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Monotone Targets

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RISK APPORTIONMENT AND MULTIPLY MONOTONE TARGETS

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Abstract

This short note proposes a new interpretation of risk apportionment, in the target-oriented decision-making model. It is shown that these risk attitudes translate into smaller targets, expressing the decision maker's conservative behavior.

Keywords: Risk aversion, prudence, temperance, edginess, stochastic dominance.

1 Introduction

Assume that decision makers act as if they maximize their respective expected utility of wealth. Specifically, a decision maker with utility function u prefers the random end-of-period wealth Y over X if $E[u(X)] \leq E[u(Y)]$, that is, if the expected utility for Y exceeds that for X .

In the expected utility setting, many risk attitudes are defined by properties of successive derivatives $u^{(1)}, u^{(2)}, u^{(3)}, \dots$ of the utility function u . According to Eeckhoudt and Schlesinger (2006), preferences are said to satisfy risk apportionment of degree k if

$$(-1)^{k+1}u^{(k)} \geq 0$$

for some positive integer k . This risk attitude is justified by means of comparison of specific lotteries and is shown to correspond to the preference for “pain disaggregation”, i.e. the decision maker prefers to spread the pains among different states of the world instead of concentrating them in a single state. Later on, Eeckhoudt, Schlesinger and Tsetlin (2009) justified risk apportionment based on the tendency to combine “good with bad”, where “good” and “bad” are defined by means of stochastic dominance rules.

For low degrees, risk apportionment corresponds to classical risk attitudes. Recall that in the expected utility model risk aversion, prudence, temperance, and edginess are defined respectively by a negative second derivative, by a positive third derivative, by a negative fourth derivative, and by a positive fifth derivative of the utility function. Thus, these notions appear to be particular cases of risk apportionment of degrees 2 to 5. Prudence is justified after Kimball (1990) by reference to the decision of building up precautionary savings in order to better face future income risk. Kimball (1992) and Gollier and Pratt (1996) justified temperance based on the demand for risky assets in the presence of background risks, when an unavoidable background risk leads the decision maker to reduce exposure to another risk even if the two risks are statistically independent. Edginess was defined by Lajeri-Chaherli (2004) in a context of multiple risks in a two-period model, capturing the reactivity to multiple risks on precautionary motives. In Denuit and Rey (2010), the concept of risk apportionment is linked to the tendency of decision makers to become less sensitive to an increase in correlation when richer. See also Denuit and Rey (2013).

In this paper, we propose a new interpretation of these concepts in the target-based counterpart to the expected utility model.

2 Multiply monotone targets

2.1 Risk apportionment and multiply monotone probability density functions

Let us restrict our analysis to bounded, continuous utility functions. Following Borch (1968) and Berhold (1973), let us normalize the utility function u so that it can be interpreted as the distribution function of a random target T to reach, i.e.

$$u(x) = P[T \leq x] \text{ and } E[u(X)] = P[X \geq T]$$

where the random wealth X and the target T are mutually independent. As pointed out by Castagnoli and LiCalzi (1996) and Bordley and LiCalzi (2000), maximizing expected utility is equivalent to maximizing the probability to meet the random target corresponding to the normalized utility function.

Completely monotone functions, i.e. non-increasing functions with derivatives of all degrees alternating in signs, correspond to scale mixtures of negative exponentials. They are related to mixed risk aversion in the sense of Caballe and Pomansky (1996). Functions on $\mathbb{R}^+ = (0, \infty)$ satisfying this property up to some degree $k < \infty$ have been studied in applied probability, under the name k -monotone. This leads us to the following elementary result.

Property 2.1. *In the target-based approach to decision making, preferences exhibiting risk apportionment of degrees $k = 1$ to $s \in \{2, 3, 4, \dots\}$ corresponds to targets with $(s - 1)$ -monotone probability density function, i.e. a decreasing density over $\mathbb{R}^+$, with derivatives of degrees 1 to $s - 1$ alternating in signs.*

This means that a risk-averse decision maker ($s = 2$) possesses a target with a decreasing density over $\mathbb{R}^+$. Prudent, risk-averse decision makers ($s = 3$) then have a target with a decreasing and convex probability density function. And so on. In the next section, based on a representation theorem for random variables with multiply monotone probability density function, we show that the corresponding targets become smaller when s increases, providing a new interpretation of risk apportionment.

2.2 Risk apportionment and scaled minima of uniform random variables

Khinchine representation theorem (see, e.g., Theorem 1.3 in Dharmadhikari and Joag-Dev, 1988) ensures that a target T has a decreasing probability density function over $\mathbb{R}^+$ if, and only if, there exist independent random variables U and Z , such that U is uniformly distributed on $[0, 1]$ and the product UZ is distributed as Y . This result extends to higher degrees and provides us with the following representation of targets possessing multiply monotone probability density functions.

Proposition 2.2. *Consider a sequence of independent random variables $U_1, U_2, \dots$, uniformly distributed over the unit interval and $s \in \{2, 3, 4, \dots\}$. In the target-based approach to decision making, preferences exhibit risk apportionment of degrees 1 to s if, and only if, the target T is distributed as $Z \min\{U_1, \dots, U_{s-1}\}$ for some positive random variable Z independent of $U_1, \dots, U_{s-1}$.*

Proof. Recall from Lefevre and Loisel (2013, Property 2.1) that the target T possesses a $(s - 1)$ -monotone probability density function if, and only if, it is distributed as $(1 - U^{\frac{1}{s-1}})Z$ where U is uniformly distributed over $[0, 1]$ and Z is valued in $\mathbb{R}^+$ and independent of U . Such targets are known to correspond to preferences exhibiting risk apportionment of degrees

1 to s according to Property 2.1. Now,

$$\begin{aligned}
P[T \leq t] &= P \left[\left(1 - U^{\frac{1}{s-1}} \right) Z \leq t \right] \\
&= P \left[U \geq \left(1 - \frac{t}{Z} \right)_+^{s-1} \right] \\
&= E \left[1 - \left(1 - \frac{t}{Z} \right)_+^{s-1} \right] \\
&= P[Z \min\{U_1, \dots, U_{s-1}\} \leq t],
\end{aligned}$$

as announced. $\square$

Proposition 2.2 provides us with a new interpretation of risk apportionment in the target-based version of expected utility. It clearly shows that increasing the degree of risk apportionment leads to more conservative decision-making as the target to reach decreases with s . For $s = 2$, we recover the classical Khinchine representation theorem recalled above. As $s \rightarrow \infty$, we see that $(s-1)(1 - U^{\frac{1}{s-1}})$ converges in distribution to $-\ln U$ which obeys the Negative Exponential distribution with unit mean. Thus, we recover at the limit weighted averages of exponential utilities corresponding to mixed risk aversion as defined by Caballe and Pomansky (1996).

2.3 Stochastic dominance rules

Stochastic dominance rules are based on iterated integrals of the distribution functions. The next result shows that in the target-based approach to decision making, it suffices to restrict to a subset of targets possessing a $(s-1)$ -monotone probability density function.

Proposition 2.3. *Consider a sequence of independent random variables $U_1, U_2, \dots$, uniformly distributed over the unit interval and $s \in \{2, 3, 4, \dots\}$. Consider random end-of-period wealths X and Y such that $E[X^k] = E[Y^k]$ for $k = 1, \dots, s-1$. Then, Y is preferred over X by all the decision makers whose preferences exhibit risk apportionment of degrees 1 to s if, and only if, the probability of reaching any target*

$$T = z \min\{U_1, \dots, U_{s-1}\}, \quad z \geq 0,$$

is larger for Y compared to X .

Proof. Repeated integrals of distribution functions correspond to basic utility functions expressing risk apportionment. Specifically, for any distribution function F with $F(0) = 0$, let us denote $F^{[0]}(t) = F(t)$, and, for $k \geq 1$, denote

$$F^{[k]}(t) = \int_0^t F^{[k-1]}(x) dx.$$

It is well known that

$$F^{[s-1]}(t) = \frac{E[(t - X)_+^{s-1}]}{(s-1)!}$$

where X is distributed according to F . As X and Y share the same first $s - 1$ moments, Y is preferred over X by all the decision makers whose preferences exhibit risk apportionment of degrees 1 to s if, and only if,

$$F^{[s-1]}(t) \geq G^{[s-1]}(t) \text{ for all } t \geq 0 \Leftrightarrow E[(t - X)_+^{s-1}] \geq E[(t - Y)_+^{s-1}] \text{ for all } t \geq 0$$

where G denotes the distribution function of Y . For any $z > 0$, the distribution function of $\min\{zU_1, \dots, zU_k\}$ is given by

$$P[\min\{zU_1, \dots, zU_k\} \leq x] = 1 - \left(1 - \frac{x}{z}\right)_+^k.$$

Hence,

$$E[(t - X)_+^{s-1}] \geq E[(t - Y)_+^{s-1}] \text{ for all } t \geq 0$$

$$\Leftrightarrow P[X \geq \min\{zU_1, \dots, zU_k\}] \leq P[Y \geq \min\{zU_1, \dots, zU_k\}] \text{ for all } z \geq 0,$$

which ends the proof. □

Proposition 2.3 shows that increasing the degree of risk apportionment makes the targets smaller (as they correspond to minima taken over a larger number of independent unit uniform random variables). As smaller targets express more conservative behavior, this result provides us with a new interpretation of risk apportionment. To recover all bounded utility functions u satisfying $(-1)^{k+1}u^{(k)} \geq 0$ for $k = 1, 2, \dots, s$, we have to consider scale mixtures, i.e. to replace the positive constant z with a positive random variable Z , independent of the U_k , as shown in Proposition 2.2.

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