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Abstract

This article studies hedging strategies of crop harvest with futures and options on indexes of cumulated average temperatures (CAT). To account for the time and space dependence, temperatures and crop yields are modeled by three dimensions Gaussian fields. In this framework, we study the features and dynamics of CAT futures and CAT basket options. Next, we find the portfolio of CAT futures minimizing the variance of incomes from crop in different regions. We compare this hedging strategy to the portfolio maximizing the expected exponential utility of incomes. Furthermore, we assess the impact of CAT basket options on the variance of crop incomes. We conclude this work by a realistic case study in which the harvest of green maize in two Belgian regions is hedged against adverse deviations of temperatures with CAT futures or options.

KEYWORDS: weather derivatives, crop insurance, Gaussian fields, dynamic hedging

1 Introduction

Agricultural insurance markets have experienced rapid expansion in the last 50 years as detailed in Smith and Glauber (2012). Today, U.S. crop insurance program is the world's largest with more than \$100 billion of liability in place in 2015. The US "shallow loss" farm income safety net programs make payments to farmers when farm revenues decline modestly from their expected levels. The development of such products in the US is linked to the shift from farm subsidy programs toward subsidized agricultural insurance. Subsidization is partly justified by the lack of diversification of weather risk: bad weather conditions affect in the same manner crop yields over a large geographical area. This issue could hinder the development of the crop insurance market in some countries.

In parallel, the deregulation of the energy sector leads to a competition among electricity producers and is at the origin of the development of temperature derivatives at the end of the 1990s. The electricity price being highly correlated to weather conditions, energy producers trade temperature derivatives to hedge their risks. To fulfil the demand for weather indexes instruments, the Chicago Board of Trade (CBOT) proposes futures and options on cumulated average temperatures (CAT) and on cooling degree days (CDD).

Apart from the energy sector, weather fluctuations have a huge impact on the agricultural industry. It is then legitimate to ask if it is possible to hedge crop yields with weather derivatives like CAT futures or options. If so, another question concerns the determination of the best investment strategy. Partial answers are provided by Mahul (2003) who studies the optimal static hedging strategies under the assumption that futures and options on crop yields are available. Vedenov et al. (2004) and Collier et al. (2009) emphasize that a weather index derivative underwrites a risk highly correlated with agricultural production losses. This instrument, although subject to basis risk, should reduce costs over traditional agricultural insurance. Our article complements

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previous research by proposing a joint spatial model both for temperatures and crop yields. We find in this framework the crop hedging strategies that reduce the exposure to adverse deviations of temperatures.

The literature on the pricing of weather derivatives is large. Alaton et al. (2002) use a single mean reverting process with a time varying drift in order to price CAT options. Benth (2003) extends this model by considering a fractional Brownian motion. Benth and Saltyte-Benth (2005) model the temperatures with a mean reverting process driven by an hyperbolic Lévy process. Zappanis and Alexandridis (2008) use neural networks to examine the time dependence of the speed of mean reversion. Benth et al. (2011) fit mean reverting processes to temperatures of Japanese cities. Whereas cited articles propose models for the temperature measured at a single location, we develop a multivariate approach in order to replicate the spatial dependence of weather conditions.

Another facet of the literature deals with the crop yield modeling. Ozaki et al. (2013) and Annan et al. (2013) study the influence of spatial pattern of yield distributions in the US and their implications for crop insurances. These works mainly focus on the correlation between yields of multiple geographical areas without explicitly modeling the role of temperature and precipitations. Schlenker and Roberts (2009) analyze the non-linear effect of weather on crop yields. Tack et al. (2012) underline the effect of climate on the higher order moments of crop yields. Whereas Tack et al. (2017) evaluate the impact of a global warming on the US crop insurance program.

Our article complements then previous research in several directions and lies at the crossroad of the financial and agricultural literature. First, it proposes a continuous-time approach based on Gaussian fields that allows for modeling time and spatial dependence between temperatures. Second, we model interactions between accumulated temperatures and crop yields. Third, the high analytical tractability of the model allows us to find closed form expressions for the portfolio of CAT futures that minimizes the variance of crop incomes. Next, we find the strategies optimizing the expected utility of harvest incomes. Furthermore, we evaluate the impact on the variance of crop incomes of holding a position in CAT basket options. Finally, we illustrate our results with a realistic case study.

Our article is structured as follows. In the next section, temperatures are modeled as a time-changed Brownian sheet. We show that this Gaussian sheet may be represented as a multivariate mean-reverting process. Next, we infer the joint-dynamics of CAT indexes at different locations and price CAT futures. In the third section, we evaluate basket options on CAT. In section 4, we model the crop yield as a linear function of the CAT and of a non-mean reverting Gaussian field. Next, we deduce the multivariate dynamics of expected crop yields. In section 5, we find the minimum variance hedging strategy. In section 6, we establish the dynamic strategy maximizing the expected utility of crop incomes. Next, we calculate the variance of incomes when CAT basket options are used for hedging. Section 8 concludes with a realistic case study. We determine the optimal strategies with futures and options on CAT measured in Paris and Frankfurt, for hedging crop harvest near two Belgian cities (Ieper and Arlon).

2 CAT indexes and CAT futures

CME CAT Indexes measure the accumulation of daily average temperatures over a calendar month, with the accumulation period beginning on the first calendar day of the contract month and ending with the last day of the contract calendar month. Cumulative Average Temperature (CAT) futures are available for summer months in the US and Europe, allowing businesses to hedge against monthly volatility by tracking average daily temperature in a given city. Today, CAT indexes are computed for only two European cities: Amsterdam and London. However, the uncertainty about climate change will probably lead to the development of CAT indexes for other locations. CAT futures are mainly designed for hedging positions in the energy market. However, these contracts may also be used for hedging harvest incomes given the evident link between temperatures and crop yields. Before proposing a model for crop yields, we propose a multivariate model based on a

Gaussian field, for temperatures at different locations. Next, we discuss the pricing of CAT futures and establish their dynamics under the risk neutral measure.

We assume that CAT indexes are measured at p geographical locations. The information about their positions is contained in a list $\{\mathbf{x}_1^h, \dots, \mathbf{x}_p^h\}$ where $\mathbf{x}_j^h$ is a vector of coordinates (latitude and longitude), $\mathbf{x}_j^h = (x_{j,1}^h, x_{j,2}^h)$ and the superscript h means ‘‘hedging’’ (in order to differentiate later with locations of crop fields). The temperature at time t and location of coordinates $\mathbf{x}_j^h$ is assumed to be the sum of one deterministic function $a_w(t, \mathbf{x}) : \mathbb{R}^+ \times \mathbb{R}^2 \rightarrow \mathbb{R}$ and of one Gaussian random field $Z_w(t, \mathbf{x})$:

$$w(t, \mathbf{x}_j^h) = a_w(t, \mathbf{x}_j^h) + Z_w(t, \mathbf{x}_j^h) \quad j = 1, \dots, p \quad (1)$$

In our model, the field $Z_w(t, \mathbf{x})$, where the subscript w is for ‘weather’, is defined a time-changed Brownian sheet. Notice that random field models are infinite-factor models in the sense that we associate a stochastic process to all geographical coordinates. A real valued Brownian sheet is a family of random variables $(W_u)_{u \in \mathbb{R}^m}$ indexed by $u \in \mathbb{R}^m$ with a collection of distribution functions of the form F_u that satisfy

$$F_{u_1, u_2}(b_1, b_2) = \Phi(W_{u_1} \leq b_1, W_{u_2} \leq b_2)$$

where $b_1, b_2 \in \mathbb{R}$ and $\Phi(\cdot)$ is the cumulative distribution function of a bivariate normal random variable. The covariance function is given by

$$\text{Cov}(W_{u_1}, W_{u_2}) = \prod_{i=1}^m (u_{1,i} \wedge u_{2,i})$$

where $u_1 = (u_{1,1}, \dots, u_{1,m})$ and $u_2 = (u_{2,1}, \dots, u_{2,m})$. The Gaussian field $Z_w(t, \mathbf{x})$ is here a time changed Brownian sheet with three dimensions ($m = 3$) defined as follows:

$$Z_w(t, \mathbf{x}) := \frac{\sigma_w(\mathbf{x})}{\sqrt{2\theta_w}} e^{-\theta_w t} e^{-\alpha_1^w x_1 - \alpha_2^w x_2} W_{\nu_1^w(t), \nu_2^w(x_1), \nu_3^w(x_2)} \quad (2)$$

$\sigma_w(\mathbf{x}), \theta_w, \alpha_1^w, \alpha_2^w \in \mathbb{R}^+$. $\nu_1^w(\cdot)$, $\nu_2^w(\cdot)$ and $\nu_3^w(\cdot)$ are time-changes defined by

$$\nu_1^w(t) = e^{2\theta_w t}, \quad \nu_2^w(x_1) = e^{2\alpha_1^w x_1},$$

$$\nu_3^w(x_2) = e^{2\alpha_2^w x_2}.$$

θ_w, α_1^w and α_2^w are positive and constant. The random field $Z_w(t, \mathbf{x})$ is defined on a probability space $(\Omega, \mathcal{F}, P)$ endowed with its natural filtration $(\mathcal{F}_t)_{t \geq 0}$ satisfying the usual conditions. Given that $(W_u)_{u \in \mathbb{R}^3}$ is a 3 dimensions Brownian sheet, we directly infer that the covariance of the Gaussian field at locations $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$ is equal to

$$\begin{aligned} \text{Cov}(Z_w(t, \mathbf{x}), Z_w(s, \mathbf{y})) &= \frac{\sigma_w(\mathbf{x})\sigma_w(\mathbf{y})}{2\theta_w} e^{-\theta_w(t+s)} e^{-\alpha_1^w(x_1+y_1) - \alpha_2^w(x_2+y_2)} \\ &\quad \times e^{2(\theta_w(t \wedge s) + \alpha_1^w(x_1 \wedge y_1) + \alpha_2^w(x_2 \wedge y_2))} \\ &= \frac{\sigma_w(\mathbf{x})\sigma_w(\mathbf{y})}{2\theta_w} e^{-\theta_w|t-s| - \alpha_1^w|x_1-y_1| - \alpha_2^w|x_2-y_2|}. \end{aligned} \quad (3)$$

The covariance decays exponentially and proportionally to time and distance between the two locations. The next proposition shows that we can reformulate the Gaussian field observed at p locations as a system of p mean-reverting (also called Ornstein-Uhlenbeck) processes.

Proposition 2.1. *Let us consider p geographical positions $(\mathbf{x}_1^h, \dots, \mathbf{x}_p^h)$. We denote by σ_w^h the diagonal matrix of $\sigma_w(\mathbf{x}_k^h)$ for $k = 1, \dots, p$, $\sigma_w^h = \text{diag}(\sigma_w(\mathbf{x}_1^h), \dots, \sigma_w(\mathbf{x}_p^h))^\top$. If the correlation matrix of $Z_w(t, \mathbf{x}_k^h)$ for $k = 1 \dots p$ admits the Choleski decomposition*

$$\begin{aligned} (\text{Corr}(Z_w(t, \mathbf{x}_k^h), Z_w(t, \mathbf{x}_l^h)))_{k,l=1,\dots,p} &= \left(e^{-\alpha_1|x_{k,1}^h - x_{l,1}^h| - \alpha_2|x_{k,2}^h - x_{l,2}^h|} \right)_{k,l=1,\dots,p} \\ &= \Sigma_w^h \Sigma_w^{h\top}, \end{aligned}$$

then the processes $Z_w(t, \mathbf{x}_k^h)$ admit the following representation:

$$\begin{pmatrix} Z_w(t, \mathbf{x}_1^h) \\ \vdots \\ Z_w(t, \mathbf{x}_p^h) \end{pmatrix} = e^{-\theta_w t} \begin{pmatrix} Z_w(0, \mathbf{x}_1^h) \\ \vdots \\ Z_w(0, \mathbf{x}_p^h) \end{pmatrix} + \boldsymbol{\sigma}_w^h \Sigma_w^h \begin{pmatrix} \int_0^t e^{-\theta_w(t-u)} dW_u^{h,1} \\ \vdots \\ \int_0^t e^{-\theta_w(t-u)} dW_u^{h,p} \end{pmatrix} \quad (4)$$

where $(W_u^{h,k})_{k=1,\dots,p}$ are independent standard Brownian motions and $(Z_w(0, \mathbf{x}_k^h))_{k=1,\dots,p}$ are distributed according to a multivariate normal random variable: $N\left(0, \frac{1}{2\theta_w} \boldsymbol{\sigma}_w^h \Sigma_w^h \Sigma_w^{h\top} \boldsymbol{\sigma}_w^h\right)$.

The proof is reported in appendix. Let us adopt the following notations in the remainder of this section: $\mathbf{Z}_w^h(t) = (Z_w(t, \mathbf{x}_k^h))_{k=1,\dots,p}$ and $\mathbf{W}_t^h := (W_t^{h,k})_{k=1,\dots,p}$. An alternative formulation of eq. (4) consists of providing the infinitesimal dynamics of the field:

$$d\mathbf{Z}_w^h(t) = -\theta_w \mathbf{Z}_w^h(t) dt + \boldsymbol{\sigma}_w^h \Sigma_w^h d\mathbf{W}_t^h. \quad (5)$$

This last expression clearly emphasizes that $\mathbf{Z}_w^h(t)$ reverts at a speed θ_w to zero.

In finance, derivatives are priced under an equivalent measure, called the risk neutral measure and denoted by Q . The absence of arbitrage entails the existence of at least one equivalent measure eligible for pricing. The next proposition introduces a family of changes of measure for a Gaussian field.

Proposition 2.2. *Let us consider p geographical positions $(\mathbf{x}_1^h, \dots, \mathbf{x}_p^h)$ and a $\mathcal{F}_t$ -adapted process $\varphi(t)$ of dimension p . If $\int_0^T \|\varphi(s)\|_2^2 d\mathbf{W}_s^h < \infty$, then the process*

$$L_w(t) = \frac{dQ}{dP} \Big|_t := \exp \left(-\frac{1}{2} \int_0^t \varphi(s)^\top \varphi(s) ds - \int_0^t \varphi(s)^\top d\mathbf{W}_s^h \right), \quad (6)$$

called the Radon-Nykodym derivative, defines a new probability measure Q under which $\tilde{\mathbf{W}}_t^h = \mathbf{W}_t^h + \int_0^t \varphi(s) ds$ is a standard p -Brownian motion. The dynamics of the Gaussian field under Q is given by:

$$d\mathbf{Z}_w^h(t) = (-\boldsymbol{\sigma}_w^h \Sigma_w^h \varphi(t) - \theta_w \mathbf{Z}_w^h(t)) dt + \boldsymbol{\sigma}_w^h \Sigma_w^h d\tilde{\mathbf{W}}_t^h. \quad (7)$$

For a proof of this result, we refer the interested reader to Lamberton Lapeyre (1997). If $\varphi(t) = -\Sigma_w^{h-1} \boldsymbol{\sigma}_w^{h-1} \theta_w \gamma_w(t)$, the dynamics of the Gaussian field under Q is equal to

$$d\mathbf{Z}_w^h(t) = \theta_w (\gamma_w(t) - \mathbf{Z}_w^h(t)) dt + \boldsymbol{\sigma}_w^h \Sigma_w^h d\tilde{\mathbf{W}}_t^h. \quad (8)$$

If we denote by $\mathbf{1}_p$ the p -vector of ones, then the processes $Z_w(t, \mathbf{x}_k^h)$ admit the following representation under Q :

$$\mathbf{Z}_w^h(t) = e^{-\theta_w t} \mathbf{Z}_w^h(0) + \theta_w \int_0^t \gamma_w(u) e^{-\theta_w u} du \mathbf{1}_p + \boldsymbol{\sigma}_w^h \Sigma_w^h \int_0^t e^{-\theta_w(t-u)} d\tilde{\mathbf{W}}_u^h \quad (9)$$

In later developments, we need to integrate the Gaussian field along the time in order to propose a model in continuous time for CAT indexes. Let us adopt the following notations

$$\begin{aligned} \int_t^T \mathbf{Z}_w^h(s) ds &= \left(\int_t^T Z_w(s, \mathbf{x}_k^h) ds \right)_{k=1,\dots,p}, \\ \int_t^T \frac{1}{\theta_w} (1 - e^{-\theta_w(T-s)}) d\mathbf{W}_s^h &= \left(\int_t^T \frac{1}{\theta_w} (1 - e^{-\theta_w(T-s)}) dW_s^{h,k} \right)_{k=1,\dots,p}. \end{aligned}$$

The next corollary provides the integral of $\mathbf{Z}_w^h(t)$ over a time interval:

Corollary 2.3. *The p vector of integrals along the time of the Gaussian fields at p different locations can be reformulated as the following sum:*

$$\int_t^T \mathbf{Z}_w^h(s) ds = \frac{1}{\theta_w} \left(1 - e^{-\theta_w(T-t)}\right) \mathbf{Z}_w^h(t) + \boldsymbol{\sigma}_w^h \Sigma_w^h \int_t^T \frac{1}{\theta_w} \left(1 - e^{-\theta_w(T-s)}\right) d\mathbf{W}_s^h \quad (10)$$

If we consider the change of measure introduced in equation (6) with $\varphi(t) = -\Sigma_w^{h-1} \boldsymbol{\sigma}_w^{h-1} \theta_w \gamma_w(t)$, the equation (10) may be rewritten under Q as follows

$$\begin{aligned} \int_t^T \mathbf{Z}_w^h(s) ds &= \frac{1}{\theta_w} \left(1 - e^{-\theta_w(T-t)}\right) \mathbf{Z}_w^h(t) + \mathbf{1}_p \int_t^T \gamma_w(u) \left(1 - e^{-\theta_w(T-u)}\right) du \\ &\quad + \frac{\boldsymbol{\sigma}_w^h}{\theta_w} \Sigma_w^h \int_t^T \left(1 - e^{-\theta_w(T-u)}\right) d\tilde{\mathbf{W}}_u^h. \end{aligned} \quad (11)$$

We have now introduced all the material for the definition of cumulated average temperatures. As mentioned earlier, CAT Indexes are the accumulation of daily average temperatures over a calendar month. In this work, the CAT indexes are modeled in continuous time as the integral of temperatures observed at locations $(\mathbf{x}_1^h, \dots, \mathbf{x}_p^h)$ over the growth period $[0, T]$. The values of the CAT indexes at time t are then defined by the sum of integrals of $a_w(t, \mathbf{x})$ and $Z_w(t, \mathbf{x})$:

$$\begin{pmatrix} CAT(t, \mathbf{x}_1^h) \\ \vdots \\ CAT(t, \mathbf{x}_p^h) \end{pmatrix} := \begin{pmatrix} \int_0^t a_w(s, \mathbf{x}_1^h) ds \\ \vdots \\ \int_0^t a_w(s, \mathbf{x}_p^h) ds \end{pmatrix} + \begin{pmatrix} \int_0^t Z_w(s, \mathbf{x}_1^h) ds \\ \vdots \\ \int_0^t Z_w(s, \mathbf{x}_p^h) ds \end{pmatrix}. \quad (12)$$

In order to lighten future calculations, we introduce the following notations for the vectors:

$$\begin{aligned} \mathbf{CAT}(t) &= (CAT(t, \mathbf{x}_k^h))_{k=1, \dots, p}, \\ \mathbf{a}_w^h(s) &= (a_w(s, \mathbf{x}_k^h))_{k=1, \dots, p}, \\ \mathbf{Z}_w^h(s) &= (Z_w(s, \mathbf{x}_k^h))_{k=1, \dots, p}. \end{aligned}$$

By construction, the vector of variations of CAT indexes satisfy the following differential equation:

$$d(\mathbf{CAT}(t)) = \mathbf{a}_w^h(t) dt + \mathbf{Z}_w^h(t) dt.$$

On the other hand, if we remember the equation (10), the CAT at time T may be rewritten as the following sum:

$$\begin{aligned} \mathbf{CAT}(T) &= \mathbf{CAT}(t) + \int_t^T \mathbf{a}_w^h(s) ds + \int_t^T \mathbf{Z}_w^h(s) ds \\ &= \mathbf{CAT}(t) + \int_t^T \mathbf{a}_w^h(s) ds + \frac{1}{\theta_w} \left(1 - e^{-\theta_w(T-t)}\right) \mathbf{Z}_w^h(t) \\ &\quad + \frac{\boldsymbol{\sigma}_w^h}{\theta_w} \Sigma_w^h \int_t^T \left(1 - e^{-\theta_w(T-u)}\right) d\mathbf{W}_u^h \end{aligned} \quad (13)$$

In the remainder of this article, we consider that crops are harvested at date T . We assume that future contracts of maturity T on CAT indexes are traded in the financial market. The initial time $t = 0$ corresponds to the planting date. These contracts allow to buy or to sell CAT indexes. The value of this future at time t , for an harvest at date T and a location $\mathbf{x}$, is denoted by $F(t, T, \mathbf{x})$. This price is equal to the expected discounted value of the CAT indexes on the harvest day:

$$F(t, T, \mathbf{x}_j^h) := e^{-r(T-t)} \mathbb{E}^Q (CAT(T, \mathbf{x}_j^h) | \mathcal{F}_t) \quad j = 1, \dots, p. \quad (14)$$

The market of weather derivatives is incomplete by nature and the choice of the risk neutral measure depends on the risk aversion of market agents. In practice, actuaries select a pricing measure under

which derivative prices contain a reasonable safety margin. If we consider the family of changes of measure introduced in equation (6) with $\varphi(t) = -\Sigma_w^h \sigma_w^{h-1} \theta_w \gamma_w(t)$, the dynamics of the CAT index under the risk neutral measure becomes

$$\begin{aligned} CAT(T) = CAT(t) &+ \int_t^T \tilde{\mathbf{a}}_w^h(s) ds + \frac{1}{\theta_w} \left(1 - e^{-\theta_w(T-t)}\right) \mathbf{Z}_w^h(t) \\ &+ \frac{\sigma_w^h}{\theta_w} \Sigma_w^h \int_t^T \left(1 - e^{-\theta_w(T-u)}\right) d\tilde{\mathbf{W}}_u^h \end{aligned} \quad (15)$$

where

$$\tilde{\mathbf{a}}_w^h(s) = \mathbf{a}_w^h(s) + \mathbf{1}_p \gamma_w(s) \left(1 - e^{-\theta_w(T-s)}\right).$$

$\gamma_w(t)$ may be interpreted as the cost of the weather risk. If $\gamma_w(t) > 0$, CAT indexes increase at a faster pace under the risk neutral measure than in reality. The next proposition establishes the joint dynamics of future prices under real and risk neutral measures.

Proposition 2.4. *If we denote by $\mathbf{F}(t, T)$ the p vector $(F(t, T, \mathbf{x}_k^h))_{k=1, \dots, p}$ of future prices on CAT indexes, we have the following relation:*

$$\begin{aligned} \mathbf{F}(t, T) &= e^{-r(T-t)} CAT(t) + e^{-r(T-t)} \int_t^T \tilde{\mathbf{a}}_w^h(s) ds \\ &+ \frac{e^{-r(T-t)}}{\theta_w} \left(1 - e^{-\theta_w(T-t)}\right) \mathbf{Z}_w^h(t). \end{aligned} \quad (16)$$

The infinitesimal variations of future prices under P and Q are respectively equal to:

$$\begin{aligned} d\mathbf{F}(t, T) &= r\mathbf{F}(t, T)dt - e^{-r(T-t)} \left(\mathbf{1}_p \gamma_w(t) \left(1 - e^{-\theta_w(T-t)}\right)\right) dt \\ &+ \frac{e^{-r(T-t)}}{\theta_w} \left(1 - e^{-\theta_w(T-t)}\right) \sigma_w^h \Sigma_w^h d\mathbf{W}_t^h \end{aligned} \quad (17)$$

and

$$d\mathbf{F}(t, T) = r\mathbf{F}(t, T)dt + \frac{e^{-r(T-t)}}{\theta_w} \left(1 - e^{-\theta_w(T-t)}\right) \sigma_w^h \Sigma_w^h d\tilde{\mathbf{W}}_t^h \quad (18)$$

Equation (18) clearly emphasizes that the expected return of the CAT future is equal to the risk free rate. Whereas equation (17) reveals that this return under P is adjusted by a risk premium. If $\gamma(t) > 0$, the CAT future earns on average less than the risk free rate. The next section introduces pricing formula of options on a basket of CAT.

3 Options on a basket of CAT

European call or put options on a basket of CAT are like stop-loss insurances. This is a derivative that promises the payment of $(\mathbf{h}^\top CAT(T, \mathbf{x}) - K)_+$, at expiry, T . $\mathbf{h} = \{h_1, \dots, h_p\} \in \mathbb{R}^p$ is a p -vector of exposures and $K \in \mathbb{R}$ is retention level. The option price is equal to the discounted payoff under the risk neutral measure:

$$C(t, \mathbf{h}) = \mathbb{E}^Q \left(e^{-r(T-t)} (\mathbf{h}^\top CAT(T) - K)_+ \mid \mathcal{F}_t \right).$$

If $\mathbf{h}$ and K are strictly positive (resp. negative), the option is a call (resp. a put). In the numerical illustration, we will also consider basket mixing short and long positions in CAT. The next corollary is a preliminary result needed for option pricing.

Corollary 3.1. *Let us consider p geographical positions $(\mathbf{x}_1, \dots, \mathbf{x}_p)$. Conditionally to $\mathcal{F}_t$, the vector of integrals $\left(\int_t^T Z_w(t, \mathbf{x}_k^h) ds\right)_{k=1, \dots, p}$ is distributed according to multivariate distributions $N(\mu_w^P(t, \mathbf{Z}_w^h(t)), V_w(t))$ under P and $N(\mu_w^Q(t, \mathbf{Z}_w^h(t)), V_w(t))$ under Q . The means $\mu_w^P(t, \mathbf{Z}_w^h(t))$ and $\mu_w^Q(t, \mathbf{Z}_w^h(t))$ are $\mathbb{R}^p$ -vectors given by*

$$\mu_w^P(t, \mathbf{Z}_w^h(t)) := \frac{1}{\theta_w} \left(1 - e^{-\theta_w(T-t)}\right) \mathbf{Z}_w^h(t), \quad (19)$$

$$\begin{aligned} \mu_w^Q(t, \mathbf{Z}_w^h(t)) &:= \frac{1}{\theta_w} \left(1 - e^{-\theta_w(T-t)}\right) \mathbf{Z}_w^h(t), \\ &+ \mathbf{1}_p \int_t^T \gamma_w(u) \left(1 - e^{-\theta_w(T-u)}\right) du. \end{aligned} \quad (20)$$

The $p \times p$ covariance matrix is equal to

$$\begin{aligned} V_w(t) &:= \frac{1}{\theta_w^2} \boldsymbol{\sigma}_w^h \Sigma_w^h \Sigma_w^{h\top} \boldsymbol{\sigma}_w^h \times \\ &\left((T-t) - \frac{2}{\theta_w} \left(1 - e^{-\theta_w(T-t)}\right) + \frac{1}{2\theta_w} \left(1 - e^{-2\theta_w(T-t)}\right) \right) \end{aligned} \quad (21)$$

This corollary is a direct consequence of equations (10) and (11). Based on this last result, the next proposition presents the closed form expression for CAT basket options.

Proposition 3.2. *Let us consider p geographical positions $(\mathbf{x}_1^h, \dots, \mathbf{x}_p^h)$. The price of a basket option on CAT with exposures $\mathbf{h} = \{h_1, \dots, h_p\} \in \mathbb{R}^p$, of maturity T and strike $K \in \mathbb{R}$ is given by*

$$\begin{aligned} C(t, \mathbf{h}) &= \left(\mathbf{h}^\top \mathbf{F}(t, T) - e^{-r(T-t)} K \right) \Phi \left(\frac{e^{r(T-t)} \mathbf{h}^\top \mathbf{F}(t, T) - K}{\sqrt{\mathbf{h}^\top V_w(t) \mathbf{h}}} \right) \\ &+ e^{-r(T-t)} \sqrt{\mathbf{h}^\top V_w(t) \mathbf{h}} \varphi \left(\frac{e^{r(T-t)} \mathbf{h}^\top \mathbf{F}(t, T) - K}{\sqrt{\mathbf{h}^\top V_w(t) \mathbf{h}}} \right), \end{aligned} \quad (22)$$

where $\Phi(\cdot)$ and $\varphi(\cdot)$ are respectively the cdf and pdf of a standard normal random variable.

In section 7, we will analyze the impact of holding basket CAT options on the variance of the total income of an agricultural company. But before we develop a model for the crop yield.

4 Crop modeling

As mentioned in the previous section, we assume that crop is harvested at date T and that crop is seed at time $t = 0$. The yield per square meter for an agricultural exploitation centered on the position $\mathbf{x}$ is denoted by $Y(\mathbf{x})$. The surface of farming is noted S . We consider here d areas of production with coordinates $\{\mathbf{x}_1^f, \mathbf{x}_2^f, \dots, \mathbf{x}_d^f\}$ where the subscript f is for 'farms'. Positions of farms differ from the locations where CAT are measured. In the remainder of this section, we assume a linear relation between the yield and the accumulation of daily average temperatures and other growth factors:

$$Y(\mathbf{x}) = a_y(\mathbf{x}) + b_w(\mathbf{x}) \int_0^T Z_w(s, \mathbf{x}) ds + Z_y(T, \mathbf{x}) \quad (23)$$

where $Z_y(T, \mathbf{x})$ is a Gaussian field. This assumption of linearity simplifies further developments and will be tested in the numerical illustration. By construction, negative values for $Y(\mathbf{x})$ are allowed. However, if variances of $Z_y(T, \mathbf{x})$ and $\int_0^T Z_w(s, \mathbf{x}) ds$ are small compared to $a_y(\mathbf{x})$, the

probability of negative realization of $Y(x)$ is negligible. The Gaussian field $Z_y(T, \mathbf{x})$ is slightly different from the one ruling temperatures. As $Z_y(T, \mathbf{x})$ gathers different type of risks, we do not consider a mean reverting feature along the time. More precisely, $Z_y(t, x)$ for $t \leq T$ is a partly time changed Brownian sheet:

$$Z_y(t, \mathbf{x}) = \sigma_y e^{-\alpha_1^y x_1 - \alpha_2^y x_2} W_{t, \nu_2^y(x_1), \nu_3^y(x_2)}$$

where $\sigma_y, \alpha_1^y, \alpha_2^y \in \mathbb{R}^+$ and

$$\nu_2^y(x_1) = e^{2\alpha_1^y x_1}, \nu_3(x_2) = e^{2\alpha_2^y x_2}.$$

$(W_u)_{u \in \mathbb{R}^3}$ being a 3 dimensions Brownian sheet, the covariance between two observations of this Brownian field is equal to

$$\text{Cov}(Z_y(t, \mathbf{x}), Z_y(s, \mathbf{y})) = \sigma_y^2 (t \wedge s) e^{-\alpha_1^y |x_1 - y_1| - \alpha_2^y |x_2 - y_2|}.$$

If we denote by $\rho_{x,y}^y = e^{-\alpha_1^y |x_1 - y_1| - \alpha_2^y |x_2 - y_2|}$, the spatial correlation between locations $\mathbf{x}$ and $\mathbf{y}$, we can rewrite the correlation between $Z_y(t, \mathbf{x})$ and $Z_y(t, \mathbf{y})$ as follows:

$$\text{Corr}(Z_y(t, \mathbf{x}), Z_y(s, \mathbf{y})) = \rho_{x,y}^y (t \wedge s).$$

As in the previous section, we adopt the following notations $\mathbf{Y} = (Y(\mathbf{x}_k^f))_{k=1, \dots, d}$,

$\mathbf{a}_y = (a_y(\mathbf{x}_k^f))_{k=1, \dots, d}$ and $\mathbf{Z}_y^f(t) = (Z_y(t, \mathbf{x}_k^f))_{k=1, \dots, d}$ to lighten further developments. The dynamics of $Z_y(t, \mathbf{x})$ is analyzed in the following proposition:

Proposition 4.1. *Let us consider d geographical positions $(\mathbf{x}_1^f, \dots, \mathbf{x}_d^f)$. If the correlation matrix of $Z_y(t, \mathbf{x}_j^f)$ for $j = 1 \dots d$ admits the Choleski decomposition*

$$\begin{aligned} \left(\text{Corr}(Z_y(t, \mathbf{x}_j^f), Z_y(t, \mathbf{x}_k^f)) \right)_{j,k=1, \dots, d} &= \left(e^{-\alpha_1^y |x_{j1}^f - x_{k1}^f| - \alpha_2^y |x_{j2}^f - x_{k2}^f|} \right)_{j,k=1, \dots, d} \\ &= \Sigma_y \Sigma_y^\top, \end{aligned}$$

then the vector of processes $\mathbf{Z}_y^f(t)$ admit the following representation

$$\mathbf{Z}_y^f(t) = \mathbf{Z}_y^f(0) + \sigma_y \Sigma_y \mathbf{W}_t^y \quad (24)$$

where $\mathbf{W}_t^y = (W_u^{y,j})_{j=1, \dots, d}$ is a d vector of independent standard Brownian motions. An alternative formulation is provided by the following differential equation:

$$d\mathbf{Z}_y^f(t) = \sigma_y \Sigma_y d\mathbf{W}_t^y. \quad (25)$$

The crop yield for a given area depends upon the temperature observed at the position of this field but is also correlated to temperatures observed at locations of other farms and where CAT are measured. To analyze this spatial dependence, we use the following vector notations:

$$\mathbf{Z}_w(t) := \begin{pmatrix} \mathbf{Z}_w^h(t) \\ \mathbf{Z}_w^f(t) \end{pmatrix} = \begin{pmatrix} (Z_w(t, \mathbf{x}_k^h))_{k=1, \dots, p} \\ (Z_w(t, \mathbf{x}_k^f))_{k=1, \dots, d} \end{pmatrix},$$

and

$$\int_t^T \mathbf{Z}_w(s) ds := \begin{pmatrix} \int_t^T \mathbf{Z}_w^h(s) ds \\ \int_t^T \mathbf{Z}_w^f(s) ds \end{pmatrix} = \begin{pmatrix} \left(\int_t^T Z_w(s, \mathbf{x}_k^h) ds \right)_{k=1, \dots, p} \\ \left(\int_t^T Z_w(s, \mathbf{x}_k^f) ds \right)_{k=1, \dots, d} \end{pmatrix}.$$

The next proposition provides additional information about the multivariate distribution of crop yields.

Proposition 4.2. We denote by Σ_w the $p+d$ matrix that is the Choleski decomposition matrix of the spatial correlation matrix of values of $Z_w(t, \mathbf{x})$ at locations $(\mathbf{x}_1^h, \dots, \mathbf{x}_p^h)$ and $(\mathbf{x}_1^f, \dots, \mathbf{x}_d^f)$:

$$\text{corr}(\mathbf{Z}_w(t), \mathbf{Z}_w(t)) := \Sigma_w \Sigma_w^\top$$

where

$$\Sigma_w := \begin{pmatrix} \Sigma_w^h & 0 \\ \Sigma_w^{fh} & \Sigma_w^f \end{pmatrix}.$$

We also note $\mathbf{b}_w = \text{diag}(b_w(x_1^f), \dots, b_w(x_d^f))$, $\boldsymbol{\sigma}_w^h = \text{diag}(\sigma_w(x_k^h))_{k=1, \dots, p}$, $\boldsymbol{\sigma}_w^f = \text{diag}(\sigma_w(x_k^f))_{k=1, \dots, d}$ and $\boldsymbol{\sigma}_w = \text{diag}(\boldsymbol{\sigma}_w^h, \boldsymbol{\sigma}_w^f)$. By construction, crop yields are distributed according to a multivariate Normal distribution. The expected yield and the variance of the yield conditionally to the information available at time t are respectively equal to

$$\mathbb{E}(\mathbf{Y}|\mathcal{F}_t) = \mathbf{a}_y + \mathbf{b}_w \int_0^t \mathbf{Z}_w^f(s) ds + \frac{\mathbf{b}_w}{\theta_w} \left(1 - e^{-\theta_w(T-t)}\right) \mathbf{Z}_w^f(t) + \mathbf{Z}_y^f(t) \quad (26)$$

$$\mathbb{V}(\mathbf{Y}|\mathcal{F}_t) = \frac{1}{\theta_w^2} \int_t^T \left(1 - e^{-\theta_w(T-s)}\right)^2 ds \quad (27)$$

$$\begin{aligned} & \times \mathbf{b}_w \boldsymbol{\sigma}_w^f (\Sigma_w^{fh} \Sigma_w^{fh\top} + \Sigma_w^f \Sigma_w^{f\top}) \boldsymbol{\sigma}_w^f \mathbf{b}_w \\ & + \sigma_y^2 (T-t) \Sigma_y \Sigma_y^\top \end{aligned} \quad (28)$$

The expected crop yields form a stochastic process of dimension d that we denote by $\mathbf{Y}(t) := \left(\mathbb{E}(Y(x_k^f)|\mathcal{F}_t)\right)_{k=1, \dots, d}$. From the previous proposition, this process is driven by the next infinitesimal dynamics:

$$\begin{aligned} \mathbf{Y}(t) &= \mathbb{E}(\mathbf{Y}|\mathcal{F}_t) \\ &= \mathbf{a}_y + \mathbf{b}_w \int_0^t \mathbf{Z}_w^f(s) ds + \frac{1}{\theta_w} \left(1 - e^{-\theta_w(T-t)}\right) \mathbf{b}_w \mathbf{Z}_w^f(t) + \mathbf{Z}_y^f(t). \end{aligned} \quad (29)$$

We deduce from this last relation that the dynamics of the predicted crop yield is ruled by the following stochastic differential equation

$$\begin{aligned} d\mathbf{Y}(t) &= \left(1 - e^{-\theta_w(T-t)}\right) \mathbf{b}_w \mathbf{Z}_w^f(t) dt + \frac{1}{\theta_w} \left(1 - e^{-\theta_w(T-t)}\right) \mathbf{b}_w d\mathbf{Z}_w^f(t) \\ &+ d\mathbf{Z}_y^f(t) \end{aligned}$$

in which we can inject the infinitesimal dynamics of Gaussian fields :

$$\begin{pmatrix} d\mathbf{Z}_w^h(t) \\ d\mathbf{Z}_w^f(t) \end{pmatrix} = -\theta_w \begin{pmatrix} \mathbf{Z}_w^h(t) \\ \mathbf{Z}_w^f(t) \end{pmatrix} dt + \boldsymbol{\sigma}_w \begin{pmatrix} \Sigma_w^h & 0 \\ \Sigma_w^{fh} & \Sigma_w^f \end{pmatrix} \begin{pmatrix} d\mathbf{W}_t^h \\ d\mathbf{W}_t^f \end{pmatrix}$$

and $d\mathbf{Z}_y^f(t) = \sigma_y \Sigma_y d\mathbf{W}_t^y$ to finally infer the dynamics of the d vector of expected crop yields:

$$d\mathbf{Y}(t) = \frac{1}{\theta_w} \left(1 - e^{-\theta_w(T-t)}\right) \mathbf{b}_w \boldsymbol{\sigma}_w^f \left(\Sigma_w^{fh} d\mathbf{W}_t^h + \Sigma_w^f d\mathbf{W}_t^f\right) + \sigma_y \Sigma_y d\mathbf{W}_t^y. \quad (30)$$

By construction, this process converges toward $\mathbf{Y}$ when $t \rightarrow T$ and is an anticipation of the future production.

5 Dynamic minimum variance hedging

In this section, we determine the optimal investment policy in CAT futures reducing the variance of crop incomes of an agricultural company. This company purchases the production of p farms and manages a portfolio of cash and d futures. The cash account is denoted by B_t and earns the risk free rate r . As in previous sections, locations of farms and CAT indexes are respectively noted $(\mathbf{x}_1^f, \dots, \mathbf{x}_d^f)$ and $(\mathbf{x}_1^h, \dots, \mathbf{x}_p^h)$.

At time t , the investment strategy is contained in a $p+1$ vector, $\omega(t) = (\omega_1(t), \dots, \omega_p(t))$ that are the number of CAT futures. $\omega_0(t)$ is the amount of treasury. The cash account is denoted by $B(t)$ and earns a constant interest rate r . To ensure that the portfolio is self-financed, we add the constraint $\omega_0(t) = -\sum_{k=1}^p \omega_k(t) F(t, T, \mathbf{x}_k^h)$. If the value of the futures portfolio at time t is denoted by R_t , its dynamics is then ruled by the following differential equation:

$$dR_t = \omega_0(t) \frac{dB_t}{B_t} + \sum_{k=1}^p \omega_k(t) dF(t, T, \mathbf{x}_k^h). \quad (31)$$

If we remember that the infinitesimal variation of the p vector of future prices is defined by equation (17), the dynamics of the future portfolio becomes:

$$\begin{aligned} dR_t = & -e^{-r(T-t)} \left(\gamma_w(t) \left(1 - e^{-\theta_w(T-t)} \right) \right) \omega(t)^\top \mathbf{1}_p dt \\ & + \frac{e^{-r(T-t)}}{\theta_w} \left(1 - e^{-\theta_w(T-t)} \right) \omega(t)^\top \boldsymbol{\sigma}_w^h \Sigma_w^h d\mathbf{W}_t^h. \end{aligned} \quad (32)$$

Let us consider that the selling price of crop per square meter, noted P , is constant¹. The total income from the harvest is then equal to $\sum_{k=1}^d S_k P Y(T, \mathbf{x}_k^f)$. If the objective of the agricultural company consists in minimizing the variance of the sum of its portfolio of derivatives and of harvest incomes:

$$V(t, \mathbf{Y}(t), R_t) = \min_{\omega(t)} \mathbb{V} \left(\left(R_T + \sum_{k=1}^d S_k P Y(T, \mathbf{x}_k^f) \right) | \mathcal{F}_t \right), \quad (33)$$

the optimal trading strategy is detailed in the following proposition.

Proposition 5.1. *The CAT portfolio minimizing the variance of the company earnings is equal to*

$$\omega_{min. var.}(t) = -e^{r(T-t)} P \boldsymbol{\sigma}_w^{h-1} \Sigma_w^{h-1\top} \Sigma_w^{fh\top} \boldsymbol{\sigma}_w^f \mathbf{b}_w \mathbf{S}. \quad (34)$$

In this case, the farmer's position only depends upon the risks that cannot be hedged by financial derivatives:

$$\begin{aligned} dR_t + P \mathbf{S}^\top d\mathbf{Y}(t) = & -e^{-r(T-t)} \left(\gamma_w(t) \left(1 - e^{-\theta_w(T-t)} \right) \right) \omega_{min. var.}(t)^\top \mathbf{1}_p dt \\ & + \frac{P}{\theta_w} \left(1 - e^{-\theta_w(T-t)} \right) \mathbf{S}^\top \mathbf{b}_w \boldsymbol{\sigma}_w^f \left(\Sigma_w^f d\mathbf{W}_t^f \right) + \sigma_y P \mathbf{S}^\top \Sigma_y d\mathbf{W}_t^y. \end{aligned} \quad (35)$$

Notice that the optimal strategy does not depend on expectations about crop yields.

6 Dynamic maximum utility hedging

In this section we assume that the agricultural company aims at maximizing the utility of its incomes from crop harvest and from positions in CAT futures. In this case, we consider a valued function that is defined by

$$V(t, \mathbf{Y}(t), R_t) = \max_{\omega(t)} \mathbb{E} \left(U \left(R_T + P \sum_{k=1}^d S_k Y(T, \mathbf{x}_k^f) \right) | \mathcal{F}_t \right) \quad (36)$$

¹In practice, the selling prices may be fixed with short positions in futures on commodities

where $U(\cdot) : \mathbb{R} \rightarrow \mathbb{R}$ is an exponential utility function with parameter $\beta \in \mathbb{R}^+$

$$U(z) = -\frac{1}{\beta} \exp(-\beta z).$$

This choice of utility function implies that the economic agent has a constant absolute risk aversion, determined by the coefficient β . Remember that the dynamics of state variables R_t and $\mathbf{Y}(t)$ are given by equations (31) and (30). In order to lighten further developments, we define denote by $g(t)$, the following function:

$$g(t) := \frac{1}{\theta_w} \left(1 - e^{-\theta_w(T-t)} \right).$$

This allows us to rewrite the dynamics of $\mathbf{Y}(t)$ and R_t as follows

$$d\mathbf{Y}(t) = \begin{pmatrix} g(t)\mathbf{b}_w\boldsymbol{\sigma}_w^f\Sigma_w^{fh} & g(t)\mathbf{b}_w\boldsymbol{\sigma}_w^f\Sigma_w^f & \sigma_y\Sigma_y \end{pmatrix} \begin{pmatrix} d\mathbf{W}_t^h \\ d\mathbf{W}_t^f \\ d\mathbf{W}_t^y \end{pmatrix},$$

and

$$dR_t = - \left(\gamma_w(t)\theta_w g(t)e^{-r(T-t)} \right) \omega(t)^\top \mathbf{1}_p dt \\ + g(t)e^{-r(T-t)} \omega(t)^\top \boldsymbol{\sigma}_w^h \Sigma_w^h d\mathbf{W}_t^h.$$

Furthermore, we denote the covariance matrix of $d\mathbf{Y}(t)$ by $\Upsilon(t)$. This is a matrix of dimension $d \times d$, function of time such that:

$$\begin{aligned} \Upsilon(t) &:= \begin{pmatrix} g(t)\mathbf{b}_w\boldsymbol{\sigma}_w^f\Sigma_w^{fh} & g(t)\mathbf{b}_w\boldsymbol{\sigma}_w^f\Sigma_w^f & \sigma_y\Sigma_y \end{pmatrix} \begin{pmatrix} g(t)\Sigma_w^{fh\top}\boldsymbol{\sigma}_w^f\mathbf{b}_w \\ g(t)\Sigma_w^{f\top}\boldsymbol{\sigma}_w^f\mathbf{b}_w \\ \sigma_y\Sigma_y^\top \end{pmatrix} \\ &= g(t)^2\mathbf{b}_w\boldsymbol{\sigma}_w^f\Sigma_w^{fh}\Sigma_w^{fh\top}\boldsymbol{\sigma}_w^f\mathbf{b}_w + g(t)^2\mathbf{b}_w\boldsymbol{\sigma}_w^f\Sigma_w^f\Sigma_w^{f\top}\boldsymbol{\sigma}_w^f\mathbf{b}_w \\ &\quad + \sigma_y^2\Sigma_y\Sigma_y^\top. \end{aligned}$$

Using the theory of stochastic control (see e.g. Fleming and Rishel, 1975), we know that the value function such as defined in equation (36) is solution of the following Hamilton Jacobi Bellman (HJB) equation:

$$\begin{aligned} 0 = \max_{\omega} & \left(\frac{\partial V}{\partial t} - \left(\gamma_w(t)\theta_w e^{-r(T-t)} g(t) \right) \omega(t)^\top \mathbf{1}_p \frac{\partial V}{\partial R} + \frac{1}{2} \text{tr} \left(\Upsilon(t) \frac{\partial^2 V}{\partial Y^2} \right) \right. \\ & + \frac{1}{2} e^{-2r(T-t)} g(t)^2 \omega(t)^\top \boldsymbol{\sigma}_w^h \Sigma_w^h \Sigma_w^{h\top} \boldsymbol{\sigma}_w^h \omega(t) \frac{\partial^2 V}{\partial R^2} \\ & \left. + \left[e^{-r(T-t)} g(t)^2 \omega(t)^\top \boldsymbol{\sigma}_w^h \Sigma_w^h \Sigma_w^{fh\top} \boldsymbol{\sigma}_w^f \mathbf{b}_w \right] \frac{\partial V}{\partial R \partial Y} \right). \end{aligned} \quad (37)$$

As stated in the next proposition, this HJB equation admits an analytical solution.

Proposition 6.1. *The value function solving equation (37) is equal to*

$$V(t, R, Y) = -\frac{1}{\beta} \exp \left(A(t) - \beta R_t - \beta P S^\top \mathbf{Y}_t \right) \quad (38)$$

where $A(t) : \mathbb{R}^+ \rightarrow \mathbb{R}$ is a function of time solution of an ordinary differential equation:

$$\begin{aligned} \frac{\partial}{\partial t} A(t) &= \frac{1}{2} \gamma_w(t)^2 \theta_w^2 \mathbf{1}_p^\top \left[\boldsymbol{\sigma}_w^{h-1} \Sigma_w^{h\top-1} \Sigma_w^{h-1} \boldsymbol{\sigma}_w^{h-1} \right] \mathbf{1}_p \\ &\quad - \frac{\beta^2 P^2}{2} \text{tr} \left(\Upsilon(t) S S^\top \right) + \frac{\beta^2 P^2}{2} g(t)^2 \left[S^\top \mathbf{b}_w \boldsymbol{\sigma}_w^f \Sigma_w^{fh} \Sigma_w^{fh\top} \boldsymbol{\sigma}_w^f \mathbf{b}_w S \right] \\ &\quad + (\beta \gamma_w(t) \theta_w g(t)) P S^\top \left[\mathbf{b}_w \boldsymbol{\sigma}_w^f \Sigma_w^{fh} \Sigma_w^{h-1} \boldsymbol{\sigma}_w^{h-1} \right] \mathbf{1}_p. \end{aligned}$$

with the terminal condition $A(T) = 0$. Furthermore, the strategy maximizing the expected utility of terminal incomes is given by

$$\begin{aligned} \omega_{expo}(t) = & -e^{r(T-t)} P [\sigma_w^{h-1} \Sigma_w^{h-1 \top} \Sigma_w^{fh \top} \sigma_w^f b_w] S \\ & - \frac{\gamma_w(t) \theta_w^2 e^{r(T-t)}}{\beta (1 - e^{-\theta_w(T-t)})} [\sigma_w^{h-1} \Sigma_w^{h-1 \top} \Sigma_w^{h-1} \sigma_w^{h-1}] \mathbf{1}_p. \end{aligned} \quad (39)$$

We observe that the strategy maximizing the utility of total incomes is the sum of two terms. The first one is the portfolio minimizing the variance of incomes, established in the previous section. The second term is a speculative position that takes advantage of the risk premium on CAT futures:

$$\begin{aligned} \omega_{expo}(t) = & \underbrace{-e^{r(T-t)} P [\sigma_w^{h-1} \Sigma_w^{h-1 \top} \Sigma_w^{fh \top} \sigma_w^f b_w] S}_{\omega_{min\,var}} \\ & - \underbrace{\frac{\gamma_w(t) \theta_w^2 e^{r(T-t)}}{\beta (1 - e^{-\theta_w(T-t)})} [\sigma_w^{h-1} \Sigma_w^{h-1 \top} \Sigma_w^{h-1} \sigma_w^{h-1}] \mathbf{1}_p}_{\omega_{spec}}. \end{aligned}$$

If the risk premium $\gamma_w(t)$ is positive, ω_{spec} consists in short positions in futures. Another interesting observation concerns the evolution of risk premium of CAT futures. If $\gamma_w(t)$ is constant and positive, we have that $\lim_{t \rightarrow T} \omega_{spec}(t) = -\infty$. Given that the uncertainty on the evolution of CAT indexes decreases with time, the company increases its position in futures when contracts arrive to maturity. A constant risk premium is then not a relevant assumption. We must impose the following constraint on $\gamma_w(t)$

$$\lim_{t \rightarrow T} \frac{\gamma_w(t)}{(1 - e^{-\theta_w(T-t)})} < \infty$$

in order to avoid infinite short positions at expiry of contracts. From this constraint we infer that the risk premium decay at least proportionally to $1 - e^{-\theta_w(T-t)}$. In numerical applications, we will assume that $\gamma_w(t) = \gamma_1 (1 - e^{-(\theta_w + \gamma_2)(T-t)})$ where $\gamma_1, \gamma_2 \in \mathbb{R}^+$. We may easily check that in this case positions in futures do not explode. Indeed, using the L'Hospital's rule allows us to infer that

$$\begin{aligned} \lim_{t \rightarrow T} \frac{\gamma_1 (1 - e^{-(\theta_w + \gamma_2)(T-t)})}{(1 - e^{-\theta_w(T-t)})} &= \frac{\gamma_1 (\theta_w + \gamma_2) e^{-(\theta_w + \gamma_2)(T-t)}}{\theta_w e^{-\theta_w(T-t)}} \Big|_{t=T} \\ &= \gamma_1 \left(1 + \frac{\gamma_2}{\theta_w} \right). \end{aligned}$$

In this case, the optimal portfolio converges toward the following positions at maturity:

$$\begin{aligned} \lim_{t \rightarrow T} \omega(t) = & -P [\sigma_w^{h-1} \Sigma_w^{h-1 \top} \Sigma_w^{fh \top} \sigma_w^f b_w] S \\ & - \frac{\gamma_1}{\beta} \left(1 + \frac{\gamma_2}{\theta_w} \right) \theta_w^2 [\sigma_w^{h-1} \Sigma_w^{h-1 \top} \Sigma_w^{h-1} \sigma_w^{h-1}] \mathbf{1}_p. \end{aligned}$$

7 Static hedging with option

In this section, we evaluate the impact of purchasing a CAT basket option on the expected income from harvest and on its variance. The global income at time T is the sum of the option payoff of crop yields, reduced by the option fee :

$$R_T = \left(\mathbf{h}^\top \mathbf{CAT}(T) - K \right)_+ - C(t, \mathbf{h}) e^{rT} + P \mathbf{S}^\top \mathbf{Y}.$$

In the numerical illustration, we will determine numerically the basket of CAT that minimizes the variance of this terminal income. Unfortunately, there does not exist any closed form solution for this optimal basket. However, we will see in following propositions that variance and expectation of the terminal income admit analytical expressions. Then, the numerical search of the CAT basket minimizing the variance does not present any particular difficulties. The next proposition introduce the expected income at maturity from time $t = 0$.

Proposition 7.1. *To lighten notations, let us denote by V_w the covariance matrix $V_w(0)$ such as defined in equation (21). The expected terminal income is equal to*

$$\begin{aligned} \mathbb{E}(R_T|\mathcal{F}_0) &= P \mathbf{S}^\top \left(\mathbf{a}_y + \frac{\mathbf{b}_w}{\theta_w} (1 - e^{-\theta_w T}) \mathbf{Z}_w^f(0) + \mathbf{Z}_y^f(0) \right) + \left(\mathbf{h}^\top \mathbb{E}(\mathbf{CAT}(T)|\mathcal{F}_0) - K \right) \\ &\quad \times \Phi \left(\frac{\mathbf{h}^\top \mathbb{E}(\mathbf{CAT}(T)|\mathcal{F}_0) - K}{\sqrt{\mathbf{h}^\top V_w \mathbf{h}}} \right) + \sqrt{\mathbf{h}^\top V_w \mathbf{h}} \varphi \left(\frac{\mathbf{h}^\top \mathbb{E}(\mathbf{CAT}(T)|\mathcal{F}_0) - K}{\sqrt{\mathbf{h}^\top V_w \mathbf{h}}} \right) \\ &\quad - \left(e^{rT} \mathbf{h}^\top \mathbf{F}(0, T) - K \right) \Phi \left(\frac{e^{rT} \mathbf{h}^\top \mathbf{F}(0, T) - K}{\sqrt{\mathbf{h}^\top V_w \mathbf{h}}} \right) - \sqrt{\mathbf{h}^\top V_w \mathbf{h}} \varphi \left(\frac{e^{rT} \mathbf{h}^\top \mathbf{F}(0, T) - K}{\sqrt{\mathbf{h}^\top V_w \mathbf{h}}} \right) \end{aligned} \quad (40)$$

where

$$\mathbb{E}(\mathbf{CAT}(T)|\mathcal{F}_0) = \frac{1}{\theta_w} (1 - e^{-\theta_w T}) \mathbf{Z}_w^h(0) + \int_0^T \mathbf{a}_w^h(s) ds.$$

The variance of the terminal income is equal to the sum of the variance of the option payoff, of crop incomes, and of their covariances:

$$\begin{aligned} \mathbb{V}(R_T|\mathcal{F}_0) &= \mathbb{V} \left(\left(\mathbf{h}^\top \mathbf{CAT}(T) - K \right)_+^2 | \mathcal{F}_0 \right) \\ &\quad + 2P \mathbb{C} \left(\left(\mathbf{h}^\top \mathbf{CAT}(T) - K \right)_+, \mathbf{S}^\top \mathbf{Y} | \mathcal{F}_0 \right) \\ &\quad + P^2 \mathbb{V} \left(\mathbf{S}^\top \mathbf{Y} | \mathcal{F}_0 \right) \end{aligned} \quad (41)$$

In the next three propositions, we develop each term involved in this sum. We start by presenting the variance of crop incomes:

Corollary 7.2. *The variance of crop incomes is the following function:*

$$\begin{aligned} \mathbb{V}(\mathbf{S}^\top \mathbf{Y} | \mathcal{F}_0) &= \frac{1}{\theta_w^2} \left(T - \frac{2}{\theta_w} (1 - e^{-\theta_w T}) + \frac{1}{2\theta_w} (1 - e^{-2\theta_w T}) \right) \\ &\quad \times \mathbf{S}^\top \mathbf{b}_w \boldsymbol{\sigma}_w^f (\Sigma_w^{fh} \Sigma_w^{fh^\top} + \Sigma_w^f \Sigma_w^{f^\top}) \boldsymbol{\sigma}_w^f \mathbf{b}_w \mathbf{S} + \sigma_y^2 T \mathbf{S}^\top \Sigma_y \Sigma_y^\top \mathbf{S}. \end{aligned} \quad (42)$$

Next, we determine the variance of the option payoff:

Proposition 7.3. *The variance under P of the payoff promised by a basket option on CAT measured at locations $\mathbf{x}^h = (\mathbf{x}_1^h, \dots, \mathbf{x}_p^h)$ is given by*

$$\begin{aligned} \mathbb{V} \left(\left(\left(\mathbf{h}^\top \mathbf{CAT}(T) - K \right)_+ \right)^2 | \mathcal{F}_t \right) &= \sqrt{\mathbf{h}^\top V_w \mathbf{h}} \left(\mathbf{h}^\top \mu_w^P - K_m \right) \varphi \left(\frac{\mathbf{h}^\top \mu_w^P - K_m}{\sqrt{\mathbf{h}^\top V_w \mathbf{h}}} \right) \\ &\quad + \left(\mathbf{h}^\top V_w \mathbf{h} + \left(\mathbf{h}^\top \mu_w^P \right)^2 - 2K_m \mathbf{h}^\top \mu_w^P + K_m^2 \right) \Phi \left(\frac{\mathbf{h}^\top \mu_w^P - K_m}{\sqrt{\mathbf{h}^\top V_w \mathbf{h}}} \right) \\ &\quad - \left(\left(\mathbf{h}^\top \mu_w^P - K_m \right) \Phi \left(\frac{\mathbf{h}^\top \mu_w^P - K_m}{\sqrt{\mathbf{h}^\top V_w \mathbf{h}}} \right) + \sqrt{\mathbf{h}^\top V_w \mathbf{h}} \varphi \left(\frac{\mathbf{h}^\top \mu_w^P - K_m}{\sqrt{\mathbf{h}^\top V_w \mathbf{h}}} \right) \right)^2 \end{aligned} \quad (43)$$

where , $\mu_w^P = \mu_w^P(0, \mathbf{Z}_w^h(0)) = \frac{1}{\theta_w} (1 - e^{-\theta_w T}) \mathbf{Z}_w^h(0)$ and $K_m = K_m(0) = K - \mathbf{h}^\top \int_0^T \mathbf{a}_w^h(s) ds$.

The next proposition details the covariance between the option payoff and crop yield. The valuation of this covariance requires a numerical integration.

Proposition 7.4. *Let us adopt the following notations for the expectation and variance of $\mathbf{S}^\top \mathbf{Z}_w^f(T)$:*

$$\mu_f := \mathbb{E}(\mathbf{Z}_w^f(T) | \mathcal{F}_0) = \frac{1}{\theta_w} (1 - e^{-\theta_w T}) \mathbf{S}^\top \mathbf{b}_w \mathbf{Z}_w^f(0), \quad (44)$$

$$\begin{aligned} \sigma_f^2 &:= \mathbb{V}(\mathbf{Z}_w^f(T) | \mathcal{F}_0) = \frac{1}{\theta_w^2} \left(T - \frac{2}{\theta_w} (1 - e^{-\theta_w T}) + \frac{1}{2\theta_w} (1 - e^{-2\theta_w T}) \right) \\ &\quad \times \mathbf{S}^\top \mathbf{b}_w \boldsymbol{\sigma}_w^f (\Sigma_w^f \Sigma_w^{f\top} + \Sigma_w^{fh} \Sigma_w^{fh\top}) \boldsymbol{\sigma}_w^f \mathbf{b}_w \mathbf{S}. \end{aligned} \quad (45)$$

Given that the conditional random variable $\left(\int_0^T \mathbf{h}^\top \mathbf{Z}_w^h(s) ds \mid \int_0^T \mathbf{S}^\top \mathbf{b}_w \mathbf{Z}_w^f(s) ds = z \right)$ is normal $N(\mu_{h|f}(z), \sigma_{h|f}^2)$, where

$$\mu_{h|f}(z) = \frac{1}{\theta_w} (1 - e^{-\theta_w T}) \mathbf{h}^\top \mathbf{Z}_w^h(0) + \frac{\mathbf{S}^\top \mathbf{b}_w \boldsymbol{\sigma}_w^f \Sigma_w^{fh} \Sigma_w^{h\top} \boldsymbol{\sigma}_w^h \mathbf{h}}{\mathbf{S}^\top \mathbf{b}_w \boldsymbol{\sigma}_w^f (\Sigma_w^f \Sigma_w^{f\top} + \Sigma_w^{fh} \Sigma_w^{fh\top}) \boldsymbol{\sigma}_w^f \mathbf{b}_w \mathbf{S}} \quad (46)$$

$$\begin{aligned} &\times \left(z - \frac{1}{\theta_w} (1 - e^{-\theta_w T}) \mathbf{S}^\top \mathbf{b}_w \mathbf{Z}_w^f(0) \right) \\ \sigma_{h|f}^2 &= \left(T - \frac{2}{\theta_w} (1 - e^{-\theta_w T}) + \frac{1}{2\theta_w} (1 - e^{-2\theta_w T}) \right) \\ &\times \left(\mathbf{h}^\top \boldsymbol{\sigma}_w^h \Sigma_w^{hh} \Sigma_w^{h\top} \boldsymbol{\sigma}_w^h \mathbf{h} - \frac{(\mathbf{S}^\top \mathbf{b}_w \boldsymbol{\sigma}_w^f \Sigma_w^{fh} \Sigma_w^{h\top} \boldsymbol{\sigma}_w^h \mathbf{h})^2}{\mathbf{S}^\top \mathbf{b}_w \boldsymbol{\sigma}_w^f (\Sigma_w^f \Sigma_w^{f\top} + \Sigma_w^{fh} \Sigma_w^{fh\top}) \boldsymbol{\sigma}_w^f \mathbf{b}_w \mathbf{S}} \right) \end{aligned} \quad (47)$$

The covariance between the option payoff and crop yield is equal to:

$$\begin{aligned} &\mathbb{C} \left(\left(\mathbf{h}^\top \mathbf{C} \mathbf{A} \mathbf{T}(T) - K \right)_+, \mathbf{S}^\top \mathbf{Y} \mid \mathcal{F}_0 \right) \\ &= \int_{-\infty}^{\infty} \left(z \mu_{h|f}(z) \Phi \left(\frac{\mu_{h|f}(z) - K_m}{\sigma_{h|f}} \right) + z \sigma_{h|f} \varphi \left(\frac{\mu_{h|f}(z) - K_m}{\sigma_{h|f}} \right) \right. \\ &\quad \left. - K_m z \sigma_{h|f} \Phi \left(\frac{\mu_{h|f}(z) - K_m}{\sigma_{h|f}} \right) \right) \times \varphi \left(\frac{z - \mu_f}{\sigma_f} \right) dz \\ &\quad - \left(\left(\mathbf{h}^\top \mu_w^P - K_m \right) \Phi \left(\frac{\mathbf{h}^\top \mu_w^P - K_m}{\sqrt{\mathbf{h}^\top V_w \mathbf{h}}} \right) + \sqrt{\mathbf{h}^\top V_w \mathbf{h}} \varphi \left(\frac{\mathbf{h}^\top \mu_w^P - K_m}{\sqrt{\mathbf{h}^\top V_w \mathbf{h}}} \right) \right) \\ &\quad \times \frac{1}{\theta_w} (1 - e^{-\theta_w T}) \mathbf{S}^\top \mathbf{b}_w \mathbf{Z}_w^f(0). \end{aligned} \quad (48)$$

8 Numerical Illustration

In this section, we test the efficiency of strategies found in previous sections for hedging incomes from the harvest of green maize near two Belgian cities, Ieper and Arlon. We choose Ieper and Arlon because these cities face different weather conditions. Ieper is closer to the sea, the landscape is flatter and the altitude is lower than Arlon. All over the year, temperatures are higher in Ieper. Whereas the variation in temperature between summer and winter times is more moderated than in Arlon. We consider two hedging instruments: futures on CAT measured in Paris (more precisely at the Orly Airport) and at the Frankfurt airport. Today, such contracts are not traded. However if the European weather derivatives market expands, it seems likely that CAT futures will be traded in these two big marketplaces. We don't have any information about prices of these contracts but it is possible to reconstruct past prices from historical data, using the valuation formula of section 2.

From the Eurostat portal², we retrieve the yield of cultures of green maize (tons per hectare) for two Belgian provinces: the "West Vlandereen" and "Luxembourg" regions. The dataset contains historical yields from 1975 to 2016. We use these figures to respectively approximate green

²<http://ec.europa.eu/eurostat/>

maize yields in Ieper and Arlon, that are located in these regions. From the website of ECAD³, we download daily average temperatures (in Celsius) measured over the same period in Lille (France), Paris, Frankfurt and at the Luxembourg Airport. Given that Lille and the Luxembourg Airport are respectively less than 32 kilometers away from Ieper and Arlon, we use these temperatures as approximation for the two Belgian cities.

The seasonality of temperatures is captured by the function $a_w(t, \mathbf{x})$ which is the expected temperature on day t at coordinates $\mathbf{x}$. We fit this function in two steps. First, we calculate the average over the period 1975 to 2016, of daily temperatures for Paris, Frankfurt, Ieper and Arlon. Next, we smooth the curves with the Whittaker-Henderson method. The result is displayed in figure 1.

Figure 1: Functions $a_w(t, \mathbf{x})$ for Paris, Frankfurt, Ieper and Arlon.

Next, we calibrate the parameters of the Gaussian field $Z_w(t, \mathbf{x})$ by the method of moments. Instead of using latitudes and longitudes for locating cities, we work with vertical-horizontal Cartesian coordinates, measured on a plane surface centered on Paris⁴. These coordinates are reported in table 1, that are next used to determine the parameters α_1^w and α_2^w . Remember that these parameters define the spatial correlation of temperatures through the relation $\rho_{x,y} = e^{-\alpha_1^w |x_1 - y_1| - \alpha_2^w |x_2 - y_2|}$. Tables 2 and 3 respectively present the observed and modeled spatial correlations between temperatures measured on the same day. We do not fully reconstruct the observed spatial correlations. However, the spread between observed and modeled correlations does not exceed 0.03.

	Horizontal coordinates	Vertical coordinates
Paris	0	0
Frankfurt	448.02	220.55
Ieper	7.39	225.89
Arlon	230.15	107.40

Table 1: Cartesian coordinates of cities in kilometers on a plane surface centered on Paris (Only airport).

³<http://www.ecad.eu/dailydata/>

⁴As in Norton et al. (2012), we calculate the distance between two longitude-latitude positions, $(lat1, lon1)$ and $(lat2, lon2)$ with the formula $d = \text{acos}(\sin(lat1) \sin(lat2) + \cos(lat1) \cos(lat2) \cos(lon1 - lon2))$.

	P.	F.	I.	A.
P.	1.00	0.81	0.91	0.90
F.	0.81	1.00	0.83	0.92
I.	0.90	0.83	1.00	0.89
A.	0.90	0.92	0.89	1.00

Table 2: Observed correlations between temperatures on the same day, in Paris (P.), Frankfurt (F.), Ieper (I.) and Arlon (A.).

	P.	F.	I.	A.
P.	1.00	0.80	0.93	0.89
F.	0.80	1.00	0.85	0.89
I.	0.93	0.85	1.00	0.89
A.	0.89	0.89	0.89	1.00

Table 3: Modeled correlations between temperatures on the same day, in Paris (P.), Frankfurt (F.), Ieper (I.) and Arlon (A.), computed with the formula: $e^{-\alpha_1^w |x_1 - y_1| - \alpha_2^w |x_2 - y_2|}$.

The next step consists in the calibration of the speed of mean reversion θ_w of $Z_w(t, \mathbf{x})$. This speed of mean reversion is obtained by minimization of the spread between observed and modeled auto-correlations (in our model this auto-correlation is equal to $e^{-\theta_w |t-s|}$), over 40 days. Figure 2 confirms the quality of the fit. Finally, we find the vector of standard deviations, σ_w , by matching $\frac{\sigma_w^2}{2\theta_w}$ to variances of temperatures. Table 4 reports the parameters of $Z_w(t, \mathbf{x})$. We notice a high speed of mean reversion. This is partly linked to the chosen scale of time, measured on a yearly basis. On a daily basis, this speed of mean reversion is equal to 0.22, which is comparable to values obtained by Alaton et al. (2002).

Figure 2: Modeled and observed autocorrelations of temperatures in Orly, Frankfurt, Ieper and Arlon.

Parameters	Values	Parameters	Values
$\sigma_w(Paris)$	43.36	θ_w	81.10
$\sigma_w(Frankfurt)$	46.70	α_1^w	3.534176e-4
$\sigma_w(Ieper)$	42.50	α_2^w	3.102098e-4
$\sigma_w(Arlon)$	47.02		

Table 4: Parameters of the Gaussian field $Z_w(t, \mathbf{x})$.

After having fit that the Gaussian field driving temperatures, we calculate cumulated average

temperatures (CAT) from the beginning of March to the end of October as follows:

$$\int_0^T \mathbf{Z}_w^f(s) ds \approx \sum_{day=1st\ March}^{day=31th\ October} \mathbf{Z}_w^f(day) \frac{1}{365}. \quad (49)$$

March is the month during which green maizes is seeded in Belgium. Whereas green maize is usually harvested in October. We assume that futures on CAT cover this period of 8 months. In order to calibrate the model for crop yields, we regress the yields at Ieper and Arlon on observed CAT over the period 1975-2016. The results of this regression are reported in table 5. The CAT is a highly significant explanatory variable of the crop yield in Arlon whereas it is significant at less than 97.58% of confidence for the yield in Ieper. The intercept for Arlon is smaller of nearly 10 tons per hectare than for Ieper. Furthermore, crop yields in Ieper and Arlon are respectively inversely proportional and proportional to cumulated temperatures. The low R^2 tell us that data contains an inherently higher amount of unexplained variability. This variability will be captured by the Gaussian field $Z_y(t, \mathbf{x})$.

	Estimate	Std Error	t value	Pr(> t)
$a_y(Ieper)$	49.804	0.76	65.519	<2e-16***
$b_w(Ieper)$	-3.624	1.547	-2.342	0.0242*
R^2	0.121			
$a_y(Arlon)$	39.9492	0.6878	58.082	<2e-16***
$b_w(Arlon)$	4.340	1.2166	3.567	9.45e-4***
R^2	0.241			

Table 5: Calibration of the relation between crop yields and cumulated average temperatures from the 1st of March till the 31th of October.

Parameter estimates of $Z_y(t, \mathbf{x})$ are obtained by moments matching and are reported in table 6. Historical prices of CAT futures for Paris and Frankfurt are reconstructed with the valuation formula (16). The risk premium is a decreasing function of time to maturity: $(\gamma_w(t) = \gamma_1 (1 - e^{-(\theta_w + \gamma_2)(T-t)}))$ with $\gamma_1 = 3$ and $\gamma_2 = 2$. We choose a relatively high risk premium to clearly underline the difference between minimum variance and maximum utility strategies. We have considered three configurations of agricultural companies: a field of 10 hectares in Ieper ($S = (10, 0)$), a field of 10 hectares in Arlon ($S(0, 10)$) and two fields of 5 hectares in Ieper and Arlon ($S(5, 5)$). The parameter of risk aversion β is set to 0.5, the risk free rate r is 1% and P is equal to 1. We report in table 7 the amount of futures at time $t = 0$ minimizing the variance or maximizing the utility of incomes.

Parameter estimates		
σ_y	5.86	
Σ_y	Ieper	Arlon
Ieper	1.0000	0
Arlon	0.4225	0.9064
α_1^y	19.82024e-4	
α_2^y	35.45717e-4	

Table 6: Parameter estimates for $Z_y(t, \mathbf{x})$

Farms around Ieper take exclusively long positions in futures whereas farms located in the area of Arlon have exclusively short positions. This difference of strategy is directly related to the sign of b_w , the coefficient of regression of crop yield on CAT. In Arlon, the yield is directly proportional to the CAT. The only solution to decrease the variance of income consists then to sell CAT futures. If CAT grow less than expected, the gain from sold futures partly compensates the reduction of crop yields. The situation in Ieper is exactly the opposite: as the crop yield is inversely proportional, a long position in CAT futures decreases the exposure to temperature risk. This observation pleads in favor of the development of the market of CAT futures. Such a market would allow the transfer

of risks between agricultural exploitations that are not geographically diversified. For a company harvesting green maize both in Ieper and Arlon, the positions in futures are significantly smaller in absolute value than for exploitations located in Ieper or Arlon.

Hedge with futures				
Position	Min. Var. Hedge		Max. Expo.	
	$\omega_{Paris}(0)$	$\omega_{Frankfurt}(0)$	$\omega_{Paris}(0)$	$\omega_{Frankfurt}(0)$
$S = (10, 0)$	24.35	8.92	9.21	2.23
$S = (0, 10)$	-20.42	-24.91	-35.56	-31.60
$S = (5, 5)$	1.97	-7.99	-13.17	-14.69

Table 7: Initial portfolio of CAT futures for three configurations

If we recall expressions (34) , we observe that the minimum variance strategy depends upon time through a discount factor. When the risk free rate is close to zero, the strategy is nearly static and does not require further re-balancing till maturity. On the other hand, according to equation (39), the utility maximizing portfolio is the sum of the minimum variance hedge and of a speculative position:

$$\omega_{spec}(t) = -\frac{\gamma_w(t)\theta_w^2 e^{r(T-t)}}{\beta(1 - e^{-\theta_w(T-t)})} [\sigma_w^{h-1} \Sigma_w^{h-1 \top} \Sigma_w^{h-1} \sigma_w^{h-1}] \mathbf{1}_p.$$

However, in our case study the speed of mean reversion θ_w is high ($\theta_w = 81.10$) and the risk premium decays at a faster pace than $(1 - e^{-\theta_w(T-t)})$. The ratio $\frac{\gamma_w(t)\theta_w^2}{\beta(1 - e^{-\theta_w(T-t)})}$ is then nearly constant up to maturity.

Hedges with futures : empirical test from 1975 to 2016						
		Without Hedge (1)	Min. Var. Hedge (2)	Relative Change (1)-(2)	Max. Utility (3)	Relative Change (1)-(3)
Std.	$S = (10, 0)$	47.42	44.43	-6.29%	45.90	-3.20%
	$S = (0, 10)$	47.78	42.63	-10.77%	43.42	-9.10%
	$S = (5, 5)$	37.04	36.55	-1.31%	37.90	2.34%
Avg.	$S = (10, 0)$	490.81	397.02	-19.10%	458.51	-6.58%
	$S = (0, 10)$	407.51	534.66	+31.20%	596.14	+46.29%
	$S = (5, 5)$	449.16	465.84	-3.71%	527.33	+17.40%

Table 8: This table report the average (Avg.) and standard deviation (Std.) of global incomes $(R_T + P \sum_{k=1}^d S_k Y(T, \mathbf{x}_k^f))$, over the period 1975-2016. Three strategies are considered: without hedge, the minimum variance hedge and the maximum utility strategy.

In order to compare performances of min. variance and max. utility strategies, we test them over the period 1975-2016. Table 8 reports the average and standard deviation of total incomes $((R_T + P \sum_{k=1}^d S_k Y(T, \mathbf{x}_k^f))$ over this period. The minimum variance strategy reduces well the standard deviation of incomes of farms located either in Arlon (-10.77%) or either in Ieper (-6.29%). For an agricultural company operating in Ieper and Arlon, the minimum variance hedge has a negligible impact on the standard deviation (only 1.31% of reduction). However, we observe that the income volatility for this type of exploitations is already much lower than in the two other configurations (37.04 instead of 47.42 or 47.78). This clearly confirms that there exists a natural diversification between crops in Ieper and Arlon. The amplitude of this diversification is directly related to signs and sizes of coefficients b_w linking crop yields to cumulated temperatures. In parallel, the minimum variance strategy also has an impact on the average income. For farms located in Ieper, this income falls by 19.10%. In the two other configurations, the average income steps up by a 46.29% and 17.40%. These high variations of incomes are related to the risk premium which is initially substantial (to clearly differentiate min. variance and max. utility strategies).

A consequence of this high risk premium is that short (resp. long) positions are profitable (resp. expensive). Compared to the minimum variance portfolio, the maximum utility strategy raises the average income by at least 12.52%. In parallel to this increase of incomes, we observe a moderate raise of the volatility.

Finally, we determinate the composition of the CAT basket option (vector $\mathbf{h}$) that minimizes the variance of terminal incomes. For this exercise, we set the risk premium to zero $\gamma_1 = 0$. We consider only “at the money” basket options for which the strike is a linear combination of future prices: $K = \mathbf{h}^\top F(0, T)$. Table 10 reports the CAT baskets and strikes K minimizing the variance of incomes for the three configurations of farms previously considered. The variances are calculated with formula (41), under the assumption that initial values of Gaussian fields are null ($\mathbf{Z}_w^{\mathbf{h}} = 0$, $\mathbf{Z}_w^{\mathbf{f}} = 0$ and $\mathbf{Z}_y = 0$). Figure 3 presents the surface of standard deviation of R_T for different combinations of $\mathbf{h}$ and for the $S = (5, 5)$ farm. The surface being convex, it admits an unique solution. For companies exclusively exposed to crop risk in Ieper or Arlon, the options are respectively a call and a put basket option. For the companies exposed to both risks, the best hedge is a call option, mainly on Frankfurt CAT.

Minimum variance hedge with CAT basket options			
	CAT basket		Strike
	h_{Paris}	$h_{Frankfurt}$	K
$S = (10, 0)$	50	20	629.47
$S = (0, 10)$	-55	-70	-1114.24
$S = (5, 5)$	3	11	-69.45

Table 9: This table report the composition of CAT basket options minimizing the variance of incomes.

Figure 3: Surface of standard deviations of the terminal wealth, for different CAT baskets. The agricultural manages fields in Arlon and Ieper, $S = (5, 5)$.

Table 10 compares expectations and standard deviations of incomes with and without the CAT basket option (computed with equations (41) and (40)). The highest reduction of volatility is obtained for an agricultural exploitation in the neighborhood of Arlon (-20.19%). We can reduce the standard deviation by nearly 6% for a farm located in Ieper. A company exposed to crop risks in Ieper and Arlon benefits from a natural diversification that makes useless the hedging with the basket option. These figures are obtained with our theoretical model. In order to check the reliability of our conclusions, we test the performance of options hedging strategies with historical

yields and CAT over the period 1975-2016. The results are reported in table 11. We first observe that historical standard deviations, with or without hedge, are very close to these forecast by our model. Second, realized and planned reductions of volatility induced by the hedge are in the same range. Finally, a comparison with figures of table 8 reveals that a static hedge with options is slightly more efficient than a minimum variance strategy with futures.

CAT basket options: theoretical moments					
	With Hedge		Without Hedge		Relative Change (1)-(2)
	$\mathbb{E}(R_T \mathcal{F}_0)$	$\sqrt{\mathbb{V}(R_T \mathcal{F}_0)}$ (1)	$\mathbb{E}(R_T \mathcal{F}_0)$	$\sqrt{\mathbb{V}(R_T \mathcal{F}_0)}$ (2)	
$S = (10, 0)$	498.04	44.18	498.04	46.99	-5.98%
$S = (0, 10)$	399.49	38.79	399.49	48.60	-20.19%
$S = (5, 5)$	448.76	37.96	448.76	38.04	-0.21%

Table 10: This table report the average and standard deviation of global incomes, hedged with the CAT basket option minimizing the variance.

CAT basket options: empirical test from 1975 to 2016				
		Without Hedge (1)	Min. Var. Hedge (2)	Relative Change (1)-(2)
Std.	$S = (10, 0)$	47.42	43.47	-8.32%
	$S = (0, 10)$	47.78	39.48	-17.35%
	$S = (5, 5)$	37.04	36.45	-1.59%
Avg.	$S = (10, 0)$	490.82	500.47	+1.96%
	$S = (0, 10)$	407.51	400.56	-1.70%
	$S = (5, 5)$	449.16	448.49	-0.14%

Table 11: This table report the average (Avg.) and standard deviation (Std.) of global incomes hedged with the minimum variance basket option, over the period 1975-2016.

9 Conclusion

This article proposes a continuous-time model based on Gaussian fields that duplicates time and spatial dependence between temperatures and crop yields. Due to its high analytical tractability, CAT basket options are priced analytically and hedging portfolios admit closed form expressions. When we hedge crop with futures, the strategy maximizing the utility of incomes is the sum of the portfolio minimizing the variance and of a speculative position. This second position takes advantage of the risk premium used for the valuation of CAT futures. From the model, we deduce that this risk premium decreases proportionally with time-to-expiry.

In the case study, we compare performances of hedging strategies with derivatives on cumulated temperatures in Paris and Frankfurt. We consider agricultural exploitations with an exposure to crop risk in Ieper and/or Arlon. We find that exploitations in Arlon and Ieper should take opposite positions in futures for reducing the volatility of incomes. This result confirms that the CAT market would be a leverage for transferring risks between agricultural companies geographically not diversified. We benchmark strategies with backtests over the period 1975-2016. On average, the deviation of incomes is reduced by 6% and 11% for farms located in Ieper and Arlon. Furthermore, we observe a natural diversification for companies operating both in Ieper and Arlon areas. On the other hand, the strategy maximizing the expected utility of wealth raises the average income if the risk premium on CAT futures is positive. Finally, hedging with options is an efficient alternative to futures. In our case study, CAT basket options reduce by 8% and 17% the standard deviation of crop incomes.

Initially CAT futures or options were designed for hedging positions of energy suppliers, mainly

in the US. However our analysis confirms that these instruments can hedge crop incomes against adverse fluctuations of temperatures. This observation pleads in favor of the development of a market for weather derivatives in Europe. Such a market would allow insurers to propose new products helping the agricultural industry to stabilize its incomes. This is a very timely issue given the uncertainty about the impact of climate changes on crop yields and more generally on food autonomy.

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Appendix

Proof of proposition 2.1.

By definition of the Gaussian field and relation (3), the covariance matrix of Brownian fields at locations $(\mathbf{x}_1^h, \dots, \mathbf{x}_p^h)$ is given by:

$$\begin{aligned} & (Cov(Z_w(t, \mathbf{x}_k^h), Z_w(s, \mathbf{x}_l^h)))_{k,l=1,\dots,p} \\ &= \frac{1}{2\theta_w} e^{-\theta_w|t-s|} \left(\sigma_w(\mathbf{x}_k) \sigma_w(\mathbf{x}_l) e^{-\alpha_1^w |x_{k,1}^h - x_{l,1}^h| - \alpha_2^w |x_{k,2}^h - x_{l,2}^h|} \right)_{k,l=1,\dots,p} \\ &= \frac{1}{2\theta_w} e^{-\theta_w|t-s|} \boldsymbol{\sigma}_w^h \Sigma_w^h \Sigma_w^{h\top} \boldsymbol{\sigma}_w^h. \end{aligned}$$

To prove that $(Z_w(t, \mathbf{x}_k^h))_{k=1,\dots,p}$ are OU processes satisfying equation (5), we consider a multivariate process driven by the following dynamics:

$$\begin{pmatrix} dU_t^1 \\ \vdots \\ dU_t^p \end{pmatrix} = -\theta_w \begin{pmatrix} U_t^1 \\ \vdots \\ U_t^p \end{pmatrix} dt + \boldsymbol{\sigma}_w^h \Sigma_w^h \begin{pmatrix} dW_t^{h,1} \\ \vdots \\ dW_t^{h,p} \end{pmatrix} \quad (50)$$

where $\mathbf{W}_t^h := (W_t^{h,k})_{k=1,\dots,p}$ denotes a vector of p independent Brownian motions. if we apply the Itô's lemma to processes $V_t^k := -e^{\theta_w t} U_t^k$, we have that

$$\begin{aligned} dV_t^k &= -\theta_w e^{\theta_w t} U_t^k dt - e^{\theta_w t} dU_t^k \\ &= -\boldsymbol{\sigma}_w^h e^{\theta_w t} \Sigma_w^h(k, :) d\mathbf{W}_t^h \end{aligned}$$

where $\Sigma_w^h(k, :)$ is the k^{th} lines of Σ_w^h and $\mathbf{W}_t^h$ is the p -vector of Brownian motions. We infer from this last relation that $(V_t^k)_{k=1,\dots,p}$ are equal to:

$$\begin{pmatrix} V_t^1 \\ \vdots \\ V_t^p \end{pmatrix} = \begin{pmatrix} V_0^1 \\ \vdots \\ V_0^p \end{pmatrix} - \boldsymbol{\sigma}_w^h \Sigma_w^h \begin{pmatrix} \int_0^t e^{\theta_w u} dW_u^{h,1} \\ \vdots \\ \int_0^t e^{\theta_w u} dW_u^{h,p} \end{pmatrix}.$$

Thus, the vector $(U_t^k)_{k=1,\dots,p}$ is given by:

$$\begin{pmatrix} U_t^1 \\ \vdots \\ U_t^p \end{pmatrix} = e^{-\theta_w t} \begin{pmatrix} U_0^1 \\ \vdots \\ U_0^p \end{pmatrix} - \boldsymbol{\sigma}_w^h \Sigma_w^h \begin{pmatrix} \int_0^t e^{-\theta_w(t-u)} dW_u^{h,1} \\ \vdots \\ \int_0^t e^{-\theta_w(t-u)} dW_u^{h,p} \end{pmatrix}.$$

Let us assume that $(U_0^k)_{k=1,\dots,p}$ are distributed according to a multivariate normal random variable $N\left(0, \frac{1}{2\theta_w} \boldsymbol{\sigma}_w^h \Sigma_w^h \Sigma_w^{h\top} \boldsymbol{\sigma}_w^h\right)$. We have that $\sigma_w(\mathbf{x}_k^p) \Sigma_w^h(k, :) \left(\int_0^t e^{-\theta_w(t-u)} dW_u^{h,k}\right)_{k=1,\dots,p}$ is a normal random of null mean and variance equal to

$$\sigma_w^2(\mathbf{x}_k^p) \Sigma_w^h(k, :) \Sigma_w^h(k, :)^{\top} \int_0^t e^{-2\theta_w(t-u)} du = \frac{\sigma_w^2(\mathbf{x}_k^p)}{2\theta_w} (1 - e^{-2\theta_w t})$$

and on the other hand $U_0^k \sim N\left(0, \frac{\sigma_w^2(\mathbf{x}_k^p)}{2\theta_w}\right)$. The variance of U_t^k is then given by the sum of $e^{-2\theta_w t} \frac{\sigma_w^2(\mathbf{x}_k^p)}{2\theta_w} + \frac{\sigma_w^2(\mathbf{x}_k^p)}{2\theta_w} (1 - e^{-2\theta_w t}) = \frac{\sigma_w^2(\mathbf{x}_k^p)}{2\theta_w}$. Furthermore, the covariance between U_s^k and U_t^l is equal to

$$\begin{aligned} \text{Cov}(U_s^k, U_t^l) &= \sigma_w(\mathbf{x}_k^p) \sigma_w(\mathbf{x}_l^p) \mathbb{E} \left(\Sigma_w^h(k, :) \int_0^s e^{-\theta_w(s-u)} d\mathbf{W}_u^h \Sigma_w^h(l, :) \int_0^t e^{-\theta_w(t-u)} d\mathbf{W}_u^h \right) \\ &\quad + \sigma_w(\mathbf{x}_k^p) \sigma_w(\mathbf{x}_l^p) e^{-\theta_w(t+s)} \mathbb{E} (U_0^k U_0^l) . \end{aligned}$$

Then, if we denote by $\rho_{k,l} = e^{-\alpha_1^w |x_{k,1}^h - x_{l,1}^h| - \alpha_2^w |x_{k,2}^h - x_{l,2}^h|} = \Sigma_w^h(k, :) \Sigma_w^h(l, :)^\top$, the spatial correlation between locations k and l , we can rewrite the previous covariance as follows

$$\begin{aligned} \text{Cov}(U_s^k, U_t^l) &= \frac{\sigma_w(\mathbf{x}_k^p) \sigma_w(\mathbf{x}_l^p) \rho_{k,l}}{2\theta_w} e^{-\theta_w(s+t)} (e^{2\theta_w s \wedge t} - 1) + \frac{\sigma_w(\mathbf{x}_k^p) \sigma_w(\mathbf{x}_l^p) \rho_{k,l}}{2\theta_w} e^{-\theta_w(t+s)} \\ &= \frac{\sigma_w(\mathbf{x}_k^p) \sigma_w(\mathbf{x}_l^p) \rho_{k,l}}{2\theta_w} e^{-\theta_w |s-t|} . \end{aligned}$$

We infer from this last relation that U_t^k and $(Z_w(t, \mathbf{x}_k^h))_{k=1, \dots, p}$ are Gaussian processes with the same covariance structure. Then, it follows that Z_t^k solves eq. (50). ■

Proof of corollary 2.3.

From eq. (4), we know that

$$\int_t^T \mathbf{Z}_w^h(s) ds = \int_t^T e^{-\theta_w(s-t)} ds \mathbf{Z}_w^h(t) + \sigma_w^h \Sigma_w^h \int_t^T \int_t^s e^{-\theta_w(s-u)} d\mathbf{W}_u^h ds$$

Given that

$$\begin{aligned} \int_t^T \int_t^s e^{-\theta_w(s-u)} dW_u^{h,j} ds &= \int_t^T \int_u^T e^{-\theta_w(s-u)} ds dW_u^{h,j} \\ &= \int_t^T \frac{1}{\theta_w} (1 - e^{-\theta_w(T-u)}) dW_u^{h,j} \end{aligned}$$

we immediately deduce equation (10). ■

Proof of proposition 2.4.

Eq. (16) is a direct consequence of eq. (15). Eq. (17) is obtained by deriving eq. (16):

$$\begin{aligned} d\mathbf{F}(t, T) &= r\mathbf{F}(t, T)dt + e^{-r(T-t)} d\mathbf{C}\mathbf{A}\mathbf{T}(t) - e^{-r(T-t)} \tilde{\mathbf{a}}_w^h(t) dt \\ &\quad + e^{-r(T-t)} \left(-e^{-\theta_w(T-t)} \mathbf{Z}_w^h(t) \right) dt + \frac{e^{-r(T-t)}}{\theta_w} (1 - e^{-\theta_w(T-t)}) d\mathbf{Z}_w^h(t) . \end{aligned}$$

Given that $d\mathbf{C}\mathbf{A}\mathbf{T}(t) = \mathbf{a}_w^h(t)dt + \mathbf{Z}_w^h(t)dt$, we obtain that

$$\begin{aligned} d\mathbf{F}(t, T) &= r\mathbf{F}(t, T)dt + e^{-r(T-t)} (\mathbf{a}_w^h(t) - \tilde{\mathbf{a}}_w^h(t)) dt \\ &\quad + e^{-r(T-t)} (1 - e^{-\theta_w(T-t)}) \mathbf{Z}_w^h(t) dt + \frac{e^{-r(T-t)}}{\theta_w} (1 - e^{-\theta_w(T-t)}) d\mathbf{Z}_w^h(t) . \end{aligned}$$

On the other hand, the drift under Q is equal to

$$\tilde{\mathbf{a}}_w^h(t) = \mathbf{a}_w^h(t) + \mathbf{1}_p \gamma_w(t) (1 - e^{-\theta_w(T-t)}) .$$

and the infinitesimal variations of the Gaussian field under P and Q are respectively

$$d\mathbf{Z}_w^h(t) = -\theta_w \mathbf{Z}_w^h(t) dt + \sigma_w^h \Sigma_w^h d\mathbf{W}_t^h ,$$

$$d\mathbf{Z}_w^h(t) = \theta_w (\gamma_w(t) - \mathbf{Z}_w^h(t)) dt + \sigma_w^h \Sigma_w^h d\tilde{\mathbf{W}}_t^h,$$

we infer then equations (17) and (18). ■

Proof of proposition 3.2.

If we remember the expression (13) of the CAT, then the option price, which is the value of the discounted payoff under Q , may be rewritten as follows

$$\begin{aligned} \mathbb{E}^Q \left(e^{-r(T-t)} \left(\mathbf{h}^\top \mathbf{CAT}(T) - K \right)_+ \mid \mathcal{F}_t \right) \\ = \mathbb{E}^Q \left(e^{-r(T-t)} \left(\mathbf{h}^\top \int_t^T \mathbf{Z}_w^h(s) ds - K_m(t) \right)_+ \mid \mathcal{F}_t \right), \end{aligned}$$

where $K_m(t)$ is a modified strike temperature defined by:

$$\begin{aligned} K_m(t) &:= K - \mathbf{h}^\top \mathbf{CAT}(t) - \mathbf{h}^\top \int_t^T \mathbf{a}_w^h(s) ds \\ &= K - \mathbf{h}^\top \int_0^t \mathbf{Z}_w^h(s) ds - \mathbf{h}^\top \int_0^T \mathbf{a}_w^h(s) ds. \end{aligned} \quad (51)$$

Let us define the random variable $U := \int_t^T \mathbf{h}^\top \mathbf{Z}_w^h(s) ds \sim N(\mathbf{h}^\top \mu_w^Q(t, \mathbf{Z}_w^h(t)), \mathbf{h}^\top V^w(t) \mathbf{h})$. The basket option price becomes a sum of two integrals with respect to the density of U , denoted $f_U(u)$:

$$\begin{aligned} \mathbb{E}^Q \left(e^{-r(T-t)} \left(\mathbf{h}^\top \int_t^T \mathbf{Z}_w^h(s) ds - K_m(t) \right)_+ \mid \mathcal{F}_t \right) \\ = e^{-r(T-t)} \int_{K_m(t)}^\infty u f_U(u) du - e^{-r(T-t)} K_m(t) \int_{K_m(t)}^\infty f_U(u) du. \end{aligned}$$

The symmetry of the Gaussian distribution allows us to rewrite the second integral of this last integral as follows:

$$\int_{K_m(t)}^\infty f_U(u) du = \Phi \left(- \left(\frac{K_m - \mathbf{h}^\top \mu_w^Q}{\sqrt{\mathbf{h}^\top V^w \mathbf{h}}} \right) \right), \quad (52)$$

(where we temporarily forget the reference to time to lighten notations). On the other hand, we have that:

$$\int_{K_m(t)}^\infty u f_U(u) du = \frac{1}{\sqrt{2\pi \mathbf{h}^\top V^w \mathbf{h}}} \int_{K_m(t)}^\infty u \exp \left(- \frac{1}{2} \frac{(u - \mathbf{h}^\top \mu_w^Q)^2}{\mathbf{h}^\top V^w \mathbf{h}} \right) du$$

We operate a change of variable $v := \frac{u - \mathbf{h}^\top \mu_w^Q}{\sqrt{\mathbf{h}^\top V^w \mathbf{h}}}$ such that $\mathbf{h}^\top \mu_w^Q + \sqrt{\mathbf{h}^\top V^w \mathbf{h}} v = u$ and $du = \sqrt{\mathbf{h}^\top V^w \mathbf{h}} dv$. Then, we can rewrite this last integral as follows:

$$\begin{aligned} \frac{1}{\sqrt{2\pi \mathbf{h}^\top V^w \mathbf{h}}} \int_{K_m(t)}^\infty u \exp \left(- \frac{1}{2} \frac{(u - \mathbf{h}^\top \mu_w^Q)^2}{\mathbf{h}^\top V^w \mathbf{h}} \right) du &= \\ \frac{1}{\sqrt{2\pi \mathbf{h}^\top V^w \mathbf{h}}} \int_{K_m(t)}^\infty (u - \mathbf{h}^\top \mu_w^Q) \exp \left(- \frac{1}{2} \frac{(u - \mathbf{h}^\top \mu_w^Q)^2}{\mathbf{h}^\top V^w \mathbf{h}} \right) du &+ \\ + \mathbf{h}^\top \mu_w^Q \Phi \left(- \frac{K_m - \mathbf{h}^\top \mu_w^Q}{\sqrt{\mathbf{h}^\top V^w \mathbf{h}}} \right). \end{aligned} \quad (53)$$

The first term in this last equation is equal to

$$\begin{aligned}
& \frac{1}{\sqrt{2\pi \mathbf{h}^\top V_w \mathbf{h}}} \int_{K_m(t)}^{\infty} \left(u - \mathbf{h}^\top \mu_w^Q \right) \exp \left(-\frac{1}{2} \frac{\left(u - \mathbf{h}^\top \mu_w^Q \right)^2}{\mathbf{h}^\top V_w \mathbf{h}} \right) du \\
&= \frac{\sqrt{\mathbf{h}^\top V_w \mathbf{h}}}{\sqrt{2\pi}} \left[-\exp \left(-\frac{1}{2} v^2 \right) \right]_{\frac{K_m - \mathbf{h}^\top \mu_w^Q}{\sqrt{\mathbf{h}^\top V_w \mathbf{h}}}}^{\infty} \\
&= \sqrt{\mathbf{h}^\top V_w \mathbf{h}} \varphi \left(\frac{K_m - \mathbf{h}^\top \mu_w^Q}{\sqrt{\mathbf{h}^\top V_w \mathbf{h}}} \right).
\end{aligned} \tag{54}$$

Combining equations (52), (53) and (54) leads to the following intermediate expression of the CAT option price:

$$\begin{aligned}
& \mathbb{E}^Q \left(e^{-r(T-t)} \left(\mathbf{h}^\top \mathbf{CAT}(T) - K \right)_+ \mid \mathcal{F}_t \right) \\
&= e^{-r(T-t)} \left(\mathbf{h}^\top \mu_w^Q - K_m \right) \Phi \left(-\frac{K_m - \mathbf{h}^\top \mu_w^Q}{\sqrt{\mathbf{h}^\top V_w \mathbf{h}}} \right) \\
&+ e^{-r(T-t)} \sqrt{\mathbf{h}^\top V_w \mathbf{h}} \varphi \left(\frac{K_m - \mathbf{h}^\top \mu_w^Q}{\sqrt{\mathbf{h}^\top V_w \mathbf{h}}} \right)
\end{aligned} \tag{55}$$

If we remember the definitions (20), (51) and (16) of future prices, we immediately infer the equality

$$\begin{aligned}
e^{-r(T-t)} \left(\mathbf{h}^\top \mu_w^Q - K_m(t) \right) &= e^{-r(T-t)} \mathbf{h}^\top \mathbf{CAT}(t) + e^{-r(T-t)} \mathbf{h}^\top \int_t^T \tilde{\mathbf{a}}_w^h(s) ds \\
&+ e^{-r(T-t)} \frac{1}{\theta_w} \left(1 - e^{-\theta_w(T-t)} \right) \mathbf{h}^\top \mathbf{Z}_w^h(t) - e^{-r(T-t)} K \\
&= \mathbf{h}^\top \mathbf{F}(t, T) - e^{-r(T-t)} K.
\end{aligned}$$

Combining this relation with equation (55) and using the symmetry of $\varphi(\cdot)$ allows us to conclude. ■

Proof of proposition 4.1.

Contrary to $Z_w(t, \mathbf{x})$, increments of $Z_y(t, \mathbf{x})$ are independent. Indeed for $t \leq s$, the covariance is null

$$\begin{aligned}
\text{Cov}(Z_y(t, \mathbf{x}), Z_y(s, \mathbf{y}) - Z_y(t, \mathbf{x})) &= \mathbb{E}(Z_y(t, \mathbf{x}), Z_y(s, \mathbf{y})) - \mathbb{E}(Z_y(t, \mathbf{x}), Z_y(t, \mathbf{x})) \\
&= t\rho_{x,y}^y - t\rho_{x,y}^y = 0.
\end{aligned}$$

Furthermore, Increments are Gaussian by construction. Their covariance is equal to

$$\left(\text{Cov}(Z_y(s, \mathbf{x}_j^f) - Z_y(t, \mathbf{x}_j^f), Z_y(s, \mathbf{x}_k^f) - Z_y(t, \mathbf{x}_k^f)) \right)_{j,k=1,\dots,d} = \sigma_y^2(s \wedge t) \Sigma_y \Sigma_y^\top$$

Given that the multivariate process satisfying the eq. (25) and $\left(Z_y(t, \mathbf{x}_j^f) \right)_{j=1,\dots,d}$ are Gaussian processes with the same covariance structure, we can conclude. ■

Proof of proposition 4.2.

By definition of crop yields, we have the following decomposition:

$$\mathbf{Y} = \mathbf{a}_y + \mathbf{b}_w \int_0^t \mathbf{Z}_w^f(s) ds + \mathbf{b}_w \int_t^T \mathbf{Z}_w^f(s) ds + \mathbf{Z}_y^f(T).$$

If we remember the corollary (2.3) and equation (10), we have that

$$\begin{aligned} \begin{pmatrix} \int_t^T \mathbf{Z}_w^h(s) ds \\ \int_t^T \mathbf{Z}_w^f(s) ds \end{pmatrix} &= \frac{1}{\theta_w} \left(1 - e^{-\theta_w(T-t)}\right) \begin{pmatrix} \mathbf{Z}_w^h(t) \\ \mathbf{Z}_w^f(t) \end{pmatrix} \\ &\quad + \int_t^T \left(1 - e^{-\theta_w(T-s)}\right) \boldsymbol{\sigma}_w \Sigma_w \begin{pmatrix} d\mathbf{W}_s^h \\ d\mathbf{W}_s^f \end{pmatrix} \end{aligned}$$

where $\mathbf{W}_s^f = (W_s^{f,j})_{j=1,\dots,d}$ is a d vector of independent Brownian motions. We infer from this last relation, the following relation:

$$\begin{aligned} \int_t^T \mathbf{Z}_w^f(s) ds &= \frac{1}{\theta_w} \left(1 - e^{-\theta_w(T-t)}\right) \mathbf{Z}_w^f(t) + \Sigma_w^{fh} \int_t^T \left(1 - e^{-\theta_w(T-u)}\right) d\mathbf{W}_u^h \\ &\quad + \Sigma_w^f \int_t^T \left(1 - e^{-\theta_w(T-u)}\right) d\mathbf{W}_u^f. \end{aligned}$$

This last integral has an expectation and variance equal to

$$\begin{aligned} \mathbb{E} \left(\int_t^T \mathbf{Z}_w^f(s) ds | \mathcal{F}_t \right) &= \frac{1}{\theta_w} \left(1 - e^{-\theta_w(T-t)}\right) \mathbf{Z}_w^f(t), \\ \mathbb{V} \left(\int_t^T \mathbf{Z}_w^f(s) ds | \mathcal{F}_t \right) &= \frac{1}{\theta_w^2} \int_t^T \left(1 - e^{-\theta_w(T-s)}\right)^2 ds \boldsymbol{\sigma}_w^f \left(\Sigma_w^{fh} \Sigma_w^{fh\top} + \Sigma_w^f \Sigma_w^{f\top} \right) \boldsymbol{\sigma}_w^f. \end{aligned} \tag{56}$$

On the other hand, the increments of the Gaussian field $\mathbf{Z}_y^f(t)$ are independent. The conditional expectation and variance of $\mathbf{Z}_y^f(T)$ are respectively equal to

$$\begin{aligned} \mathbb{E} \left(\mathbf{Z}_y^f(T) | \mathcal{F}_t \right) &= \mathbb{E} \left(\mathbf{Z}_y^f(t) | \mathcal{F}_t \right) + \mathbb{E} \left(\mathbf{Z}_y^f(T) - \mathbf{Z}_y^f(t) | \mathcal{F}_t \right) \\ &= \mathbf{Z}_y^f(t). \end{aligned} \tag{57}$$

and

$$\begin{aligned} \mathbb{V} \left(\mathbf{Z}_y^f(T) | \mathcal{F}_t \right) &= \mathbb{V} \left(\mathbf{Z}_y^f(T) - \mathbf{Z}_y^f(t) | \mathcal{F}_t \right) \\ &= \sigma_y^2(T-t) \Sigma_y \Sigma_y^\top. \end{aligned} \tag{58}$$

Combining equations (56), (57) and (58) allows us to conclude. ■

Proof of proposition 5.1.

From eq. (32) and (30), we infer that the infinitesimal variation of farms earnings is given by:

$$\begin{aligned} dR_t + P \mathbf{S}^\top d\mathbf{Y}(t) &= -e^{-r(T-t)} \left(\gamma_w(t) \left(1 - e^{-\theta_w(T-t)}\right) \right) \omega(t)^\top \mathbf{1}_p dt \\ &\quad + \frac{e^{-r(T-t)}}{\theta_w} \left(1 - e^{-\theta_w(T-t)}\right) \omega(t)^\top \boldsymbol{\sigma}_w^h \Sigma_w^h d\mathbf{W}_t^h \\ &\quad + \frac{P}{\theta_w} \left(1 - e^{-\theta_w(T-t)}\right) \mathbf{S}^\top \mathbf{b}_w \boldsymbol{\sigma}_w^f \left(\Sigma_w^{fh} d\mathbf{W}_t^h + \Sigma_w^f d\mathbf{W}_t^f \right) + \sigma_y P \mathbf{S}^\top \Sigma_y d\mathbf{W}_t^y \end{aligned} \tag{59}$$

From this last equation, we directly infer that the variance is minimized if the following equality holds:

$$\begin{aligned} \frac{e^{-r(T-t)}}{\theta_w} \left(1 - e^{-\theta_w(T-t)}\right) \omega(t)^\top \boldsymbol{\sigma}_w^h \Sigma_w^h \\ = -\frac{P}{\theta_w} \left(1 - e^{-\theta_w(T-t)}\right) \mathbf{S}^\top \mathbf{b}_w \boldsymbol{\sigma}_w^f \Sigma_w^{fh}. \end{aligned}$$

We infer from this last relation the optimal mean variance strategy (34). ■

Proof of proposition 37.

Let us assume that the value function has the following form

$$V(t, R_t, \mathbf{Y}_t) = -\frac{1}{\beta} \exp \left(A(t) + B(t)R_t + \sum_{i=1}^d C_i(t)Y_t^i \right)$$

where $A(t)$, $B(t)$ and $\mathbf{C} = (C_i(t))_{i=1\dots d}$ are functions from $\mathbb{R}^+$ to $\mathbb{R}$ with terminal conditions: $A(T) = 0$, $B(T) = -\beta$ and $\mathbf{C}(T) = -\beta \mathbf{S}$. Under this assumption, the partial derivatives of the value function are equal to

$$\begin{aligned} \frac{\partial V}{\partial t} &= \left(\frac{\partial}{\partial t} A(t) + \frac{\partial}{\partial t} B(t)R_t + \sum_{i=1}^d \frac{\partial}{\partial t} C_i Y_t^i \right) V(t, R, Y), \\ \frac{\partial V}{\partial R} &= \frac{\partial^2 V}{\partial R^2} B(t) V(t, R, Y) = B^2(t) V(t, R, Y), \\ \frac{\partial V}{\partial Y} &= \frac{\partial V}{\partial R \partial Y} \begin{pmatrix} C_1(t) V \\ \vdots \\ C_d(t) V \end{pmatrix} = \mathbf{C}(t) V = \begin{pmatrix} C_1(t) B(t) V \\ \vdots \\ C_d(t) B(t) V \end{pmatrix} = \mathbf{C}(t) B(t) V, \end{aligned}$$

and

$$\frac{\partial^2 V}{\partial Y^2} = \begin{pmatrix} C_1^2(t) & \dots & C_1(t)C_d(t) \\ \vdots & \ddots & \vdots \\ C_1(t)C_d(t) & \dots & C_d^2(t) \end{pmatrix} = \mathbf{C}(t)\mathbf{C}^\top(t)V.$$

Injecting these expressions in the HJB equation (37) leads to:

$$\begin{aligned} 0 &= \max_{\omega} \left((\partial_t A + \partial_t B R_t + \partial_t \mathbf{C}^\top \mathbf{Y}_t) - (\gamma_w(t)\theta_w e^{-r(T-t)}g(t)) \omega(t)^\top \mathbf{1}_p B \right. \\ &\quad + \frac{1}{2} e^{-2r(T-t)} g(t)^2 \omega(t)^\top \boldsymbol{\sigma}_w^h \Sigma_w^h \Sigma_w^{h\top} \boldsymbol{\sigma}_w^h \omega(t) B^2 + \frac{1}{2} tr \left(\Upsilon(t) \mathbf{C} \mathbf{C}^\top \right) \\ &\quad \left. + \left[e^{-r(T-t)} g(t)^2 \omega(t)^\top \boldsymbol{\sigma}_w^h \Sigma_w^h \Sigma_w^{fh\top} \boldsymbol{\sigma}_w^f \mathbf{b}_w \right] \mathbf{C} B \right). \end{aligned} \quad (60)$$

If we derive with respect to ω , we infer that the optimal hedging strategy satisfies the following relation:

$$\begin{aligned} 0 &= - \left(\gamma_w(t)\theta_w e^{-r(T-t)}g(t) \right) B + e^{-2r(T-t)} g(t)^2 \boldsymbol{\sigma}_w^h \Sigma_w^h \Sigma_w^{h\top} \boldsymbol{\sigma}_w^h \omega(t) B^2 \\ &\quad + \left[e^{-r(T-t)} g(t)^2 \boldsymbol{\sigma}_w^h \Sigma_w^h \Sigma_w^{fh\top} \boldsymbol{\sigma}_w^f \mathbf{b}_w \right] \mathbf{C} B. \end{aligned}$$

from which we infer that

$$\frac{\gamma_w(t)\theta_w e^{r(T-t)}}{g(t)B} [\boldsymbol{\sigma}_w^{h-1} \Sigma_w^{h\top-1} \Sigma_w^{h-1} \boldsymbol{\sigma}_w^{h-1}] \mathbf{1}_p - \frac{e^{r(T-t)}}{B} [\boldsymbol{\sigma}_w^{h-1} \Sigma_w^{h\top-1} \Sigma_w^{fh\top} \boldsymbol{\sigma}_w^f \mathbf{b}_w] \mathbf{C} = \omega(t)$$

Injecting the expression of $\omega(t)$ into the different terms of the HJB equation (60) leads to the following intermediate results:

$$\begin{aligned} \left(\gamma_w(t)\theta_w e^{-r(T-t)}g(t) \right) B \omega(t)^\top \mathbf{1}_p &= \gamma_w^2 \theta_w^2 \mathbf{1}_p^\top [\boldsymbol{\sigma}_w^{h-1} \Sigma_w^{h\top-1} \Sigma_w^{h-1} \boldsymbol{\sigma}_w^{h-1}] \mathbf{1}_p \\ &\quad - (\gamma_w \theta_w g(t)) \mathbf{C}^\top [\mathbf{b}_w \boldsymbol{\sigma}_w^f \Sigma_w^{fh\top} \Sigma_w^{h-1} \boldsymbol{\sigma}_w^{h-1}] \mathbf{1}_p. \end{aligned} \quad (61)$$

The third term of the HJB equation is:

$$\begin{aligned}
& \frac{1}{2} e^{-2r(T-t)} g(t)^2 B^2 \omega(t)^\top \sigma_w^h \Sigma_w^h \Sigma_w^{h\top} \sigma_w^h \omega(t) \\
&= \frac{\gamma_w^2(t) \theta_w^2}{2} \mathbf{1}_p^\top [\sigma_w^{h-1} \Sigma_w^{h\top-1} \Sigma_w^{h-1} \sigma_w^{h-1}] \mathbf{1}_p - \frac{\gamma_w(t) \theta_w}{2} g(t) C^\top [b_w \sigma_w^f \Sigma_w^{fh} \Sigma_w^{h-1} \sigma_w^{h-1}] \mathbf{1}_p \\
&\quad - \frac{\gamma_w(t) \theta_w}{2} g(t) \mathbf{1}_p^\top [\sigma_w^{h-1} \Sigma_w^{h\top-1} \Sigma_w^{fh\top} \sigma_w^f b_w] C + \frac{1}{2} g(t)^2 C^\top [b_w \sigma_w^f \Sigma_w^{fh} \Sigma_w^{fh\top} \sigma_w^f b_w] C,
\end{aligned} \tag{62}$$

whereas the last term becomes

$$\begin{aligned}
& e^{-r(T-t)} g(t)^2 [\omega(t)^\top \sigma_w^h \Sigma_w^h \Sigma_w^{fh\top} \sigma_w^f b_w] C B \\
&= \gamma_w \theta_w g(t) [\mathbf{1}_p^\top \sigma_w^{h-1} \Sigma_w^{h\top-1} \Sigma_w^{fh\top} \sigma_w^f b_w C] \\
&\quad - g(t)^2 [C^\top b_w \sigma_w^f \Sigma_w^{fh} \Sigma_w^{fh\top} \sigma_w^f b_w C].
\end{aligned} \tag{63}$$

Finally, combining expressions (62), (61) and (63) with the HJB equation (60) gives us:

$$\begin{aligned}
0 &= (\partial_t A + \partial_t B R_t + \partial_t C^\top Y_t) + \frac{1}{2} \text{tr} \left(\Upsilon(t) C C^\top \right) \\
&\quad - \gamma_w^2 \theta_w^2 \mathbf{1}_p^\top [\sigma_w^{h-1} \Sigma_w^{h\top-1} \Sigma_w^{h-1} \sigma_w^{h-1}] \mathbf{1}_p + (\gamma_w \theta_w g(t)) C^\top [b_w \sigma_w^f \Sigma_w^{fh} \Sigma_w^{h-1} \sigma_w^{h-1}] \mathbf{1}_p \\
&\quad + \frac{\gamma_w^2 \theta_w^2}{2} \mathbf{1}_p^\top [\sigma_w^{h-1} \Sigma_w^{h\top-1} \Sigma_w^{h-1} \sigma_w^{h-1}] \mathbf{1}_p - \frac{\gamma_w \theta_w}{2} g(t) C^\top [b_w \sigma_w^f \Sigma_w^{fh} \Sigma_w^{h-1} \sigma_w^{h-1}] \mathbf{1}_p \\
&\quad - \frac{\gamma_w \theta_w}{2} g(t) \mathbf{1}_p^\top [\sigma_w^{h-1} \Sigma_w^{h\top-1} \Sigma_w^{fh\top} \sigma_w^f b_w] C + \frac{1}{2} g(t)^2 C^\top [b_w \sigma_w^f \Sigma_w^{fh} \Sigma_w^{fh\top} \sigma_w^f b_w] C \\
&\quad + \gamma_w \theta_w g(t) [\mathbf{1}_p^\top \sigma_w^{h-1} \Sigma_w^{h\top-1} \Sigma_w^{fh\top} \sigma_w^f b_w C] - g(t)^2 [C^\top b_w \sigma_w^f \Sigma_w^{fh} \Sigma_w^{fh\top} \sigma_w^f b_w C].
\end{aligned}$$

This equation depends upon random processes R_t and Y_t and is then satisfied if and only if terms multiplying random processes are null. We infer then that that $B = -\beta$, $C = -\beta P S$ and that

$$\begin{aligned}
0 &= (\partial_t A + \partial_t B R_t + \partial_t C^\top Y_t) + \frac{1}{2} \text{tr} \left(\Upsilon(t) C C^\top \right) \\
&\quad - \frac{1}{2} \gamma_w^2 \theta_w^2 \mathbf{1}_p^\top [\sigma_w^{h-1} \Sigma_w^{h\top-1} \Sigma_w^{h-1} \sigma_w^{h-1}] \mathbf{1}_p \\
&\quad - \frac{1}{2} g(t)^2 [C^\top b_w \sigma_w^f \Sigma_w^{fh} \Sigma_w^{fh\top} \sigma_w^f b_w C] \\
&\quad + (\gamma_w \theta_w g(t)) C^\top [b_w \sigma_w^f \Sigma_w^{fh} \Sigma_w^{h-1} \sigma_w^{h-1}] \mathbf{1}_p
\end{aligned}$$

with the terminal condition $A(T) = 0$. ■

Proof of proposition 7.1.

At time $t = 0$, the expected income is

$$\mathbb{E}(R_T | \mathcal{F}_0) = P S^\top \mathbb{E}(Y | \mathcal{F}_0) + \mathbb{E} \left(\left(h^\top C A T(T) - K \right)_+ | \mathcal{F}_0 \right) - C(0, h) e^{rT}. \tag{64}$$

The option price $C(0, h)$ is given by expression (22) whereas according to equation (26), the expected yield is equal to

$$\mathbb{E}(Y | \mathcal{F}_0) = a_y + \frac{b_w}{\theta_w} (1 - e^{-\theta_w T}) Z_w^f(0) + Z_y^f(0).$$

Using similar arguments to these for deriving equation (55) in proposition 3.2, we prove that the expectation of the payoff under the real measure P is given by

$$\begin{aligned}
& \mathbb{E} \left(\left(h^\top C A T(T) - K \right)_+ | \mathcal{F}_0 \right) = \\
& \left(h^\top \mu_w^P - K_m \right) \Phi \left(\frac{h^\top \mu_w^P - K_m}{\sqrt{h^\top V_w h}} \right) + \sqrt{h^\top V_w h} \varphi \left(\frac{h^\top \mu_w^P - K_m}{\sqrt{h^\top V_w h}} \right)
\end{aligned}$$

where , $\mu_w^P = \mu_w^P(0, \mathbf{Z}_w^h(0)) = \frac{1}{\theta_w} (1 - e^{-\theta_w T}) \mathbf{Z}_w^h(0)$ and $K_m = K_m(0) = K - \mathbf{h}^\top \int_0^T \mathbf{a}_w^h(s) ds$. Given that by construction, we have

$$\mathbf{h}^\top \mu_w^P - K_m = \mathbf{h}^\top \mathbb{E}(\mathbf{C} \mathbf{A} \mathbf{T}(T) | \mathcal{F}_0) - K,$$

we can conclude. ■

Proof of proposition 7.3.

The second order moment of the option payoff can be rewritten as follows:

$$\begin{aligned} & \mathbb{E} \left(\left(\left(\mathbf{h}^\top \mathbf{C} \mathbf{A} \mathbf{T}(T) - K \right)_+ \right)^2 | \mathcal{F}_0 \right) \\ &= \mathbb{E} \left(\left(\left(\int_0^T \mathbf{h}^\top \mathbf{Z}_w^h(s) ds - K_m \right)_+ \right)^2 | \mathcal{F}_0 \right), \end{aligned}$$

Let us define the random variable $U := \int_0^T \mathbf{h}^\top \mathbf{Z}_w^h(s) ds \sim N(\mathbf{h}^\top \mu_w^P, \mathbf{h}^\top V_w \mathbf{h})$. The second moment can thus be rewritten as the sum of three integrals with respect to the density of U :

$$\begin{aligned} & \mathbb{E} \left(\left(\left(\int_0^T \mathbf{h}^\top \mathbf{Z}_w^h(s) ds - K_m \right)_+ \right)^2 | \mathcal{F}_0 \right), \\ &= \int_{K_m}^\infty u^2 f_U(u) du - 2K_m \int_{K_m}^\infty u f_U(u) du + K_m^2 \int_{K_m}^\infty f_U(u) du. \end{aligned} \quad (65)$$

The last integral in this equation is equal to

$$\int_{K_m}^\infty f_U(u) du = \Phi \left(\frac{\mathbf{h}^\top \mu_w^P - K_m}{\sqrt{\mathbf{h}^\top V_w \mathbf{h}}} \right). \quad (66)$$

As shown in proposition 3.2, the second integral in equation (65) is given by:

$$\int_{K_m}^\infty u f_U(u) du = \mathbf{h}^\top \mu_w^P \Phi \left(\frac{\mathbf{h}^\top \mu_w^P - K_m}{\sqrt{\mathbf{h}^\top V_w \mathbf{h}}} \right) + \sqrt{\mathbf{h}^\top V_w \mathbf{h}} \varphi \left(\frac{\mathbf{h}^\top \mu_w^P - K_m}{\sqrt{\mathbf{h}^\top V_w \mathbf{h}}} \right). \quad (67)$$

In order to calculate the last term of equation (65), we do the following change of variable $v = \frac{u - \mathbf{h}^\top \mu_w^P}{\sqrt{\mathbf{h}^\top V_w \mathbf{h}}}$, such that $\mathbf{h}^\top \mu_w^P + \sqrt{\mathbf{h}^\top V_w \mathbf{h}} v = u$ and $du = \sqrt{\mathbf{h}^\top V_w \mathbf{h}} dv$. We infer that

$$\begin{aligned} & \int_{K_m}^\infty u^2 f_U(u) du = \frac{1}{\sqrt{2\pi \mathbf{h}^\top V_w \mathbf{h}}} \int_{K_m}^\infty (u - \mathbf{h}^\top \mu_w^P)^2 \exp \left(-\frac{1}{2} \frac{(u - \mathbf{h}^\top \mu_w^P)^2}{\mathbf{h}^\top V_w \mathbf{h}} \right) du \\ &+ 2 \mathbf{h}^\top \mu_w^P \left(\mathbf{h}^\top \mu_w^P \Phi \left(\frac{\mathbf{h}^\top \mu_w^P - K_m}{\sqrt{\mathbf{h}^\top V_w \mathbf{h}}} \right) + \sqrt{\mathbf{h}^\top V_w \mathbf{h}} \varphi \left(\frac{\mathbf{h}^\top \mu_w^P - K_m}{\sqrt{\mathbf{h}^\top V_w \mathbf{h}}} \right) \right) \\ &- (\mathbf{h}^\top \mu_w^P)^2 \Phi \left(\frac{\mathbf{h}^\top \mu_w^P - K_m}{\sqrt{\mathbf{h}^\top V_w \mathbf{h}}} \right). \end{aligned} \quad (68)$$

As $(\mathbf{h}^\top V_w \mathbf{h}) v^2 = (u - \mathbf{h}^\top \mu_w^P)^2$ and $\sqrt{\mathbf{h}^\top V_w \mathbf{h}} dv = d(u - \mathbf{h}^\top \mu_w^P)$, the first term in this last equation becomes:

$$\begin{aligned} & \frac{1}{\sqrt{2\pi \mathbf{h}^\top V_w \mathbf{h}}} \int_{K_m}^\infty (u - \mathbf{h}^\top \mu_w^P)^2 \exp\left(-\frac{1}{2} \frac{(u - \mathbf{h}^\top \mu_w^P)^2}{\mathbf{h}^\top V_w \mathbf{h}}\right) du \\ &= \frac{(\mathbf{h}^\top V_w \mathbf{h})}{\sqrt{2\pi}} \int_{\frac{K_m - \mathbf{h}^\top \mu_w^P}{\sqrt{\mathbf{h}^\top V_w \mathbf{h}}}}^\infty v^2 \exp\left(-\frac{1}{2} v^2\right) dv \\ &= (\mathbf{h}^\top V_w \mathbf{h}) \left(\frac{K_m - \mathbf{h}^\top \mu_w^P}{\sqrt{\mathbf{h}^\top V_w \mathbf{h}}} \varphi\left(\frac{K_m - \mathbf{h}^\top \mu_w^P}{\sqrt{\mathbf{h}^\top V_w \mathbf{h}}}\right) + \Phi\left(-\frac{K_m - \mathbf{h}^\top \mu_w^P}{\sqrt{\mathbf{h}^\top V_w \mathbf{h}}}\right) \right) \end{aligned} \quad (69)$$

given that

$$\begin{aligned} \frac{1}{\sqrt{2\pi}} \int_c^\infty v^2 \exp\left(-\frac{1}{2} v^2\right) dv &= \frac{1}{\sqrt{2\pi}} \left[-v \exp\left(-\frac{1}{2} v^2\right) \right]_c^\infty + \frac{1}{\sqrt{2\pi}} \int_c^\infty \exp\left(-\frac{1}{2} v^2\right) dv \\ &= c\varphi(c) + \Phi(-c). \end{aligned}$$

Regrouping equations (66), (67), (68) and (69) allows us to conclude. ■

Proof of proposition 7.4.

Given that $\mathbf{Y} = \mathbf{a}_y + \mathbf{b}_w \int_0^T \mathbf{Z}_w^f(s) ds + \mathbf{Z}_y^f(T)$, the cross moment of the option payoff and crop yield can be developed as follows

$$\begin{aligned} & \mathbb{E} \left(\left(\int_0^T \mathbf{h}^\top \mathbf{Z}_w^h(s) ds - K_m \right)_+ \mathbf{S}^\top \mathbf{Y} \mid \mathcal{F}_0 \right) = \\ & \mathbb{E} \left(\left(\int_0^T \mathbf{h}^\top \mathbf{Z}_w^h(s) ds - K_m \right)_+ \mid \mathcal{F}_t \right) \mathbf{S}^\top \left(\mathbf{a}_y + \mathbb{E} \left(\mathbf{Z}_y^f(T) \mid \mathcal{F}_0 \right) \right) \\ & + \mathbb{E} \left(\left(\int_0^T \mathbf{h}^\top \mathbf{Z}_w^h(s) ds - K_m \right)_+ \int_0^T \mathbf{S}^\top \mathbf{b}_w \mathbf{Z}_w^f(s) ds \mid \mathcal{F}_0 \right). \end{aligned}$$

From which we infer that the covariance is equal to the following cross expectation:

$$\begin{aligned} & \mathbb{C} \left(\left(\int_0^T \mathbf{h}^\top \mathbf{Z}_w^h(s) ds - K_m \right)_+ \mathbf{S}^\top \mathbf{Y} \mid \mathcal{F}_0 \right) \\ &= \mathbb{E} \left(\left(\int_0^T \mathbf{h}^\top \mathbf{Z}_w^h(s) ds - K_m \right)_+ \int_0^T \mathbf{S}^\top \mathbf{b}_w \mathbf{Z}_w^f(s) ds \mid \mathcal{F}_0 \right) \\ & \quad - \mathbb{E} \left(\left(\int_0^T \mathbf{h}^\top \mathbf{Z}_w^h(s) ds - K_m \right)_+ \mid \mathcal{F}_t \right) \mathbb{E} \left(\mathbf{S}^\top \mathbf{b}_w \int_0^T \mathbf{Z}_w^f(s) ds \mid \mathcal{F}_t \right) \end{aligned}$$

Using nested conditional expectations, we can rewrite the cross moment as follows:

$$\begin{aligned} & \mathbb{E} \left(\left(\int_0^T \mathbf{h}^\top \mathbf{Z}_w^h(s) ds - K_m \right)_+ \int_0^T \mathbf{S}^\top \mathbf{b}_w \mathbf{Z}_w^f(s) ds \mid \mathcal{F}_0 \right) \\ &= \mathbb{E} \left(\mathbb{E} \left(\left(\int_0^T \mathbf{h}^\top \mathbf{Z}_w^h(s) ds - K_m \right)_+ \mid \mathcal{F}_t \cup \int_0^T \mathbf{S}^\top \mathbf{b}_w \mathbf{Z}_w^f(s) ds = z \right) \mid \mathcal{F}_0 \right). \end{aligned}$$

If we remember that $\int_0^T \mathbf{Z}_w^h(s)ds$ and $\int_0^T \mathbf{Z}_w^f(s)ds$ are distributed according to a multivariate normal random distribution, we deduce that

$$\begin{pmatrix} \mathbb{E} \left(\int_0^T \mathbf{h}^\top \mathbf{Z}_w^h(s)ds | \mathcal{F}_0 \right) \\ \mathbb{E} \left(\int_0^T \mathbf{S}^\top \mathbf{b}_w \mathbf{Z}_w^f(s)ds | \mathcal{F}_0 \right) \end{pmatrix} = \frac{1}{\theta_w} (1 - e^{-\theta_w T}) \begin{pmatrix} \mathbf{h}^\top \mathbf{Z}_w^h(0) \\ \mathbf{S}^\top \mathbf{b}_w \mathbf{Z}_w^f(0) \end{pmatrix}.$$

Furthermore, the covariance function of integrals is equal to

$$\begin{aligned} \mathbb{C} \left(\int_0^T \mathbf{Z}_w(s)ds, \int_0^T \mathbf{Z}_w(s)ds \right) &:= \int_0^T (1 - e^{-\theta_w(T-s)})^2 ds \\ &\times \sigma_w \begin{pmatrix} \Sigma_w^h \Sigma_w^{h\top} & \Sigma_w^h \Sigma_w^{fh\top} \\ \Sigma_w^{fh} \Sigma_w^{h\top} & \Sigma_w^{fh} \Sigma_w^{fh\top} + \Sigma_w^f \Sigma_w^{f\top} \end{pmatrix} \sigma_w \end{aligned}$$

and then, the covariance between $\int_0^T \mathbf{h}^\top \mathbf{Z}_w^h(s)ds$ and $\int_0^T \mathbf{S}^\top \mathbf{b}_w \mathbf{Z}_w^f(s)ds$ is given by:

$$\begin{aligned} \mathbb{C} \left(\int_0^T \mathbf{h}^\top \mathbf{Z}_w^h(s)ds, \int_0^T \mathbf{S}^\top \mathbf{b}_w \mathbf{Z}_w^f(s)ds \right) &= \int_0^T (1 - e^{-\theta_w(T-s)})^2 ds \times \\ &\begin{pmatrix} \mathbf{h}^\top \sigma_w^h \Sigma_w^h \Sigma_w^{h\top} \sigma_w^h \mathbf{h} & \mathbf{h}^\top \sigma_w^h \Sigma_w^h \Sigma_w^{fh\top} \sigma_w^f \mathbf{b}_w \mathbf{S} \\ \mathbf{S}^\top \mathbf{b}_w \sigma_w^f \Sigma_w^{fh} \Sigma_w^{h\top} \sigma_w^h \mathbf{h} & \mathbf{S}^\top \mathbf{b}_w \sigma_w^f (\Sigma_w^{fh} \Sigma_w^{fh\top} + \Sigma_w^f \Sigma_w^{f\top}) \sigma_w^f \mathbf{b}_w \mathbf{S} \end{pmatrix}. \end{aligned}$$

Using the properties of multivariate Gaussian distributions, we infer that $\int_0^T \mathbf{h}^\top \mathbf{Z}_w^h(s)ds$, conditioned by $\int_0^T \mathbf{S}^\top \mathbf{b}_w \mathbf{Z}_w^f(s)ds$ is a normal random variable with a mean and variance defined by equations (46) and (47).

$$\mu_{h|f}(z) = \frac{1}{\theta_w} (1 - e^{-\theta_w T}) \mathbf{h}^\top \mathbf{Z}_w^h(0) + \frac{\mathbf{h}^\top \sigma_w^h \Sigma_w^h \Sigma_w^{fh\top} \sigma_w^f \mathbf{b}_w \mathbf{S}}{\mathbf{S}^\top \mathbf{b}_w \sigma_w^f (\Sigma_w^{fh} \Sigma_w^{fh\top} + \Sigma_w^f \Sigma_w^{f\top}) \sigma_w^f \mathbf{b}_w \mathbf{S}} \quad (70)$$

$$\begin{aligned} &\times \left(z - \frac{1}{\theta_w} (1 - e^{-\theta_w T}) \mathbf{S}^\top \mathbf{b}_w \mathbf{Z}_w^f(0) \right) \\ \sigma_{h|f}^2 &= \left(T - \frac{2}{\theta_w} (1 - e^{-\theta_w T}) + \frac{1}{2\theta_w} (1 - e^{-2\theta_w T}) \right) \\ &\times \left(\mathbf{h}^\top \sigma_w^h \Sigma_w^h \Sigma_w^{h\top} \sigma_w^h \mathbf{h} - \frac{(\mathbf{h}^\top \sigma_w^h \Sigma_w^h \Sigma_w^{fh\top} \sigma_w^f \mathbf{b}_w \mathbf{S})^2}{\mathbf{S}^\top \mathbf{b}_w \sigma_w^f (\Sigma_w^{fh} \Sigma_w^{fh\top} + \Sigma_w^f \Sigma_w^{f\top}) \sigma_w^f \mathbf{b}_w \mathbf{S}} \right) \end{aligned} \quad (71)$$

Let us define a random variable $U = \int_0^T \mathbf{h}^\top \mathbf{Z}_w^h(s)ds \sim N(\mu_{h|f}, \sigma_{h|f}^2)$. If $f_U(\cdot)$ is the density of U , the cross moment becomes:

$$\begin{aligned} &\mathbb{E} \left(\left(\int_0^T \mathbf{h}^\top \mathbf{Z}_w^h(s)ds - K_m \right) \middle| \mathcal{F}_t \cup \int_0^T \mathbf{S}^\top \mathbf{b}_w \mathbf{Z}_w^f(s)ds = z \right) \\ &= z \left(\int_{K_m}^\infty u f_U(u) du - K_m \int_{K_m}^\infty f_U(u) du \right). \end{aligned} \quad (72)$$

The symmetry of the Gaussian distribution allows us to rewrite the second integral of this last integral as follows:

$$\int_{K_m}^\infty f_U(u) du = \Phi \left(\frac{\mu_{h|f}(z) - K_m}{\sigma_{h|f}} \right). \quad (73)$$

Using the same reasoning as in proposition 3.2, we can show that

$$\int_{K_m}^\infty u f_U(u) du = \mu_{h|f}(z) \Phi \left(\frac{\mu_{h|f}(z) - K_m}{\sigma_{h|f}} \right) + \sigma_{h|f} \varphi \left(\frac{\mu_{h|f}(z) - K_m}{\sigma_{h|f}} \right). \quad (74)$$

Combining equations (72), (73), (74) and as $\int_0^T \mathbf{S}^\top \mathbf{b}_w \mathbf{Z}_w^f(s)ds$ is distributed as a normal $N(\mu_f, \sigma_f^2)$, we deduce the expression (48). ■