MEAN REVERSION IN STOCHASTIC MORTALITY : WHY AND HOW?

Zeddouk, Fadoua and Pierre Devolder

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Mean reversion in stochastic mortality : why and how?

Fadoua Zeddouk*, Pierre Devolder*
ISBA, *Université Catholique de Louvain, Belgium*

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Abstract

Life insurance companies use stochastic models to forecast mortality. We know from literature that non mean reversion models seem to be more suitable for mortality in comparison with mean reversion models having a fixed long term target. In this paper, we adopt stochastic affine processes for the force of mortality, and study the impact of adding a time dependant long term mean reversion level to two non-mean reverting processes. We calibrate the models to different generations in the Belgian population, and assess their ability to predict mortality by using different statistical methodologies. The backtest shows that the survival curves provided by the mean reverting processes are closer to reality. We conclude that incorporating a time-dependant target to the considered models improves significantly their performance.

Keywords : Stochastic mortality, affine process, mean reversion, model calibration.

1 Introduction

In life insurance companies and pension funds, mortality have traditionally been estimated by using simplified approach to reality. Actuarial practice used to consider deterministic life tables that are projected from historical mortality and add safety margins, which is a prudent risk management scheme that doesn't take into account the high uncertainty related to mortality evolution. Namely, mortality is translated into a deterministic scenario that is used as a basis for risk management and pricing purpose, which may lead to a bad evaluation of life insurance products. In addition, under the Solvency II directive, a solvency capital has to be held to avoid the risk of ruin. This capital is related to the risk of deviation of future mortality from expectation, and using deterministic methodology to evaluate it is no longer sufficient. As a consequence, stochastic models for mortality are needed to measure not only the expected improvement in mortality over the next few decades, but also the associated uncertainty in the projections.

Also, by modeling mortality as a stochastic process, reserves can be expanded to include the uncertainty associated with the future development of mortality intensity. This should allow for an even more accurate assessment of future commitments that take into account the systematic risk of mortality.

Moreover, the development of mortality / longevity-related securities requires the use of stochastic models for the pricing of financial instruments related to demographic risks.

To better understand and predict longevity risk, research and development efforts are now focused on stochastic mortality models. Lee-Carter model (Lee and Carter (1992)), which is a stochastic discrete-time model, provides the very first model for modeling the mortality rate with stochastic time and age mortality dynamics. It has been used as the basis of stochastic forecasts of the finances of the U.S. social security system and other aspects of the U.S. federal budget (see Congressional Budget Office of the United States¹).

Even though data are typically reported at discrete intervals (e.g. annually), it is natural to consider the development of mortality rates in continuous time.

Since the 2000s, and with Solvency II's requirements for the assessment of mortality risk, insurance companies and academic researchers have become increasingly interested in continuous-time stochastic mortality models, which have been the subject of several publications (see for example: Biffis (2005), Cairns et al. (2006), Denuit and Devolder (2010)).

*Postal adress : Voie du Roman Pays 20, 1348 Louvain-La-Neuve Belgium . E-mail to : pierre.devolder(at)uclouvain.be, E-mail to : fadoua.zeddouk(at)uclouvain.be

¹Congressional Budget Office Of The United States, "Long-Term Budgetary Pressures and Policy Options." (1998)

Various continuous-time stochastic mortality models have been proposed in literature. In our framework, we will focus on mortality and modelling the survival function of the individuals using one factor short-rate models, which are used in finance to model changes in the term structure of interest rates. The similarities between mortality and interest rate dynamics were exploited by Milevsky and Promislow (2001), Cairns et al. (2006), Biffis (2005), Dahl (2004), and Schrager (2006) to model the force of mortality using tools and techniques developed in interest rate modelling.

Milevsky and Promislow (2001) were the first to propose a stochastic force of mortality model, and several stochastic affine models are considered assuming mean-reverting characteristics. However, according to Cairns et al. (2006), affine stochastic models shouldn't incorporate mean reverting elements. Luciano and Vigna (2005) states that non-mean reverting affine processes for modeling the force of mortality are better. These conclusions have been based on reversion models with a fixed target (for instance Vasicek model), and it's not so surprising that a fixed time target is not convenient to represent the evolution on time of mortality intensity increasing naturally with age. In this context, it could be more relevant to look at mean reverting models with a floating target. We will study the impact of adding a time-dependent long-term mean element to the Ornstein Uhlenbeck and Feller processes, and see whether it improves the models's performance in predicting mortality.

This paper is organized as follows: In section 2 we give a brief review of the longevity risk, and present the considered time-homogeneous affine processes. In section 3 we calibrate the models to Belgian population and study their ability to predict mortality. Finally, in section 4 we conclude.

2 Models presentation

We will model mortality using a single explanatory variable: the intensity of mortality μ_x at age x defined by :

$$\mu_x = \lim_{\Delta \rightarrow 0} \frac{\mathbb{P}[x < \bar{T} \leq x + \Delta \mid \bar{T} > x]}{\Delta} \quad (1)$$

Where $\bar{T}$ is a random variable that represents the future lifetime of an individual.

Within the framework of our study, we are going to model the intensity of mortality $\mu_x(t)$ as a continuous stochastic process, defined in the real space of probability $(\Omega, F, \mathbb{P})$ and adapted to its natural filtration $(F_t)_{0 \leq t \leq T}$. Thus μ_{x+t} written as : $\mu_x(t, \omega)$, with ω an event of the space of probability $(\Omega, F, \mathbb{P})$. Therefore, the starting point of our modelling will be to describe the dynamics of the mortality through the specification of a stochastic differential equation (SDE). Afterward, we will focus on the survival process, that represents the index at time t of an individual initially aged x , and alive at time t to survive $T - t$ years more. For one individual, the survival process is defined as follows :

$$S(x, t, T, \omega) = e^{-\int_t^T \mu_x(u, \omega) du} \quad (2)$$

At time t , the survival probability is obtained by taking the expectation of the survival process (2), i.e. :

$${}_{T-t}p_{x+t} = E_{\mathbb{P}}[S(x, t, T, \omega) \mid F_t] \quad (3)$$

Several stochastic mortality models have been proposed in the literature. According to Luciano and Vigna (2005), the constant mean reversion models are not adapted to describe the mortality of the individuals, and non mean reversion models describe it better. Our aim is to study two mean reversion models, but this time we will incorporate a moving mean reverting target, and to compare their performances with a mean reversion model that have a constant mean level and a non mean reversion model. To do so, we will consider the following five affine stochastic processes:

- Ornstein Uhlenbeck
- Feller
- Vasicek
- Hull & White
- Cox-Ingersoll-Ross extended

In the following, and for the sake of simplification, we will write $\mu_x(t, \omega) = \mu_x(t)$ which represents the intensity of mortality of an individual having the age $x + t$ at time t , and $w(t, \omega) = w(t)$, a standard Brownian motion.

The advantage of considering affine models for modeling $\mu_x(t)$ is that it is easy to find an explicit expression for the survival probability (3). In fact, for the affine processes, the properties of the classical stochastic process (see for example Duffie et al. (2003)) allow to write:

$${}_{T-t}p_{x+t} = E_{\mathbb{P}}[S(x, t, T, \omega) \mid F_t] = e^{\alpha(t, T) - \beta(t, T)\mu_x(t)} \quad (4)$$

Where $\alpha(\cdot)$ and $\beta(\cdot)$ are two deterministic functions, solutions of the Riccati generalized ordinary differential equations that can be determined numerically, and in some cases analytically (Duffie et al. (2000)).

2.1 Non mean reversion models

2.1.1 Ornstein Uhlenbeck

The force of mortality is assumed to satisfy the stochastic differential equation :

$$d\mu_x(t) = a\mu_x(t)dt + \sigma dw(t) \quad (5)$$

With $a > 0$, $\sigma > 0$ and $w(t)$ a standard Brownian motion.

As the OU model is Gaussian, we can easily find the expression of the survival probability by determining its distribution. We can also use the affine framework that gives the following system of ODEs for $\alpha(t, T)$ and $\beta(t, T)$:

$$\begin{cases} \alpha(t, T) = \frac{\sigma^2(T-t)}{2a^2} + \frac{\sigma^2}{4\sigma^3}e^{2a(T-t)} - \frac{\sigma^2}{a^3}e^{a(T-t)} + \frac{3\sigma^2}{4a^3} \\ \beta(t, T) = \frac{1}{a}(1 - e^{a(T-t)}) \end{cases} \quad (6)$$

With boundary conditions $\alpha(T, T) = \beta(T, T) = 0$

Let $0 \leq s \leq t \leq T$. The solution $\mu_x(t)$ of the SDE (5) is:

$$\mu_x(t) = \mu_x(s)e^{a(t-s)} + \sigma \int_s^t e^{a(t-u)} dw(u) \quad (7)$$

In this model, the force of mortality has always a stricly positive probability to become negative. This probability is given by :

$$\mathbb{P}[\mu_x(t) < 0 \mid F_s] = \Phi\left(-\frac{\mu_x(s)e^{a(t-s)}}{\sigma \sqrt{\frac{e^{2a(t-s)} - 1}{2a}}}\right) = \Phi(\zeta(a, \sigma)) \quad (8)$$

Where $\Phi(\cdot)$ is the distribution function of a standard normal random variable.

The function $\zeta(\cdot)$ is an increasing function of σ and a decreasing function of a . In practice, this probability tends to be very small, since the values of σ and a are respectively small and high enough. This is confirmed by calibration results in the simulation part of this paper.

2.1.2 Feller

The stochastic differential equation (SDE) of the Feller process describing the force of mortality dynamics is given by:

$$d\mu_x(t) = a\mu_x(t)dt + \sigma\sqrt{\mu_x(t)}dw(t) \quad (9)$$

With $a > 0$, $\sigma > 0$ and $w(t)$ a standard Brownian motion.

The solution of the SDE (9) is :

$$\mu_x(t) = \mu_x(s)e^{a(t-s)} + \sigma \int_s^t e^{a(t-u)} \sqrt{\mu_x(u)} dw(u) \quad (10)$$

Compared to the Ornstein Uhlenbeck process, the Feller process avoids to have negative values for the force of mortality, but can reach zero with a positive probability.

In a Feller setting, $\alpha(\cdot)$ and $\beta(\cdot)$ can be determined by using the Feynman-Kac representation theorem (1949) (4) is solution of the following differential equation (PDE):

$$\begin{cases} \frac{\partial p}{\partial t} + a\mu_x \frac{\partial p}{\partial \mu_x} + \frac{1}{2}\sigma^2\mu_x \frac{\partial^2 p}{\partial \mu_x^2} - \mu_x p = 0 \\ 0p_{x+t} = 1 \end{cases} \quad (11)$$

By substituting (4) in the PDE (11), we obtain a system of Riccati's ODE that $\alpha(t, T)$ and $\beta(t, T)$ must verify:

$$\begin{cases} \frac{d\alpha(t, T)}{dt} = 0 \\ \frac{d\beta(t, T)}{ds} = 1 - a\beta(t, T) - \frac{1}{2}\sigma^2\beta^2(t, T) \\ \alpha(T, T) = \beta(T, T) = 0 \end{cases} \quad (12)$$

The unique solution of the system (12) is given by:

$$\begin{cases} \alpha(t, T) = 0 \\ \beta(t, T) = \frac{1 - \exp(b(T-t))}{c + d\exp(b(T-t))} \\ \beta(T, T) = 0 \end{cases} \quad (13)$$

With $b = \sqrt{a^2 + 2\sigma^2}$, $c = \frac{b+a}{2}$ and $d = \frac{b-a}{2}$

2.2 Mean reversion models

2.2.1 Vasicek model

The Vasicek model of mortality is given by :

$$d\mu_x(t) = b\left(\frac{a}{b} - \mu_x(t)\right)dt + \sigma dw(t) \quad (14)$$

With $a > 0, b > 0, \sigma > 0, w(t, \omega)$ standard Brownian motion

The drift of this model presents a mean reversion to a constant target $\frac{a}{b}$, realized with a speed b .

As the Vasicek model is affine, the survival probability takes the form : ${}_T p_{x+t} = e^{\alpha(t, T) - \beta(t, T)\mu_x(t)}$
 $\alpha(t, T)$ and $\beta(t, T)$ are given by :

$$\begin{cases} \alpha(t, T) = \left(\frac{a}{b} - \frac{\sigma^2}{2b^2}\right)\left(\frac{1}{b}(1 - \exp(-b(T-t))) - T + t\right) - \frac{\sigma^2}{4b^3}(1 - \exp(-b(T-t)))^2 \\ \beta(t, T) = \frac{1}{b}(1 - \exp(-b(T-t))) \end{cases} \quad (15)$$

2.2.2 Hull & White model

The model of Hull and White is a time-dependent extension of the Vasicek model, where the force of mortality tends to return to a moving target $\frac{\xi(t)}{b}$ with a restoring force b . The expression of the model is as follows:

$$d\mu_x(t) = (\xi(t) - b\mu_x(t))dt + \sigma dw(t) \quad (16)$$

with $b > 0, \sigma > 0, w(t, \omega)$ standard Brownian motion, and $\xi(t)$ a deterministic function.

In our study, we choose $\xi(t) = Ae^{Bt}$ with A and B two strictly positive constants. $\xi(t)$ is none other than the function of Gompertz which expresses that the force of mortality increases exponentially with age, where A is the baseline mortality at age x , and B is the senescent component.²

²The Gompertz model is an exponential process given by :

$$\begin{aligned} \mu_x(t) &= \mu_{x+t} = A' e^{B(x+t)} \\ &= A' e^{Bx} e^{Bt} \\ &= \mu_x(0) e^{Bt} \end{aligned}$$

The HW given by (16) can also be written as :

$$d\mu_x(t) = b\left(\frac{A}{b} e^{Bt} - \mu_x(t)\right)dt + \sigma dw(t)$$

If we consider that $\frac{A}{B} = \mu_x(0)$, then this equation becomes :

$$d\mu_x(t) = b(\mu_x(0)e^{Bt} - \mu_x(t))dt + \sigma dw(t)$$

This shows that the target is well given by the Gompertz law.

Since the Hull and White model is Gaussian, we can easily find the expression of the survival probability by determining its distribution. We can also use the affine framework that gives the following system of ODEs for $\beta(t, T)$ and $\alpha(t, T)$:

$$\begin{cases} \frac{d\alpha(t, T)}{dt} = \beta(t, T)\xi(t) - \frac{1}{2}\sigma^2\beta^2 \\ \frac{d\beta(t, T)}{dt} = \beta(t, T)b - 1 \end{cases} \quad (17)$$

With the boundary conditions $\alpha(T, T) = \beta(T, T) = 0$

$\alpha(t, T)$ and $\beta(t, T)$ are given by :

$$\begin{cases} \alpha(t, T) = \frac{A}{b} \left[e^{-bT} \frac{e^{(B+b)T} - e^{(B+b)t}}{B+b} - \frac{e^{BT} - e^{Bt}}{B} \right] - \frac{\sigma^2}{2b^2} \left[\frac{1}{b}(1 - e^{-b(T-t)}) - T + t \right] - \frac{\sigma^2}{4b^3} (1 - e^{-b(T-t)})^2 \\ \beta(t, T) = \frac{1}{b}(1 - e^{-b(T-t)}) \end{cases} \quad (18)$$

Let $0 \leq s \leq t \leq T$. From short calculations we find that the force of mortality in HW model is given by

$$\mu_x(t) = \mu_x(s)e^{-b(t-s)} + \frac{A}{b+B}(e^{Bt} - e^{Bs-b(t-s)}) + \sigma e^{-bt} \int_s^t e^{bu} dw(u) \quad (19)$$

and is, conditionally on $F(s)$, normally distributed with the following expectation and variance :

$$E_{\mathbb{P}}[\mu_x(t) | F_s] = \mu_x(s)e^{-b(t-s)} + \frac{A}{b+B}(e^{Bt} - e^{Bs-b(t-s)}) \quad (20)$$

$$Var_{\mathbb{P}}[\mu_x(t) | F_s] = \frac{\sigma^2}{2b}(1 - e^{-2b(t-s)}) \quad (21)$$

As in the Ornstein Uhlenbeck model, for each time t , the intensity of mortality can be negative with a positive probability : respectively by

$$\Phi \left(- \frac{\mu_x(s)e^{-b(t-s)} + \frac{A}{b+B}(e^{Bt} - e^{Bs-b(t-s)})}{\sqrt{\frac{\sigma^2}{2b}(1 - e^{-2b(t-s)})}} \right) = \Phi(\psi(A, B, b, \sigma)) \quad (22)$$

which is often negligible in practice.

2.2.3 Cox-Ingersoll-Ross (CIR) extended

The CIR model was developed by Cox et al. (1985) and known as square root process. We consider here an extension of the CIR, by considering a time dependant long-term mean. The CIR extended process for the force of mortality is characterized by:

$$d\mu_x(t) = (\xi(t) - b\mu_x(t))dt + \sigma\sqrt{\mu_x(t)}dw(t) \quad (23)$$

With $b > 0$, $\sigma > 0$, $w(t, \omega)$ standard Brownian motion

Unlike the model of Hull and White, this process has the advantage of ensuring the positivity of the intensity of mortality : thanks to the presence of the square root, the intensity process will never be negative. It can reach zero but this probability is negligible in the practice. Nevertheless, this improvement has a price in the sense that the intensity of mortality cannot be handled analytically.

To be able to compare the two models, we choose the same expression for $\xi(t)$ as for the model of Hull and White (Gompertz model).

The CIR model is affine but not Gaussian, (23) is a complicated stochastic process with a distribution difficult to evaluate. We use the Feynman-Kac representation to determine the survival function. From short calculations we get :

$$\begin{cases} \frac{d\alpha(t, T)}{dt} = \xi(t)\beta(t, T) \\ \frac{d\beta(t, T)}{dt} = b\beta(t, T) + \frac{1}{2}\sigma^2\beta^2(t, T) - 1 \end{cases} \quad (24)$$

$\alpha(t, T)$ and $\beta(t, T)$ are given by :

$$\begin{cases} \alpha(t, T) = 2A \int_t^T \frac{e^{Bs}(e^{\gamma(T-s)} - e^{-\gamma(T-s)})}{(2\gamma+b)e^{\gamma(T-s)} + (2\gamma-b)e^{-\gamma(T-s)}} ds \\ \beta(t, T) = \frac{\sinh(\gamma(T-t))}{\gamma \cosh(\gamma(T-t) + \frac{1}{2}b \sinh(\gamma(T-t)))} \end{cases} \quad (25)$$

With the boundary conditions $\alpha(T, T) = \beta(T, T) = 0$, and $\gamma = \frac{(b^2 + 2\sigma^2)^{\frac{1}{2}}}{2}$

3 Simulation and tests

In this section, we will evaluate the effectiveness of these models to predict mortality by using some statistical methodologies.

We will begin our study by calibrating the models on historical and projected mortality data, then we will assess the quality of these calibrations by performing a robustness test. Finally, we will use the models to predict mortality.

3.1 Calibration

Our aim is to calibrate our models on different historical survival processes, corresponding to four different cohorts of individuals born in 1900, 1915, 1965 and 1970 respectively, and initially aged 50 years old.

To do so, we will calibrate our models on mortality data of the Belgian population (unisex data) considering the following assumptions :

- We consider the 1900 and 1915 generations, we use historical data from Human Mortality Database ³, we have unisex annual death rates for the ages from 0 to 110, from year 1841 to 2015.
- We consider the 1965 and 1970 generations, we use projected data from the IABE projected generational mortality table ⁴ starting from year 2015 to 2060.
- Initial age of observation $x_0=50$
- $\mu_{50}(0) = -\ln(s_{50})$, where s_{50} is the one-year survival probability for an individual initially aged $x_0 = 50$

We have chosen these particular generations and $x_0=50$ as initial age of observation in order to have more or less same number of observations (50 for historical data and 45 for projected data).

Let $(s_1, s_2, \dots, s_n)$ be the annual survival probabilities assumed to be functions of both the age and the calendar year extracted from both life tables. We denote ${}_t\hat{p}_{50}$ the probability that an individual aged $x_0 = 50$ reaches age $x_0 + t$. ${}_t\hat{p}_{50}$ is computed as follows :

$${}_t\hat{p}_{50} = \prod_{i=50}^{50+t-1} s_i \quad (26)$$

And let ${}_t p_x(\theta)$ be the function to calibrate and defined by equation (3), where θ is the vector of the model's parameters. To estimate θ , we will use the Least Square Estimation (LSE). It should be noted that there are many different ways to calibrate each of these models, and different calibration techniques will yield different results. For instance, we could have estimated the parameters using the maximum likelihood (MLE) or method of moments techniques. However, for the purposes of this study, the least square method was the easiest to implement.

We minimise the least square errors on the survival probabilities computed between the observed survival probabilities, around which we want to calibrate our models, and the predicted mortality by the stochastic model :

$$R = \min_{\theta} \sum_{j=0}^n ({}_j p_x(\theta) - {}_j \hat{p}_x)^2 \quad (27)$$

For the Least Square minimization, it is necessary to specify the initial parameters where the R routine starts searching the minimum of (27). As we have a non linear system to solve, we will use the generalized Newton-Raphson method provided by R. Since the final result is highly sensitive to these start values, we generate various initial parameters randomly and calculate some optimization steps. The values with minimal residual are taken as start parameters for the final LS optimization which supplies the final parameters.

We use this kind of algorithm as well for all other calibration issues that we describe in the following sections.

3.1.1 Non mean reversion models

We calibrate the Ornstein Uhlenbeck and Feller models to the data using the above described algorithm. We find the following optimal parameters :

³ Available at www.mortality.org

⁴ Mortality projection for the Belgian population (2015), available at <https://www.iabe.be>

	1900	1915	1965	1970
$\mu_{50}(0)$	0.007909195	0.005820909	0.002850792	0.002600332
OU-error	0.005885259	0.003403252	0.055996630	0.01865938
a-OU	0.080696570	0.082023980	0.07827748	0.07411661
σ -OU	0.000000129	0.000000130	0.000000177	0.0000000359
FELL-error	0.005885342	0.003403291	0.05599667	0.0186594
a-FELL	0.080687850	0.082019520	0.07827293	0.0741147
σ -FELL	0.000001527	0.000010657	0.000001104	0.00001162

Table 1: Optimal parameters values and residuals for the OU and FELL models

The parameters a has decreasing trend for both OU and Feller, σ are very small wich confirms that the volatility of the mortality is much lower compared to the financial market.

In both models the mean square errors are not very low, indicating an acceptable fit to the historical mortality table and a bad fit to projected data. Also, the residuals are more important for the younger generations.

We see that the values of the parameters are quite coherent in both models.

Figures 1 and 2 present calibration results graphically. As predicted by Table 1 above, we see that the fit is acceptable for old generations and poor for young generations .

Figure 1: Survival functions, 1900,1915,1965 and 1970 generations for OU model

Figure 2: Survival functions, 1900,1915,1965 and 1970 generations for Feller model

3.1.2 Mean reversion models

The following table reports the optimal calibration parameters for Vasicek, HW and CIR models

	1900	1915	1965	1970
$\mu_{50}(0)$	0.007909195	0.005820909	0.002850792	0.002600332
Vasi-error	0.3552288	0.3792649	0.3455769	0.145937
Vasi-a	0.002379388	0.00194506	0.0008525974	0.0009971052
Vasi-b	0.02379912	0.02061802	0.0099139901	0.0553056376
Vasi- σ	0.00002780	0.00000962	0.0003079058	0.0006527820
Vasi-a/b	0.00997798	0.09433787	0.08599942	0.018029
HW-error	0.00004950	0.002591606	0.006926344	0.003449854
HW-A	0.004333549	0.004015638	0.0002397676	0.0003075705
HW-B	0.084061184	0.084626391	0.1093956381	0.1006279164
HW-b	0.472769826	0.650482593	0.1041555426	0.1385505877
HW- σ	0.030988143	0.000444231	0.0004082868	0.0005196101
HW-A/b	0.009166298	0.006173321	0.002302015	0.002219915
CIR-error	0.0001098416	0.002511401	0.007417984	0.003175387
CIR-A	0.006388952	0.002355749	0.0002696357	0.0002631539
CIR-B	0.088292773	0.084880513	0.1061992059	0.1018886261
CIR-b	0.850550891	0.340179406	0.1066316055	0.1059934492
CIR- σ	0.162230659	0.084985972	0.0425582771	0.0417149602
CIR-A/b	0.007511546	0.006925019	0.002528666	0.002482737

Table 2: Optimal parameters values and residuals for the Vasicek, HW and CIR models

We see that the residuals of the Vasiček model are very big for the four generations (10^{-1}), and low for HW and CIR models in comparison with non mean reversion models OU and Feller above. We also notice that Vasi- σ is very low in comparison with the other two models.

As expected, the starting point of the target is strongly decreasing with the generations, but what is less intuitive is that B seems more important for the young generations, and could be explained by the fact that the target is much lower for the younger generations.

Figures 3,4 and 5 present the calibration results graphically. We see that the fit is very poor for Vasiček model and very satisfying for the HW and CIR models, as we cannot distinguish the two survival functions.

Figure 3: Survival functions, 1900,1915,1965 and 1970 generations for Vasiček model

Figure 4: Survival functions, 1900,1915,1965 and 1970 generations for HW model

Figure 5: Survival functions, 1900,1915,1965 and 1970 generations for CIR model

3.1.3 Models comparison

To conclude, we plot the calibration errors i.e. the differences between the observed survival probabilities and their theoretical counterparts for all generations.

	1900	1915	1965	1970
OU	0.005885259	0.003403252	0.05599663	0.01865938
FELL	0.005885342	0.003403291	0.05599667	0.0186594
Vasi	0.3552288	0.3792649	0.3455769	0.145937
HW	0.00004950659	0.002591606	0.006926344	0.003449854
CIR	0.0001098416	0.002511401	0.007417984	0.003175387

Table 3: Calibration errors-all models

The 4 graphes in figure 6 show that the Vasiček fitting error is very big which confirm the conclusions of the authors mentioned in the introduction: Vasiček model is not appropriate to describe the mortality. In general, OU and Feller models give similar acceptable errors. HW and CIR models were able to fit the observed data with smaller error. We can conclude that adding a variable long-term mean level leads to a significant improvement in the fit, and the mean reversion models perform better than non mean reversion ones, on the condition of using a variable target.

Figure 6: Calibration errors Vasicek, OU, Feller, HW and CIR, different generations

In order to compare the quality of each model relative to each of the other models, we can use the AIC (Akaike Information Criterion) and BIC (Bayesian Information Criterion), two model selection methods that allow to select the model that will be a good fit for the data without losing generality from over-fitting the data.

Namely, they reward the model that present the best balance between the goodness of fit to data and the complexity.

AIC is defined as follows :

$$AIC = n \ln\left(\frac{R}{n}\right) + 2k \quad (28)$$

Where n is the number of observations, k is the number of parameters and R the residual found with the LSE method. The best model is the one with the lowest AIC.

BIC is given by :

$$BIC = n \ln\left(\frac{R}{n}\right) + k \ln(n) \quad (29)$$

We can see from (29) that BIC penalizes model complexity more heavily.

In the table 4 we report the AIC criteria for all models and different generations :

	1900	1915	1965	1970
OU-AIC	-448.3664	-475.7523	-297.0107	-346.4631
FELL-AIC	-448.3657	-475.7518	-297.0106	-346.463
Vasi-AIC	-241.3508	-238.0772	-213.1141	-251.9059
HW-AIC	-683.2714	-485.375	-387.0589	-418.4239
CIR-AIC	-643.4247	-486.9469	-383.973	-422.1545

Table 4: AIC for all models and different generations

Table 5 reports the BIC criteria for all models and different generations :

	1900	1915	1965	1970
OU-BIC	-444.5423	-471.9283	-293.3974	-342.8498
FELL-BIC	-444.5416	-471.9277	-293.3973	-342.8497
Vasi-BIC	-235.6147	-232.3411	-207.6941	-246.4859
HW-BIC	-675.6233	-477.7269	-379.8322	-411.1972
CIR-BIC	-635.7766	-479.2988	-376.7463	-414.9278

Table 5: BIC for all mdels and different generations

The two non mean reversion models OU and Feller provide almost the same values of AIC/ BIC for all generations, which is normal because both models have the same number of parameters and give close residuals. The AIC and BIC of HW and CIR models are close too for the same reason. We can see clearly the gap between the AIC/ BIC of the two mean reversion and non mean reversion models : for all generations and in 100% of cases, the non mean reversion models have the highest AIC/ BIC value in comparison with HW and CIR models, the Vasiček model is by far the model with the highest AIC/BIC criteria, it should be eliminated. We keep the Hull and White and CIR models as reference, OU and Feller models for further tests.

3.2 Best estimate of future mortality

To look at mortality trend, we compare the best estimate of $\mu_x(t)$ for different generations in order to illustrate the longevity pattern. We have simulated the process $\mu_{50}(t)$ for the generations 1900,1915,1965 and 1975 from the age of 50 until the age of 95, using the optimal parameters found in section 3.1

We use the Euler discretization for Feller and CIR processes, with a daily time step.

3.2.0.1 Non mean reversion models

The following graph reports the mean of $\mu_{50}(t)$ for the four generations for the OU and Feller models :

Figure 7: Mean of $\mu_{50}(t)$ for generations 1900,1915,1965 and 1970, OU and Feller models

3.2.0.2 Mean reversion models

The following graph reports the mean of $\mu_{50}(t)$ for the four generations for the HW and CIR models :

Figure 8: Mean of $\mu_{50}(t)$ for generations 1900,1915,1965 and 1970, HW and CIR models

For all models, the older the generation, the higher the mean of $\mu_x(t)$, but we observe that for HW and CIR models, the mean of $\mu_x(t)$ seems to increase more rapidly with age.

3.3 Robustness

Studying the quality of adjustment of the four models to the data is only a preliminary step to their real use: forecasting mortality. We will now test the quality of the calibrations performed before.

In the previous section we considered different generations and an initial observation age of 50 years old. If this age changes, the profile of the survival function will change as well. It is interesting to test the robustness of our models according to the profile of the historical data curve.

To do so, we consider the cohort born in 1900 (historical data), and two initial observation ages, 35 and 65 years old. These two ages frame the age of 50 at which previous simulations were conducted: In order to make the results comparable, we use the same number of observations for each of the three historical survival functions (31 observations for each generation).

Namely, consider the following :

- $x_0 = 35$: individuals born in 1900 aged 35 years old in 1935 until 1965
- $x_0 = 50$: individuals born in 1900 aged 50 years old in 1950 through 1980⁵
- $x_0 = 65$: individuals born in 1900 aged 65 years old in 1965 to 1995

3.3.1 Non mean reversion models

After calibration we obtain the following optimal parameters :

$\mu_{35}(0)$	0.004578465	$\mu_{50}(0)$	0.007909195	$\mu_{65}(0)$	0.0267383
OU-error	0.001200412	OU-error	0.003062471	OU-error	0.00032768
OU-a	0.04378364	OU-a	0.07884322	OU-a	0.08358668
OU- σ	0.000000017	OU- σ	0.000000196	OU- σ	0.000002802
FELL-error	0.001200413	FELL-error	0.00306248	FELL-error	0.000327738
FELL-a	0.04378221	FELL-a	0.07884492	FELL-a	0.083574609
FELL- σ	0.0000173123	FELL- σ	0.000029223	FELL- σ	0.000071488

Table 6: Robustness test optimal parameters, Feller and OU models, generation 1900

Figure 9 presents the calibration results graphically.

⁵Even if it was done in the calibration section, we re-calibrate the models to the historical data for $x = 50$ because here we consider only 31 observations contrary to 51 observations used previously

Figure 9: Survival functions of the OU and Feller model for three initial ages of observations, generation 1900

3.3.2 Mean reversion models

After calibration we obtain the following optimal parameters :

$\mu_{35}(0)$	0.004578465	$\mu_{50}(0)$	0.007909195	$\mu_{65}(0)$	0.0267383
HW-error	0.0002805102	HW-error	0.0000385808	HW-error	0.000043424
HW-A	0.0007943429	HW-A	0.004064894	HW-A	0.02076235
HW-B	0.0604713862	HW-B	0.084706914	HW-B	0.08665132
HW-b	0.1711649859	HW-b	0.449641795	HW-b	0.72009322
HW- σ	0.0027935774	HW- σ	0.027014776	HW- σ	0.00192661
CIR-error	0.000236459	CIR-error	0.0000685425	CIR-error	0.00004430
CIR-A	0.0005175401	CIR-A	0.004202289	CIR-A	0.01420525
CIR-B	0.0655478420	CIR-B	0.089652319	CIR-B	0.08684733
CIR-b	0.1049618344	CIR-b	0.545048072	CIR-b	0.46521531
CIR- σ	0.0199655405	CIR- σ	0.091630589	CIR- σ	0.04344126

Table 7: Robustness test optimal parameters, HW and CIR models, generation 1900

Figure 10 presents calibration results graphically :

Figure 10: Survival functions of the HW and CIR models for three initial ages of observations, generation 1900

We can conclude that once again, the HW and CIR models are able to adapt to different curve profiles, which is a good performance.

3.4 Backtesting

In this part, we will check if after the calibration of our models on a portion of the historical survival curve, they can extend it correctly. To do this, we will perform a backtesting. We will use historical data of the generation 1900 and we consider :

- The initial age of observation $x_0 = 50$
- $N = 50$, the total number of historical data
- N_0 the portion of historical data we use for calibration
- $\delta = \frac{N_0}{N}$ the ratio of data used to total data

The backtest consists on varying δ from 20% to 100%, and see how the survival curve provided by the models meet reality.

3.4.1 Non mean reversion models

3.4.1.1 Ornstein Uhlenbeck

We calibrate the models to the portion of observed survival function, then we stock the optimal parameters and the global error, that refers to the the difference between the curve projected by the model and the historical one. For the OU model we obtain the following results:

N_0	Age	δ	global error	a	σ
10	50-60	20%	0.7792664	0.05791558	0.0000013042
20	50-70	40%	0.07666702	0.07319190	0.00000023774
30	50-80	60%	0.00991389	0.07884375	0.000000006161
40	50-90	80%	0.005931298	0.08049631	0.000001238150
50	50-100	100%	0.005885259	0.080696570	0.000000129

Table 8: Backtesting OU model, generation 1900

Figure 11: Evolution of the fitting quality of the OU model according to the portion of observed data considered

3.4.1.2 Feller

For the Feller model we obtain the following results :

N_0	Age	δ	global error	a	σ
10	50-60	20%	0.77910120	0.05791906	0.00020321040
20	50-70	40%	0.07672630	0.07318895	0.00003899174
30	50-80	60%	0.009921382	0.07884205	0.00001075077
40	50-90	80%	0.005931992	0.08049489	0.00002490427
50	50-100	100%	0.005885342	0.080687850	0.000001527

Table 9: Backtesting Feller model, generation 1900

Figure 12: Evolution of the fitting quality of the Feller model according to the portion of observed data considered

Both models OU and Feller, need an important volume ($\delta \geq 80\%$) of information to complete correctly the historical curve. Below this value both models overestimate the survival function.

3.4.2 Mean reversion models

3.4.2.1 Hull & White model

For the HW model we obtain the following results :

N_0	Age	δ	global error	A	B	b	σ
10	50-60	20%	0.003464492	0.002733086	0.088094733	0.334805279	0.001641703
20	50-70	40%	0.0005948094	0.003718993	0.085592503	0.400761822	0.027073209
30	50-80	60%	0.0002458885	0.006525992	0.081024406	0.690103707	0.053059593
40	50-90	80%	0.00005137036	0.004631080	0.08392772	0.50978892	0.03330665
50	50-100	100%	0.00004950648	0.004333549	0.084061184	0.472769826	0.030988143

Table 10: Backtesting HW generation 1900

Figure 13: Evolution of the fitting quality of the HW model according to the portion of observed data considered

3.4.2.2 CIR model

For the CIR model we obtain the following results :

N_0	Age	δ	global error	A	B	b	σ
10	50-60	20%	0.009092467	0.003205452	0.086116506	0.397389283	0.051463759
20	50-70	40%	0.002097292	0.004808043	0.091721513	0.604706446	0.283849703
30	50-80	60%	0.0002585306	0.006302467	0.089327738	0.858853267	0.115886515
40	50-90	80%	0.0001394693	0.003613061	0.088649272	0.450780512	0.068252594
50	50-100	100%	0.0001098416	0.006388952	0.088292773	0.850550891	0.162230659

Table 11: Backtesting CIR generation 1900

Figure 14: Evolution of the fitting quality of the CIR model according to the portion of observed data considered

In the backtesting strategy also, HW and CIR models perform better than OU and Feller : we can observe that the fit is very satisfying when $\delta \geq 40\%$ for both models in comparison with non mean reversion models that need more data to fit correctly the historical survival function.

3.5 Forecasting mortality

In the previous sections, we have seen that the models of HW and CIR give very satisfactory results in comparison with Feller and OU:

- They managed to perfectly fit the survival curve regardless of its profiles
- They were able to properly extrapolate the end of a survival curve even with a small initial portion of information.

We will now assess the prediction quality of the four model. to do so, we will perform two tests:

3.5.1 Test 1 :

We will here predict the end of a survival curve of a generation not yet extinct.

To assess the ability of these models to predict mortality, we will simulate a particular situation:

- We consider the generation born in 1930 not yet extinct
- We calibrate our models to historical data from the age of 50 until the age of 80
- We use the model to predict the survival curve from the age of 50 until the age of 110
- We use the first 30 values to calibrate the model and we keep the 5 remaining values to see the trend of the historical curve. (we then use 30 historical values for calibration ($\delta = 50\%$))

3.5.1.1 Non mean reversion models

For the OU and Feller models we obtain the following results :

Model	Prediction error	Calibration error	a	σ
OU	0.004527336	0.0005230701	0.06982033	0.000003889523
FELL	0.004529387	0.0005230677	0.06981939	0.000031189980

Table 12: Prediction error and optimal parameters Feller and OU models

Figure 15: Mortality prediction generation 1930 Feller and OU models

We find that the profile of the curves is realistic, and that both models manage to prolong the survival function correctly. However, they seem to overestimate it.

3.5.1.2 Mean reversion models

For the HW and CIR models we obtain the following results :

Model	prediction error	Calibration error	A	B	b	σ
HW	0.001117123	0.00002846756	0.002236015	0.076146491	0.396080528	0.002092511
CIR	0.001122078	0.00002968514	0.002433378	0.076125462	0.437412206	0.017971028

Table 13: Prediction error and optimal parameters HW and CIR models

Model	Difference projected data - prediction
OU(50-95)	3.924573
FELL(50-95)	3.924629
OU(65-110)	7.247965
FELL(65-110)	7.246802

Table 14: Comparison between Projected data and predicted survival function of OU and FELLER models, two initial ages of observation

Figure 16: Mortality prediction generation 1930 HW and CIR models

Both CIR and HW models slightly overestimate the historical survival curve, but their prediction errors are lower than the those found when using OU and Feller models.

3.5.2 Test 2 :

In order to have a better idea on the prediction quality, we will now compare the predicted survival function with the projected mortality data.

We consider two cohorts of individuals aged simultaneously 50 and 65 years old in 2015, and we follow these steps :

- $x_0 = 50$:
 1. Calibration from 20 to 50 years old
 2. Projection with the model from 50 to 95 years old
 3. Comparison with IABE projected data from age 50 to 95
- $x_0 = 65$
 1. Calibration from 20 to 65 years old
 2. Projection from 65 to 110 years old
 3. Comparison with IABE data from age 65 to 110

We then stock the difference between predicted and projected data⁶

⁶The “Difference” reported by the tables and the graphes represents the least squared errors between the projected data from IABE and the one predicted by the model

Figure 17: Comparison between projected data and predicted survival function of OU and FELLER models, two initial ages of observation

3.5.2.1 Non mean reversion models

As we use the survival historical data from the age of 20 to calibrate the models, it's normal to have a poor prediction and as a result very big residuals. But it's interesting to compare the behaviour of the mean reversion and non mean reversion models when we use the beginning of the historical survival function for calibration in order to forecast longevity

3.5.2.2 Mean reversion models

The following table reports the comparison between projected data and survival function of HW and CIR models :

Model	Difference projected data - prediction
HW(50-95)	1.393081
CIR(50-95)	0.4070036
HW(65-110)	1.003617
CIR(65-110)	0.3578838

Table 15: Comparison between Projected data and predicted survival function of HW and CIR models, two initial ages of observation

Figure 18: Comparison between projected data and predicted survival function of HW and CIR models, two initial ages of observation, generation 1965

The distance between the prediction and the projected data is very important for all models, but CIR and HW models provide survival functions that are more close to projected data.

3.6 Mortality surface

In this section we will plot the mortality surface to represent simultaneously the survival functions of different cohorts. In order to avoid the problem mentioned in 3.5.2, instead of calibrating the models to historical data, we will consider the projected data.

To plot the mortality surface, we follow these steps :

- We consider 10 cohorts of individuals initially aged from 55 to 65 in 2015
- We calibrate the models on projected data for the different cohorts (from the age x_0 until the end of the projected life table)
- We consider that all individuals survive until the age of 65
- We use the models to project the survival functions for the 10 cohorts from the age of 65 until the age of 105
- We plot the mortality surface in 3D

Let define $S^f(x_0, 65, z)$ the survival forward probability between age 65 and age z for an individual aged x_0 in 2015 at time 0, and assumed to be alive at age 65. We will define here this probability as the ratio between the two corresponding survival probabilities :

$$S^f(x_0, 65, z) = \frac{z - x_0 p_{x_0}}{65 - x_0 p_{x_0}} \quad (30)$$

This definition is similar to the the concept of forward rates in interest rates theory.

This probability is given explicitly, taking into account (2), (3) and (4) by :

$$S^f(x_0, 65, z) = S^f(x_0, z) = \frac{E_{\mathbb{P}}(S(x_0, 0, z - x_0))}{E_{\mathbb{P}}(S(x_0, 0, 65 - x_0))} = \frac{e^{\alpha(0, z - x_0) - \beta(0, z - x_0)\mu_{x_0}(0)}}{e^{\alpha(0, 65 - x_0) - \beta(0, 65 - x_0)\mu_{x_0}(0)}} \quad (31)$$

3.6.1 Non mean reversion models :

Figure 19: Forward survival surface for individuals aged from 55 to 65 in 2015, projected from age 65 to 105, OU and Feller models

Projection age z	$S^f(55, z)$	$S^f(58, z)$	$S^f(61, z)$	$S^f(64, z)$
70	0.94270999	0.94086362	0.94000700	0.94073165
75	0.86324605	0.85867284	0.85584838	0.85594137
80	0.75692340	0.74867934	0.74240724	0.73964739
85	0.62208501	0.60956896	0.59843903	0.59017254
90	0.46416890	0.44786853	0.43160361	0.41629487
95	0.29981629	0.28209663	0.26297026	0.24269607
100	0.15614449	0.14104335	0.12407613	0.10538469
105	0.05896950	0.04987813	0.03973097	0.02901933

Table 16: Predicted forward survival probabilities with OU model, different cohorts

Projection age z	$S^f(55, z)$	$S^f(58, z)$	$S^f(61, z)$	$S^f(64, z)$
70	0.94270779	0.94059967	0.93981972	0.94066064
75	0.86323974	0.85786186	0.85519102	0.85559036
80	0.75691057	0.74695390	0.74089639	0.73871205
85	0.62206344	0.60658881	0.59570571	0.58833943
90	0.46413823	0.44358153	0.42757920	0.41348398
95	0.29978022	0.27708191	0.25826448	0.23939465
100	0.15611136	0.13656292	0.12000066	0.10264470
105	0.05894795	0.04713311	0.03741897	0.02763406

Table 17: Predicted forward survival probabilities with Feller model, different cohorts

3.6.2 Mean reversion models

Figure 20: Forward survival surface for individuals aged from 55 to 65 in 2015, projected from age 65 to 105, HW and CIR models

The following tables present the annual survival probabilities given by the HW and CIR models, and used to plot the 3D figure above :

Projection age z	$S^f(55, z)$	$S^f(58, z)$	$S^f(61, z)$	$S^f(64, z)$
70	0.962835443	0.960325946	0.957746038	0.950056248
75	0.904339900	0.899711357	0.897777534	0.884516037
80	0.811135375	0.803912194	0.801521846	0.784853820
85	0.669693860	0.659217546	0.652808569	0.635412171
90	0.477145956	0.463747219	0.451074084	0.435174359
95	0.261767768	0.248404861	0.233137192	0.220321818
100	0.090383387	0.081991900	0.072303009	0.064728824
105	0.013744286	0.011454980	0.009139222	0.007139386

Table 18: Predicted forward survival probabilities with HW model, different cohorts

Projection age z	$S^f(55, z)$	$S^f(58, z)$	$S^f(61, z)$	$S^f(64, z)$
70	0.96289906	0.960181543	0.955302546	0.949143243
75	0.90423742	0.899746640	0.892639081	0.882913764
80	0.81075621	0.804358436	0.796005973	0.783807917
85	0.66912780	0.659968649	0.649885642	0.635573943
90	0.47668261	0.464236350	0.451509308	0.435762844
95	0.26168739	0.248033054	0.234128786	0.219823304
100	0.09058580	0.081125847	0.071543132	0.063408897
105	0.01386676	0.011058342	0.008409237	0.006615424

Table 19: Predicted forward survival probabilities with CIR model, different cohorts

A comparison between table 17 and table 18 shows that HW and CIR models are once again more performant: the forward survival probabilities provided by HW and CIR models, are more in line with the improvement of longevity for younger generations.

4 Summary and further research

In this paper, we have described the evolution of mortality with the intensity of mortality using time-homogeneous affine processes. We have studied four types of stochastic processes which are different in their drift and stochastic part. Their survival probabilities have been provided in closed form and studied analytically. The intensity processes have been calibrated to the Belgian population, using historical data for old generations and projected mortality tables for the younger ones. For each model, we have verified the robustness of the models and their ability to forecast mortality. As a result, we showed that adding a time dependent long-term mean to the selected processes seems to be more appropriate to describe the death intensity of individuals. They satisfy most of the reasonable criteria that a good model for mortality should have. To conclude, we have proposed procedures for mortality forecasting and mortality trend assessment, which describe future evolution of mortality within a single cohort and between different cohorts, respectively. Recall that we have voluntarily confined ourselves to one-factor, non-jump basic models in order to focus on our research question concerning the mean reversion or not mortality aspect. Our conclusion in this respect is that while the mean reversion aspect does not seem to be suitable for mortality when using a fixed target, it is quite appropriate when using a variable target linked to the increase in mortality with age.

The models used in this paper can be used for pricing mortality-based derivative products.

List of Tables

1	Optimal parameters values and residuals for the OU and FELL models	7
2	Optimal parameters values and residuals for the Vasicek, HW and CIR models	8
3	Calibration errors-all models	11
4	AIC for all models and different generations	12
5	BIC for all mdels and different generations	13
6	Robustness test optimal parameters, Feller and OU models, generation 1900	14
7	Robustness test optimal parameters, HW and CIR models, generation 1900	16
8	Backtesting OU model, generation 1900	18
9	Backtesting Feller model, generation 1900	20
10	Backtesting HW generation 1900	21
11	Backtesting CIR generation 1900	22
12	Prediction error and optimal parameters Feller and OU models	24
13	Prediction error and optimal parameters HW and CIR models	24
14	Comparison between Projected data and predicted survival function of OU and FELLER models, two initial ages of observation	25
15	Comparison between Projected data and predicted survival function of HW and CIR models, two initial ages of observation	26
16	Predicted forward survival probabilities with OU model, different cohorts	28
17	Predicted forward survival probabilies with Feller model, different cohorts	28
18	Predicted forward survival probabilities with HW model, different cohorts	29
19	Predicted forward survival probabilities with CIR model, different cohorts	29
20	Optimal parameters values for the 10 different cohorts, OU model	34
21	Optimal parameters values for the 10 different cohorts, Feller model	34
22	Optimal parameters values for the 10 different cohorts, HW model	34
23	Optimal parameters values for the 10 different cohorts, CIR model	35

List of Figures

1	Survival functions, 1900,1915,1965 and 1970 generations for OU model	7
2	Survival functions, 1900,1915,1965 and 1970 generations for Feller model	8
3	Survival functions, 1900,1915,1965 and 1970 generations for Vasicek model	9
4	Survival functions, 1900,1915,1965 and 1970 generations for HW model	10
5	Survival functions, 1900,1915,1965 and 1970 generations for CIR model	11
6	Calibration errors Vasicek, OU, Feller, HW and CIR, different generations	12
7	Mean of $\mu_{50}(t)$ for generations 1900,1915,1965 and 1970, OU and Feller models	13
8	Mean of $\mu_{50}(t)$ for generations 1900,1915,1965 and 1970, HW and CIR models	14
9	Survival functions of the OU and Feller model for three initial ages of observations, generation 1900	15
10	Survival functions of the HW and CIR models for three initial ages of observations, generation 1900	17
11	Evolution of the fitting quality of the OU model according to the portion of observed data considered	19
12	Evolution of the fitting quality of the Feller model according to the portion of observed data considered	20
13	Evolution of the fitting quality of the HW model according to the portion of observed data considered	22
14	Evolution of the fitting quality of the CIR model according to the portion of observed data considered	23
15	Mortality prediction generation 1930 Feller and OU models	24
16	Mortality prediction generation 1930 HW and CIR models	25
17	Comparison between projected data and predicted survival function of OU and FELLER models, two initial ages of observation	26
18	Comparison between projected data and predicted survival function of HW and CIR models, two initial ages of observation, generation 1965	27
19	Forward survival surface for individuals aged from 55 to 65 in 2015, projected from age 65 to 105, OU and Feller models	28
20	Forward survival surface for individuals aged from 55 to 65 in 2015, projected from age 65 to 105, HW and CIR models	29

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Appendix

Age in 2015	a	σ
55	8.010534e-02	5.217524e-07
56	8.065272e-02	2.321505e-07
57	8.08262e-02	1.42989e-07
58	8.103744e-02	4.971963e-07
59	8.190360e-02	3.951415e-07
60	8.203626e-02	5.383440e-08
61	8.321991e-02	5.389058e-07
62	8.391121e-02	5.651989e-07
63	8.511538e-02	3.265036e-07
64	8.713319e-02	2.899020e-07
65	8.842545e-02	5.081601e-07

Table 20: Optimal parameters values for the 10 different cohorts, OU model

Age in 2015	a	σ
55	8.010873e-02	1.172081e-05
56	8.066222e-02	1.391401e-07
57	8.082895e-02	7.656239e-06
58	8.156720e-02	2.367449e-05
59	8.247819e-02	1.128001e-05
60	8.265352e-02	1.397672e-05
61	8.378668e-02	1.677087e-05
62	8.444280e-02	7.012742e-06
63	8.561437e-02	2.721347e-05
64	8.759366e-02	2.753745e-05
65	8.880231e-02	5.920618e-06

Table 21: Optimal parameters values for the 10 different cohorts, Feller model

Age in 2015	A	B	b	σ
55	0.0004225868	0.1142818793	0.1166911385	0.0020011325
56	0.0004656116	0.1125119345	0.1056348209	0.0018553666
57	0.0006613847	0.1096522468	0.1410822493	0.0016365113
58	0.000678605	0.114823970	0.143197561	0.001384317
59	0.0006588964	0.1165563409	0.1149121062	0.0027556387
60	0.0008922615	0.1157183667	0.1535578708	0.0016601581
61	0.001128205	0.112672878	0.150326645	0.012011195
62	0.0008.619701	0.119757300	0.1048798	0.006039723
63	0.001372316	0.114974004	0.156962038	0.008329491
64	0.001486815	0.117535425	0.163190841	0.001592163
65	0.002398110	0.115379365	0.261814487	0.001864268

Table 22: Optimal parameters values for the 10 different cohorts, HW model

Age in 2015	A	B	b	σ
55	0.0004489031	0.1141072098	0.1285368140	0.0250095154
56	0.0006005366	0.1133099142	0.1271696029	0.1349026874
57	0.000593937	0.114826738	0.119950472	0.092577021
58	0.0006321663	0.1156945248	0.1303883272	0.0145121586
59	0.0006309432	0.1173244128	0.1073439044	0.0303527446
60	0.0009030778	0.1159504836	0.1472126549	0.0692949778
61	0.0008401468	0.1182312437	0.1107459520	0.0606005408
62	0.001127561	0.116916665	0.152460091	0.032818536
63	0.001799053	0.114855971	0.243510139	0.042243629
64	0.001197032	0.119606162	0.118714525	0.006008059
65	0.002317753	0.115622207	0.250629489	0.017700069

Table 23: Optimal parameters values for the 10 different cohorts, CIR model