\mathbb{E} \left[e^{\gamma^{-1}(\xi + \Pi^m(t_i, \lambda_j + \eta \xi))} \right]$. This is done with a discretization of the random variable ξ :

$$\begin{aligned}
& \mathbb{E} \left[e^{\gamma^{-1}(\xi + \Pi^m(t_i, \lambda_j + \eta \xi))} \right] \\
& \approx \sum_{k=j+1}^{n_\lambda} \exp \left\{ \gamma^{-1} \left(\frac{(k-j)}{\eta} \Delta_\lambda + p_{i,k}^m \right) \right\} P \left(\frac{(k-j-1)}{\eta} \Delta_\lambda \leq \xi < \frac{(k-j)}{\eta} \Delta_\lambda \right) \\
& + \exp \left\{ \gamma^{-1} \left(\frac{n_\lambda-j}{\eta} \Delta_\lambda + p_{i,n_\lambda}^m \right) \right\} P \left(\xi \geq \frac{n_\lambda-j}{\eta} \Delta_\lambda \right).
\end{aligned} \tag{6.9}$$

We can now proceed inductively as before: $P_{\star k}^m$ is deduced from $P_{\star(k+1)}^m$ by solving numerically a system of equations.

6.3 Numerical Illustration

6.3.1 Convergence

In this subsection, we present and discuss the numerical results obtained by solving the PIDE's with the method described in the previous section. The values of the parameters we fixed for the computations are given in Table 6.1. Recall that the parameters α , β and γ are parameters that are specific to the variance, standard-deviation and exponential principle respectively. Note that in the section devoted to the variance principle, we pointed out that the variance term should be divided by a reference amount L . This reference amount L is here chosen to be the average default cost, ρ^{-1} , as indicated in Table 6.1 (parameter α).

¹that is actually the column number $n_t + 1$ of P^v , since there are $n_t + 1$ points in the time grid.

θ	λ_0	κ	σ	η	ρ	r	T	β	α	γ
0.1	0.1	0.24	0.05	0.0001	0.001	0	10	0.1	0.1ρ	10000

Table 6.1: Parameters Used for the Computations

The maturity is fixed to $T = 10$ years. We start by checking the convergence and stability of the results. The convergence with respect to Δ_t is fast and the results are not modified by the choice of $\lambda_{\max}$, as we can observe on Figure 6.1. In this Figure, we represent the evolution of the time-consistent price for the 3 principles with an intensity kept constant and equal to $\lambda_0 = \theta = 0.1$. Note that in order to obtain Figure 6.1, Δ_λ was fixed to 0.0025.

Figure 6.1: Evolution of the time-consistent variance, standard-deviation (SD) and exponential principles.

Figure 6.2 shows the convergence with respect to Δ_λ , which is visibly slower than the convergence with respect to Δ_t .

Now that we have checked the convergence of our method, we will use it to assess the impact of time-consistency and self-excitation on the price.

6.3.2 Joint Impact of Time-Consistency and Self-Excitation

The goal of this subsection is to compare the results of the usual time inconsistent evaluations with their time consistent counterparts. This is done in two different settings: With and without contagion. The time inconsistent evaluations represented in the upcoming graphs correspond to the functions

$$t \in [0, T] \mapsto \mathbb{E}[P_T | \mathcal{F}_t] + \alpha \text{Var}(P_T | \mathcal{F}_t), \quad (6.10)$$

$$t \in [0, T] \mapsto \mathbb{E}[P_T | \mathcal{F}_t] + \beta \sqrt{T - t} \sqrt{\text{Var}(P_T | \mathcal{F}_t)} \quad (6.11)$$

and

$$t \in [0, T] \mapsto \gamma \ln (\mathbb{E}[\exp\{\gamma^{-1} P_T\} | \mathcal{F}_t]) \quad (6.12)$$

Figure 6.2: The convergence is slower for Δ_λ , in particular for the exponential principle.

respectively for the variance, standard-deviation and exponential principles. Note that in equation (6.11), the standard-deviation term is multiplied by $\sqrt{T-t}$ so that the expectation term and the standard-deviation term are on the same time-scale, see the discussion at the beginning section 5.2. The prices in the time inconsistent cases are obtained numerically with the help of equation (4.4), i.e. via the Laplace transform of $(N_t)_{t \geq 0}$. For the variance and standard-deviation principle, we need to evaluate $\mathbb{E}[P_t]$, $\text{Var}(P_t)$. This is done with the following computations

$$\mathbb{E}[P_t] = \mathbb{E}[\mathbb{E}[P_t|N_t]] = \rho^{-1}\mathbb{E}[N_t], \quad (6.13)$$

$$\begin{aligned} \text{Var}(P_t) &= \text{Var}(\mathbb{E}[P_t|N_t]) + \mathbb{E}[\text{Var}(P_t|N_t)] \\ &= \text{Var}(\rho^{-1}N_t) + \mathbb{E}[N_t \text{Var}(\xi)] \\ &= \rho^{-2}\text{Var}(N_t) + \mathbb{E}[N_t]\text{Var}(\xi). \end{aligned} \quad (6.14)$$

From there, the moments of $(P_t)_{t \geq 0}$ are obtained by applying finite difference approximations on the Laplace transform of $(N_t)_{t \geq 0}$. For the exponential principle, we need to compute $\mathbb{E}[\exp\{\gamma^{-1}P_t\}]$, which is again done with the help of the Laplace transform of $(N_t)_{t \geq 0}$:

$$\begin{aligned} \mathbb{E}[\exp\{\gamma^{-1}P_t\}] &= \mathbb{E}[\mathbb{E}[\exp\{\gamma^{-1}P_t\}|N_t]] \\ &= \mathbb{E}[\mathbb{E}[\exp\{\gamma^{-1}\xi\}]^{N_t}] = \mathbb{E}[\exp\{N_t \ln(\mathbb{E}[\exp\{\gamma^{-1}\xi\}])\}]. \end{aligned} \quad (6.15)$$

The right hand side of equation (6.15) is simply the Laplace transform of N_t evaluated at $-\ln(\mathbb{E}[\exp\{\gamma^{-1}\xi\}])$.

Computing prices that ignore contagion also requires some comments. Cancelling the phenomenon of contagion in the portfolio can be simply done in our model. Indeed, it suffices to set $\eta = 0$. However, if we do so, the new model is not anymore fairly comparable to the previous one (i.e. the one with $\eta = 0.0001$, see Table 6.1). As a matter of fact, setting $\eta = 0$ will lead to a lower expected number of defaults and therefore to lower prices.

This is why we will adjust the parameter θ in order to keep the average number of defaults unchanged. This is illustrated at Figure 6.3 and the new "corrected" value for θ is 0.15325. The abbreviation SE stands for "Self-Exciting", meaning that "SE=no" corresponds to the model without contagion ($\eta = 0$, $\theta = 0.15325$) and, as already noted before, to lower expected numbers of default.

Figure 6.3: Expected number of defaults $\mathbb{E}[N_{10}]$ in function of θ .

Figure 6.4 compares the prices in four different settings. The first setting combine contagion and time-consistency (TC = yes, SE = yes) and is represented by a continuous dark curve. The second setting is time-consistent but ignores contagion (TC = yes, SE = no). It corresponds to the dashed dark curve. The third and fourth settings are not time-consistent. The third takes the phenomenon of contagion into account and is represented by a clear continuous curve whereas the fourth, represented by a clear dashed curve, does not. Note that for in all three cases, the dashed curves are superimposed on each other, so that the clear dashed curve is not visible. This means that when there is no contagion, the difference between the time-consistent price and the time inconsistent ones is negligible. Thus it seems that in absence of contagion risk, working with a time-consistent evaluation is not worth it. However, if a risk of contagion exists, we observe that working with a time-consistent evaluation systematically leads to higher prices, compared to an evaluation that is not time-consistent. From this observation, we conclude that time-consistent evaluations seem to be more appropriate when a phenomenon of contagion exists because the extra-cost caused by the contagion is better taken into account. In other words, working with a time-inconsistent evaluation leads to an underestimation of the cost of the risk of contagion.

Conclusions

In this paper we propose a model for credit insurance portfolio allowing to take the risk of contagion into account. This is done with the help of self-exciting processes. We start by providing closed-form formula for the two first moments of such processes. Then

Figure 6.4: Evolution of the time-consistent and time-inconsistent prices, with and without contagion.

we derive semi-closed form expressions for the Laplace transform of these self-exciting processes. These semi-closed form expressions are then used to derive the distribution of the time of default in a credit insurance model relying on self-exciting processes. We derive PIDE's for time-consistent evaluations of credit insurance portfolio, extending the method of Pelsser and Ghalehjooghi (2016) to models which incorporate self-excitation. In order to numerically solve these PIDE's, we describe a method allowing to find the solution, in spite of the lack of boundary conditions of our problem. By solving numerically these PIDE's, we are able to bring to light possible interactions between time-consistency of evaluations and the existence of a risk of contagion. More precisely, we showed that the time-consistency of evaluations makes a significant difference in the prices when there is a risk of contagion. However, this is not the case when such a risk does not exist.

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