National debt sustainability and the dynamics in the economy of Diamond

David de la Croix* Philippe Michel†

April 13, 1999

Abstract

We analyze the effect of constant debt policy on capital accumulation and provide an in depth treatment of the dynamics in the economy of Diamond. We derive the conditions for avoiding debt crisis in both the short-run and the long-run and provide geometrical tools to analyze the issue of sustainability. There is always a level of debt ensuring that any level of capital between 0 and an upper limit, and in particular the Golden rule equilibrium, exists as a steady-state equilibrium, but is not necessarily stable. Poverty trap issues are also discussed.

JEL Classification numbers: E32, O41.
Keywords: debt, sustainability, over-accumulation, Golden rule, poverty trap.

*National Fund for Scientific Research and IRES, Université catholique de Louvain, Place Montesquieu 3, B-1348 Louvain-la-Neuve, Belgium. E-mail: delacroix@ires.ucl.ac.be. The financial support of the PAI programme P4/01 is gratefully acknowledged.

†IUF, Université de la Méditerranée and GREQAM, Centre de la Vieille Charité, rue de la Charité 2, F-13002 Marseille, France.

Introduction

The effect of government debt on growth and welfare is an old debate. According to the view of Ricardo (1817), "it is not by the payment of the interest on the national debt that a country is distressed, nor is it by the exoneration from payment that it can be relieved. It is only by saving from income, and retrenching in expenditure, that the national capital can be increased; and neither the income would be increased, nor the expenditure diminished by the annihilation of the national debt". This famous neutrality result hold in the basic neo-classical growth model but breaks down in the overlapping-generations model of Diamond (1965) and Blanchard (1985) in which the finitely-lived agents consider government debt as net wealth (Barro, 1974). As pointed out by Weil (1989) it is not the assumption of infinite horizon which is responsible for the breakdown of the Ricardian equivalence, but well the presence of unborn future generations whose interests are not taken for by the present generations. The overlapping generations model thus provides a relevant framework to study the role of public debt.

In his seminal contribution, Diamond (1965) examines the effect of government debt on the long-run competitive equilibrium of an economy with overlapping generations; he shows that internal positive debt lowers utility when the equilibrium is efficient (under-accumulation case) but may raise utility in the inefficient case (over-accumulation). In Diamond's work,[1] the analysis is limited to steady state equilibria and global stability is implicitly assumed.

However, even in the case where the debt per capita is constant, which is the one considered by Diamond, the dynamic effects of debt on capital accumulation are complex. Indeed, even if debt can be beneficial in an long-run equilibrium, nothing guarantees that, given its initial endowment in capital, the economy will converge to this equilibrium rather than to an unsustainable outcome. Hence, the issue of the sustainability of constant debt per capita is central to the dynamic analysis.

The renewed interest for sustainability issues has been fostered by the Maastricht Treaty, imposing on the European governments the duty to reach a maximum level of debt of 60% of GDP. A recent paper by Rankin and Roffia (1999) has a comparable motivation: "There is a need to investigate another aspect of unsustainability: namely, the possibility that, even with a constant stock of debt, fiscal policy may be unsustainable because a steady state equilibrium (...) may not exists. (...) it is necessary to have a good understanding of the technical limits to debt before we can make serious progress in modelling its political limits." As we shall show, contrary to the claim of Rankin and Roffia (1999), the existence of steady state is not sufficient to guarantee sustainability, stability is also required. Notice also that the analysis of Rankin and Roffia (1999) is carried out in a Cobb-Douglas world, while we would like to analyze this issue with general preferences and production functions.[2]

Accordingly, we analyze the effect of constant debt policy on capital accumulation and provide an in depth treatment of the dynamics in the economy of Diamond. We extent in some directions the general approach of Galor and Ryder (1989) to overlapping

[1] the contribution of Diamond (1965) often serves as a building block for further studies on debt and related issues. See for instance King (1992), Grossman and Yanagawa (1993), King and Ferguson (1993), Bertocchi (1994), Uhlig (1996), Azariadis and Smith (1998).

[2] Without, e.g., imposing Inada conditions.

generations model with constant debt.[3] We show that the essential information can be gathered in a single planar diagram thereby allowing us to answer the issues of global stability and sustainability in a comprehensive way by means of a graphical exposition.

The paper is organized as follows. Diamond's model is introduced in section 1. The conditions for short-term sustainability are derived in section 2. Long-term sustainability is studied in section 3. A global characterization of inter-temporal equilibria with constant debt is provided in section 4. The implications for policy are drawn in section 5.

1 Diamond's model

Two-period lived individuals Time t is discrete and goes from 0 to ∞ with $t \in \mathbb{N}$. In each period t, N_t agents are born and live for two periods. N_t grows at a constant rate n: $N_{t+1} = (1+n)N_t$. In their first period of life t (when "young"), agents are endowed with one unit of labor that they supply inelastically to firms. Their income is equal to the real wage w_t minus the lump-sum tax τ_t levied by the government. They allocate this income between current consumption c_t and savings s_t which are invested in the firms' capital and government debt. The budget constraint of period t is:

$$w_t \Leftrightarrow \tau_t = c_t + s_t. \tag{1}$$

In their second period of life $t+1$ (when "old"), they are retired. Their income comes from the return on the savings made at time t. As in Diamond (1965) the government does not impose taxes on the older generation.[4] They consume their income entirely. Denoting R_{t+1} the gross rate of return on savings from time t to time $t+1$, the income of an old individual is $R_{t+1}s_t$ and his consumption is

$$d_{t+1} = R_{t+1}\, s_t. \tag{2}$$

The preferences of the agents are defined over their consumption bundle (c_t, d_{t+1}) and are represented by a life-cycle utility function $U(c,d)$, strictly concave, increasing, twice continuously differentiable and that verifies $\lim_{c\to 0} U'_c(c,d) = +\infty$ and $\lim_{d\to 0} U'_d(c,d) = +\infty$. The hypothesis of an infinite marginal utility of zero consumption implies that the agent always chooses positive consumption levels when he/she maximizes his/her life-cycle utility (as long as his disposable income is positive). We also assume that c and d are normal commodities, i.e. that their demands will be non-decreasing in wealth. At each period $t \geqslant 1$, $N_t + N_{t-1}$ individuals are alive, including N_t young agents born in t and N_{t-1} old agents born in $t \Leftrightarrow 1$. At the first period $t = 0$, there is, in addition to the N_0 young agents, N_{-1} old agents. Each of these N_{-1} old agents is the owner of the same fraction of the installed capital stock K_0 and of the existing debt B_0. We assume that $K_0 > 0$ and $B_0 > \Leftrightarrow K_0$. Their wealth is thus $s_{-1} = (K_0 + B_0)/N_{-1}$, and their income is equal to $R_0 s_{-1}$.[5] The problem of the individual is to maximize its utility function subject

[3] Although their main concern was with Inada conditions, the paper of Galor and Ryder (1989) provides a general treatment of the OLG model of Diamond (1965) without debt.

[4] An extension of Diamond (1965) to a framework with two different positive taxes is proposed in Ihori (1978).

[5] As we do not exclude negative debt, debt should be thought as debt net of the capital stock detained by the public sector.

to its budget constraints (1) and (2). It admits a solution if and only if taxes are not greater than wage income, i.e. $w_t > \tau_t$. The solution

$$s_t = s(w_t \Leftrightarrow \tau_t, R_{t+1}) = \arg\max U(w_t \Leftrightarrow \tau_t \Leftrightarrow s_t, R_{t+1} s_t) \tag{3}$$

is interior and is characterized by the first-order condition:

$$U'_c(w_t \Leftrightarrow \tau_t \Leftrightarrow s_t) = R_{t+1} U'_d(R_{t+1} s_t). \tag{4}$$

The function $s(.)$ is called the savings function. Since the two goods are normal, the marginal propensity to save out of income is between 0 and 1: $0 < s'_w < 1$.

Neo-classical technology The production technology is the same for all periods. It is represented by the neo-classical production function $F(K, L)$. F is homogeneous of degree one with respect to its arguments capital K and labor L. During the production process, the capital stock depreciates physically at a rate $\delta \in [0, 1]$.After the production process, the part of capital that is not depreciated is identical to the good produced, so that we may define a net production function in intensive terms

$$f(k) = F(k, 1) + (1 \Leftrightarrow \delta)k \tag{5}$$

We make the following hypothesis on the function $f(.)$: it is defined on the set $\mathbb{R}_{++}$ and twice continuously differentiable. It satisfies For all $k > 0$, $f(k) > 0$, $f'(k) > 0$ and $f''(k) < 0$.[6] The marginal productivity of labor is:

$$\omega(k) = f(k) \Leftrightarrow k\ f'(k) = F'_L(K, L) > 0.$$

Competitive firms At time t, the representative *producing firm* has an installed stock of capital K_t and maximizes its profits

$$\pi_t = \max_{L_t} F(K_t, L_t) \Leftrightarrow w_t L_t.$$

The labor demand L_t which maximizes this expression is obtained by equalizing the marginal productivity of labor with the wage rate:

$$F'_L(K_t, L_t) = \omega\left(\frac{K_t}{L_t}\right) = w_t. \tag{6}$$

The profit is distributed to the owners of the capital stock

$$\pi_t = F(K_t, L_t) \Leftrightarrow F'_L(K_t, L_t)\ L_t = F'_K(K_t, L_t) K_t = f'\left(\frac{K_t}{L_t}\right) K_t \tag{7}$$

---

[6]This hypothesis amounts to assume that the function F is positive valued, increasing and strictly concave with respect to K.

The government At any period t, the government has a debt B_t coming from the past; the principal plus the accrued interests is $R_t B_t$. The government also collects lump-sum taxes paid by the N_t young agents. The level of debt carried from time t to time $t+1$ is thus

$$B_{t+1} = R_t B_t \Leftrightarrow N_t \tau_t \tag{8}$$

The policy of the government is to keep constant the level of debt per young individual. Hence, $B_t/N_t = b \quad \forall t$, where b is the target level of debt per capita. The adequate level of taxes to maintain this level of debt is thus

$$\tau_t = (R_t \Leftrightarrow (1+n))b \tag{9}$$

The short term constraint faced by the government is of course that total savings is able to absorb the current debt:

$$B_{t+1} < N_t s_t$$

which implies that investment in productive capital I_t is positive:

$$I_t = N_t s_t \Leftrightarrow B_{t+1} > 0 \tag{10}$$

2 Debt crisis in the short-run

To study the short-run issues, we first define the temporary equilibrium. The temporary equilibrium of period t will gives the equilibrium value of the current variables as a function of the past and of the expectations about the future.

2.1 The temporary equilibrium

At time t, the physical capital is already installed and investment I_t results from the decision of the young individuals in. Nevertheless, the gross rate of return R_t depends on the decisions of the firm that produces at time t and this decision depends on the equilibrium wage w_t. The equilibrium conditions are:

1. **Labor market equilibrium:** At time t the inelastic labor supply is N_t; the labor demand L_t is given by the solution to (6). Hence, the equilibrium wage equalizing supply and demand is defined by $L_t = N_t$ or with $k_t = K_t/N_t$:

$$w_t = \omega(k_t) = f(k_t) \Leftrightarrow k_t f'(k_t) \tag{11}$$

2. **Capital markets equilibrium:** Given the equilibrium on the labor market we have $\pi_t = f'(k_t)K_t$. Hence, by the no-arbitrage condition between capital and debt, the rate of return on total savings is equal to the marginal productivity of capital:

$$R_t = f'(k_t). \tag{12}$$

3. **Good market equilibrium:** The equilibrium on the goods market is

$$Y_t = N_{t-1}d_t + N_t(c_t + s_t). \tag{13}$$

It results from the equilibrium in the labor and capital markets and from the budget constraints of the agents.

4. **Balanced budget:** It results from the tax policy of the government (9) and from equation (12) that the tax should be equal to

$$\tau_t = (f'(k_t) \Leftrightarrow (1+n))b \equiv \tau(k_t, b) \tag{14}$$

The temporary equilibrium can now be defined as follows.

Definition 1 (temporary equilibrium) *Given a fixed level of debt b, the variables from the previous period $\{s_{t-1}, I_{t-1}\}$, the number of agents, N_t and N_{t+1}, and the expected rate of return on savings R_{t+1}, the temporary equilibrium of time t is defined by the wage rate w_t, the gross rate of return R_t, the aggregate variables K_t, L_t, Y_t, k_t and I_t, the individual variables c_t, s_t, and d_t, all positive quantities, and the government variable τ_t, that satisfy the optimality conditions of the agents and the equilibrium conditions (10), (11), (12),(13) and (14).*

A temporary equilibrium exists if and only if taxes are smaller than first-period income and investment in productive capital is positive. These two conditions are equivalent to

$$\omega(k_t) > \tau(k_t, b) \tag{15}$$

$$\frac{s_t}{1+n} > b \tag{16}$$

If it exists, it is unique and can be expressed as a function of k_t and R_{t+1}.

The two conditions (15) and (16) are closely related to the sustainability of debt policy. We now study them in turn.

2.2 The scope for positive net income and investment

Let us define the net income of the young agents as

$$\tilde{w}(k_t, b) = \omega(k_t) \Leftrightarrow \tau(k_t, b) = \omega(k_t) \Leftrightarrow \varrho(k_t)b \tag{17}$$

where $\varrho(k) \equiv f'(k) \Leftrightarrow (1+n)$. We study the set $\mathcal{E}_{\tilde{w}}$ of pairs $(k, b) \in \mathbb{R}_{++} \times \mathbb{R}$ which satisfy $\tilde{w}(k, b) > 0$. We first introduce assumption **C1**

C1. $f'(0) > 1+n > f'(+\infty)$,

which allows us to define the Golden Rule capital stock $k_{\rm GR}$ as the capital stock that maximizes the steady state net production $f(k) \Leftrightarrow (1+n)k$:

$$f'(k_{\rm GR}) = 1+n$$

• When $k < k_{\rm GR}$ (under-accumulation of capital), $\varrho(k) > 0$ and the condition $\tilde{w}(k, b) > 0$ is equivalent to

$$b < \frac{\omega(k)}{\varrho(k)} \equiv \overline{b}(k)$$

$\overline{b}(k)$ is an increasing function of k ($\omega' > 0$, $\varrho' < 0$), positive valued and its limits are:

$$\lim_{k\to 0} \overline{b}(k) = \frac{\omega(0)}{\varrho(0+)} \text{ and } \lim_{k\to k_{\rm GR}} \overline{b}(k) = +\infty$$

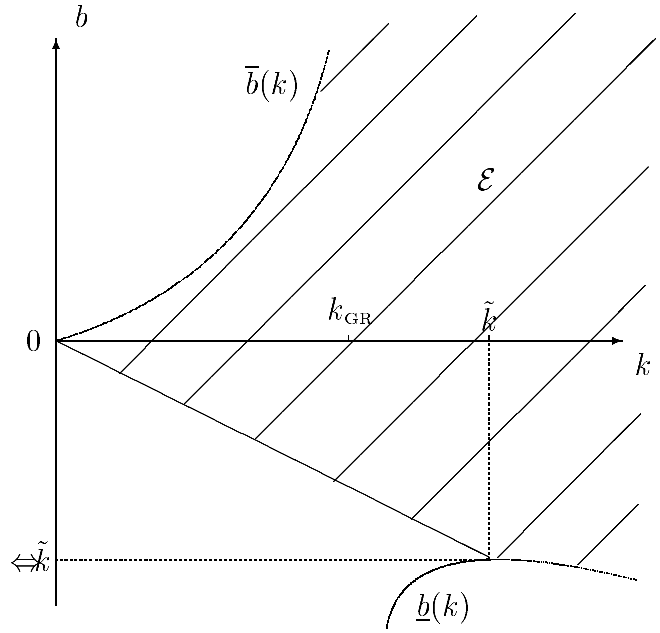

Figure 1: Domain of existence

The limit $\overline{b}(0+)$ is zero if $\omega(0) = 0$ or $f'(0+) = +\infty$. If $\omega(0) > 0$ and $f'(0+)$ is finite, $\overline{b}(0+)$ is positive.

• When $k > k_{\mathrm{GR}}$ (over-accumulation of capital), $\varrho(k) < 0$ and the condition $\tilde{w}(k, b) > 0$ is equivalent to

$$b > \frac{-\omega(k)}{-\varrho(k)} = \frac{-\omega(k)}{1 + n - f'(k)} \equiv \underline{b}(k)$$

$\underline{b}(k)$ is negative and its limits are

$$\lim_{k \to k_{\mathrm{GR}}} \underline{b}(k) = -\infty \text{ and } \lim_{k \to +\infty} \underline{b}(k) = \frac{-\omega(+\infty)}{1 + n - f'(+\infty)}$$

The derivative of $\underline{b}(k)$ is

$$\frac{f''(k)((1 + n)k - f(k))}{(1 + n - f'(k))^2}$$

Let us define $\tilde{k}$ as the positive root of $f(k) - (1 + n)k$.[7] We then have that $f(k) > (1 + n)k$ if and only if $k < \tilde{k}$. The function $\underline{b}(k)$ is increasing in $(0, \tilde{k})$ and decreasing in $(\tilde{k}, +\infty)$.

The functions $\overline{b}(k)$ and $\underline{b}(k)$ are represented in figure 1. The interpretation is the following. When k is below the Golden Rule level, running a positive debt requires to levy positive taxes, as the interest rate is above the rate of growth (of population). Short-term sustainability thus requires an upper bound $\overline{b}$ on the debt. When k is above the Golden Rule level, running a negative debt requires to levy positive taxes, as the interest rate that benefits to the government assets is below the rate of growth (of population). Solvability thus requires a lower bound $\underline{b}$ on the debt.

The scope for positive investment is given by the set $\mathcal{E}_I$ of pairs (k, b) such that $b > -k$.

---

[7] If this solution does not exist, i.e. $f(k) > (1 + n)k$ for all k, we set $\tilde{k} = +\infty$.

Figure 1 gathers the important information. The scope for positive net income is given by the set $\mathcal{E}_{\tilde{w}}$ of pairs $(k, b) \in \mathbb{R}_{++} \times \mathbb{R}$ which satisfy $\tilde{w}(k, b) > 0$ is the surface between the two curves $\overline{b}(k)$ and $\underline{b}(k)$. The scope for positive investment is given by the set of pairs (k, b) such that $b > \Leftrightarrow k$. The set $\mathcal{E}$, on which the temporary equilibrium is defined, is given by the intersection of the two.

3 Debt crisis in the long-run

We shall study the inter-temporal equilibria with perfect foresight:

$$R_{t+1} = f'(k_{t+1}) \tag{18}$$

$$k_{t+1} = \frac{1}{1+n}\, s(\tilde{\omega}(k_t, b), R_{t+1}) \Leftrightarrow b \tag{19}$$

where the last equation links the two periods by equating investment and savings.

Definition 2 (inter-temporal equilibrium) *Given an initial capital stock $k_0 = K_0/N_{-1}$, an inter-temporal equilibrium with perfect foresight is a sequence of temporary equilibria that satisfies for all $t \geqslant 0$ the conditions (18) and (19).*

Hence, at the inter-temporal equilibrium with perfect foresight, the stock of capital of period $t+1$ should verify the following implicit equation:

$$\varphi(k_{t+1}, \tilde{w}(k_t, b), b) = (1+n)(k_{t+1} + b) \Leftrightarrow s(\tilde{w}(k_t, b), f'(k_{t+1})) = 0. \tag{20}$$

The equilibrium sequence $(k_t)_{t \geqslant 0}$ is called an equilibrium trajectory. b and $w > 0$ being given, we wonder whether there exists $k > 0$ solution to

$$\varphi(k, w, b) = (1+n)(b+k) \Leftrightarrow s(w, f'(k)) = 0 \tag{21}$$

If the net income is w, savings with perfect foresight $s(w, f'(k_{t+1}))$ should be equal to $(1+n)(b + k_{t+1})$ when the debt per capita b is constant. Notice that as $s(w, f'(k_{t+1}))$ is bounded from above by w we have:

$$\lim_{k \to +\infty} \varphi(k, w, b) = +\infty \tag{22}$$

We should note that the uniqueness of the equilibrium is a fundamental property to interpret the perfect foresight assumption in a non ad–hoc way. Only when the equilibrium is unique, it can be analyzed as an equilibrium with rational perfect foresight. In the case of uniqueness, there is no exogenous problem of coordinating expectations, as each agent can solve the model and calculate the next period equilibrium k_{t+1} which is necessary to obtain the rate of return $R_{t+1} = f'(k_{t+1})$. Under the rational perfect foresight hypothesis, the agents use all available information, including the model describing the economy. In the case of multiplicity of equilibria k_{t+1} agents do not know what will be the expectations of the others and thus face non-unique R_{t+1}, unless some exogenous coordination device is assumed (like, e.g., sunspots). In the sequel we use the following terminology: A rational inter-temporal equilibrium is an inter-temporal equilibrium with perfect foresight such that there exists no other inter-temporal equilibrium with perfect foresight having the same initial capital stock.

The condition of rational foresight supposes thus that equation (21) has a unique solution in k. Accordingly we make the following hypothesis: Given b, for all $w > 0$, equation (21) has at most one solution $k > 0$. This assumption means that either there is no solution (and we study the existence later), or there exists only one. This amounts to impose that, if there is a solution k_1, then the function $\varphi(k, w, b)$ is necessarily locally increasing at k_1. Indeed, if it is not, then we have $\varphi(k_2, w, b) \leqslant 0$ at $k_2 > k_1$, and the continuous function φ for which (22) holds has necessarily a root at $k_3 \geqslant k_2 > k_1$. We thus have the following assumption:

C2. $\forall w > 0$, $\forall (k, b) \in \mathcal{E}$,

$$\varphi(k, w, b) = 0 \Rightarrow \frac{\partial \varphi}{\partial k}(k, w, b) > 0$$

A sufficient condition is obviously that $\varphi(k, w, b)$ is always increasing. As we have

$$\frac{\partial \varphi}{\partial k}(k, w, b) = 1 + n \Leftrightarrow s'_R(w, f'(k))f''(k)$$

$s'_R \geqslant 0$ is a sufficient condition, i.e., the inter-temporal elasticity of substitution is greater or equal than one implying that the substitution effect is not dominated by the income effect. Assuming that **C2** holds, for all $k_t > 0$ such that $\tilde{w}(k_t, b) > 0$, there exists at most one solution $k_{t+1} > 0$ to equation (20).

3.1 Monotonicity of equilibrium trajectories

The derivative of $\tilde{w}(k, b)$ with respect to k is

$$\frac{\partial \tilde{w}(k, b)}{\partial k} = \omega'(k) \Leftrightarrow f''(k)b = \Leftrightarrow f''(k)(k + b)$$

which is positive for $k > \Leftrightarrow b$. Along a trajectory solution to (20) this condition obviously holds since date $t = 0$ as we have $b + k_0 > 0$.

Lemma 1 *Assume* **C2** *holds. The set of values of w such that the equation (21) has a root $k(w)$ is an interval $(\underline{w}, +\infty)$. The solution $k(w)$ is an increasing function of w.*

Proof: Consider $w_1 > 0$ such that there exists a solution $k_1 > 0$: $\varphi(k_1, w_1, b) = 0$. As φ is a decreasing function of w ($\varphi'_w = \Leftrightarrow s'_w < 0$), we have for all $w_2 > w_1$: $\varphi(k_1, w_2, b) < 0$. Given (22), the continuous function $\varphi(k, w, b)$ of k has at least one root between k_1 and $+\infty$: $\exists k_2$ such that $\varphi(k_2, w, b) = 0$. By assumption **C2**, this root is unique.
Moreover, if (21) has a root with w_1, $k(w_1) = k_1$, then for all $w_2 > w_1$, (21) has a unique root $k(w_2)$ with w_2 and we have $k(w_2) > k(w_1)$. As any union of intervals of the form $(w_1, +\infty)$ is an interval of this form, the proof is completed. ■

Notice that this proposition does not say if the interval is open or closed. It can even be empty for certain values of b. We shall thus pursue the analysis of existence under assumption **C2**.

Proposition 1 *Assume* **C2** *and let* $(k_0, b) \in \mathcal{E}$. *Then we have*

1. *if* $\varphi(k_0, \tilde{w}(k_0, b), b) \leqslant 0$, *there exists an inter-temporal equilibrium with perfect foresight with initial capital* k_0*; the sequence* (k_t) *is non-decreasing.*
2. *if* $\varphi(k_0, \tilde{w}(k_0, b), b) > 0$, *then equation (20) defines a decreasing sequence* (k_t). *Either there is a date at which equation (20) has no solution, or there is a solution for all* t *and the inter-temporal equilibrium exists.*

Proof:

1. By induction, for all $t \geqslant 0$ such that $\varphi(k_t, \tilde{w}(k_t, b), b) \leqslant 0$ there exists $k_{t+1} \geqslant k_t$ such that $\varphi(k_{t+1}, \tilde{w}(k_t, b), b) = 0$ as the continuous function of k, $\varphi(k, .)$ is negative or nil at k_t and positive for k large enough (equation (22)). As we have $k_{t+1} > k_t > b$, we have $\tilde{w}(k_{t+1}, b) \geqslant \tilde{w}(k_t, b)$. As $\varphi(.)$ is decreasing with respect to $\tilde{w}$, we deduce that
$$\varphi(k_{t+1}, \tilde{w}(k_{t+1}, b), b) \leqslant 0.$$
The first result is proven: the sequence (k_t) exists and is non-decreasing. Moreover, it is strictly increasing if it is not constant (case $\varphi(k_0, \tilde{w}(k_0, b), b) < 0$).

2. Assume now that $\varphi(k_t, \tilde{w}(k_t, b), b) > 0$ at $t \geqslant 0$ with $(b, k_t) \in \mathcal{E}$ and assume that there exists k_{t+1} such that (20) holds. Let us use the simplified notation $\tilde{\varphi}_t(k) = \varphi(k, \tilde{w}(k_t, b), b))$. If k_{t+1} was greater than k_t, we would have, following assumption **C2**:
$$\frac{\partial \tilde{\varphi}_t}{\partial k}(k_{t+1}) > 0 \quad \text{and} \quad \tilde{\varphi}_t(k_{t+1}) = 0$$
As φ is increasing in k, there exists ϵ such that $k_{t+1} \Leftrightarrow \epsilon > k_t$ and $\tilde{\varphi}_t(k_{t+1} \Leftrightarrow \epsilon) < 0$. Then, $\tilde{\varphi}(k)$ crosses the horizontal axis between k_t and $k_{t+1} \Leftrightarrow \epsilon$. We would then have two solutions which is excluded. We deduce that k_{t+1} should necessarily be smaller than k_t: as long at it is defined, the sequence k_{t+1} is decreasing and satisfies: $\tilde{w}(k_{t+1}, b) < w(k_t, b)$ and, if $\tilde{w}(k_{t+1}, b) > 0$, $\varphi(k_{t+1}, \tilde{w}(k_{t+1}, b), b) > 0$.

■

We have thus established the existence of the inter-temporal equilibrium in the case $\varphi(k_0, \tilde{w}(k_0, b), b) \leqslant 0$ and the monotonicity of the dynamics in that case. However, the inter-temporal equilibrium may not exists for certain initial conditions such that $\varphi(k_0, \tilde{w}(k_0, b), b) > 0$. In that case, at a given date, either the net income becomes negative, or savings are not sufficient to finance the debt. The debt b is therefore not sustainable.

3.2 The steady state curve $\hat{b}(k)$

To characterize this problem further, we analyze the curve of the steady states
$$\psi(k, b) = \varphi(k, \tilde{w}(k, b), b) = (1+n)(b+k) \Leftrightarrow s(\tilde{w}(k, b), f'(k)) \tag{23}$$
in the set $\mathcal{E}$, and wonder whether it is possible to define a function $\hat{b}(k)$ such that $\psi(k, b) = 0$.

We have the following proposition:

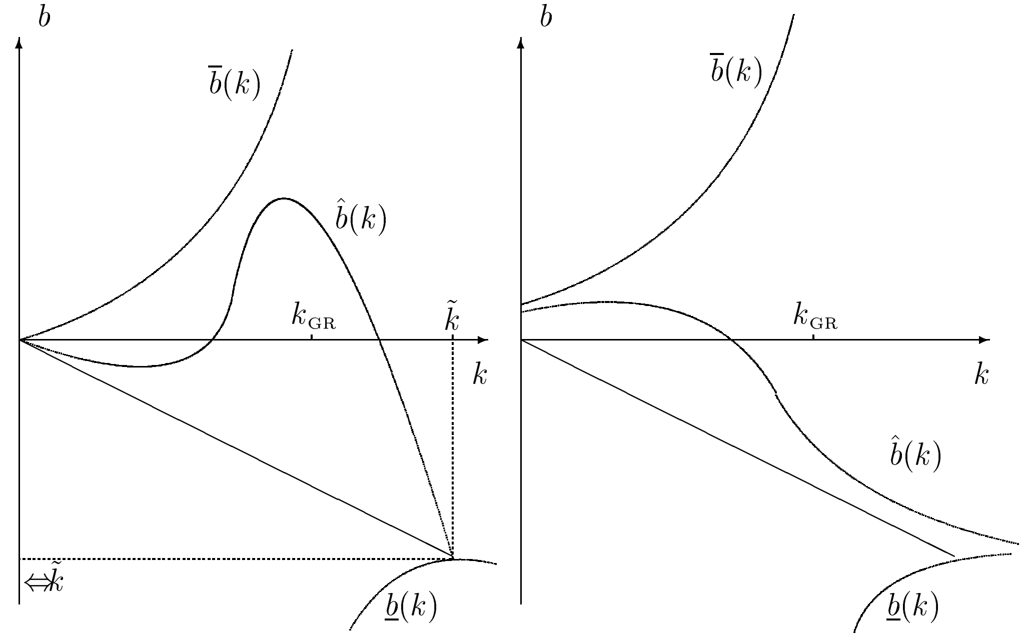

Figure 2: The steady state curve

Proposition 2 *The equation $\psi(k, b)$ has a unique solution $\hat{b}(k)$ for all k such that $0 < k < \tilde{k}$ and $(k, \hat{b}(k)) \in \mathcal{E}$. The function $\hat{b}(.)$ is continuously differentiable on $(0, \tilde{k})$ and its limits are: $\hat{b}(\tilde{k}) = \Leftrightarrow \tilde{k}$, $\hat{b}(0+) \geqslant 0$; moreover, $\hat{b}(0+) > 0$ if and only if $\overline{b}(0+) > 0$.*

Proof: We first study the existence and uniqueness of $\hat{b}$ and compute its limits. The partial derivative of $\psi(k, b)$ with respect to b is:

$$\begin{aligned} \psi'_b = \varphi'_{\tilde{w}} \tilde{w}'_b + \varphi'_b &= s'_w (f'(k) \Leftrightarrow (1+n)) + 1 + n \\ &= s'_w f'(k) + (1+n)(1 \Leftrightarrow s'_w) > 0 \end{aligned}$$

as $0 < s'_w < 1$ (normal goods).

- When $k < k_{\mathrm{GR}}$ (under-accumulation of capital), $\tilde{w}(k, b) > 0 \Leftrightarrow b < \overline{b}(k)$. As $\psi(b, k) < (1+n)(b+k)$ we have

  $$\lim_{b \to -\infty} \psi(b, k) = \Leftrightarrow \infty$$

  Moreover, by definition of $\overline{b}(k)$ we have $\lim_{b \to \overline{b}(k)} \tilde{w}(k, b) = 0$ which implies

  $$\lim_{b \to \overline{b}(k)} \psi(b, k) = (1+n)(\overline{b}(k) + k) > 0$$

  as $\overline{b}(k) = \omega(k)/\varrho(k)$ is positive and greater than $\Leftrightarrow k$. As a consequence, for all $k > 0$ the function of b, $\psi(k, b)$ is increasing from $\Leftrightarrow \infty$ to a positive value. There thus exists $\hat{b}(k) < \overline{b}(k)$ unique such that $\psi(k, \hat{b}(k)) = 0$. $\hat{b}(k)$ is the value of the debt such that k is a steady state. The limits of the function $\hat{b}(k)$ are:

  – when $\overline{b}(0) = 0$ ($\omega(0) = 0$ or $f'(0+) = +\infty$) we deduce from $\hat{b}(k) < \overline{b}(k)$ that

  $$\limsup_{k \to 0} \hat{b}(k) \leqslant 0$$

We also have

$$\liminf_{k\to 0} \hat{b}(k) = \liminf_{k\to 0} s(\tilde{w}(k,\hat{b}(k)), f'(k)) \geqslant 0$$

As $\liminf \leqslant \limsup$, the two limits are equal at 0 and

$$\lim_{k\to 0} \hat{b}(k) = 0$$

– when $\overline{b}(0) > 0$ ($\omega(0) > 0$ and $f'(0+) < +\infty$), any accumulation point $\breve{b}$ of $\hat{b}(k)$ when k tends to zero satisfies $\psi(0,\breve{b}) = 0$, where

$$\psi(0,\breve{b}) = (1+n)\breve{b} \Leftrightarrow s(\omega(0) \Leftrightarrow \varrho(0)\breve{b}, f'(0))$$

The limit when $k \to 0$ of $\psi(k,\breve{b})$ is defined for $\breve{b} < \overline{b}(0)$. We next deduce from $\psi'_b(0,\breve{b}) > 0$ and $\psi(0,0) = \Leftrightarrow s(\omega(0), f'(0)) < 0$ that $\breve{b} > 0$ is the unique solution to $\psi(0,\breve{b}) = 0$ and that it verifies $0 < \breve{b} < \overline{b}(0)$. Hence, the limit of $\hat{b}(k)$ when k tends to zero exists and is equal to $\breve{b}$.

Concerning the limit when k tends to k_{GR}, $\tilde{w}(k,b)$ tends to $\omega(k_{\mathrm{GR}})$ and any accumulation point b of $\hat{b}(k)$ verifies $b = b_{\mathrm{GR}}$.

- When $k > k_{\mathrm{GR}}$ (over-accumulation of capital), $\tilde{w}(k,b) > 0 \Leftrightarrow b > \underline{b}(k)$. As

$$\begin{aligned}\psi(b,k) &> (1+n)(b+k) \Leftrightarrow \tilde{w}(b,k)\\ &> (1+n)(b+k) \Leftrightarrow \omega(k) + (f'(k) \Leftrightarrow (1+n))b\\ &> (1+n)k \Leftrightarrow \omega(k) + f'(k)b\end{aligned}$$

we have

$$\lim_{b\to+\infty} \psi(b,k) = +\infty$$

Moreover, we have $\lim_{b\to\underline{b}(k)} \tilde{w}(k,b) = 0$ which implies

$$\lim_{b\to\underline{b}(k)} \psi(b,k) = (1+n)(\underline{b}(k)+k) = \frac{1+n}{\varrho(k)}(\omega(k) \Leftrightarrow \varrho(k)k) = \frac{1+n}{\Leftrightarrow \varrho(k)}((1+n)k \Leftrightarrow f(k))$$

This limit is negative if and only if $k < \tilde{k}$. As a consequence, since the function of b, $\psi(k,b)$, is increasing, there exists, for $k < \tilde{k}$, a $\hat{b}(k) > \underline{b}(k)$ unique such that $\psi(k,\hat{b}(k)) = 0$. The limits of the function $\hat{b}(k)$ are:

$$\lim_{k\to k_{\mathrm{GR}}} \hat{b}(k) = b_{\mathrm{GR}} = \frac{s(\omega(k_{\mathrm{GR}}), 1+n)}{1+n} \Leftrightarrow k_{\mathrm{GR}}$$

and

$$\lim_{k\to\tilde{k}} \hat{b}(k) = \underline{b}(\tilde{k}) = \Leftrightarrow \tilde{k}$$

For $k > \tilde{k}$ we have $\forall b > \underline{b}(k), \psi(k,b) > 0$. Finally, the continuity and differentiability of $\hat{b}$ results from the theorem of implicit functions.

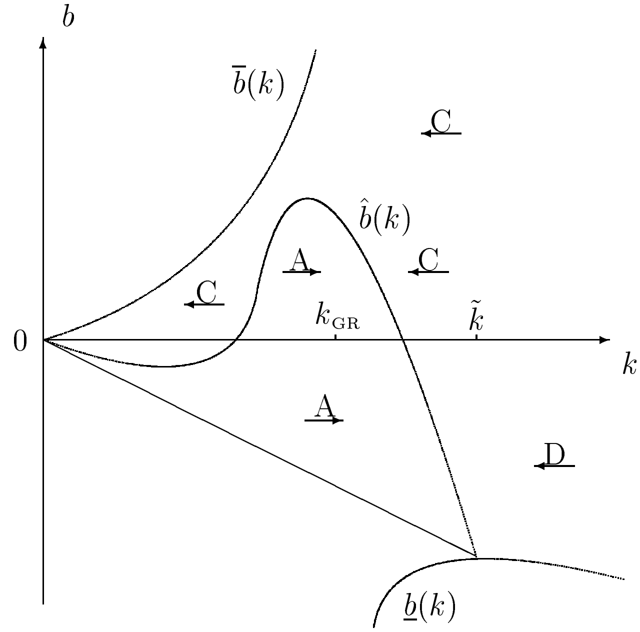

Figure 3: Characteristics of inter-temporal equilibria

■

Figure 2 shows two possible configurations (among others). On the left panel the curve $\hat{b}(k)$ is plotted for the case where $\hat{b}(0+) = 0$ and $\tilde{k}$ is finite. This case arises, e.g., with a logarithmic utility function and a CES production function with low possibilities of substitution. On the right panel, the curve $\hat{b}(k)$ is plotted for the case where $\hat{b}(0+) > 0$ and $\tilde{k}$ is infinite. This case arises, e.g., with a logarithmic utility function and a CES production function with high possibilities of substitution.

One important implication of Proposition 2 is that any level of capital between 0 and an upper limit given by $f(k) = (1+n)k$ can be implemented as an equilibrium. In particular, one can reach the Golden rule with either $b \geqslant 0$ and $b < 0$.

4 Characteristics of inter-temporal equilibria

It is now possible to characterize the inter-temporal equilibria, their existence and the nature of their dynamics by using the function $\hat{b}(k)$.

Proposition 3 *Assume* **C2** *and let* $(b, k_0) \in \mathcal{E}$. *Then*

When $k_0 < \tilde{k}$

A. *if* $b < \hat{b}(k_0)$, *the inter-temporal equilibrium with initial state* k_0 *exists, is unique and the sequence* $(k_t)_{t\geqslant 0}$ *is increasing and converges to a steady state* $\bar{k}$; $\bar{k} > k_0$ *is the smallest value for which* $\hat{b}(\bar{k}) = 0$ *holds.*

B. *if* $b = \hat{b}(k_0)$, k_0 *is a steady state equilibrium: the inter-temporal equilibrium exists and the sequence* $(k_t)_{t\geqslant 0}$ *is constant and equal to* (k_0).

C. *if* $b > \hat{b}(k_0)$*, the inter-temporal equilibrium exists if and only if there exists a steady state equilibrium* $\bar{k} < k_0$*,* $\bar{k} \geqslant 0$ *for which* $\hat{b}(\bar{k}) = 0$ *holds. The sequence* $(k_t)_{t \geqslant 0}$ *is then decreasing and converges to the largest steady state which is smaller than* k_0*. If there does not exist a steady state* $\bar{k}$*,* $0 \leqslant \bar{k} < k_0$*, then the sequence* (k_t)*, which is decreasing, ceases to be defined after a finite number of periods, because either* $\tilde{w}(k_t, b)$ *becomes negative, or* $\varphi(k, \tilde{w}(k_t, b), b) = 0$ *has no solution.*

D. *When* $k_0 \geqslant \tilde{k}$*, the condition* $\tilde{w}(k_0, b) > 0$ *implies that* $\psi(k_0, b)$ *is positive and we have the same properties as in case C.*

Proof: A. If $b < \hat{b}(k_0)$, then $\psi(k_0, b) < \psi(k_0, \hat{b}(k_0)) = 0$. We are in case 1 of proposition 2 (with strict inequality). Then, the inter-temporal equilibrium exists, is unique according to **C2** and the sequence (k_t) is increasing. Let us show that it is bounded. We have:

$$(1+n)(k_{t+1} + b) < \tilde{w}(k_t, b) = \omega(k_t) \Leftrightarrow \varrho(k_t) b$$

and

$$\frac{k_{t+1}}{k_t} < \frac{1}{1+n} \left[\frac{\omega(k_t)}{k_t} \Leftrightarrow \frac{\varrho(k_t) b}{k_t} \right] \Leftrightarrow \frac{b}{k_t}$$

The limit of the right-hand-side is nil when k_t tends to $+\infty$, as $\varrho(+\infty) = f'(+\infty) \Leftrightarrow (1+n)$ is finite and the limit of $\omega(k)/k$ when k tends to $+\infty$ is equal to 0 (see e.g. Jones and Manuelli (1990)). Thus we have that, for k_t large enough, $k_{t+1} < k_t$. The increasing sequence (k_t) is bounded.

We deduce that the sequence (k_t) converges to limit $\bar{k}$ which is a steady state equilibrium. This implies that there exists $\bar{k} > k_0$ such that $b = \hat{b}(\bar{k})$. Let us denote $\bar{k}_0$ the smallest steady state equilibrium greater than k_0. By induction, if we have for all $t \geqslant 0$ $k_t < \bar{k}_0$ and $\varphi(k_t, \tilde{w}(k_t, b), b) < 0 = \varphi(\bar{k}_0, \tilde{w}(\bar{k}_0, b), b)$ then $\tilde{w}(\bar{k}_0, b) > \tilde{w}(k_t, b)$ and ψ'_w imply $\varphi(\bar{k}_0, \tilde{w}(k_t, b), b) > 0$. The function φ has thus a root at k_{t+1} such that $k_t < k_{t+1} < \bar{k}_0$ and we deduce from $\varphi(k_{t+1}, \tilde{w}(k_t, b), b) = 0$ and $\tilde{w}(k_{t+1}, b) > \tilde{w}(k_t, b)$ that we have $k_{t+1} < \bar{k}_0$ and $\varphi(k_{t+1}, \tilde{w}(k_{t+1}, b), b) < 0$. The first conclusion is proved.

B. In the case where If $b = \hat{b}(k_0)$, the conclusion results directly from the definition of $\hat{b}(.)$.

C. If $b > \hat{b}(k_0)$, then $\psi(k_0, b) > \psi(k_0, \hat{b}(k_0)) = 0$. We are in case 2 of proposition 2. As long as it exists, the equilibrium sequence (k_t) is decreasing.

sufficient condition: if there exists a steady state equilibrium $0 \leqslant \bar{k}_0 < k_0$, we apply the same method as in the proof of case A. By induction, for $t \geqslant 0$, if

$$k_t > k_0 \quad \text{and} \quad \varphi(k_t, \tilde{w}(k_t, b), b) > \varphi(\bar{k}_0, \tilde{w}(\bar{k}_0, b), b) = 0$$

then we deduce from $\tilde{w}(k_t, b) > \tilde{w}(\bar{k}_0, b)$ that $\varphi(\bar{k}_0, \tilde{w}(k_t, b), b) < 0$. There exists k_{t+1}, $k_t > k_{t+1} > \bar{k}_0$ such that $\varphi(k_{t+1}, \tilde{w}(k_t, b), b) = 0$ as the continuous function of k: $\varphi(k, \tilde{w}(k_t, b), b)$ is positive at k_t and negative at $\bar{k}_0$. We then have $\varphi(k_{t+1}, \tilde{w}(k_{t+1}, b), b) > 0$. The inter-temporal equilibrium exists and converges to the largest steady state $\bar{k}_0$ which is smaller than k_0.

necessary condition: if the inter-temporal equilibrium exists, the sequence (k_t) is defined and decreasing. It has thus a limit $\bar{k}_0 \geqslant 0$ when t tends to $+\infty$. Let us show that $\bar{k}_0$ is a steady state equilibrium.

- if $\bar{k}_0 > 0$, $\bar{k}_0$ is the largest steady state equilibrium which is smaller than k_0: indeed, the previous proof can be applied to the largest steady state equilibrium smaller than k_0 when it exists.

- if $\bar{k}_0 = 0$, there is no positive steady state equilibrium smaller than k_0 and we have by assumption: $\forall t, \tilde{w}(k_t, b) > 0$ and $\phi(k_{t+1}, \tilde{w}(k_t, b), b) = 0$ and $\lim k_t = 0$. When there is no debt, $b = 0$, we have $\tilde{w}(k_t, b) = \omega(k_t)$ and 0 is a corner steady state. When debt is non-zero, $b \neq 0$, the consumption of old agents at t is $d_t = f'(k_t)s_{t-1} = f'(k_t)(1+n)(k_{t+1} + b)$ and

$$\lim_{t \to 0} d_t = f'(0)b(1+n)$$

As along equilibrium trajectory d_t is positive and bounded ($d_t < (1+n)f(k_t)$), we have necessarily that b is positive and $f'(0)$ is finite. Then $\tilde{w}(k, b)$ and φ have limits when k tends to 0 and we have:

$$b = s(\tilde{w}(0, b), f'(0)) \quad \text{with} \quad \tilde{w}(0, b) = \omega(0) \Leftrightarrow (f'(0) \Leftrightarrow (1+n))b \tag{24}$$

The condition $\tilde{w}(k, b) > 0$ imposes $\omega(0) > (f'(0) \Leftrightarrow (1+n))b > 0$. Equation (24) means that $\bar{k}_0 = 0$ is a steady state equilibrium.

D. If $k_0 \geqslant \tilde{k}$ and $(k_0, b) \in \mathcal{E}$ we have $\psi(k_0, b) > 0$. Then we are in case 2 of proposition 3, and the analysis developed in C. can be applied. ■

The conditions of proposition 3 can be compared with the requirement often found in the literature (see e.g. Blanchard (1990)) that the inter-temporal budget constraint of the government should hold for its debt to be sustainable.

Corollary: *Let us consider a stable stationary equilibrium. Then, the inter-temporal government budget constraint holds if and only if under-accumulation prevails at the equilibrium.*

Obviously, if over-accumulation prevails, the growth rate of debt, which equals the growth rate of population, is larger than the interest rate, and the inter-temporal government budget constraint does not hold. In that case, though, the inter-temporal equilibrium exists and debt is sustainable.

As shown in figure 3, Proposition 3 provides us with a simple geometrical tool to analyze the dynamics of capital. Locally stable (resp. unstable) steady states are located on the downward (resp. upward) sloping branches of the function $\hat{b}(.)$.

5 Policy implications

The main policy issues are as follows. On the one hand, any level of debt is not compatible with the existence of capital. On the other hand, any level of capital between 0 and an upper limit given by $f(k) = (1+n)k$ can be implemented as an equilibrium. In particular, one can reach the Golden rule with either $b \geqslant 0$ and $b < 0$. Debt can also be used to avoid a poverty trap. We detail these three issues in turn.

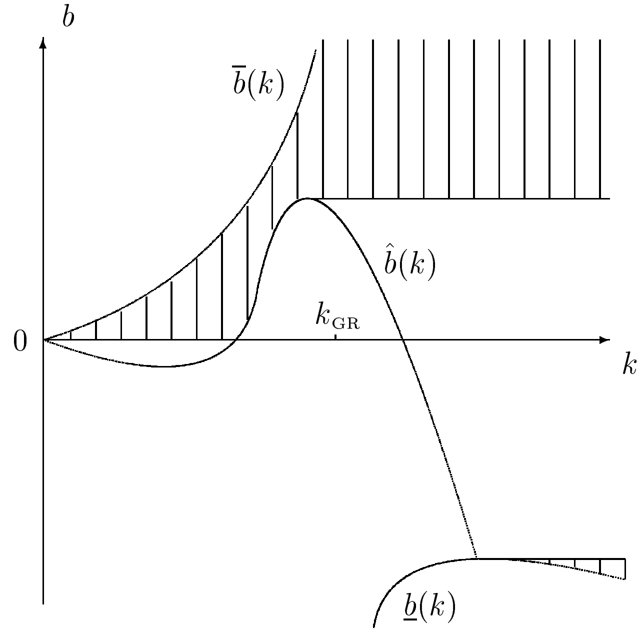

Figure 4: Unsustainable debts.

5.1 Sustainability

Sustainability means here that an inter-temporal equilibrium with constant debt exists. As shown in figure 4, three different unsustainable situation can occur.

In the first situation, the economy starts above the maximum of the steady state curve. The debt is too large implying that the over-taxed young households do not save enough to maintain constant the private capital stock. This is the case where the burden of debt is excessive. As productive capital falls, wage incomes become at some point insufficient to cover the tax payments and the temporary equilibrium with this level of debt no longer exists.

In the second situation, the economy starts at the left of the steady state curve. The debt is too large with respect to the initial capital stock and the conclusion is the same as in the previous case.

The third case arises when the debt is negative and initial capital is large. In that situation, the interest rate is lower than the growth rate of population and households have to pay taxes to finance the government investment program. The stock of capital (and hence wage income) tends to decrease and at some point households are no longer able to sustain this situation. Note in the figure the whole range at the right of the $\hat{b}(k)$ curve where over-accumulation prevails implying that the growth rate of debt is larger than the interest rate. In this range, the inter-temporal government budget constraint does not hold but the inter-temporal equilibrium exists and debt is sustainable.

5.2 Poverty traps

We have seen in the proof of proposition 3 that 0 can be a corner steady state (poverty trap) only if debt is positive or nil. Indeed, when the government holds private assets

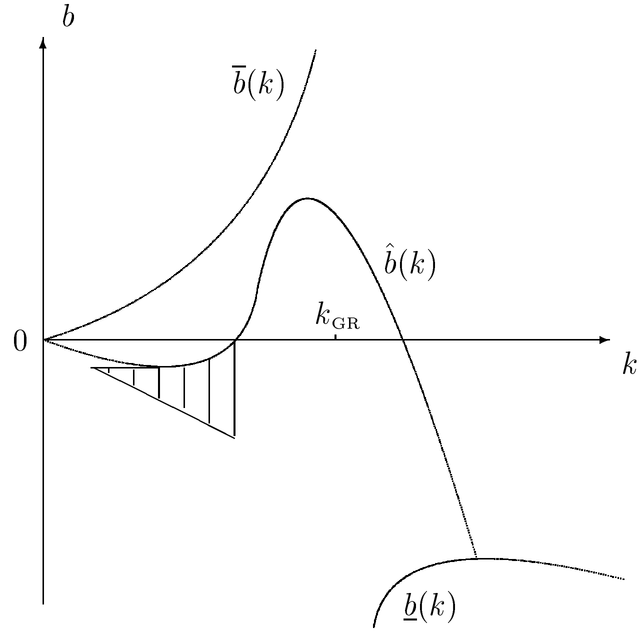

Figure 5: Escapes from a poverty trap.

$(b < 0)$, we have

$$(1+n)k_{t+1} = s_t \Leftrightarrow b > \Leftrightarrow b > 0$$

and capital is bounded below by the constant quantity held by the government.

Moreover, a negative public debt can be used to escape from a poverty trap under certain circumstances. As illustrated in figure 5, a negative debt allows to escape from a poverty trap when (i) the corner steady state is locally stable in the economy without debt, (ii) the initial capital stock lies in the range of stability of the corner steady state and (iii) the curve $\hat{b}(k)$ is increasing at k_0, which guarantees that there exists some level of debt $b^\star$ such that $b^\star < \hat{b}(k_0)$. In that case, running a negative debt would put the economy on a path that converges to a high steady state level of capital. This policy amounts to "nationalize" part of the capital stock detained by the first old generation, thereby allowing to distribute the dividends of this capital to the young generation by the means of a negative tax. This inter-generational transfer enables a high level of investment, allowing the capital stock to grow.

Notice also that the size of the "negative" debt that is necessary to escape from the poverty trap depends on the distance between the actual stock of capital and the lowest (unstable) steady state. Near this steady state, a very small amount of negative debt is sufficient. The further one get from the steady state the higher the necessary level of negative debt will be. At some point, it becomes impossible to escape from the trap, as the required level of debt would violate the positivity constraint on private investment.

5.3 Golden rule

As the curve $\hat{b}(k)$ is always defined at k_{GR}, there always exists a level of debt such that the golden rule capital stock is a steady state. This is the central point of Diamond's analysis. However, nothing says that this steady state equilibrium is locally or globally stable. Our analysis allows us to obtain conditions where it is the case. When the Golden rule equilibrium is locally stable, i.e. when $\hat{b}'(k_{\mathrm{GR}}) < 0$, the constant debt policy with $b = \hat{b}(k_{\mathrm{GR}})$ leads to the Golden rule only if k_0 is close enough to k_{GR}.

A noteworthy situation arises when the Golden rule steady state equilibrium is unstable. This case is characterized by $\hat{b}'(k_{\mathrm{GR}}) > 0$. In that case, although the level of debt such that the golden rule capital stock is a steady state always exists, the constant debt policy is always unable to lead to the Golden rule, except if the initial capital stock is already at the Golden rule level. From proposition 3, if the debt is set such that $b = \hat{b}(k_{\mathrm{GR}})$ and the initial capital stock is larger that k_{GR}, we have $b < \hat{b}(k_0)$, the inter-temporal equilibrium exists, is unique and the sequence (k_t) is increasing. It converges to a steady state greater than the Golden Rule level, thereby characterized by over-accumulation of capital. If the initial capital stock is smaller that k_{GR}, $b > \hat{b}(k_0)$, and, as long as it exists, the equilibrium sequence (k_t) is decreasing. It either converges to a steady state characterized by under-accumulation, or the equilibrium ceases to be defined after a finite number of periods.

The framework developed above can be used to obtain the conditions under with the Golden Rule equilibrium is unstable. Let us take a Cobb-Douglas production function $f(k) = Ak^\alpha$ and a logarithmic utility function $U(c,d) = \ln c + \beta \ln d$. Then the steady state curve is given by

$$\hat{b}(k) = \frac{k((1 \Leftrightarrow \alpha)\beta k^\alpha \Leftrightarrow k(1+n)(1+\beta))}{k(1+n) + k^\alpha \alpha\beta}$$

and the condition for unstable Golden Rule equilibrium is

$$\hat{b}'(k_{\mathrm{GR}}) > 0 \Longleftrightarrow \beta > \frac{\sqrt{\alpha}}{1 \Leftrightarrow \alpha \Leftrightarrow \sqrt{\alpha}}$$

i.e., if the weight attached to future consumption is large enough, Diamond's Golden Rule equilibrium is unstable. Notice that the condition does not depend on n.

6 Conclusion

In his seminal contribution, Diamond (1965) examines the effect of government debt on the long-run competitive equilibrium of an economy with overlapping generations; he shows that internal positive debt lowers utility when the equilibrium is efficient (under-accumulation case) but may raise utility in the inefficient case (over-accumulation). In Diamond's work global stability is implicitly assumed.

In this article, we analyze the conditions under which this assumption holds and provide an in depth treatment of the dynamics in the Diamond economy. This analysis is central to the issue of debt sustainability as, even if debt can be beneficial in an long-run equilibrium, nothing guarantees that, given its initial endowment in capital, the economy will converge to this equilibrium rather than to an unsustainable outcome.

We have studied the sustainability of national debt in the sense that an inter-temporal equilibrium with constant debt exists. Two different unsustainable situation can occur. Either the debt is too large implying that the over-taxed young households do not save enough to maintain constant the private capital stock. Or the debt is negative and initial capital is large. In that case, the interest rate is lower than the growth rate of population and households have to pay taxes to finance the government investment program. At some point households are no longer able to sustain this situation.

Moreover, the dynamic analysis has pointed out a series of issues that are often neglected in the literature:

- When over-accumulation prevails, the growth rate of debt is larger than the interest rate. In this range, the inter-temporal government budget constraint does not hold but the inter-temporal equilibrium may exist and debt may thus be sustainable.
- If initial capital stock is low and the economy is threatened by a poverty trap, a negative debt can be used to escape from the trap if we are not too far from the lowest positive steady state.
- Although the level of debt such that the golden rule capital stock is a steady state always exists, the constant debt policy can be unable to lead to the Golden rule.

One of the contribution of this paper is to generalize the second aspect of the global analysis of Galor and Ryder (1989) to cases with constant debt and gather the essential information in a single planar diagram that can be used to answer the issues of global stability in a comprehensive way.

References

Azariadis, C. and B. Smith (1998) "Financial intermediation and regime switching in business cycles". *American Economic Review*, 88:516–536.

Barro, R. (1974) "Are government bonds net wealth?" *Journal of Political Economy*, 81:1095–1117.

Bertocchi, G. (1994) "Safe debt, risky capital". *Economica*, 61:493–508.

Blanchard, O. (1985) "Debts, deficits and finite horizon". *Journal of Political Economy*, 93:223–247.

Blanchard, O. (1990) "The sustainability of fiscal policy: new answers to an old question". *OECD Economic Studies*, 15:7–36.

Diamond, P. (1965) "National debt in a neoclassical growth model". *American Economic Review*, 55:1126–1150.

Galor, O. and H. Ryder (1989) "Existence, uniqueness and stability of equilibrium in overlapping generations model with productive capital". *Journal of Economic Theory*, 49:360–375.

Grossman, G. and N. Yanagawa (1993) "Asset bubbles and endogenous growth". *Journal of Monetary Economics*, 31:3–19.

Ihori, T. (1978) "The golden rule and the role of government in a life-cycle growth model". *American Economic Review*, 68:389–396.

Jones, L. and R. Manuelli (1990) "A convex model of equilibrium growth: theory and policy implications". *Journal of Political Economy*, 98:1008–1038.

King, I. (1992) "Endogenous growth and government debt". *Southern Economic Journal*, 59:15–21.

King, I. and D. Ferguson (1993) "Dynamic inefficiency, endogenous growth and ponzi games". *Journal of Monetary Economic*, 32:79–104.

Rankin, N. and B. Roffia (1999) "Maximum sustainable government debt in the overlapping generations model". Working Paper, CEPR.

Ricardo, D. (1817) *On the principles of political economy and taxation.* Cambridge University Press.

Uhlig, H. (1996) "Capital income taxation and the sustainability of permanent primary deficits". Working Paper, CentER, Tilburg.

Weil, P. (1989) "Overlapping families of infinitely-lived agents". *Journal of Public Economics*, 38:183–198.