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A simple model for now-casting volatility series

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A simple model for now-casting volatility series

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November 19, 2014

Abstract

Nowcasting volatility of financial time series appears difficult with classical volatility models. This paper proposes a simple model, based on an ARMA representation of the log-transformed squared returns, that allows to estimate current volatility, given past and current returns, in a very simple way. The model can be viewed as a degenerate case of the stochastic volatility model with perfect correlation between the two error terms. It is shown that the volatility nowcasts do not depend on this correlation, so that both models provide the same nowcasts for given parameter values. A simulation study suggests that the ARMA and SV models have a similar performance, but that in cases of moderate persistence the ARMA model is preferable. An extension of the ARMA model is proposed that takes into account the so-called leverage effect. Finally, the alternative models are applied to a long series of daily S&P 500 returns.

Keywords: EGARCH, stochastic volatility, ARMA, realized volatility, leverage

JEL Classification: C22, C58

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Abstract

Nowcasting volatility of financial time series appears difficult with classical volatility models. This paper proposes a simple model, based on an ARMA representation of the log-transformed squared returns, that allows to estimate current volatility, given past and current returns, in a very simple way. The model can be viewed as a degenerate case of the stochastic volatility model with perfect correlation between the two error terms. It is shown that the volatility nowcasts do not depend on this correlation, so that both models provide the same nowcasts for given parameter values. A simulation study suggests that the ARMA and SV models have a similar performance, but that in cases of moderate persistence the ARMA model is preferable. An extension of the ARMA model is proposed that takes into account the so-called leverage effect. Finally, the alternative models are applied to a long series of daily S&P 500 returns.

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1 Introduction

The literature on volatility models continues to grow steadily, driven mainly by the success that these models encounter in modelling financial time series, but also by the non-exhausted understanding of some of their properties and their estimators. The main benchmark remains the classical GARCH model, introduced by Engle (1982) and Bollerslev (1986), due to its simplicity in estimation and widespread availability in software packages. The GARCH model is essentially a model for *predicting* volatility for today, given past observations. It does so quite well, as demonstrated by Andersen and Bollerslev (1998) by using a realized volatility target instead of the commonly used daily squared returns. However, the GARCH model does not offer the possibility to update a prediction with today's observed data. In other words, *nowcasting* volatility in the GARCH model corresponds to using predicted volatility, ignoring today's observation.

Consider the ordinary GARCH(1,1) specification for the volatility process $\tilde{h}_t$

$$y_t = \sqrt{\tilde{h}_t} \xi_t \quad (1)$$

$$\tilde{h}_t = \hat{y}_{t|t-1}^2 \equiv \mathbb{E}(y_t^2 | y_{t-1}^2, y_{t-2}^2, \dots) \quad (2)$$

$$= \mu + \alpha y_{t-1}^2 + \phi \tilde{h}_{t-1} \quad (3)$$

where ξ_t is i.i.d. with $\mathbb{E}(\xi_t) = 0$ and $\mathbb{E}(\xi_t^2) = 1$. Letting $y_t^2 = \tilde{h}_t + v_t$ we can replace $\tilde{h}_t$ by $y_t^2 - v_t$ yielding the ARMA representation of y_t^2 :

$$y_t^2 = \mu + (\alpha + \phi) y_{t-1}^2 + v_t - \phi v_{t-1}. \quad (4)$$

Accordingly, the volatility process is equivalent to the linear forecast of y_t^2 conditional on $\{y_{t-1}, y_{t-2}, \dots\}$ and the variance process results from a filtration of the form

$$\tilde{h}_t = \frac{\mu}{1 - \phi} + \alpha \sum_{i=1}^{\infty} \phi^i y_{t-i}^2. \quad (5)$$

An important drawback of the standard GARCH model is that the observation y_t^2 does not enter the variance process, which is arguably the most important information about current volatility. This pitfall has been noted, for instance, by Politis (2007) among others. In this paper we propose a simple variant of the (exponential) GARCH model that exploits the information in the current observation y_t . Assuming $\varepsilon_t = \log \xi_t^2 \stackrel{iid}{\sim} \mathcal{N}(\mu, \sigma_\varepsilon^2)$ the model parameters can be estimated efficiently by fitting an ARMA(1,1) model to the transformed series $x_t = \log y_t^2$. In contrast to the GARCH(1,1) the log variance process in our model

results from the filtration

$$E(\log \tilde{h}_t | x_t, x_{t-1}, \dots) = c + \left(1 - \frac{\theta}{\beta}\right) \sum_{i=0}^{\infty} \theta^i x_{t-i}, \quad (6)$$

where θ and β are typically positive parameters, close to unity with $\theta < \beta$ and c is a constant.

To appreciate the importance of the current observation for estimating (“nowcasting”) volatilities we computed realized volatilities as the quadratic variation of the squared returns from 5-minute intervals of the S&P 500 index obtained from the Oxford Man Institute Realized Library. To investigate how well these volatilities ($\mathcal{V}_t$) can be estimated by fitting a linear filter of the daily returns (r_{t-i} , the log differences of the opening and closing price) we fit a distributed lag model of the form

$$\mathcal{V}_t = \mu + \sum_{i=0}^8 \beta_i r_{t-i}^2 + e_t \quad (7)$$

$$\text{or } \log \mathcal{V}_t = \beta_0 + \sum_{i=0}^8 \beta_i \log r_{t-i}^2 + e_t^*. \quad (8)$$

It should be noted that the realized volatilities are computed as $\mathcal{V}_t = \sum_{j=1}^n r_{j,t}^2$, whereas the squared daily return result as $r_t^2 = \left(\sum_{j=1}^n r_{j,t}\right)^2$, where $r_{j,t}$ denotes the 5-minute intraday return. The $n(n-1)$ cross-terms $r_{j,t}r_{k,t}$ with $j \neq k$ give rise to a very noisy estimator of daily variances and, thus, some smoothing is required to obtain reliable results. Table 1 presents the estimates for $\beta_0, \dots, \beta_8$. It turns out that the current observation contributes substantially to the variance process in particular for the log realized volatility series. The GARCH(1,1) and the EGARCH model provide a reasonable but less accurate approximation to the weight function.

2 The nowcasting model

To exploit the information in the current observation y_t we consider the following model for a series of financial returns y_t ,

$$y_t = \exp(h_t/2)\xi_t, \quad \xi_t \sim i.i.d.(0, 1) \quad (9)$$

$$h_t = \alpha + \beta h_{t-1} + \kappa \varepsilon_t, \quad (10)$$

where $\varepsilon_t = \log(\xi_t^2) - C \sim i.i.d.(0, \sigma_\varepsilon^2)$ and $C = \mathbb{E}[\log(\xi_t^2)]$. Log volatility h_t in (10) follows an AR(1) process, but unlike in stochastic volatility models where this process is

independent of ξ_t , the error term ε_t in (10) is an explicit function of the innovation term ξ_t in (9). Some more comparisons with the stochastic volatility model will be given in Section 4.

Let us first discuss some properties of model (9)-(10). If the distribution of ξ_t is known, then the parameter C is identified. For example, for Gaussian ξ_t , $C \approx -1.27$. In what follows we assume that C is unknown and absorb $E[\log(\xi_t^2)]$ into the constant α . At the end of Section 3 we discuss how to estimate this constant. Note also that mean and variance of log volatility are given by, respectively, $\mathbb{E}[h_t] = \alpha/(1 - \beta)$ and $\text{Var}(h_t) = \kappa^2 \sigma_\varepsilon^2 / (1 - \beta^2)$, and σ_ε^2 depends on the distribution of ξ_t . If ξ_t is Gaussian, then $\sigma_\varepsilon^2 = \pi^2/2$.

Under the assumption that ξ_t has a symmetric distribution it follows that y_t is a martingale difference series, i.e. $\mathbb{E}_{t-1}[y_t] = 0$, where $\mathbb{E}_{t-1}[\cdot]$ denotes expectation conditional on $\{y_{t-1}, y_{t-2}, \dots\}$. This results from

$$\mathbb{E}_{t-1}[y_t] = \mathbb{E}_{t-1}[\exp\{(\alpha + \beta h_{t-1})/2\}] \mathbb{E}[\exp(\kappa/2 \log \xi_t^2) \xi_t],$$

where the second expectation on the right hand side is zero since it is the expectation of an odd function of ξ_t . Thus, as in classical ARCH or stochastic volatility models, the return series y_t has a conditional mean of zero, and all temporal dependence is captured via the log volatility process h_t .

We now transform model (9) – (10) to obtain a linear process for the transformed variable. Defining $x_t = \log y_t^2$, we have

$$x_t = h_t + \varepsilon_t \tag{11}$$

and, replacing h_{t-1} in (10) by $x_{t-1} - \varepsilon_{t-1}$,

$$h_t = \alpha + \beta x_{t-1} + \kappa \varepsilon_t - \beta \varepsilon_{t-1} \tag{12}$$

$$x_t = \alpha + \beta x_{t-1} + (1 + \kappa) \varepsilon_t - \beta \varepsilon_{t-1} . \tag{13}$$

Indeed, the transformed returns x_t in (13) follow an ARMA(1,1) process.

It is interesting to compare this model specification with two popular GARCH alternatives: First, the (symmetric version of the) EGARCH model suggested by Nelson (1991) replaces (10) by the equation

$$h_t = \alpha + \beta h_{t-1} + \psi |\xi_{t-1}|. \tag{14}$$

Here, log-volatilities are driven by lagged values ξ_{t-1} instead of the current values ξ_t . Moreover, by rewriting (10) as $h_t = \alpha + \beta h_{t-1} + \kappa \log(\xi_t^2)$ it becomes obvious that large

shocks ξ_t have a much stronger effect in model (14). The proposed model (10) is actually closer to the so-called log-GARCH model, introduced independently by Geweke (1986) and Pantula (1986), where x_t is as in (11) with h_t given by

$$h_t = \alpha + \beta h_{t-1} + \psi \log y_{t-1}^2 \quad (15)$$

which leads to the ARMA representation

$$x_t = \alpha + (\psi + \beta)x_{t-1} + \varepsilon_t - \beta\varepsilon_{t-1}. \quad (16)$$

Notice the difference with respect to the ARMA representation (13), in which the coefficient κ captures the impact of the current observation on volatility in the moving average part, which is shifted to a lagged effect ψx_{t-1} in the autoregressive part of (16).

3 The reduced form ARMA representation

An observationally equivalent ARMA(1,1) model for x_t is obtained from

$$\begin{aligned} x_t &= \alpha + \beta x_{t-1} + (1 + \kappa)\varepsilon_t - \frac{\beta}{1 + \kappa}(1 + \kappa)\varepsilon_{t-1} \\ &= \alpha + \beta x_{t-1} + u_t - \theta u_{t-1}, \end{aligned} \quad (17)$$

where $u_t = (1 + \kappa)\varepsilon_t$ is white noise with variance $\sigma_u^2 = (1 + \kappa)^2\sigma_\varepsilon^2$ and $\theta = \beta/(1 + \kappa)$. The relationship between the reduced form parameters θ , $\sigma_u^2 = \mathbb{E}(u_t^2)$, and the structural parameters κ , σ_ε^2 is given by

$$\kappa = \beta/\theta - 1 \quad (18)$$

$$\sigma_\varepsilon^2 = \left(\frac{\theta}{\beta}\right)^2 \sigma_u^2. \quad (19)$$

Since $\varepsilon_t = u_t/(1 + \kappa) = (\theta/\beta)u_t$, the variance component h_t can be estimated from the reduced form as

$$\hat{h}_t = x_t - \frac{\theta}{\beta}u_t. \quad (20)$$

Note that (20) is measurable w.r.t. present and past values of x_t , because the reduced form is invertible and we have $u_t = -\alpha/(1 - \theta) + \phi(L)x_t$ with $\phi(L) = (1 - \theta L)^{-1}(1 - \beta L)$. By comparing coefficients of the lag polynomials, one obtains $\phi(L) = 1 + (1 - \beta/\theta) \sum_{j=1}^{\infty} \theta^j L^j$. Inserting this result into (20), we obtain

$$\hat{h}_t = \frac{\theta\alpha}{\beta(1 - \theta)} + \left(1 - \frac{\theta}{\beta}\right) \sum_{j=0}^{\infty} \theta^j x_{t-j}. \quad (21)$$

This shows that the filtered volatility is a linear combination of present and past values of x_t with exponentially declining weights.

(Pseudo) ML estimators of the structural parameters $(\beta, \kappa, \sigma_\varepsilon^2)$ are obtained by inserting the ML estimators of the reduced form $(\beta, \theta, \sigma_u^2)$ into (18) and (19). Based on the consistency and asymptotic normality of the reduced form maximum likelihood estimators, we can find similar results for the estimators of the structural form using the delta method. This gives closed form expressions for the asymptotic variances of $\sqrt{n}(\hat{\beta} - \beta)$ and $\sqrt{n}(\hat{\kappa} - \kappa)$, see Appendix A.1. Note that in practice, θ is often close to unity. Therefore, an exact ML estimation method rather than a conditional ML estimator (treating the starting value as fixed) should be employed. Whenever $\varepsilon_t = \log(\xi_t^2)$ is normally distributed, then the ML estimator is asymptotically efficient.

Estimation of the constant. The ARMA approach yields an estimator for $h_t^* = C + h_t$ and therefore an estimator for C is required to estimate h_t . From (11) it follows that

$$y_t^2 = e^{h_t^* - C} \xi_t^2 = c e^{h_t^*} \xi_t^2 ,$$

where $c = \exp(-C)$ and

$$\xi_t^2 = \frac{y_t^2}{c e^{h_t^*}} .$$

Since we assume that $\mathbb{E}(\xi_t^2) = 1$ we can estimate the constant from the estimated values of ξ_t^2 as

$$\begin{aligned} \frac{1}{T} \sum_{t=1}^T \hat{\xi}_t^2 &= 1 \\ \Leftrightarrow \hat{c} &= \frac{1}{T} \sum_{t=1}^T \frac{y_t^2}{e^{\hat{h}_t^*}} , \end{aligned}$$

where $\hat{h}_t^* = x_t - (\hat{\theta}/\hat{\beta})\hat{u}_t$ denotes the ARMA estimator of the volatility series.

4 Relationship to the stochastic volatility model

It is interesting to compare our approach to the stochastic volatility (SV) model, where (10) is replaced by

$$h_t = \alpha + \beta h_{t-1} + \eta_t \tag{22}$$

assuming that ξ_t and η_t are independent. The ARMA representation results as

$$x_t = \alpha^* + \beta x_{t-1} + \eta_t + \varepsilon_t - \beta \varepsilon_{t-1} . \quad (23)$$

where $\varepsilon_t = \log(\xi_t^2) - C$. Again we can find a second-order equivalent reduced form ARMA model as in (17), i.e.

$$x_t = \alpha^* + \beta x_{t-1} + u_t - \theta u_{t-1} , \quad (24)$$

that is, the autocovariance functions of x_t in (23) and (24) are identical. Accordingly, the model parameters of the SV model can be seen as transformations of the reduced form parameters in (24). Specifically we have

$$\sigma_u^2(1 + \theta^2) = \sigma_\eta^2 + \sigma_\varepsilon^2(1 + \beta^2) \quad (25)$$

$$\theta \sigma_u^2 = \beta \sigma_\varepsilon^2 . \quad (26)$$

It follows that

$$\sigma_\varepsilon^2 = \frac{\theta}{\beta} \sigma_u^2 \quad (27)$$

$$\sigma_\eta^2 = \left[1 - \frac{\theta}{\beta} - \theta(\beta - \theta) \right] \sigma_u^2 . \quad (28)$$

The Kalman filter applied to the state space representation of this model delivers the filtered volatility

$$h_{t|t} = (1 - \theta/\beta)x_t + (\theta/\beta)h_{t|t-1}$$

where the predicted volatility $h_{t|t-1}$ is given by

$$h_{t|t-1} = \alpha + (\beta - \theta)x_{t-1} + \theta h_{t-1|t-2}$$

Hence, we obtain

$$\begin{aligned} h_{t|t} &= \frac{\theta}{\beta} \left(\frac{\alpha}{1 - \theta} \right) + \frac{\kappa}{1 + \kappa} \sum_{j=0}^{\infty} \theta^j x_{t-j} \\ &= \frac{\theta \alpha}{\beta(1 - \theta)} + \left(1 - \frac{\theta}{\beta} \right) \sum_{j=0}^{\infty} \theta^j x_{t-j} \end{aligned}$$

which shows that the SV filtered volatility is equivalent to the filtered volatility using the ARMA model given by (21).

In the next proposition we show that this result extends to the class of models with an arbitrary error correlation:

Proposition 1 *Let $x_t = h_t + \varepsilon_t$, where $h_t = \alpha + \beta h_{t-1} + \eta_t$, $\eta_t \sim i.i.d.(0, \sigma_\eta^2)$, $\varepsilon_t \sim i.i.d.(0, \sigma_\varepsilon^2)$ and arbitrary covariance $\mathbb{E}(\eta_t \varepsilon_t) = \rho \sigma_\varepsilon \sigma_\eta$ with $\rho \in [-1, 1]$. It follows that*

$$h_{t|t} = \frac{\theta \alpha}{\beta(1 - \theta)} + \left(1 - \frac{\theta}{\beta}\right) \sum_{j=0}^{\infty} \theta^j x_{t-j}$$

The proof is provided in Appendix A.2.

Our model in (10) corresponds to the case $\rho = 1$, while the classical SV model (22) results from setting $\rho = 0$. It follows from Proposition 1 that for the estimation of h_t based on the information set $x_t, x_{t-1}, \dots$ the correlation between ε_t and η_t does not matter. Therefore, there is no need to invoke Kalman filter recursions to estimate the variance process.

Note that a non-zero correlation between ε_t and η_t does not imply that ξ_t and η_t are correlated. The latter case attracted some interest to model the so-called leverage effect in stochastic volatility, see e.g. Harvey and Shephard (1996). For example, consider our model (9)-(10), i.e. the degenerate case of Proposition 1 with $\rho = 1$, $\eta_t = \kappa \varepsilon_t$, and suppose that the distribution of ξ_t is symmetric. Then, the correlation between ξ_t and η_t is zero even though ε_t and η_t are perfectly correlated. To include a leverage effect, the model needs to be extended, which we will do in Section 6.

Note also that, according to Harvey and Koopman (2000), the model (22) with correlated ε_t and η_t is said to be in the contemporaneous state form. If one replaces (22) by $h_{t+1} = \alpha + \beta h_t + \eta_t$ with correlated ε_t and η_t , then the model is said to be in the future state form. As Harvey and Koopman (2000) show for the case of a random walk plus noise (i.e. $\beta = 1$), the filtered estimator of h_t depends on ρ in the future state model, but not in the contemporaneous state model.

Finally, if the distribution of ξ_t is symmetric, then it can be shown that the white noise u_t of the reduced form ARMA representation (24) is serially uncorrelated. In general, however, it is not a martingale difference, as e.g. $\mathbb{E}[u_t x_{t-1}^2] \neq 0$, see Francq and Zakoian (2006). The fact that u_t in the ARMA representation of the SV model is neither i.i.d. nor a martingale difference also has implications for inference. The general sandwich type formula for the asymptotic covariance matrix of QMLE estimators remains valid, but it is not available in closed form and it is different from the asymptotic covariance matrix of our model, given in Appendix A.1. Thus, although for given parameters both models yield the same filtered volatility estimates, estimation and inference are different due to the different properties of the error term u_t .

5 Finite sample properties

In this section we compare the finite sample properties of alternative estimators for volatilities. The data are generated as

$$y_t = e^{h_t/2} \xi_t \quad t = 1, \dots, T,$$

where h_t is either

$$\text{ARMA: } h_t = \alpha + \beta h_{t-1} + \kappa \varepsilon_t \quad (29)$$

$$\text{or SV: } h_t = \alpha + \beta h_{t-1} + \eta_t. \quad (30)$$

The error process $\varepsilon_t = \log(\xi_t^2) + 1.27$ with $\xi_t \stackrel{iid}{\sim} N(0, 1)$ is independent of $\eta_t \stackrel{iid}{\sim} N(0, \sigma_\eta^2)$. Accordingly, in the stochastic volatility model (SV) $x_t = \log y_t^2$ is composed of two independent processes, whereas in the ARMA model x_t is driven by a single stochastic process ε_t .

First, consider the case where the generated volatility is a classical stochastic volatility process. We follow Sandmann and Koopman (1998) in specifying the parameters of the SV model. Defining the coefficient of variation, $CV = \text{Var}[\exp(h_t)]/\mathbb{E}[\exp(h_t)]^2$, one obtains the expression $CV = \exp(\sigma_\eta^2/(1 - \beta^2)) - 1$. The coefficient of variation for this model is directly related to the kurtosis of y_t , which is given by $\kappa = 3(CV + 1)$. Here, α is an irrelevant scaling parameter, but Sandmann and Koopman (1998) determine α such that $E[h_t] = 0.0009$, which gives a realistic annualized standard deviation of 22%. To distinguish between highly and moderately persistent volatility processes, we fix β alternatively at 0.98 and 0.90. Similarly, to evaluate the effects of high versus low coefficients of variation (or, equivalently, high versus low kurtosis), we fix CV alternatively at 10 and 1, with corresponding kurtosis coefficients 33 and 6, respectively. This gives four different parameterizations. The sample sizes are $T = 500$ and 2000. Each process is simulated $k = 1000$ times.

The volatilities of the process y_t are estimated by fitting a symmetric EGARCH model, a symmetric SV model, and the ARMA approach proposed in Section 3, where the constant is estimated as suggested in Section 3. The performance is measured by an R^2 type

criterion computed as

$$\tilde{R}_h^2 = 1 - \frac{\sum_{t=1}^T (h_t - \hat{h}_t)^2}{\sum_{t=1}^T (h_t - \bar{h})^2}$$

where $\bar{h} = T^{-1} \sum_{t=1}^T h_t$. This variant of the usual R^2 imposes a zero constant and a unit scaling coefficient in order to measure the correspondence of the estimates with the original volatility process. Table 2 reports the results.

Not surprisingly, the R^2 measures of the EGARCH fit are substantially smaller in all cases, due to the smaller information set that is used in estimation. Also not surprisingly, the R^2 of SV and ARMA are close to each other, since both models deliver the same filtered volatility estimate for given parameters. Hence, the differences between them are solely due to differences in parameter estimates. Note that the ARMA R^2 tends to be higher when the persistence is moderate ($\beta = 0.9$). Note also that for increasing sample size, the R^2 does not need to improve, because the sample size affects the estimation error but not the signal to noise ratio determined essentially by σ_η^2 and σ_ε^2 .

In the second simulation setup we generate reduced form ARMA processes for h_t with parameters chosen analogously to the SV case. More precisely, the persistence parameter β and the intercept α are the same as in SV. The moving average parameter θ is chosen such that $CV \in \{1, 10\}$, as before, by expressing θ as a function of σ_η , β , and σ_ε . Results are reported in Table 3. Overall, the R^2 tends to be higher than in the SV case, which is plausible as there is no second noise term in the volatility equation. Furthermore, volatility is a measurable function of today's and lagged information. Thus, for increasing sample size we expect the R^2 to converge to unity, which happens for both estimation methods based on ARMA and SV. Again we observe the same effect as for a true SV process where the ARMA R^2 is higher for moderate persistence.

6 An asymmetric extension

In order to account for the leverage effect that is often encountered in empirical applications, we define the dummy variable $d_t = I(y_t > \tau)$, where $I(\cdot)$ is the indicator function, and τ is a predefined threshold (which is typically zero), the mean of y_t , or some other value of interest.

An asymmetric extension of the above model is given by

$$h_t = \alpha + \beta x_{t-1} + \kappa^+ d_t \varepsilon_t + \kappa^- (1 - d_t) \varepsilon_t - \beta \varepsilon_{t-1}, \quad (31)$$

which we call ARMA model with leverage, or ARMA-L. Note that in contrast to Nelson's EGARCH model and other asymmetric GARCH models, the asymmetric effect in this model is contemporaneous and not lagged.

The structural form for x_t results as

$$x_t = \alpha + \beta x_{t-1} + (1 + \kappa^+) d_t \varepsilon_t + (1 + \kappa^-) (1 - d_t) \varepsilon_t - \beta \varepsilon_{t-1}. \quad (32)$$

Denote again the MA part of this model by $v_t = (1 + \kappa^+) d_t \varepsilon_t + (1 + \kappa^-) (1 - d_t) \varepsilon_t - \beta \varepsilon_{t-1}$. We have the following conditional second order moment structure.

$$\text{Var}(v_t | d_t) = ((1 + \kappa^+)^2 d_t + (1 + \kappa^-)^2 (1 - d_t) + \beta^2) \sigma_{\varepsilon_t}^2 \quad (33)$$

$$\mathbb{E}[v_t v_{t-1} | d_t] = -\{(1 + \kappa^+) d_t + (1 + \kappa^-) (1 - d_t)\} \beta \sigma_{\varepsilon_t}^2 \quad (34)$$

We can find an observationally equivalent ARMA(1,1) process

$$x_t = \alpha + \beta x_{t-1} + u_t - \theta^+ d_t u_{t-1} - \theta^- (1 - d_t) u_{t-1}. \quad (35)$$

This process has the same conditional second order moment structure provided that

$$\kappa^+ = \beta / \theta^+ - 1 \quad (36)$$

$$\kappa^- = \beta / \theta^- - 1 \quad (37)$$

$$\sigma_{\varepsilon_t}^2 = \{(1 + \kappa^+)^2 d_t + (1 + \kappa^-)^2 (1 - d_t)\}^{-1} \sigma_u^2 \quad (38)$$

Note that the error term ε_t is conditionally heteroskedastic. If the estimated model (35) is invertible, then it is easy to check that the model (32) with parameters given by (36)-(38) will also be invertible.

We could have chosen the alternative solution

$$\kappa^+ = \beta \theta^+ - 1 \quad (39)$$

$$\kappa^- = \beta \theta^- - 1 \quad (40)$$

$$\sigma_{\varepsilon}^2 = \frac{\sigma_u^2}{\beta^2} \quad (41)$$

which is conditionally homoskedastic. However, if the estimated model (35) is invertible, then the model (32) with parameters given by (39)-(41) will not be invertible, and is therefore excluded.

Note that the process (35) is similar to the asymmetric ARMA model proposed by Brännäs and De Gooijer (1994), the difference being that in their model, the indicator variable is specified as $d_t = I(u_{t-1} > 0)$. The model (35) can be estimated by quasi maximum likelihood. To obtain the information matrix, the Hessian can be approximated by the sum of the outer products of the gradient as in Brännäs and De Gooijer (1994).

7 An empirical application

We apply our model on a large dataset, the demeaned daily (close to close) return on the S&P 500 index from 1/1/1950 — 25/10/2012, a total of 16,058 observations. We first estimate the classical EGARCH(1,1) with $N(0, 1)$ innovations, as proposed by Nelson (1991):

$$\begin{aligned} y_t &= \exp(h_t/2)\xi_t, \quad \xi_t \sim N(0, 1) \\ h_t &= \alpha + \beta h_{t-1} - \theta \xi_{t-1} + \gamma |\xi_{t-1}| \end{aligned}$$

This model is estimated by maximum likelihood, and the results are shown in Table 4.

We estimate the SV model (22) by QMLE and the Kalman filter, assuming $\eta_t \sim N(0, \sigma_\eta^2)$ and $\varepsilon_t \sim N(0, \sigma_\varepsilon^2)$. Results are also presented in Table 4. The estimators of σ_η^2 and σ_ε^2 correspond to (28) and (27).

The ARMA(1,1) model in (17) is estimated using nonlinear least squares with numerical optimization. For the nonlinear ARMA model with leverage (ARMA-L), see equation (31), we choose a threshold $\tau = -0.01$, corresponding to one negative unconditional standard deviation of returns y_t . Estimation results for the three models are also reported in Table 4. All three models pass portmanteau specification tests applied to the squared residuals $\hat{\xi}_t^2$.

The persistence of shocks to volatility measured by β is even higher in the ARMA models than for EGARCH. The parameter estimate of κ implied by the estimates of θ and β is given by $\hat{\kappa} = \hat{\beta}/\hat{\theta} - 1 = 0.0391$ for the ARMA model, and $\hat{\kappa}^+ = \hat{\beta}\hat{\theta}^+ - 1 = 0.0353$ and $\hat{\kappa}^- = \hat{\beta}\hat{\theta}^- - 1 = 0.0605$ for the ARMA-L model. The estimated volatility process $\hat{h}_t^*$ is adjusted by the estimated constant $\hat{C} = -\log(\hat{c})$, where $\hat{c}$ is the sample mean of $y_t^2/\exp(\hat{h}_t^*)$, see Section 3. Table 4 also presents the sample variances of the error terms $\varepsilon_t = x_t - h_t$, which are clearly smaller for the ARMA models, as expected.

Figure 1 shows the nowcast of log volatility using the ARMA-L model (31), and the predicted log-volatility of the EGARCH model. The sample correlation between both

volatility series is 91%. The predicted EGARCH volatility was higher after the October 1987 crash than after the Lehman crisis 2008, while the updated ARMA volatility was higher for the Lehman crisis. An explanation might be that the 1987 crash was mainly driven by an exceptionally severe one-day drop of returns, while absolute returns were exceptionally high during a longer time period around the Lehman crisis.

8 Conclusions

The proposed ARMA representation of log squared returns provides a simple method for estimating current volatility given the past and current information on the underlying returns. Our results suggest that it outperforms predictions of GARCH-type models, and similarly to stochastic volatility models while being easier to estimate.

We have proposed an important extension of the model to incorporate the so-called leverage effect. Many other extensions are possible and indeed object of future work. For example, it is straightforward to include a "GARCH-in-mean"-type risk premium in the conditional mean of returns, where the risk premium would depend on the current volatility, not on the predicted one. Second, multivariate extensions are possible. For example, one could use a factorization as in the orthogonal GARCH model of Alexander (2001). We believe that these are important topics of future research.

Appendix

A.1 Asymptotic distribution of estimators

Under our conditions, the maximum likelihood estimator $\hat{\gamma} = (\hat{\beta}, \hat{\theta})$ of the reduced form ARMA model (17) is consistent and asymptotically normal with asymptotic distribution given by

$$\sqrt{n}(\hat{\gamma} - \gamma) \rightarrow_d N \left(0, \frac{1 - \beta\theta}{(\beta - \theta)^2} \begin{bmatrix} (1 - \beta^2)(1 - \beta\theta) & -(1 - \theta^2)(1 - \beta^2) \\ -(1 - \theta^2)(1 - \beta^2) & (1 - \theta^2)(1 - \beta\theta) \end{bmatrix} \right)$$

see e.g. Brockwell and Davis (1991). This gives directly the asymptotic variance of $\sqrt{n}(\hat{\beta} - \beta)$, while that of $\sqrt{n}(\hat{\kappa} - \kappa)$ is obtained via the delta method. Straightforward calculations yield

$$n\text{Var}(\hat{\kappa}) \rightarrow \frac{(1 - \beta\theta)^2}{(\beta - \theta)^2} \frac{1 - \theta^2}{\theta^2} (1 - \beta^2) \left(\frac{1}{1 - \theta^2} + \frac{2\beta}{\theta(1 - \theta\beta)} + \frac{\beta^2}{\theta^2(1 - \beta^2)} \right).$$

A.2 Proof of Proposition 1

Denote $X_t = \sigma(x_t, x_{t-1}, x_{t-2}, \dots)$ and let $h_{t|t-1} = \mathbb{E}[h_t|X_{t-1}]$, $h_{t|t} = \mathbb{E}[h_t|X_t]$, and $\mathcal{V}_t = \text{Var}(h_t|X_{t-1})$. First we note that

$$\begin{aligned} u_t &= x_t - \mathbb{E}(x_t|X_{t-1}) = h_t - h_{t|t-1} + \varepsilon_t \\ &= \alpha + \eta_t + \beta(h_{t-1} - h_{t-1|t-1}) + \varepsilon_t. \end{aligned}$$

The estimator of the log-variance process is

$$h_{t|t} = h_{t|t-1} + \frac{\mathcal{V}_t + \mathbb{E}[(h_t - h_{t|t-1})\varepsilon_t]}{\mathcal{V}_t + 2\mathbb{E}[(h_t - h_{t|t-1})\varepsilon_t] + \sigma_\varepsilon^2} u_t \quad (42)$$

where $\mathbb{E}[(h_t - h_{t|t-1})\varepsilon_t] = \mathbb{E}(\eta_t\varepsilon_t)$, which follows from the Kalman filter with correlated measurement and transition errors, see e.g. Section 3.2.4 of Harvey (1989). For the case of correlated errors, equation (26) generalizes to

$$\begin{aligned} \text{Var}(u_t) &= \mathcal{V}_t + \sigma_\varepsilon^2 + 2\mathbb{E}(\eta_t\varepsilon_t) \\ &= \frac{\beta}{\theta} [\sigma_\varepsilon^2 + \mathbb{E}(\eta_t\varepsilon_t)] \end{aligned}$$

which is the denominator in the second term of the right hand side of (42). The numerator is obtained by subtracting from this expression $\sigma_\varepsilon^2 + \mathbb{E}(\eta_t\varepsilon_t)$, and we obtain

$$\begin{aligned} \mathcal{V}_t + \mathbb{E}(\eta_t\varepsilon_t) &= \left(\frac{\beta}{\theta} - 1 \right) [\sigma_\varepsilon^2 + \mathbb{E}(\eta_t\varepsilon_t)] \\ &= \kappa [\sigma_\varepsilon^2 + \mathbb{E}(\eta_t\varepsilon_t)] \end{aligned}$$

We finally obtain

$$\begin{aligned} h_{t|t} &= h_{t|t-1} + \frac{\mathcal{V}_t + \mathbb{E}(\eta_t \varepsilon_t)}{[\mathcal{V}_t + \mathbb{E}(\eta_t \varepsilon_t)] + [\sigma_\varepsilon^2 + \mathbb{E}(\eta_t \varepsilon_t)]} u_t \\ &= h_{t|t-1} + \frac{\kappa}{1 + \kappa} u_t . \end{aligned}$$

Since $h_{t|t-1}$ is identical to the forecast of x_t based on the $x_{t-1}, x_{t-2}, \dots$, the estimator $h_{t|t}$ is invariant to the covariance $\mathbb{E}(\eta_t \varepsilon_t)$. Therefore, h_t is identical to the estimator based on perfect correlation with $\varepsilon_t = \kappa \eta_t$ which is given in (21). $\square$

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(a) Nowcast log volatility using the ARMA-L model

(b) Predicted log volatility using EGARCH

Figure 1: Volatility estimates for daily S&P 500 returns.

Table 1: Dynamic regressions of (log) realized volatilities on (log) squared returns

dep.var.:	realized volatilities			log realized volatilities		
lag	nowcast	forecast	GARCH(1,1)	nowcast	forecast	EGARCH
0	0.1482	—	—	0.1207	—	—
1	0.1927	0.1926	0.0880	0.0700	0.0724	0.1117
2	0.0583	0.0927	0.0793	0.0737	0.0855	0.1094
3	0.0633	0.0615	0.0714	0.0664	0.0772	0.1072
4	0.0407	0.0543	0.0644	0.0674	0.0736	0.1050
5	0.0178	0.0464	0.0580	0.0577	0.0699	0.1029
6	0.0273	0.0468	0.0523	0.0564	0.0670	0.1008
7	−0.0339	0.0508	0.0471	0.047	0.0560	0.0987
8	0.0458	0.0493	0.0425	0.0480	0.0571	0.0967
R^2	0.6095	0.5837	0.4456	0.5449	0.4666	0.2823

Note: Entries report estimated coefficients of the unrestricted dynamic regressions (7) and (8) as well as the restricted GARCH/EGARCH model (5) resp. (14).

Table 2: Performance under the SV model (29)

		EGARCH		SV		ARMA	
β	CV	R^2	s.d.	R^2	s.d.	R^2	s.d.
$n = 500$							
0.9	1	0.360	(0.143)	0.528	(0.115)	0.588	(0.095)
0.98	1	0.746	(0.165)	0.883	(0.066)	0.874	(0.079)
0.9	10	0.271	(0.199)	0.626	(0.071)	0.641	(0.061)
0.98	10	0.590	(0.321)	0.863	(0.072)	0.859	(0.074)
$n = 2000$							
0.9	1	0.314	(0.099)	0.392	(0.078)	0.438	(0.047)
0.98	1	0.737	(0.097)	0.776	(0.061)	0.771	(0.064)
0.9	10	0.297	(0.149)	0.579	(0.048)	0.592	(0.044)
0.98	10	0.679	(0.232)	0.807	(0.051)	0.806	(0.051)

Note: Pseudo- R^2 of fitted volatility models for h_t compared with true, simulated stochastic volatility series. The standard deviation of the sample R^2 is indicated as s.d. (in parentheses). The coefficient of variation is denoted by $CV = \text{Var}[\exp(h_t)]/\mathbb{E}[\exp(h_t)]^2$.

Table 3: Performance under the ARMA model (30)

β	EGARCH			SV		ARMA	
	CV	R^2	s.d.	R^2	s.d.	R^2	s.d.
$n = 500$							
0.9	1	0.453	(0.170)	0.843	(0.092)	0.917	(0.091)
0.98	1	0.794	(0.176)	0.959	(0.108)	0.953	(0.120)
0.9	10	0.263	(0.222)	0.956	(0.034)	0.971	(0.037)
0.98	10	0.623	(0.337)	0.982	(0.054)	0.978	(0.057)
$n = 2000$							
0.9	1	0.430	(0.151)	0.857	(0.124)	0.958	(0.049)
0.98	1	0.808	(0.141)	0.976	(0.101)	0.969	(0.104)
0.9	10	0.317	(0.181)	0.964	(0.097)	0.981	(0.099)
0.98	10	0.673	(0.299)	0.987	(0.055)	0.985	(0.059)

Note: Pseudo- R^2 of fitted volatility models for h_t compared with true, simulated ARMA series. Remaining notes as in Table 2.

Table 4: Parameter estimates of alternative volatility models

	EGARCH	SV	ARMA	ARMA-L
α	-0.2666 (0.0100)	-0.0755 (0.0204)	-0.0822 (0.0166)	-0.0792 (0.0148)
β	0.9839 (0.0009)	0.9932 (0.0018)	0.9926 (0.0015)	0.9930 (0.0013)
γ	0.1475 (0.0033)			
θ	-0.0647 (0.0019)		0.9552 (0.0038)	
θ^+				0.9590 (0.0036)
θ^-				0.9359 (0.0088)
σ_η^2		0.0097 (0.0022)		
σ_ε^2		5.3156 (0.0950)		
RSV	5.4219	5.1249	5.1101	5.0942

Note: The residual sample variance is given by RSV. Residuals ε_t are obtained as $x_t - h_t$, where h_t is either the predicted volatility using EGARCH, the updated volatility $h_{t|t}$ using SV, or the estimated h_t using the ARMA model. ARMA-L is the asymmetric ARMA model of section 6. Standard errors are reported in parentheses.

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