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Extreme value analysis of mortality at the oldest ages: a case study based on individual ages at death

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EXTREME VALUE ANALYSIS OF MORTALITY AT THE OLDEST AGES: A CASE STUDY BASED ON INDIVIDUAL AGES AT DEATH

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Abstract

In this paper, the force of mortality at the oldest ages is studied using the statistical tools from extreme value theory. A unique data basis recording all individual ages at death above 95 for cohorts born in Belgium between 1886 and 1904 is used to illustrate the relevance of the proposed approach. No levelling-off in the force of mortality at the oldest ages is found and the analysis supports the existence of an upper limit to human lifetime for these cohorts. Therefore, assuming that the force of mortality becomes ultimately constant, i.e that the remaining lifetime tends to the Negative Exponential distribution as the attained age grows is a conservative strategy for managing life annuities.

Key words and phrases: life insurance, extreme value theory, peaks over threshold, ultimate age, closure of the life table.

1 Introduction and motivation

Insurers need reliable estimate of mortality at oldest ages to price life annuities and reverse mortgages, for instance. Gompertz, Makeham, Logistic as well as other parametric models have often been used by actuaries to fit mortality statistics. The exponential increase in the force of mortality postulated by the Gompertz model is generally appropriate for adults and early old ages but the mortality increases tend to slow down at older ages, typically around 85, so that this model fails to describe the end of the life table. Alternative models have been proposed to account for the mortality deceleration observed at older ages in industrialized countries. The plateau that appears at these advanced ages is attributed to the earlier selection of the more robust individuals in heterogeneous cohorts. This empirical finding supports the Logistic model obtained by including a Gamma distributed frailty coefficient accounting for the heterogeneity in the Gompertz model (Manton et al. (1986)).

When the main interest is in the mortality at the oldest ages, inference is conducted in an area of the sample where there is a very small amount of data. Moreover, extrapolation beyond the range of the data is often desirable, a procedure known as the closure of the life table. In such a case, it is essential to have a good model for the longest lifetimes: Extreme Value Theory deals with statistical problems concerning the far tail of the probability distribution, being supported by strong theoretical arguments, and thus appears as the natural candidate for analyzing mortality at the oldest ages.

Models for extreme values have already been used to describe individual lifetimes but concentrating on minima. Brillinger (1961) was among the first authors who draw the attention of the actuarial community to the use of extreme value techniques in the analysis of mortality. Specifically, he pointed out that the Gompertz law naturally appears to describe lifetimes if the human body is considered as a series system made of a large number of components with independent and identically distributed failure times. The death then occurs when the first component fails, that is, at the minimum of these failure times which asymptotically obeys the Gompertz model. By construction, the approximation of the lifetime distribution by an extreme value distribution such as the Gompertz law can only be expected to hold for small or moderate ages, but not for high ages, as pointed out by Aarssen and de Haan (1994). Another approach is thus needed for high ages, such as the one proposed in this paper. Notice that some authors argue that the Gompertz model may also apply to higher ages. For instance, Gavrilov and Gavrilova (2011) concluded from a study of several extinct cohorts that mortality at advanced ages obeys the Gompertz model up to ages 102-105, without noticeable deceleration. However, this study is subject to debate. See e.g. Ouellette and Bourbeau (2014) for an opposite view.

The present paper adopts a distribution-free approach to analyze mortality at the oldest ages; only some weak regularity of the tail of the distribution function is assumed. This method is acceptable both from the theoretical and practical viewpoints. To this end, we recall the basic features of Extreme Value Theory (EVT, in short) that gives the theory for describing extremes of random phenomena. EVT, and its close link to limiting residual life distributions offers a unified approach to the modelling of the right tail of a lifetime distribution. This method is not solely based on the data at hand but includes a probabilistic argument concerning the behavior of the extreme sample values. EVT has already been successfully applied to solve various nonlife insurance problems; see e.g. Cebrian et al.

(2003) and the references therein. In this paper, we demonstrate that it also offers the appropriate framework for dealing with the highest ages at death. The analysis of residual lifetimes at high ages is in line with the Peaks Over Threshold (POT) method used to study tail behavior exhibited by nonlife insurance losses. POT is based on the convergence of the distribution of exceedances over a threshold to the Generalized Pareto (GP) distribution when the threshold increases. Transposed to the life insurance setting, this approach is justified by the convergence of the remaining lifetime distribution at high ages to the GP distribution.

Let us mention that the present paper is not the first attempt to use EVT techniques to model mortality at the oldest ages. EVT has even been initially developed to deal with such problems, as it can be seen from Gumbel (1937), Balkema and de Haan (1974) and Aarssen and de Haan (1994), for instance. Our approach is closely related to the pioneering study by Watts et al. (2006) where the usefulness of EVT techniques in life insurance data analysis is clearly established. Actuaries have also developed threshold life tables where the extrapolation of mortality to older ages (above the threshold) is done using EVT. We refer the readers to the works by Han (2005), Li et al (2008, 2010) and Bravo et al. (2008, 2012). Compared to these previous studies, the originality of our paper relies on the fact that :

- we work here with reliable individual exact ages at death, not with red annual aggregated mortality data,
- we resort to advanced statistical tools to explore the trajectory of mortality near the end of the lifetimes, questioning the choice of the appropriate threshold in the POT approach above 95 years old as well as the estimation method for the tail index,
- we conduct the analysis using cohort mortality statistics, and not period data, as the theoretical arguments supporting the use of EVT techniques are fulfilled in this case as it will become clear in the next sections.

EVT techniques have also been considered in other disciplines such as demography, as a natural tool to analyze mortality at the oldest ages. For instance, Hanayama and Sibuya (2015) estimated the ultimate age in the Japanese population by means of EVT techniques, based on aggregated mortality statistics for Japanese centenarians comprised in the Human Mortality Database (HMD). The approximate linearity of the mean residual life function lead to the use of the GP distribution to describe survival above a high age. The GP parameters are then estimated by Binomial regression on yearly death counts and a significantly negative estimated tail index resulted for male cohorts borne after 1883 and female cohorts born after 1867. Despite obvious similarities, there are also important difference between this study and the one conducted in the present paper. First, we work with individual data, recording exact ages at death for all Belgian residents born and died in Belgium at age 95 and older. Also, these authors arbitrarily selected the age range where mortality obeys the GP model, starting at age 100, and only considered maximum likelihood estimation techniques. We discuss these two choices here and we stress their great impact on the final estimates.

Several authors, including Gampe (2010), have suggested that age-specific death probabilities may level off at some point above 100 years, rather than continuing to increase with age. This proposal has been supported by the analysis of the mortality of supercentenarians (aged 110 and over) from many countries included in the International Database of Longevity (IDL). Notice that if the force of mortality tends to flatten at oldest ages then remaining lifetimes become ultimately Negative Exponential. The analysis conducted in the present paper suggests that the remaining lifetimes display a lighter tail compared to the Negative Exponential and that there is an upper bound on the human lifetime, that actuaries call ultimate age. Our results are in line with those obtained by Poulain et al. (2001) who concluded that the annual death probabilities between ages 100 and 108 rise from 0.35 to 0.5 for Belgian females and from 0.42 to 0.55 for Belgian males. These values are similar to those obtained by Kannisto (1994) from a significantly larger study population. Importantly, we work here with individual exact ages at death and never aggregate them over intervals, as this may bias the analysis.

Notice that actuaries are more interested in closing the life table in an appropriate manner, than to provide a final answer to the question of maximum lifespan. Indeed, the problem posed by the mortality pattern at oldest ages, with a still increasing, or plateauing, or even decreasing force of mortality after a maximum level has been reached, goes far beyond the actuarial expertise as almost no data is available at ages above 115. Actuaries nevertheless need an appropriate model, supported by empirical evidence, to close the life table so that actuarial calculations can be performed, typically in life annuity or reverse mortgage portfolios. To this end, this paper uses a unique database recording the ages at death for 19 extinct cohorts born in Belgium between 1886 and 1904. The survival of Belgian centenarians has been studied by Poulain et al. (2001). As explained by these authors, the National Register, a centralized, fully computerized population register for the whole population of Belgium that was established by law in 1985 (see Poulain (2010)), minimizes the risk of errors in recorded ages at death. This is why we use data extracted from the Belgian National Register to conduct the present study. As ages are recorded up to the day, we can model lifetimes on a continuous scale and apply EVT techniques to complete data, that are not censored by intervals.

The remainder of this paper is organized as follows. We start by describing the data used in the present study in Section 2. Section 3 gives a brief, user-friendly description of EVT tools that are necessary to analyze mortality at the oldest ages. The definition of the Generalized Extreme Value distribution is recalled, providing actuaries with a limit distribution for maxima subject to an appropriate standardization. Its connection to the Generalized Pareto distribution is established by taking the limit of the conditional distribution for the exceedances over a threshold. The presentation is made in terms of demographic indicators, to make it more appealing for actuaries working in life insurance. In particular, the close links existing between EVT and limiting residual lifetime distributions are made explicit. Selecting the threshold above which the exceedances obey the Generalized Pareto distribution is the key first step in actuarial application based on EVT. The analysis conducted here is based on tools proposed by Pickands (1975), Reiss and Thomas (1997) and McDonald et al. (2011). Section 4 is devoted to several applications of the model such as the estimation of ultimate age and the prediction of the highest age at death recorded among a group of individuals. The final Section 5 summarizes the main findings of the paper and briefly concludes.

2 Description of the data set

2.1 Data source and validation process

As mentioned in the introduction to this paper, a national population register serves as the centralizing database in Belgium and provides official population figures. Statistics on births and deaths are available from this register according to basic demographic characteristics (e.g., age, gender, marital status). We have at our disposal the ages at death for every individual who died at age 95 or older, plus a list of survivors aged 95 and older at the end of the observation period. There are 46,666 observations relating to individuals born in Belgium who died after age 95 since 1981. The data basis comprises 22% males and 78% females. For each individual, we know the exact birth and death dates.

Data have been validated in a number of ways. For more details, we refer to Poulain et al. (2001), so that we can be confident in the reliability of the ages at death recorded in the database. In particular, all individual who were not born in Belgium have been excluded from the data set so that migrations can be neglected (as international mobility is almost inexistent for native Belgian citizens above age 95) and date of birth is recorded from official certificates (and not reported by individuals).

2.2 Descriptive statistics

Basic descriptive statistics of the data sets are given in Table 2.1. The average age at death for females is greater than the one for males, as expected. However, the differences are rather modest, compared to the differences observed for the total lifetime from birth, except for the much higher number of females reaching age 95 compared to males. The highest ages at death observed in Belgium for these cohorts are 112.58 for females and 111.47 for males. Recall that Jeanne Calment died at the age of 122 years in 1997. After this record, only a single person, Sarah Knauss has lived for 119 years, and died in 1993. Since then, three women lived more than 117 years and three others are still alive at 116 years old.

	Males	Females
Number of observations	10,050	36,616
Mean	97.35	97.75
Std. Dev.	2.05	2.37
1st Quartile	95.77	95.91
Median	96.79	97.12
3rd Quartile	98.36	98.99
Maximum	111.47	112.58

Table 2.1: Basic descriptive statistics for the observed ages at death above 95, cohorts 1886 to 1904.

Cohort-specific data are displayed in Table 2.2 for both females and males. We can read there the initial number of individuals included in the analysis, for each extinct cohort born between 1886 and 1904 (that is, the number L_{95} of individuals born in calender year $t \in \{1886, \ldots, 1904\}$ still alive at age 95) as well as the age at death of the last survivor for each cohort. These values are given separately for males and females. We can see there that L_{95} increases as time passes, as expected. Also, the number of individuals entering the study is larger for females compared to males. The maxima m are relatively stable over cohorts, and are higher for females compared to males: For each cohort, the last survivor was a female. In addition, there is no visible trend in the shifted life expectancy at 95. It even remains stable around 97 for all male and female cohorts. This seems to suggest no potential gain of longevity trough generations at these advanced ages.

	Males			Females		
t	L_{95}	m	$95+\hat{e}_{95}$	L_{95}	m	$95+\hat{e}_{95}$
1886	359	108.17	97.35	1045	107.78	97.70
1887	418	105.13	97.41	1130	110.45	97.64
1888	412	106.33	97.20	1242	110.32	97.68
1889	462	105.58	97.53	1208	110.16	97.69
1890	425	107.70	97.50	1311	112.58	97.60
1891	428	105.81	97.35	1368	109.72	97.70
1892	444	105.44	97.29	1510	110.89	97.72
1893	492	110.29	97.46	1544	107.75	97.63
1894	498	106.19	97.22	1760	107.41	97.71
1895	526	106.62	97.24	1852	109.38	97.86
1896	545	106.27	97.43	1894	109.79	97.85
1897	568	106.43	97.33	2009	109.85	97.72
1898	572	105.74	97.26	2149	110.89	97.71
1899	630	106.88	97.35	2301	111.60	97.72
1900	576	111.47	97.27	2460	111.70	97.78
1901	635	103.77	97.23	2787	110.36	97.89
1902	632	106.79	97.42	2829	112.36	97.82
1903	669	104.71	97.46	2980	109.96	97.81
1904	759	106.15	97.32	3237	110.18	97.74

Table 2.2: Initial size for each cohort (L_{95}), observed highest age at death (m) and mean age at death ($95+\hat{e}_{95}$). Males and Females.

Box-plots are provided in Figure 2.1. Figure 2.2 displays the corresponding histograms. The boxplots reveal ages at death much above the medians and third quartile, for both genders. The same conclusion is reached on the basis of the histogram, where we see that the right tail expands on old ages, whereas the majority of deaths are concentrated before age 100.

3 Methodology

Now that the data basis has been presented, let us perform a statistical analysis using the tools from EVT.

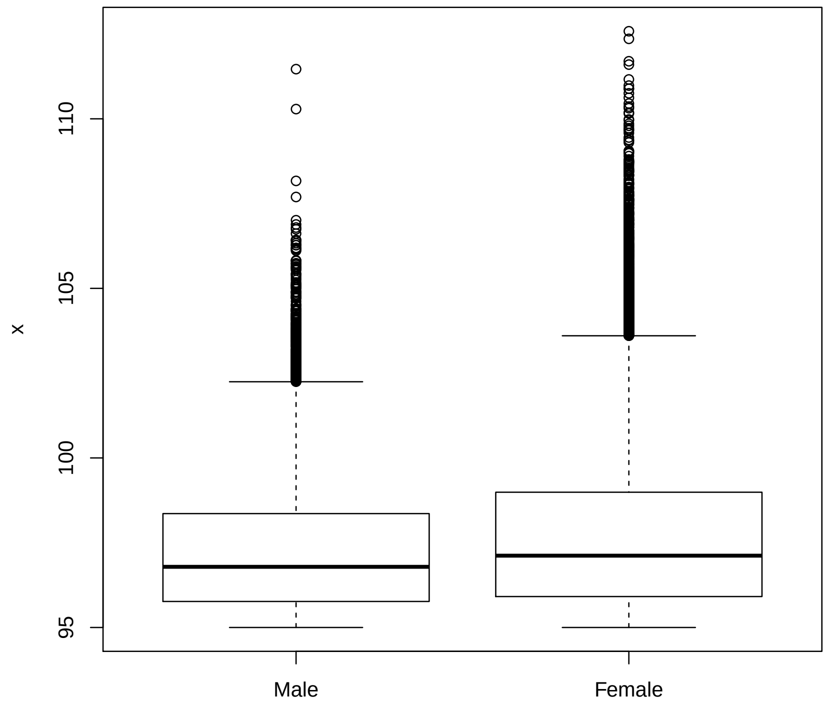


Figure 2.1: Boxplots for the observed ages at death above 95, cohorts 1886 to 1904.

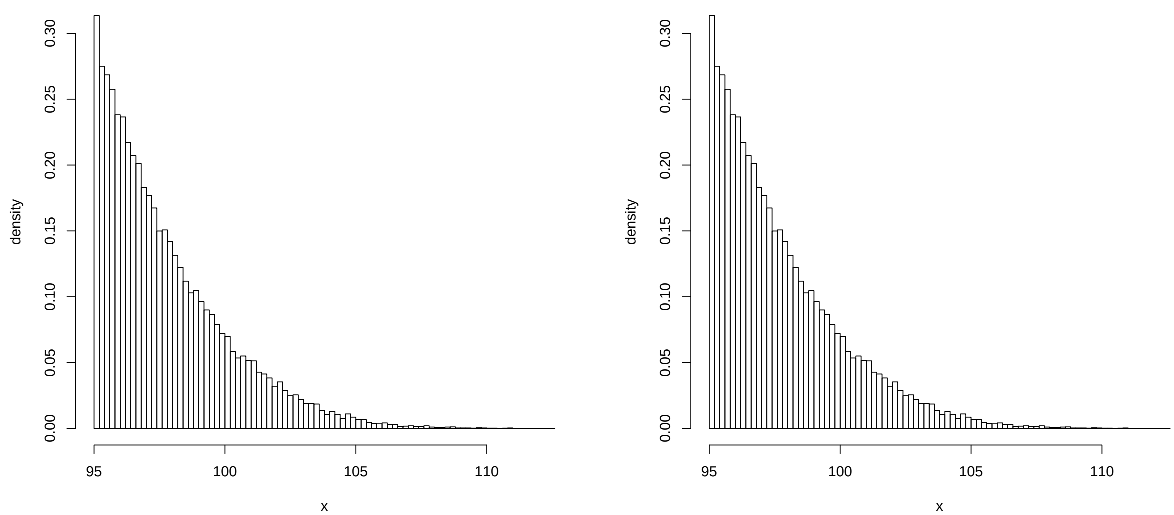


Figure 2.2: Histograms for the observed ages at death above 95 for females (left panel) and males (right panel), cohorts 1886 to 1904.

3.1 Extreme Value Theory (EVT) for mortality at the oldest ages

Consider a sequence of independent individual lifetimes $T_1, T_2, T_3, \ldots$ with common distribution function

$$ {}_xq_0 = \mathrm{P}[T_i \leq x], \;\; x \geq 0, $$

for $i = 1, \ldots, n$, satisfying ${}_0q_0 = 0$. Here, T_i represents the total lifetime, from birth to death. It may also represent the remaining lifetime after a given initial age α (equal to 95 in our study), with common distribution function ${}_xq_\alpha$.

Define a sequence of maxima $M_n = \max\{T_1, T_2, \ldots, T_n\}$. In words, M_n represents the oldest age at death observed in an homogeneous group of n individuals subject to the same life table $x \mapsto {}_xq_0$. EVT studies the asymptotic behavior of M_n when n tends to infinity and provides results analogous to the central limit theorem for maxima (rather than sums), provided some mild technical conditions on $x \mapsto {}_xq_0$ are fulfilled. Of course, without further restriction, M_n obviously approaches the upper limit of the support

$$ \omega = \sup\{x \geq 0 | {}_xq_0 < 1\}, \text{ possibly infinite.} $$

This is easily seen from

$$ \mathrm{P}[M_n \leq x] = \big({}_xq_0\big)^n \rightarrow \begin{cases} 0 \text{ if } x < \omega \\ 1 \text{ if } x \geq \omega \end{cases} \quad \text{as } n \rightarrow \infty. $$

Once M_n is appropriately centered and normalized, however, it may converge to some specific limit distribution. Precisely, if there exist sequences of real numbers $a_n > 0$ and $b_n \in \mathbb{R}$ such that the normalized sequence $(M_n - b_n)/a_n$ converges in distribution to H, i.e.

$$ \lim_{n \to +\infty} \mathrm{P}\left[\frac{M_n - b_n}{a_n} \leq x\right] = \lim_{n \to +\infty} \big({}_{a_n x + b_n}q_0\big)^n = H(x) \tag{3.1} $$

for all points of continuity of H, then H is a Generalized Extreme Value (GEV, in short) distribution, i.e. $H = H_\xi$ given by

$$ H_\xi(x) = \begin{cases} \exp\left(-(1+\xi x)_+^{-1/\xi}\right) \text{ if } \xi \neq 0, \\ \\ \exp\left(-\exp(-x)\right) \text{ if } \xi = 0, \end{cases} $$

where $y_+ = \max\{y, 0\}$ is the positive part of y. The support of H_ξ is $(-1/\xi, +\infty)$ if $\xi > 0$, $(-\infty, -1/\xi)$ if $\xi < 0$ and the whole real line $\mathbb{R}$ if $\xi = 0$. Here, the parameter ξ controlling the right tail is called the tail index or the extreme value index. The three classical extreme value distributions are special cases of the GEV family: if $\xi > 0$ we have the Fréchet distribution, if $\xi < 0$ we have the Weibull distribution and $\xi = 0$ gives the Gumbel distribution. When $\xi > 0$, we face lifetimes with heavy tails which contradicts empirical evidence available for human lifetimes (in this case, forces of mortality decrease with attained age). Thus, the cases $\xi = 0$ and $\xi < 0$ are of interest for life insurance applications. Notice that if (3.1) holds with $\xi < 0$ then $\omega < \infty$, so that a negative value of ξ supports the existence of a finite ultimate age ω.

A sufficient condition for (3.1) to hold is

$$\lim_{x\to\omega} \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{\mu_x}\right) = \xi \tag{3.2}$$

where

$$\mu_x = \frac{\frac{\mathrm{d}}{\mathrm{d}x}\,{}_xq_0}{{}_xp_0}$$

is the force of mortality at age x. Intuitively speaking, $\frac{1}{\mu_x}$ can be considered as the force of resistance to mortality, or force of vitality at age x. The resistance to mortality must thus stabilize when $\xi = 0$ or become ultimately linear. A negative ξ indicates that the resistance ultimately decreases at advanced ages. For $\xi < 0$ we have $\omega < \infty$ and condition (3.2) implies

$$\lim_{x\to\omega}\big((\omega - x)\mu_x\big) = -\frac{1}{\xi}.$$

3.2 Limiting remaining lifetime distribution

Consider the remaining lifetime $T - x$ at age x, given $T > x$, with distribution function

$$s \mapsto {}_sq_x = \mathrm{P}[T - x \le s | T > x].$$

It may happen that for large attained age x, this conditional probability distribution stabilizes after a normalization, that is, there exists a positive function a such that

$$\lim_{x\to\omega} \mathrm{P}\left[\frac{T-x}{a(x)} > s \middle| T > x\right] = 1 - G(s), \;\; s > 0, \tag{3.3}$$

where G is a non-degenerate distribution function. It is possible to establish that only a limited class of distribution functions are eligible in (3.3), namely

$$\begin{aligned} G(s) = G_\xi(s) &= \ln H_\xi(s) \\ &= \begin{cases} 1 - (1+\xi s)_+^{-1/\xi} \text{ if } \xi \neq 0, \\ 1 - \exp(-s) \text{ if } \xi = 0. \end{cases} \end{aligned}$$

Here, the support is the half positive real line if $\xi \ge 0$ and $[0, -1/\xi]$ if $\xi < 0$. The related scale family known as the Generalized Pareto (GP) distribution is then defined as

$$G_{\xi,\beta}(s) = G_\xi\left(\frac{s}{\beta}\right), \;\; \beta > 0.$$

As particular cases of the GP distribution functions $G_{\xi;\beta}$, we find some classical distributions, namely the Pareto distribution when $\xi > 0$, the type II Pareto distribution when $\xi < 0$ and the Negative Exponential distribution when $\xi = 0$. Thus, when $\xi = 0$, the remaining lifetimes at high ages become ultimately Negative Exponentially distributed so that the forces of mortality stabilize, in line with the empirical study conducted by Gampe (2010).

Clearly, extreme value analysis for maxima is thus closely connected with the study of residual lifetimes. It can be shown that (3.1) holds with $H = H_\xi$ if, and only if, (3.3) holds with $G = G_\xi$. In words, G_ξ describes the remaining lifetimes above sufficiently old ages if, and only if, H_ξ governs the sample maximum behaviour, i.e. if F belongs to the domain of attraction of a GEV distribution.

For some appropriate function $\beta(\cdot)$, the approximation

$$ {}_sq_x \approx G_{\xi;\beta(x)}(s) \text{ for } s \geq 0 \tag{3.4}$$

holds for x large enough. The approximation (3.4) is justified by the Pickands-Balkema-de Haan theorem:

$$\lim_{x\to+\infty} \sup_{s\geq 0} \left| {}_sq_x - G_{\xi,\beta(x)}(s) \right| = 0 \tag{3.5}$$

is true provided that F satisfies some rather general technical conditions. In view of (3.4) the remaining lifetimes at age x can be treated as a random sample from the GP distribution provided x is large enough.

If $\omega < \infty$, i.e. $\xi < 0$, then a suitable transformation of the extreme value index ξ possesses an intuitive interpretation. Recall that the remaining life expectancy at age x, denoted as e_x, is defined as

$$e_x = \mathrm{E}\left[T - x | T > x\right].$$

Aarssen and de Haan (1994) established that for $\xi < 0$, so that an upper bound ω exists on the life span, (3.1) is equivalent to

$$\lim_{x\to\omega} \mathrm{E}\left[\frac{T-x}{\omega - x}\Big| T > x\right] = \lim_{x\to\omega} \frac{e_x}{\omega - x} = -\frac{\xi}{1-\xi} = \alpha$$

These authors call $\alpha = \alpha(\xi)$ the perseverance parameter and provide the following explanation. Consider an individual still alive at some advanced age x. The ratio $\frac{T-x}{\omega-x}$ represents the percentage of the actual remaining lifetime $T - x$ to the maximum remaining lifetime $\omega - x$. This percentage stabilizes, on average when $x \to \omega$ and converges to α, which thus appears as the expected percentage of the maximum possible remaining lifetime effectively used by the individual.

3.3 Threshold selection

To be in a position to apply the results presented in the preceding section, we have first to determine what a threshold age $x^\star$ such that the approximation (3.4) is sufficiently accurate for $x \geq x^\star$. Thus, our aim is to determine the youngest age beyond which the GP distribution offers a reasonable approximation to the remaining lifetime distribution. Notice that Li et al. (2008) and Bravo et al. (2012) also used statistical procedures to select the threshold age $x^\star$ above which the GP behavior appears. This is done by modeling the mortality below the threshold using a standard parametric model (Gompertz in these studies) and to shift to the GP model above the threshold. The optimal threshold value is then obtained by a grid search, maximizing the likelihood of the composite model. However, this selection procedure heavily depends on the appropriateness of the parametric model describing mortality below the threshold. This introduces an additional specification risk in the analysis of mortality at the oldest ages whereas EVT is by essence a nonparametric approach to modeling the tails of a distribution. The flexible extreme value mixture model discussed in this section is also based on a composite model but it replaces the parametric specification for younger ages with a flexible nonparametric one, avoiding this drawback.

To identify the optimal threshold value, we can use the following graphical tools from EVT.

3.3.1 Empirical mean excess function plot

It is easily checked that when a lifetime has the GP distribution function $G_{\xi,\beta}$, the remaining life expectancy is a linear function in the attained age x, that is,

$$e_x = \frac{\beta}{1-\xi} + \frac{\xi}{1-\xi}x \tag{3.6}$$

provided $x < \omega \Leftrightarrow \beta + x\xi > 0$. Hence, the idea is to determine, on the basis of the graph of the empirical version $\widehat{e}_x$ of e_x, a pivot age $x^\star$ such that $\widehat{e}_x$ becomes approximately linear for older ages.

The empirical version of the remaining life expectancy $\widehat{e}_x$ viewed as a function of attained age x is displayed in Figure 3.1. These values have been obtained from the observed ages at death $\{t_1, t_2, \ldots, t_n\}$ comprised in the database by

$$\widehat{e}_x = \frac{\sum_{i=1}^n t_i \mathrm{I}[t_i > x]}{\#\{t_i : t_i > x\}} - x = \frac{\sum_{i=1}^n (t_i - x)\mathrm{I}[t_i > x]}{\#\{t_i : t_i > x\}}$$

where $\mathrm{I}[A] = 1$ if the event A did occur and 0 otherwise, and where $\#B$ is the number of elements in the set B. As usual, $\widehat{e}_x$ is only evaluated at the observations, i.e. for $x \in \{t_1, t_2, \ldots, t_n\}$. Denoting the observed ages at death arranged in ascending order as $t_{[1]} \leq t_{[2]} \leq \ldots \leq t_{[n]}$, we have in this case

$$\widehat{e}_{t_{[k]}} = \frac{1}{n-k}\sum_{j=1}^{n-k}(t_{[k+j]} - t_{[k]}).$$

Clearly, if the remaining lifetimes obey the Negative Exponential distribution, then the mean excess function is constant. Consequently, the plot of e_x versus age x will be an horizontal line. Short-tailed distributions will show a downward trend. On the contrary, an upward trend will be an indication of heavy tailed behaviour. The linear, decreasing shape of $\widehat{e}_x$ in Figure 3.1 contradicts the Negative Exponential behavior of the remaining lifetime. A downward trend is clearly visible on Figure 3.1, supporting short-tailed behavior. Therefore, a negative value of tail index ξ is expected.

According to Poulain et al. (2001), the remaining life expectancies for Belgian male and female centenarians (i.e. e_{100}) were estimated to 1.68 and 1.97, respectively. This is in line with the values displayed on Figure 3.1.

The comparison to Negative Exponentially distributed lifetimes can be performed with the help of an exponential QQ-plot. Its interpretation is easy. If the data are an independent

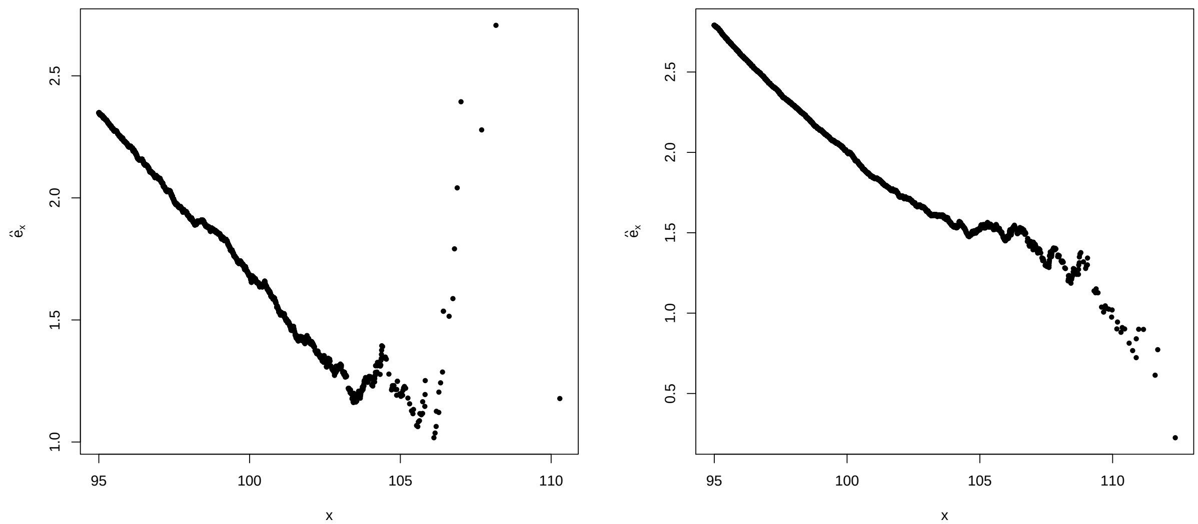


Figure 3.1: Empirical expected remaining lifetimes $\widehat{e}_x$ for males (left panel) and females (right panel).

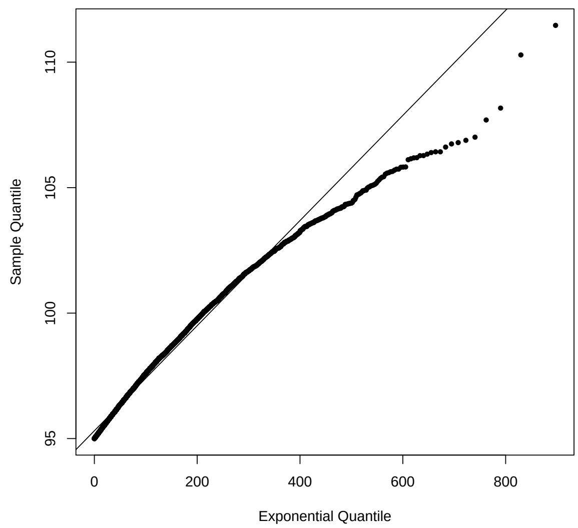
and identically distributed sample from a Negative Exponential distribution, the points should lie approximately along a straight line. A convex departure from the ideal shape indicates a heavier tailed distribution in the sense that empirical quantiles grow faster than the theoretical ones. On the contrary, concavity indicates a shorter tailed distribution. It is easy to see a concave pattern of the exponential QQ-plot on Figure 3.2 confirming the short-tailed behavior of the lifetime distribution.

Here it is tricky to detect the lowest threshold beyond which the mean excess function plot becomes approximately linear (See Figure 3.2). Consequently, we favor automatic selection procedures for the threshold suggested in the literature. We refer the reader to Scarrott et al. (2012) for a detailed review of threshold selection methods.

3.3.2 Pickands method

In its original paper, Pickands (1975) who established the theorem underlying the POT approach, also proposed a selection procedure for the threshold. This author defined $x^\star \equiv T_{[n-4M+1]}$ since the $4M$ largest observations intuitively contain information about the upper tail of distribution function. We recall that $T_{[k]}$ is the k^{th} order statistics in ascending order. Specifically, Pickands (1975) suggested to compute M in the following way. For each $l, l = 1, 2, \ldots, \lfloor n/4 \rfloor$, let $d_l = \sup_{0 \leq x < \infty} |\widehat{S}_l(x) - \widehat{\widehat{G}}_l(x)|$ where $\widehat{S}_l(x)$ is the empirical upper tail defined as

$$\hat{S}_l(x) = \frac{1}{4l} \sum_{m=1}^{4l-1} \mathrm{I}\Big[T_{[n-m+1]} - T_{[n-4l+1]} \geq x\Big] \tag{3.7}$$



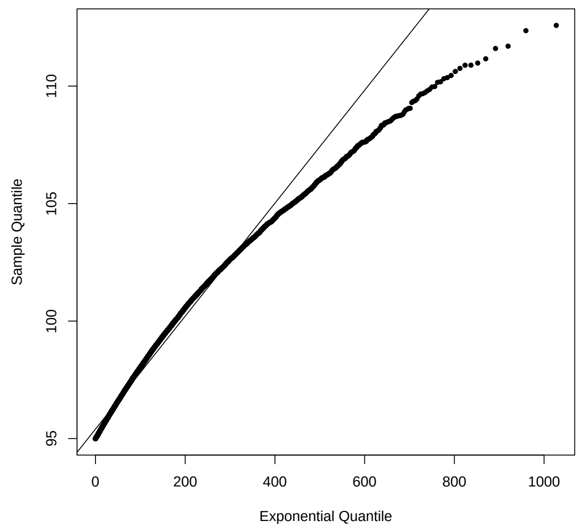


Figure 3.2: Exponential QQplot for lifetimes for males (left panel) and females (right panel).

and $\widehat{\overline{G}}(x)$ is the estimate of the survival function of the GP distribution with parameters ξ and β being estimated by the percentile method using the median and the third quartile. Notice that d_l is a kind of Kolmogorov-Smirnov test statistic. Finally, M is obtained as the smallest integer solution of

$$d_M = \min_{1 \leq l \leq \lfloor n/4 \rfloor} d_l. \tag{3.8}$$

Applying this method to our datasets we obtain the results displayed in Table 3.1.

	Male	Female
M	2,302	3,159
$4M$	9,208	32,256
$T_{[n-4M+1]}$	95.24	95.41

Table 3.1: Values of M, $4M$ and $T_{[n-4M+1]}$ obtained by the Pickands method for both males and females.

3.3.3 Reiss - Thomas method

Reiss and Thomas (1997) proposed an automatic selection procedure to choose the optimal value of the threshold $x^\star$. According to these authors $x^\star \equiv T_{[n-k^\star+1]}$ where $k^\star$ minimizes

$$\frac{1}{k-1} \sum_{i \leq k} i^\delta \left(\widehat{\xi}_i - \widehat{\xi}_k \right)^2. \tag{3.9}$$

Here, $\widehat{\xi_i}$ is a shape parameter estimator based on the upper ordrered statistics $T_{[n-i+1]}, \dots, T_{[n]}$. The tunning parameter δ satisfies $0 \leq \delta \leq 0.5$. Neves and Alves (2004) recommand to take 0.35 as value of δ for computations. In this setting, final values for $k^\star$ and $x^\star$ are recorded in Table 3.2.

	Male	Female
$k^\star$	9,959	4,108
$T_{[n-k^\star+1]}$	95.03	100.89

Table 3.2: Values of $k^\star$ and $x^\star$ for both males and females with the Reiss - Thomas method.

3.3.4 Flexible extreme value mixture model

As pointed out in Scarrott et al. (2012), more sophisticated threshold selection procedure have been developed in the last decade such as mixture model. Our goal is to approximate the upper tail of the lifetime distribution with the GP distribution without specifying any parametric form for the bulk of the distribution in order to avoid a specification risk. This differs from Li et al. (2008) who specified a Gompertz distribution below the threshold. Our purpose is achieved if the bulk of the distribution is simultaneously estimated non-parametrically. This model is precisely the one developed by MacDonald et al. (2011) with their flexible extreme value mixture distribution. Formally, if F is the common distribution of the $T_1, T_2, T_3, \dots$ then it is expressed as

$$F(x|\lambda, u, \beta, \xi) = \begin{cases} (1-\phi_u)\frac{H(x|\lambda)}{H(u|\lambda)} & x \leq u \\ (1-\phi_u) + \phi_u G(x|u, \beta, \xi) & x > u \end{cases} \tag{3.10}$$

where ϕ_u is the probability of being above the optimal threshold $u \equiv x^\star$, H is a non-parametric kernel distribution which depends on a bandwidth parameter λ and G is the GP distribution function. The likelihood of model (3.10) is expressed as

$$L(\theta) = L_K(\lambda, u) L_{PP}(u, \mu, \beta, \xi) \tag{3.11}$$

where $\theta = (\lambda, u, \mu, \beta, \xi)$ is the vector of parameters. L_K and L_{PP} respectively refer to the likelihood of the non-parametric and GP parts of the overall distribution.

The function `nlbckdengpd` from the R-package `evmix` has been used for estimation. The Epanechnikov kernel was specified for the non-parametric part as suggested by Wang et al. (1998) since it is optimal in the mean square sense. To avoid ties issue in kernel estimation, similar exact ages were uniformly spread over the same day as suggested in Einmahl et al. (2008). Maximum likelihood estimation of the model requires initial values for parameters λ, u, β and ξ for numerical optimization. As there exists no rule-of-thumb for choosing these values, a reasonable grid of values for each parameter was fixed, then the negative log-likelihood has been computed at each point of the cartesian product of those grids in order to reasonably surround the parameters space. Initial values were set as parameter values which minimized the negative log-likelihood. These values are in Table 3.3.

	Males	Females
u	99	100
ξ	-0.1	-0.1
β	2	2.2
λ	0.019	0.019

Table 3.3: Initial values for parameters of the flexible extreme value mixture model for both males and females.

Estimation results are summarised in Table 3.4. In order to asses the impact of the age range considered for estimation, we also consider the restricted dataset covering ages 98 and over. We start at age 98 because market statistics are available from the Belgian National Bank, acting as the insurance regulator, up to that age (the last category is open, gathering ages 99 and over).

	Males		Females	
x	≥ 98	≥ 95	≥ 98	≥ 95
u	98.89	97.82	99.99	100.01
$\widehat{\xi}$	-0.156	-0.140	-0.125	-0.125
$\widehat{\beta}$	2.146	2.218	2.251	2.248
$\widehat{\lambda}$	0.030	0.025	0.028	0.039

Table 3.4: Estimated values parameters of the flexible extreme value mixture model for both males and females.

We see from Table 3.4 that the estimated parameters remain stable when the age range is modified. In particular, the optimal age $x^\star$ is not affected when the initial age moves from 95 to 98.

3.4 Final choice

Thresholds selected by all selection procedures are summarized in Table 3.5.

	Males		Females	
Age	≥ 98	≥ 95	≥ 98	≥ 95
Pickands	98.40	95.24	98.60	95.41
Reiss & Thomas	98.07	95.03	100.89	100.89
Mixture model	98.89	97.82	99.99	100.01

Table 3.5: Thresholds selected by all three automatic selection procedures for both males and females.

The GP distribution enjoys the convenient threshold stability property which ensures that if the lifetime has distribution function $G_{\xi,\beta}$ then the remaining lifetime at any age x has distribution function $G_{\xi,\beta+\xi x}$. This basically says that provided the lifetime obeys the GP model, the remaining lifetime at any attained age x is still GP with the same index ξ.

Therefore, thresholds $x^\star$ was finally fixed at the maximum of those values in order to be in line with the property of threshold stability of the GP distribution. Henceforth, for the remainder of the paper, $x^\star$ is set to 98.89 for males and to 100.89 for females.This means that extreme value region for female ages appeared much later than for male ones. It also clearly reflects gender gap.

3.5 Maximum Likelihood (ML) estimation of the GP parameters

Now that the threshold age $x^\star$ has been selected, the GP parameters have to be estimated from the observed T_i exceeding $x^\star$. To this end, several methods are available, including moment estimation, probability-weighted moment estimation and maximum likelihood estimation. Maximizing the likelihood function is used here, due to its optimal statistical properties, its intuitive contents and its general acceptance in the actuarial community. Alternative methods are considered in Section 3.6.

The GP likelihood function for observed remaining lifetimes at attained age $x^\star$ is given by

$$\mathcal{L}(\xi,\beta) = \prod_{i|t_i>x^\star} \left(\frac{1}{\beta} \left(1 + \frac{\xi}{\beta}(t_i - x^\star) \right)^{-\frac{1}{\xi}-1} \right).$$

The log-likelihood to be maximized writes

$$L(\xi,\beta) = -\ln \beta \#\{t_i|t_i > x^\star\} - \left(1+\frac{1}{\xi}\right) \sum_{i|t_i>x^\star} \ln \left(1 + \frac{\xi}{\beta}(t_i - x^\star)\right) \tag{3.12}$$

where $\#\{t_i|t_i > x^\star\}$ is the number of survivors at age $x^\star$, denoted as $L_{x^\star}$. This optimization problem requires numerical algorithms. There are different approaches to get starting values for the parameters ξ and β. A natural approach consists in using moment conditions (that is, we equate sample mean and sample variance to their theoretical expressions involving ξ and β), which gives

$$\widehat{\xi}^{MOM} = \frac{1}{2}\Big(1 - \frac{\bar{x}^2}{s^2}\Big) \text{ and } \widehat{\beta}^{MOM} = \frac{1}{2}\bar{x}\Big(\frac{\bar{x}^2}{s^2} + 1\Big), \tag{3.13}$$

where $\bar{x}$ and s^2 are the sample mean and variance of the observed $t_i - x^\star$. For $\xi < 0$, the use of (3.13) does not require any additional assumption as all moments are finite.

Another convenient way to get starting values for the maximum likelihood estimates consists in fitting a straight line to the empirical remaining life expectancies, as a function of attained age x. The intercept and slope of a straight line fit to $\widehat{e}_x$ determine the estimations of ξ and β in accordance with (3.6).

It can be shown that for $\xi > -0.5$, the maximum likelihood (ML in short) regularity conditions are fulfilled. Hosking and Wallis (1987) proved that the ML estimators of the GP parameters $(\hat{\xi},\hat{\beta})$ are asymptotically Normally distributed with expected value (ξ,β) and approximate covariance matrix

$$\mathbf{\Sigma}_{(\widehat{\beta},\widehat{\xi})} = \frac{1}{n} \begin{pmatrix} (1+\xi)^2 & \beta(1+\xi) \\ \beta(1+\xi) & 2\beta^2(1+\xi) \end{pmatrix}. \tag{3.14}$$

This result allows us to obtain the standard errors for the ML estimators. For reasons of convergence of the algorithm, a reparametrization of the GP model with parameters $\sigma = \log(\beta)$ and $\alpha = 1/\xi$ is often used to maximize the likelihood.

ML estimation of GP parameters can be found in Table 3.6 together with standard errors. Since $\widehat{\xi} > -0.5$ in both cases, upper bounds of 95% confidence interval for tail index ξ can be obtained from the asymptotic Normality of the ML estimator $\widehat{\xi}$ recalled above. The upper bounds to the resulting confidence intervals are -0.065 for females and -0.102 for males. Both bounds are negative supporting the assumption $\xi < 0$ for both genders. Moreover, a Kolmogorov-Smirnov goodness-of-fit test has been performed to check the compliance of the data with the GP distributions. The resulting p-values are reported in Table 3.6. As they exceed all the usual confidence levels, the GP behavior cannot be rejected for the lifetime data under study.

Parameters	Male	Female
$x^\star$	98.89	100.89
$L_{x^\star}$	1,940	4,104
$\widehat{\xi}^{ML}$	-0.132	-0.092
s.e $(\widehat{\xi}^{ML})$	0.015	0.014
$\widehat{\beta}^{ML}$	2.098	2.019
s.e $(\widehat{\beta}^{ML})$	0.057	0.042
KS pvalue	0.828	0.951

Table 3.6: Maximum likelihood estimates for the GP parameters β and ξ, together with standard errors and Kolmogorov-Smirnov goodness-of-fit p-values.

In order to check whether the data comply with the GP distribution, we have plotted

- The histogram and the GP density function (see Figure 3.3)
- The GP QQ-plot (see Figure 3.4). In this plot the empirical quantiles versus the estimated GP quantiles are represented. If the GP model fits, a linear pattern must be visible.

Both histogram and density function exhibit the same shape. Furthermore, there is a clear linear pattern in the QQ-plot, which confirms that the GP model adequately describes the remaining lifetime distribution above $x^\star$.

3.6 Comparison with alternative estimators

Apart from the ML estimator, several other estimation procedures have been proposed in the literature for the tail index ξ. In this section, we compare ML estimation with some commonly-used alternative estimators:

moment estimator denoted as $\widehat{\xi}^{MOM}$ and $\widehat{\beta}^{MOM}$, that have been presented in Section 3.5;

probability weighted moment estimator denoted $\widehat{\xi}^{PMOM}$ and $\widehat{\beta}^{PMOM}$ that are often used in EVT studies (we refer the reader to Beirlant et al. (2005a) for theoretical details);

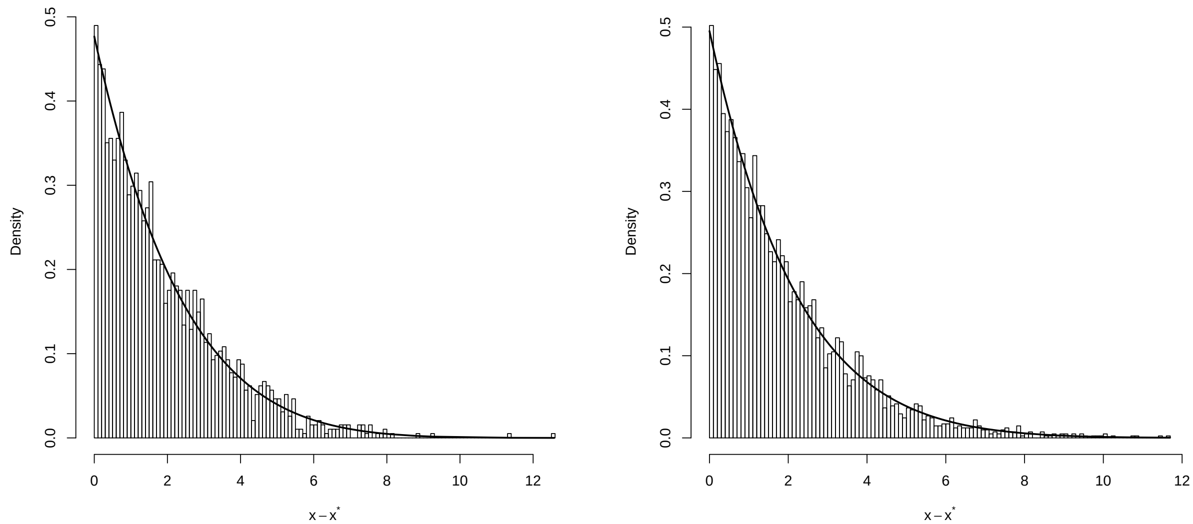


Figure 3.3: Histogram and GP density function (plain line) of the remaining lifetimes at age $x^\star = 98.89$ for males (left panel) and $x^\star = 100.89$ for females (right panel).

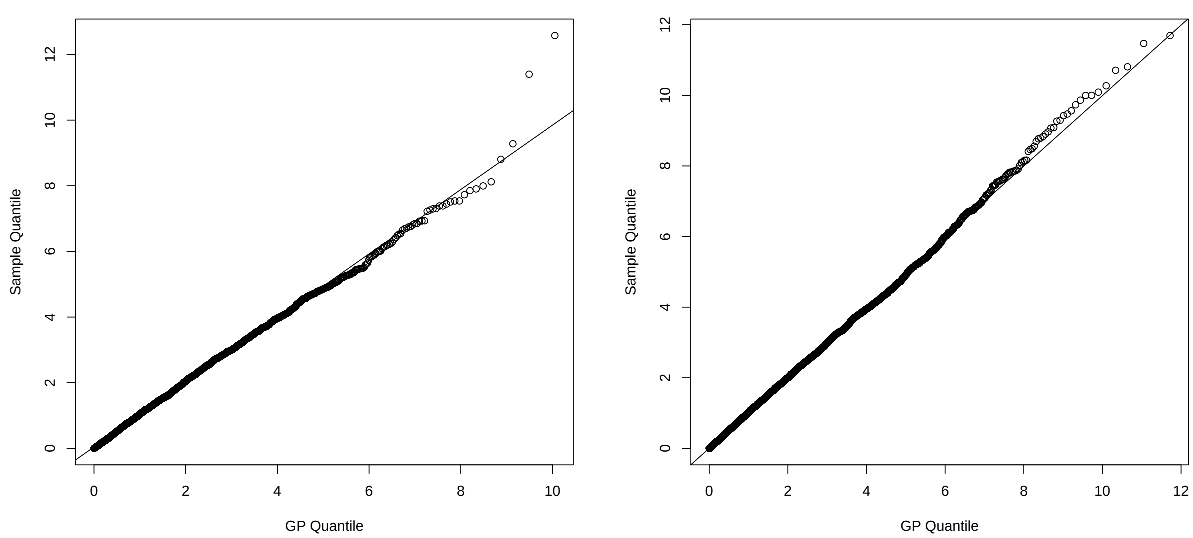


Figure 3.4: GP QQ-plots for the remaining lifetimes at age $x^\star = 98.89$ for males (left panel) and $x^\star = 100.89$ for females (right panel).

extended Hill estimator applicable to the case $\xi < 0$, denoted as $\widehat{\xi}^H$ and $\widehat{\beta}^H$ and proposed by Beirlant et al.(2005b);

moment-type estimator denoted as $\widehat{\xi}^M$ and $\widehat{\beta}^M$ and proposed by Dekkers et al. (1989).

Results from these alternative estimators are summarized in Tables 3.8-3.7 for both females and males at optimal thresholds $x^\star$. We see there that the estimated tail index for females is smaller than the one for males for all estimators while estimated scale parameter for females is higher than the ones of males. Clearly, the estimation method impacts on the estimated tail index, which in turn may lead to different conclusions, about the value of the ultimate age for instance. Despite these differences, all estimators support $\xi < 0$.

	ML	MOM	PMOM	M	H
$\widehat{\xi}$	-0.132	-0.163	-0.171	-0.163	-0.155
$\widehat{\beta}$	2.098	2.160	2.175	2.128	2.113

Table 3.7: Comparison of estimation results with optimal threshold $x^\star = 98.89$ for males.

	ML	MOM	PMOM	M	H
$\widehat{\xi}$	-0.092	-0.097	-0.105	-0.097	-0.094
$\widehat{\beta}$	2.019	2.027	2.043	1.997	1.991

Table 3.8: Comparison of estimation results with optimal threshold $x^\star = 100.89$ for females.

4 Some applications of the model

4.1 Ultimate age ω

As the negativity of tail index is supported by the data, this implies the existence of a finite ultimate age ω that can be estimated by

$$\widehat{\omega} = x^\star - \frac{\widehat{\beta}}{\widehat{\xi}}. \tag{4.1}$$

Estimated ultimate age computed from (4.1) with ML estimates are displayed in Table 4.1. Simulated confidence intervals at 95% are also shown there, based on 10,000 simulations.

	Males	Females
$\widehat{\omega}$	114.82	122.73
$\widehat{\omega}^-$	112.31	118.19
$\widehat{\omega}^+$	118.87	131.21

Table 4.1: ML estimated ultimate ages with 95% simulated confidence interval ($\widehat{\omega}^-$ and $\widehat{\omega}^+$ are respectively lower and upper bounds) for both male and female.

Female population has the highest estimated ultimate age. Estimations are in line with the highest ages at death 112.58 recorded for females and 111.47 recorded for males.

Estimated ultimate age derived from alternative estimators are also considered. For moment type estimators $\widehat{\xi}^M$ and $\widehat{\xi}^H$, we estimate ω as proposed by Einmahl and Magnus (2008):

$$\widehat{\omega}_{k^\star} = T_{[n-k^\star]}\left(1 - M_n^{(1)}(k^\star)\frac{1-\min(0,\widehat{\xi}_{k^\star})}{\widehat{\xi}_{k^\star}}\right) \tag{4.2}$$

with $k^\star = n - i^\star$. Under some mild technical conditions that can be found in Dekkers et al. (1989),

$$\frac{\sqrt{k^\star}\,(\widehat{\omega}-\omega)}{T_{[n-k^\star]}M_n^{(1)}\left(1-\widehat{\xi}\right)}$$

is Normally distributed with zero mean and variance

$$\frac{(1-\xi)^2(1-3\xi+4\xi^2)}{\xi^4(1-2\xi)(1-3\xi)(1-4\xi)}.$$

Therefore, simulated confidence intervals for ω can be provided. Results are summarized in Tables 4.3-4.2 for females and males, respectively. Notice how estimated ultimate age is sensitive to the estimator used for computation.

	ML	MOM	PMOM	M	H
$\widehat{\omega}$	114.82	112.14	111.64	111.91	112.53
$\widehat{\omega}^-$	112.31	109.81	109.05	109.07	109.29
$\widehat{\omega}^+$	118.87	116.20	116.65	114.75	115.76

Table 4.2: Alternative estimators results with optimal threshold $x^\star = 98.89$ for males.

	ML	MOM	PMOM	M	H
$\widehat{\omega}$	122.73	121.86	120.30	121.41	122.14
$\widehat{\omega}^-$	118.19	117.43	115.80	115.80	116.09
$\widehat{\omega}^+$	131.21	130.22	129.53	127.03	128.20

Table 4.3: Alternative estimators results with optimal threshold $x^\star = 100.89$ for females.

4.2 Highest age at death M_n

As pointed out by Wilmoth and Robine (2003), the maximum age at death can be modelled statistically as an extreme value of a random sample. In order to study the behavior of the highest age at death, we use the GP approximation of the exceedances. Under the conditions that yield to the GP approximation $G_{\xi;\beta}$ for the remaining lifetime distribution at age $x^\star$, the number $L_{x^\star}$ of survivors at threshold age $x^\star$ is roughly Poisson with mean $\ell_{x^\star} = \mathrm{E}[L_{x^\star}]$

(as an approximation to the Binomial distribution valid for large sizes and small success probabilities). As a consequence, the distribution function

$$\begin{aligned} s \mapsto \quad & \mathrm{P}[M_{[L_{x^\star}]} \leq s] \\ = \quad & \mathrm{P}[L_{x^\star} = 0] + \sum_{k=1}^{\infty} \mathrm{P}[L_{x^\star} = k, T_{[1]} - x^\star \leq s - x^\star, \ldots, T_{[k]} - x^\star \leq s - x^\star] \end{aligned}$$

can be approximated by

$$\begin{aligned} & \exp(-\ell_{x^\star}) + \sum_{k=1}^{\infty} \exp(-\ell_{x^\star}) \frac{\ell_{x^\star}^k}{k!} \left(1 - \left(1 + \xi \frac{s - x^\star}{\beta} \right)_+^{-1/\xi} \right)^k \\ = \quad & \exp\left(-\lambda \left(1 + \xi \frac{s - x^\star}{\beta} \right)_+^{-1/\xi} \right). \end{aligned}$$

Hence, we find the following approximation for the quantile at probability level $1 - \epsilon$ of $M_{[L_{x^\star}]}$:

$$x^\star + \frac{\beta}{\xi} \left(\left(-\frac{\ell_{x^\star}}{\ln(1 - \epsilon)} \right)^\xi - 1 \right)$$

Notice that if we define the location-scale family $H_{\xi;\mu,\psi}$ by

$$H_{\xi;\mu,\psi}(x) = H_\xi \left(\frac{x - \mu}{\psi} \right), \ \ \mu \in \mathbb{R}, \ \ \psi > 0,$$

then the distribution of the maximum $M_{L_{x^\star}}$ of the lifetimes of the $L_{x^\star}$ individuals reaching age $x^\star$ can be approached by a GEV distribution $H_{\xi;\mu,\psi}$ where

$$\mu = \beta\xi^{-1}(\ell_{x^\star}^\xi - 1) \text{ and } \psi = \beta\ell_{x^\star}^\xi.$$

The corresponding quantile function is given by

$$H_{\xi,\mu,\psi}^{-1}(\epsilon) = \begin{cases} \mu + \frac{\psi}{\xi}\Big((-\ln(\epsilon))^{-\xi} - 1 \Big) & \text{if } \xi \neq 0 \\ \mu - \psi \ln(-\ln(u)) & \text{if } \xi = 0 \end{cases}$$

So that the median of the highest age at death is

$$H_{\xi,\mu,\psi}^{-1}(0.5) = \mu + \psi \frac{(\ln(2))^{-\xi} - 1}{\xi}$$

will limit $\mu - \psi \ln(\ln 2)$ as $\xi \to 0$. If $\xi < 1$ then

$$\mathrm{E}[M_{L_{x^\star}}] = \mu + \frac{\psi}{\xi} \Big(\Gamma(1 - \xi) - 1 \Big)$$

will limit $\mu + \psi j$ as $\xi \to 0$, where j is the Euler constant (≈ 0.57721) which equals $\Gamma'(1)$, where Γ' is the digamma function, i.e the derivative of the Gamma function. If $\xi < 1/2$ then

$$\mathrm{V}[M_{L_{x^\star}}] = \frac{\psi^2\Big(\Gamma(1-2\xi) - \big((\Gamma(1-\xi)\big)^2\Big)}{\xi^2}$$

with limit $\psi^2\pi^2/6$ as $\xi \to 0$.

Notice that this suggests another estimation procedure for the tail index ξ, often referred to as the block maxima approach. See for instance Watts et al. (2006) for an application in life insurance. The idea is to fit the GEV distribution to maxima taken from independent blocks of data. Here, this amounts to record the cohort-specific maxima and to fit the 19 gender-specific observations using the GEV distribution $H_{\xi;\mu,\psi}$. Estimates are summarized in Table 4.4. They are obtained by using the R-package `ismev` which provides tools for statistical modeling of extreme values.

	$\widehat{\mu}$	$\widehat{\beta}$	$\widehat{\xi}$
Males	105.83	1.323	-0.012
	(0.334)	(0.236)	(0.139)
Females	109.78	1.477	-0.434
	(0.375)	(0.279)	(0.170)

Table 4.4: GEV estimates and their standard errors (in brackets) for both males and females.

Theoretically, tail index estimates from the block maxima and POT approach should be very close. As one can see, results are slightly different, meaning that estimation is sensitive to the chosen approach. This difference may also be linked to the block definition we adopted for the block maxima approach. In fact, we simply consider cohort block which makes more sense demographically. However, the block maxima only takes into account the 19 observed cohort maxima leading to more estimation instability.

Let us now compute the prediction intervals for the highest ages at death observed for each cohort and compare them with the observed, cohort-specific M_n. The result is visible on Figure 4.1, where we can see the observed cohort-specific maxima surrounded by 95% prediction intervals. Almost all cohort maxima are within the prediction interval.

4.3 Point estimation of high quantiles

Quantiles at usual probability levels can easily be estimated by their empirical counterparts. However, when we are interested in the very high quantiles, this approach is not longer valid since estimation based on a limited number of large observations would be strongly inaccurate. Fortunately, the EVT approach offers an efficient alternative.

From (3.4), we see that (provided $x^\star$ is sufficiently large) a potential estimator for the remaining lifetime distribution ${}_sq_{x^\star}$ is $G_{\widehat{\xi};\widehat{\beta}}(s)$. Quantile estimators derived from this curve are conditional quantile estimators which indicate the potential survival beyond the threshold age $x^\star$ when it is attained. When estimates of the unconditional quantiles are of interest,

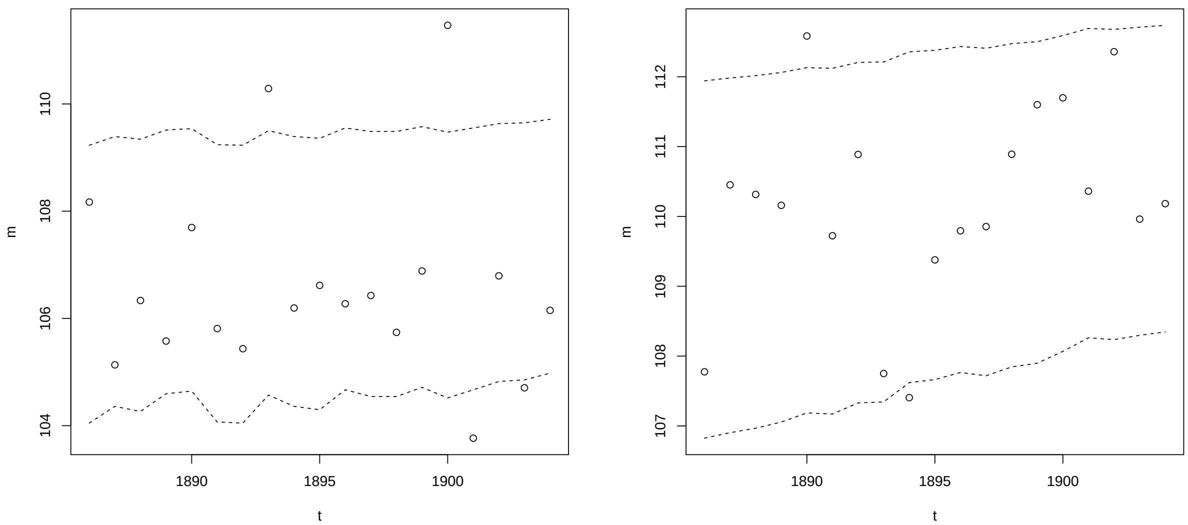


Figure 4.1: Observed cohort-specific maxima surronded by 95% prediction intervals.

relating the unconditional cumulative distribution function ${}_xq_0$ to $G_{\hat{\xi};\hat{\beta}}$ for $x > x^\star$, through

$$\begin{aligned} {}_xq_0 &= 1 - {}_xp_0 \\ &= 1 - {}_{x^\star}p_0 {}_{x-x^\star}p_{x^\star} \\ &\approx 1 - {}_{x^\star}p_0 \Big(1 - G_{\widehat{\xi};\widehat{\beta}}(x - x^\star)\Big) \end{aligned}$$

we can obtain high-level quantiles, i.e. solutions $t(\epsilon)$ to the equation

$${}_{t(\epsilon)}q_0 = 1 - \epsilon.$$

Provided the sample size is large enough, we can estimate ${}_{x^\star}p_0$ by its empirical counterpart so that estimated quantiles are estimated as

$$\widehat{t}(\epsilon) = x^\star + \frac{\widehat{\beta}}{\widehat{\xi}} \left(\left(\frac{L_{95}}{L_{x^\star}} (1 - \epsilon) \right)^{-\widehat{\xi}} - 1 \right) \tag{4.3}$$

where $L_{x^\star}$ and L_{95} are the number of survivors at the threshold age $x^\star$ and at age 95 (i.e., the total number of individuals under study for a given cohort), respectively.

The survival probability to age $x > x^\star$ can be estimated by

$$\widehat{{}_xp_0} = \frac{L_{x^\star}}{L_{95}} \left(1 - G_{\widehat{\xi};\widehat{\beta}}(x - x^\star)\right).$$

Notice that for $x \geq x^\star$, we have

$$\widehat{\mu}_x = \frac{1}{\widehat{\beta} + \widehat{\xi}(x - x^\star)}.$$

5 Discussion

Mortality at oldest ages is difficult to study because of data scarcity. In this paper, a unique Belgian database of individual lifetimes allowed us to accurately study this mortality using tools from extreme value theory. Maximum likelihood estimation gives a negative estimated tail index suggesting the existence of an ultimate age. We found an ultimate age of 114.82 for males and 122.73 for females which are consistent with observed data and demographical experience mentioned in Section 2.2. Alternative estimators for the tail index were also considered. Results clearly show the sensitivity to the selected estimation procedure.

Contrarily to previous studies by Robine et al. (2005) and Gampe (2010), we do not conclude that the force of mortality approaches a constant level (0.7 in their studies). This may be explained by the following reasons:

- the IDL gathers information from different populations whereas we conduct our analysis on a single population;

- younger ages are included in the analysis.

Our analysis also clearly rules out the Gompertz model.

As a byproduct of our analysis, we also conclude about the existence of an ultimate age. However, we do not claim that this limit must be interpreted in the biological or demographic sense, induced by genes or other natural mechanism. Rather, the ultimate age that has been obtained in the present paper serves as a working upper bound on policyholder's lifetime when actuarial calculations need to be performed. From a practical point of view, closing the life table by assuming that the missing q_x, beyond the last available age, i.e. assuming that the remaining lifetimes ultimately conform to the Negative Exponential distribution, is equally efficient and even conservative for products exposed to longevity risk.

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