ON STRONG INTEGRALITY PROPERTIES OF THE PERFECT MATCHING POLYTOPE

Roland Grappe, Mathieu Lacroix,
Francesco Pisanu

LIDAM Discussion Paper CORE
2024 / 32

CORE

Voie du Roman Pays 34, L1.03.01

B-1348 Louvain-la-Neuve

Tel (32 10) 47 43 04

Email: immaq-library@uclouvain.be

https://uclouvain.be/en/research-institutes/lidam/core/core-discussion-papers.html

On strong integrality properties of the perfect matching polytope

Roland Grappe[1][0000−0002−7093−2175], Mathieu Lacroix[2][0000−0001−8385−3890], and Francesco Pisanu[2,3][0000−0003−0799−5760]

[1] Université Paris Dauphine, PSL Research University, LAMSADE, CNRS UMR 7243
`roland.grappe@lamsade.dauphine.fr`

[2] Université Sorbonne Paris Nord, LIPN, CNRS UMR 7030, F-93430, Villetaneuse, France
`{lacroix, pisanu}@lipn.univ-paris13.fr`

[3] Université Catholique de Louvain, CORE, 1348 Ottignies-Louvain-La-Neuve, Belgium
`francesco.pisanu@uclouvain.be`

Abstract. This paper investigates integrality properties of perfect matching polytopes, focusing on box-total dual integrality and integer decomposition properties.
We begin by characterizing the graphs whose perfect matching polytope is a slice of the nonnegative orthant, identifying these as the solid graphs introduced by de Carvalho, Lucchesi, and Murty in On a Conjecture of Lovász Concerning Bricks: I. The Characteristic of a Matching Covered Graph (*Journal of Combinatorial Theory, Series B)*.
As a result, we show that the perfect matching polytope of solid graphs admits a compact description, and we establish that deciding the box-total dual integrality of a perfect matching polytope can be done in polynomial time.
Additionally, we characterize the conditions under which perfect matching polytopes of two fundamental graph classes, namely near-bricks and bicritical graphs, are box-totally dual integral. We discuss implications of these results for identifying perfect matching polytopes with the integer decomposition property.

Keywords: Perfect matching polytope · Box-totally dual integral polyhedron · Integer decomposition property.

1 Introduction

This paper investigates integrality properties of perfect matching polytopes, widely studied in combinatorial optimization, with a focus on box-total dual integrality and integer decomposition properties, two fundamental notions in integer programming.

Totally dual integral and box-totally dual integral systems were introduced as a versatile framework for establishing various min-max relations in combinatorial optimization [38]. A rational linear system $Ax \leqslant b$ is *totally dual integral (TDI)* if the minimum in the linear duality equation

$$\max\{w^\top x\colon Ax \leqslant b, x \geqslant \mathbf{0}\} = \min\{b^\top y\colon A^\top y \geqslant w, y \geqslant \mathbf{0}\}$$

has an integer optimal solution for every integer vector w such that the optimum is finite. If a system is TDI, then the right-hand side can be chosen integer if and only if the polyhedron described by the system is integer [23].

A stronger property for a system $Ax \leqslant b$ is to be *box-total dual integrality (box-TDI)*, that is when $Ax \leqslant b, \ell \leqslant x \leqslant u$ is TDI for all rational vectors ℓ and u (with possible infinite components). Classical examples of box-TDI systems are those defined by *totally unimodular* matrices, which are the matrices whose minors are all $0, \pm 1$ [37]. Examples are the classical Kőnig's Theorem [31] and the MaxFlow-MinCut Theorem of Ford and Fulkerson [24]. While every polyhedron can be described by a TDI system [26], there are polyhedra that are not described by any box-TDI system. A polyhedron is *box-TDI* if it can be described by a box-TDI system. Cook [12] proved that any TDI system describing a box-TDI polyhedron is a box-TDI system.

Box-TDI systems and polyhedra have been actively studied the past three decades. Box-Mengerian matroid ports are characterized in [7]. Series-parallel graphs form a class in which several polyhedra turn out to be box-TDI: a box-TDI system describes their 2-edge-connected spanning subgraph polyhedron [8] and is generalized for k-edge-connectivity [1]; [13] provide several other related box-TDI systems; [2] proves the box-TDIness of their flow cone. Regarding box-perfect graphs, which are the perfect graphs having a box-TDI stable set polytope, new classes of box-perfect graphs are introduced in [20], and a weak box-perfect graph theorem is given in [9]. Matricial and geometrical characterizations of box-TDI polyhedra can be found in [11]. Complexity results regarding box-TDIness are given in [10]. Graphs for which the matching polytope is box-TDI are characterized in [19].

A polyhedron P has the *integer decomposition property (IDP)* if every integer point in the k-th dilation kP of P is the sum of k integer points from P, for all $k \in \mathbb{Z}_{>0}$. If P has the integer decomposition property, then P is integer, and every face of P also has the this property [37, Section 22.10]. Originally introduced in integer programming by Baum and Trotter [3], the integer decomposition property has since been studied in fields such as algebraic geometry and combinatorial commutative algebra [27].

Several classes of polyhedra are known to have the integer decomposition property, including projections of polyhedra defined by totally unimodular matrices [39], polyhedra defined by nearly totally unimodular matrices [25], certain polyhedra defined by k-balanced matrices [42], and stable set polytopes of claw-free t-perfect graphs and h-perfect line-graphs [5]. Additional connections with Fulkerson's theory of blocking and anti-blocking polyhedra are explored in [4]. The matching polytope of a bipartite graph has the integer decomposition property, since it is described by a totally unimodular matrix. This is generalized in [40] to matchings of size $k \leqslant \lfloor n/2 \rfloor$ of a bipartite graph with n vertices.

In a graph, a *matching* is a subset of pairwise nonincident edges, and a *perfect matching* is a matching that covers all the vertices. The *matching polytope* of a graph is the convex hull of the incidence vectors of its matchings. Similarly, the *perfect matching polytope (PMP)* of a graph is the convex hull of the incidence vectors of its perfect matchings. Since the perfect matching polytope is a face of the matching polytope and as box-TDIness is preserved under taking faces, Ding, Tan, and Zang [19]'s characterization in terms of forbidden subgraphs gives sufficient conditions for the box-TDIness of the perfect matching polytope. However, as the perfect matching polytope of a subgraph needs not to be a face of the perfect matching polytope of the original graph, there is no characterization of its box-TDIness in terms of forbidden subgraphs.

In this paper, we report progress on the box-TDIness and the integer decomposition property of the perfect matching polytope of nonbipartite graphs.

Contributions. Our contributions are threefold. First, we characterize the graphs for which the perfect matching polytope is the intersection of its affine hull with the nonnegative orthant: these are precisely the so-called solid graphs. This extends a result of de Carvalho et al. [16].

Second, we characterize the box-TDIness of the perfect matching polytope of two fundamental classes of graphs in the context of perfect matchings: near-bricks and bicritical graphs. This graphic characterization involves odd intercyclicity and follows from the study of the impact of tight cut contractions on the box-TDIness of the perfect matching polytope. More precisely, we prove that contracting a tight cut preserves the box-TDI of the perfect matching polytope. We observe that the converse does not hold in general. Nevertheless, for 2-separation cuts, which are particular tight cuts, we prove that the converse holds.

It is known that a box-TDI polyhedron has a box-TDI affine hull. We prove that the converse holds for the perfect matching polytope, that is, the perfect matching polytope is box-TDI if and only if its affine hull is. As a consequence, determining whether the perfect matching polytope is box-TDI can be done in polynomial time. As another consequence, box-TDI perfect matching polytopes have the integer decomposition property. We highlight that the converse does not hold by providing a general class of graphs whose perfect matching polytopes have the integer decomposition property but is not box-TDI.

Outline. In Section 2, we provide the results from the literature that we shall use throughout: about the perfect matching polytope, matching covered graphs and their tight cut decomposition, and box-TDI polyhedra. We also discuss the difference between characterizing the box-TDIness of the matching polytope and that of the perfect matching polytope. In Section 3, we characterize the graphs for which the perfect matching polytope is the intersection of its affine hull with the nonnegative orthant. Moreover, we prove that the box-TDIness of perfect matching polytopes is characterized by that of its affine hull. This yields a polynomial-time algorithm for verifying the box-TDIness of the perfect matching polytope of any graph. In Section 4, we characterize which near-bricks and bicritical graphs have a box-TDI perfect matching polytope. In Section 5, we discuss the integer decomposition property of the perfect matching polytope.

Due to space constraints, the proofs are provided in the appendix.

2 Preliminaries

In this section, we give the results that we shall use throughout the paper: about box-TDI polyhedra, the perfect matching polytope, matching covered graphs and their tight cut decompositions, and the affine hull of perfect matchings. Finally, we discuss differences between the box-TDIness of the matching polytope and that of the perfect matching polytope.

In this work, all considered graphs are undirected. Without loss of generality, we assume all graphs to be connected with at least one edge, as our results extend immediately to general undirected loopless graphs. A special role is played by *odd intercyclic* graphs, which are the graphs having no two vertex-disjoint odd cycles.

For a given graph $G = (V, E)$ we denote by A_G the (vertex-edge) incidence matrix of G. For $C \subseteq E$, we denote by χ^C the incidence vector of C. Throughout, $\mathbf{0}$ (resp. $\mathbf{1}$) will respectively denote a zero (resp. one) entrywise matrix of appropriate size.

2.1 Box-total dual integrality, equimodularity, and integer decomposition property

A matrix is *equimodular* if it has full row rank and all maximal nonzero minors are equal up to the sign. Equimodularity is characterized in terms of total unimodularity.

Theorem 1 (Heller [28]). An $m \times n$ matrix A is equimodular if and only if $B^{-1}A$ is totally unimodular for every nonsingular $m \times m$ submatrix B of A. Equivalently, $B^{-1}A$ is a $\{0, \pm 1\}$-matrix for every nonsingular $m \times m$ submatrix B of A.

It is well-known that the incidence matrix of a graph is totally unimodular if and only if the graph is bipartite [29]. The following result characterizes the class of graphs whose incidence matrix is equimodular.

Theorem 2 (Chervet et al. [10]). The incidence matrix of a connected non-bipartite graph G is equimodular if and only if G is odd intercyclic.

Box-TDI polyhedra are characterized in terms of equimodular matrices as follows. A matrix is *face-defining* for a polyhedron P if it has full row rank and describes the affine hull of some face of P.

Theorem 3 (Chervet et al. [11]). A polyhedron is box-TDI if and only if all its face-defining matrices are equimodular. Equivalently, each of its faces admits an equimodular face-defining matrix.

Theorem 3 contains the well-known fact that polyhedra described by totally unimodular matrices are box-TDI [37, Chapter 22]. Moreover, these polyhedra have the integer decomposition property.

Theorem 4 (Baum and Trotter [3]). If A is totally unimodular and b integer, then $P = \{x \colon Ax \leqslant b\}$ has the integer decomposition property.

2.2 Perfect matchings

We refer to [33] for an extensive introduction about perfect matchings. We first recall the following well-known characterization of the existence of a perfect matching, where $\mathcal{O}(G)$ denotes the family of connected components of odd cardinality of the graph G.

Theorem 5 (Tutte [41]). A graph $G = (V, E)$ has a perfect matching if and only if $|\mathcal{O}(G \backslash S)| \leqslant |S|$ for all $S \subseteq V$.

When looking at the perfect matchings of a graph, one may restrict to *matching covered* graphs, which are the graphs in which every edge belongs to a perfect matching. By definition, a matching covered graph is 2-*connected*, that is, no vertex removal disconnects the graph. The following theorem of Lovász [34] characterizes matching covered graphs. A subset S of vertices of a graph G is a *barrier* if $|\mathcal{O}(G \backslash S)| = |S|$.

Theorem 6 (Lovász [34]). A graph having a perfect matching is matching covered if and only if each barrier is composed of pairwise nonadjacent vertices.

Let $G = (V, E)$ be a matching covered graph. For a subset X of vertices, $E(X)$ is the set of edges of G having both extremities in X, and $\delta(X)$ denotes the *cut* determined by X, that is, the set of edges having precisely one extremity in X. The *shores* of a cut $\delta(X)$ are X and $\overline{X} = V \backslash X$. A cut is *trivial* if one of its shores is a singleton. For $X \subseteq V$, *contracting* X *to a single vertex* x means replacing X by a new vertex x with $\delta(x) = \delta(X)$, and we denote the resulting graph by G/X. The two graphs G/X and $G/\overline{X}$ are referred to as the two $\delta(X)$-*contractions* of G. A cut C of G is *tight* if $|C \cap M| = 1$ for every perfect matching M of G. A cut C of G is a *separating cut* if both of the C-contractions of G are also matching covered. Every tight cut of a matching covered graph is separating, but the converse does not hold. For example, the set of edges of $\overline{C}_6$ contained in no triangle is a separating cut, and it is not tight since it is also a perfect matching. Two tight cuts $\delta(X)$ and $\delta(Y)$ are *laminar* if either $X \subseteq Y$ or $X \subseteq \overline{Y}$. A *laminar family of tight cuts* is a family of pairwise laminar tight cuts.

A graph is *solid* if it is matching covered and all its separating cuts are tight. A matching covered graph free of nontrivial tight cuts is a *brace* if it is bipartite and a *brick* otherwise. A graph is *bicritical* if removing any couple of vertices yields a graph having a perfect matching. A graph is a brick if and only if it is 3-connected and bicritical [34]. Typical examples of bricks are the complete graph on four vertices K_4, the prism $\overline{C}_6 \coloneqq K_6 \backslash E(C_6)$, which is a the complement of a cycle of length 6, and the Petersen graph [34].

For a matching covered graph, the following holds.

Theorem 7 (Edmonds et al. [22]). Let G be a matching covered graph, and $\delta(U)$ and C be two laminar tight cuts of G. If C is a cut of G/U, then C is tight for G/U.

Let $\mathcal{F}$ be a family of laminar nontrivial tight cuts of a matching covered graph G. Note that contracting a shore of some tight cut of $\mathcal{F}$ yields a smaller matching covered graph. By Theorem 7, this can be repeated with the cuts of $\mathcal{F}$ which remain nontrivial cuts in the resulting graph, until a graph with no nontrivial tight cut from $\mathcal{F}$ is obtained. The graphs obtained by this procedure are called *$\mathcal{F}$-contractions* of G. When $\mathcal{F}$ is an inclusionwise maximal family of laminar nontrivial tight cuts, the $\mathcal{F}$-contractions contain no nontrivial tight cuts, hence are either bricks or braces. Given such an $\mathcal{F}$, a *tight cut decomposition* of G is the list of all bricks and braces that are $\mathcal{F}$-contractions of G [34]. Lovász [34] proved that any tight cut decomposition of a given graph provides the same list of bricks and braces (up to edge multiplicities). Moreover, a family of laminar nontrivial tight cuts $\mathcal{F}$ has the *odd cycle property* if every $\mathcal{F}$-contraction is nonbipartite. A nontrivial tight cut $\delta(U)$ has the *odd cycle property* if $\{\delta(U)\}$ has it. For a family $\mathcal{F}$ of nontrivial laminar tight cuts and a nontrivial tight cut $\delta(U)$ that is laminar with each tight cut of $\mathcal{F}$, we denote by $\mathcal{F}_{G/U}$ the set of tight cuts in $\mathcal{F}$ that are cuts of G/U.

Let denote by $\mathfrak{F}_G$ the family whose elements are maximal inclusionwise families of laminar nontrivial tight cuts having the odd cycle property. Every $\mathcal{F} \in \mathfrak{F}_G$ has the same cardinality, which is the number of bricks of G minus one [34]. When $\mathfrak{F}_G = \varnothing$ and G is nonbipartite, G is called *near-brick*. In particular, a near-brick has a single brick, and any brick is a near-brick.

In [16], the following connection between the solidity of a graph and the one of its bricks is established.

Theorem 8 (de Carvalho et al. [16]). A matching covered graph is solid if and only if all its bricks are.

By de Carvalho et al. [16, Lemma 2.29], odd intercyclic matching covered graphs are solid, such as bipartite matching covered graphs, odd wheels, and Möbius ladders of even order [18].

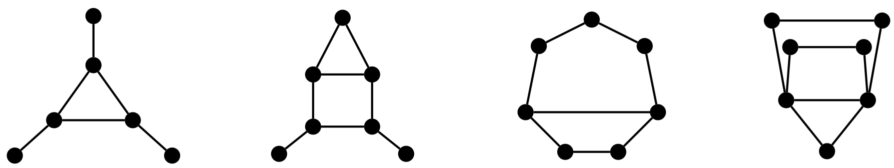
2.3 The matching polytope

Given a graph $G = (V, E)$, the matching polytope of G is denoted by $P_{\mathrm{M}}(G)$. The following system of inequalities describes $P_{\mathrm{M}}(G)$ and is known as *Edmonds' system* [21].

$$
(1)\begin{cases}
x(E(U)) \leqslant (|U|-1)/2, & \text{for each } U \subseteq V \text{ with } U| \geqslant 3 \text{ odd}, \quad \text{(1a)}\\
x(\delta(u)) \leqslant 1, & \text{for each } u \in V, \quad \text{(1b)}\\
x \geqslant \mathbf{0}. & \quad \text{(1c)}
\end{cases}
$$

Cunningham and Marsh [14] proved that Edmonds' system is always TDI. In [19], the authors characterized the graphs for which Edmonds' system is box-TDI. A graph H is a *fully odd subdivision* of a graph G if H is obtained from G by subdividing each edge of G into a path of odd length (possibly the length is one), where the *length* is the number of edges in the path. Since Edmonds' system is always TDI [15], its box-TDIness is equivalent to that of the underlying polytope [12], hence we can restate their result as follows.

Theorem 9 (Ding et al. [19]). The matching polytope of a graph G is box-TDI if and only if G has no fully odd subdivision of G_1, G_2, G_3, and G_4 of Figure 1 as a subgraph.



G_1 G_2 G_3 G_4

Fig. 1: The graphs G_i, $i = 1, 2, 3, 4$, are the forbidden subgraphs for the box-TDIness of the matching polytope (up to fully odd subdivision).

2.4 The perfect matching polytope

The perfect matching polytope $P(G)$ of a graph G is described by the TDI system obtained from Edmonds' system by setting (1b) to equality. However, the following system also describes the perfect matching polytope [21], and is more convenient to investigate its box-TDIness:

$$(2) \begin{cases} x(\delta(U)) \geqslant 1, & \text{for each } U \subseteq V \text{ with } U| \geqslant 3 \text{ odd}, \quad \text{(2a)} \\ x(\delta(u)) = 1, & \text{for each } u \in V, \quad \text{(2b)} \\ x \geqslant \mathbf{0} & \quad \text{(2c)} \end{cases}$$

Note that for bipartite graphs, inequalities (2a) are redundant, and the remaining system (2b)-(2c) is box-TDI by the total unimodularity of the incidence matrix [37, Section 19.3].

Let $\mathcal{F} \in \mathfrak{F}_G$ with $G = (V, E)$ matching covered. In their seminal works, Naddef [35], Edmonds et al. [22], and Lovász [34], proved that the matrix $M_G^{\mathcal{F}}$ whose $|V| + |\mathcal{F}|$ rows are associated with equalities (2b) and inequalities (2a) associated with the cuts of $\mathcal{F}$ has full row rank. Moreover, the maximum number of linearly independent perfect matchings is $|E| - |V| - |\mathcal{F}| + 1$. We restate the results of Naddef [35], Edmonds et al. [22], and Lovász [34] in terms of face-defining matrices.

Theorem 10 ([22,34,35]). Let G be a nonbipartite matching covered graph. Then, $M_G^{\mathcal{F}}$ is face-defining for $\mathrm{aff}(P(G))$ for every $\mathcal{F} \in \mathfrak{F}_G$.

In particular, the equality associated with any tight cut is a linear combination of equalities associated with tight cuts of $\mathcal{F}$.

2.5 Box-TDIness: matchings VS perfect matchings

The box-TDIness of the matching polytope implies that of the perfect matching polytope, since the latter is a face of the former. However, the converse does not hold. For instance, the PMP of the graph G_1 in Figure 1 is box-TDI — containing only a single point — while its matching polytope is not, as shown by Theorem 9.

This phenomenon also occurs for matching covered graphs: we provide below four infinite families of near-bricks whose PMP is box-TDI, but whose matching polytopes are not, as they contain one of the forbidden structures of Theorem 9. Indeed, by Theorem 16 and Corollary 18, the PMP of any fully odd subdivision of the graphs G_1', G_2', G_3', and G_4' of Figure 2 is box-TDI. Their matching polytope is not box-TDI, as each of them contains one of the forbidden subgraphs of Theorem 9

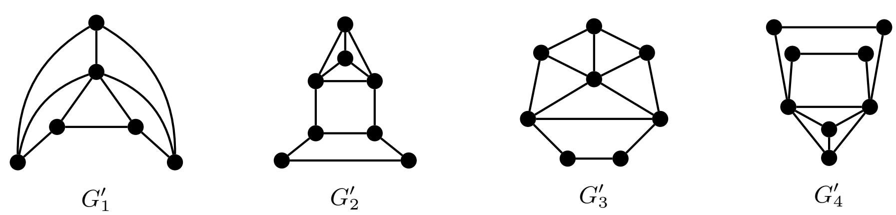


Fig. 2: The graphs G_1', G_2', G_3', and G_4' are the smallest matching covered graphs whose PMP is box-TDI, yet their matching polytope is not.

3 When is the perfect matching polytope a slice of $\mathbb{R}^{\boldsymbol{E}}_{\geqslant \mathbf{0}}$?

In [17, Theorem 2.1], de Carvalho et al. characterize the class of matching covered graphs for which the perfect matching polytope can be described by the system $x(\delta(u)) = 1$ for all $u \in V$, $x \geqslant \mathbf{0}$: they are the solid near-bricks and the bipartite graphs. In this section, following the line of their proof, we generalize this result and prove that the perfect matching polytope of a graph is a slice of nonnegative orthant if and only if the graph is solid. This yields a compact description for the perfect matching polytope of solid graphs, whereas this polytope has no compact formulation in general [36].

Theorem 11. Let $G = (V, E)$ be a matching covered graph. Then, $P(G) = \mathrm{aff}(P(G)) \cap \mathbb{R}^E_{\geqslant 0}$ if and only if G is solid.

Moreover, it turns out that the box-TDIness of a perfect matching polytope is captured by that of its affine hull.

Theorem 12. Let G be a nonbipartite matching covered graph and $\mathcal{F} \in \mathfrak{F}_G$. Then, the following statements are equivalent:

1. $P(G)$ is box-TDI;
2. $\text{aff}(P(G))$ is box-TDI;
3. $M_G^{\mathcal{F}}$ is equimodular;
4. $P(G) = \{x \colon Mx = b, x \geqslant \mathbf{0}\}$ with M totally unimodular.

Moreover, if $P(G)$ is box-TDI, then $P(G) = \text{aff}(P)(G) \cap \mathbb{R}^E_{\geqslant 0}$.

Note that when G is bipartite, $M_G^{\mathcal{F}} = A_G$ so $M_G^{\mathcal{F}}$ has not full row rank by Theorem 2 and hence, it is not equimodular. However, Statements 1, 2, and 4 of Theorem 12 hold in this case, and $P(G)$ is the intersection of its affine hull and the nonnegative orthant.

As a consequence of Theorem 12, the box-TDIness of perfect matching polytopes can be checked in polynomial time. This stands in contrast to the general case, where determining whether a given polytope is box-TDI is co-NP-complete [10]. Since testing the equimodularity of a given matrix can be done in polynomial-time [11], as well as determining a maximal inclusionwise family of laminar tight cuts [34], statement 3 of Theorem 12 implies the following.

Corollary 13. Deciding whether the perfect matching polytope of a matching covered graph is box-TDI can be done in polynomial time.

4 Near-bricks and bicritical graphs

In this section, we characterize the box-TDIness of the perfect matching polytope of near-bricks using forbidden subgraphs. We also characterize the box-TDIness of perfect matching polytopes of bicritical graphs through tight cut decomposition.

We first prove that tight cut contractions preserve the box-TDIness of the perfect matching polytope.

Lemma 14. Let G be a matching covered graph. If $P(G)$ is box-TDI, then so is $P(G/U)$ for each tight cut $\delta(U)$.

4.1 The case of near-bricks

Recall that a near-brick is a nonbipartite matching covered graph such that no tight cut has the odd cycle property.

Lemma 15. If a near-brick is odd intercyclic, then so is its brick.

Theorem 16. The perfect matching polytope of a near-brick is box-TDI if and only if the near-brick is odd intercyclic.

4.2 The case of bicritical graphs

Let u and v be two vertices of a matching covered graph G such that $G\backslash\{u, v\}$ has precisely two connected components G_1 and G_2 which are even. Then, $\delta(V(G_1) \cup \{u\})$ is a 2-*separation cut with respect to* u and v (see Figure 3). Similarly, $\delta(V(G_2) \cup \{u\})$ is a 2-separation cut. Note that a 2-separation cut is tight.

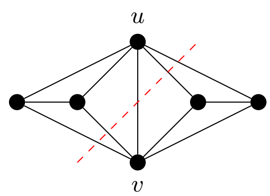


Fig. 3: A 2-separation cut with respect to u and v.

It turns out that the converse of Lemma 14 holds for 2-separation cuts.

Theorem 17. Let $\delta(X)$ be a 2-separation cut of a nonbipartite matching covered graph G. Then, $P(G)$ is box-TDI if and only if both $P(G/X)$ and $P(G/\overline{X})$ are.

Fully odd subdividing a graph preserves the nonbox-TDIness of its matching polytope, yet the converse does not hold [19]. For perfect matchings, it is an equivalence.

Corollary 18. The perfect matching polytope of a fully odd subdivision of a matching covered graph G is box-TDI if and only if $P(G)$ is.

A graph $G = (V, E)$ is *bicritical* if $G\backslash\{u, v\}$ has a perfect matching for every $u, v \in V$. Edmonds et al. [22] proved that a bicritical graph is a matching covered graph whose tight cut decomposition can be accomplished by a sequence of tight cut contractions stepping exclusively in 2-separation cuts. Then, Theorems 16 and 17 and Corollary 18 give the following.

Corollary 19. The perfect matching polytope of a fully odd subdivision of a bicritical graph is box-TDI if and only if all the bricks of the graph are odd intercyclic.

In [16], the authors proved that the only solid planar bricks are odd wheels. Thus, the following holds.

Corollary 20. The perfect matching polytope of a fully odd subdivision of a bicritical planar graph is box-TDI if and only if all the bricks of the graph are odd wheels.

4.3 A necessary unsufficient condition

Theorem 16 immediately gives the following.

Corollary 21. Let G be a matching covered graph and $\mathcal{F} \in \mathfrak{F}_G$. If $P(G)$ is box-TDI, then the near-bricks obtained by any sequence contracting all the cuts of $\mathcal{F}$ are odd intercyclic.

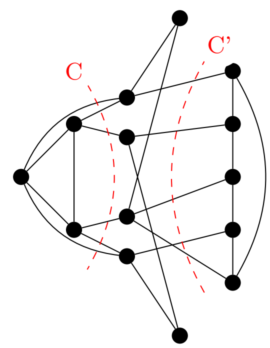


Fig. 4: The Moonfish graph $\mathcal{M}$.

The converse of Corollary 21 does not hold. Figure 4 provides the Moonfish graph $\mathcal{M}$, which is the smallest graph with two bricks illustrating this. Specifically, $\mathfrak{F}_{\mathcal{M}} = \{\{C\}, \{C'\}\}$, and all C and C'-contractions are odd intercyclic. However, by Theorems 3 and 10, $P(\mathcal{M})$ is not box-TDI, as $\{C\} \in \mathfrak{F}_{\mathcal{M}}$ and $M_{\mathcal{M}}^{\{C\}}$ is not equimodular.

5 Integer decomposition property

Lovász [33] proved that a vector u belongs to the perfect matching lattice of a matching covered graph G if and only if all restrictions of u to the bricks of G belongs to their perfect matching lattices. We observe that a similar result holds regarding the integer decomposition property. By projection, it is immediate that contracting a tight cut preserves the integer decomposition property of the perfect matching polytope. This is turned into an equivalence as follows.

Lemma 22. Let G be a matching covered graph and $\delta(U)$ a tight cut of G. Then, $P(G)$ has the integer decomposition property if and only if both $P(G/U)$ and $P(G/\overline{U})$ have it.

Applied to a tight cut decomposition, Lemma 22 give the the following.

Corollary 23. The perfect matching polytope of a matching covered graph has the integer decomposition property if and only if the perfect matching polytope of each of its bricks has this property.

In [32, Section 2.1], the authors mention that they are not aware whether $0,1$ polytopes with the integer decomposition property are all box-TDI. This is not the case, and actually it is the converse that holds for perfect matchings. Indeed, Theorem 4 and statement 4 of Theorem 12 yield that box-TDI perfect matching polytopes have the integer decomposition property. However, $P(\overline{C}_6)$ has this property but is not box-TDI by Theorem 16. We provide a more general class of graphs whose perfect matching polytope has the integer decomposition property. The following is a consequence of Theorems 4 and 16 and Lemma 22.

Corollary 24. If all the bricks of a matching covered graph are odd intercylic, then its perfect matching polytope has the integer decomposition property.

In particular, the perfect matching polytope of the Moonfish graph in Figure 4 has the integer decomposition property.

Holyer [30] showed that deciding whether a bridgeless cubic graph G is 3-edge-colorable is NP-complete, which is equivalent to deciding whether $\mathbf{1}$ belongs to the integer cone of the perfect matchings of G. Thus, Corollary 24 implies the following.

Corollary 25. Let G be a d-edge-connected d-regular graph. If the bricks of G are odd intercyclic, then G is d-edge-colorable.

Further questions

This paper explores integer properties of perfect matching polytopes and raises several further questions suggested by our findings.

Concerning the box-TDIness of perfect matching polytopes, the impact of barrier cuts remains to be investigated, as Edmonds et al. [22] essentially proves that every matching covered graph admits a tight cut decomposition consisting only of 2-separation cuts and barrier cuts. *Can one characterize which barrier cut contractions preserve the box-TDIness of perfect matching polytopes?*

Additionnaly, statement 4 of Theorem 12 prompts the following question, suggesting a potential min-max theorem: *Can an explicit totally unimodular matrix be found to describe box-TDI perfect matching polytopes?*

Lastly, our findings provide a new characterization of solid graphs, though it remains unknown whether an efficient recognition algorithm or graphic characterization exists for nonplanar solid graphs [6]. *Could this new polyhedral characterization help?*

References

1. Barbato, M., Grappe, R., Lacroix, M., Lancini, E.: Box-total dual integrality and edge-connectivity. Mathematical Programming **197**, 307–336 (2023)
2. Barbato, M., Grappe, R., Lacroix, M., Lancini, E., Wolfler Calvo, R.: The Schrijver system of the flow cone in series–parallel graphs. Discrete Applied Mathematics **308**, 162–167 (2022)

3. Baum, S., Trotter, L.E.: Integer rounding and polyhedral decomposition for totally unimodular systems. In: Henn, R., Korte, B., Oettli, W. (eds.) Optimization and Operations Research. pp. 15–23. Springer Berlin Heidelberg, Berlin, Heidelberg (1978)
4. Baum, S., Trotter, L.E.: Integer rounding for polymatroid and branching optimization problems. Siam Journal on Algebraic and Discrete Methods **2**, 416–425 (1981)
5. Benchetrit, Y.: Integer round-up property for the chromatic number of some h-perfect graphs. Mathematical Programming **164**(1), 245–262 (Jul 2017)
6. de Carvalho, M.H., Lucchesi, C.L., Murty, U.S.R.: On tight cuts in matching covered graphs. The Journal of Combinatorics **9**, 163–184 (2018)
7. Chen, X., Ding, G., Zang, W.: A characterization of box-mengerian matroid ports. Math. Oper. Res. **33**, 497–512 (2008)
8. Chen, X., Ding, G., Zang, W.: The box-TDI system associated with 2-edge connected spanning subgraphs. Discrete Applied Mathematics **157**(1), 118–125 (2009)
9. Chervet, P., Grappe, R.: A weak box-perfect graph theorem. Journal of Combinatorial Theory, Series B **169**, 367–372 (2024)
10. Chervet, P., Grappe, R., Lacroix, M., Pisanu, F., Wolfler Calvo, R.: Hard problems on box-totally dual integral polyhedra. Discrete Optimization **50** (2023)
11. Chervet, P., Grappe, R., Robert, L.: Box-total dual integrality, box-integrality, and equimodular matrices. Mathematical Programming **188**(1), 319–349 (2021)
12. Cook, W.J.: On box totally dual integral polyhedra. Mathematical Programmming **34**(1), 48–61 (1986)
13. Cornaz, D., Grappe, R., Lacroix, M.: Trader multiflow and box-TDI systems in series-parallel graphs. Discrete Optimization **31**(1) (2019)
14. Cunningham, W.H., Marsh, A.B.: A primal algorithm for optimum matching, pp. 50–72. Springer Berlin Heidelberg, Berlin, Heidelberg (1978)
15. Cunningham, W.H., Green-Krotki, J.: Dominants and submissives of matching polyhedra. Mathematical Programming **36**, 228–237 (1986)
16. de Carvalho, M.H., Lucchesi, C.L., Murty, U.: On a Conjecture of Lovász Concerning Bricks: I. The Characteristic of a Matching Covered Graph. Journal of Combinatorial Theory, Series B **85**(1), 94–136 (2002)
17. de Carvalho, M.H., Lucchesi, C.L., Murty, U.: The perfect matching polytope and solid bricks. Journal of Combinatorial Theory, Series B **92**(2), 319–324 (2004)
18. de Carvalho, M.H., Lucchesi, C.L., Murty, U.: How to build a brick. Discrete Mathematics **306**(19), 2383–2410 (2006)
19. Ding, G., Tan, L., Zang, W.: When is the matching polytope box-totally dual integral? Mathematics of Operations Research **43**(1), 64–99 (2018)
20. Ding, G., Zang, W., Zhao, Q.: On box-perfect graphs. Journal of Combinatorial Theory, Series B **128**, 17–46 (2018)
21. Edmonds, J.: Maximum matching and a polyhedron with 0,1-vertices. Journal of Research of the National Bureau of Standards **69B** (01 1965)
22. Edmonds, J., Pulleyblank, W.R., Lovász, L.: Brick decompositions and the matching rank of graphs. Combinatorica **2**, 247–274 (1982)
23. Edmonds, J., Giles, R.: A min-max relation for submodular functions on graphs. Annals of Discrete Mathematics **1**, 185–204 (1977)
24. Ford, L.R., Fulkerson, D.R.: Maximal flow through a network. Canadian Journal of Mathematics **8**, 399–404 (1956)
25. Gijswijt, D.: Integer decomposition for polyhedra defined by nearly totally unimodular matrices. SIAM Journal on Discrete Mathematics **19**, 798–806 (2005)

26. Giles, F., Pulleyblank, W.: Total dual integrality and integer polyhedra. Linear Algebra and its Applications **25**, 191–196 (1979)
27. Gubeladze, J.: Normal polytopes: between discrete, continuous, and random. Journal of Pure and Applied Algebra **227**(2), 107187 (2023)
28. Heller, I.: On linear systems with integral valued solutions. Pacific Journal of Mathematics **7**, 1351–1364 (1957)
29. Hoffman, A.J., Kruskal, J.B.: Integral boundary points of convex polyhedra. In: 50 Years of Integer Programming 1958-2008: From the Early Years to the State-of-the-Art. pp. 49–76. Springer Berlin Heidelberg, Berlin, Heidelberg (2010)
30. Holyer, I.: The NP-Completeness of edge-coloring. SIAM Journal on Computing **10**, 718–720 (1981)
31. Kőnig, D.: Gráfok és mátrixok. Matematikai és Fizikai Lapok **38**, 116–119 (1931)
32. Kleer, P., Schäfer, G.: Computation and efficiency of potential function minimizers of combinatorial congestion games. Mathematical Programming **190**, 523–560 (2020)
33. Lovász, L., Plummer, M.D.: Matching theory, Annals of Discrete Mathematics, 29, vol. 121. North-Holland Publishing Co., Amsterdam (1986)
34. Lovász, L.: Matching structure and the matching lattice. Journal of Combinatorial Theory, Series B **43**(2), 187–222 (1987)
35. Naddef, D.: Rank of maximum matchings in a graph. Mathematical Programming **22**, 52–70 (1982)
36. Rothvoss, T.: The matching polytope has exponential extension complexity. Journal of the ACM (JACM) **64**, 1 – 19 (2017)
37. Schrijver, A.: Theory of linear and integer programming. In: Wiley-Interscience series in discrete mathematics and optimization (1986)
38. Schrijver, A.: Combinatorial Optimization: Polyhedra and Efficiency, vol. B. Springer Berlin (01 2003)
39. Sebő, A.: Path partitions, cycle covers and integer decomposition. In: Lipshteyn, M., Levit, V.E., McConnell, R.M. (eds.) Graph Theory, Computational Intelligence and Thought: Essays Dedicated to Martin Charles Golumbic on the Occasion of His 60th Birthday, pp. 183–199. Springer Berlin Heidelberg, Berlin, Heidelberg (2009)
40. Torres, J.C.: Normality of k-matching polytopes of bipartite graphs. arXiv:2306.11910 (2023)
41. Tutte, W.T.: The factorization of linear graphs. Journal of The London Mathematical Society-second Series pp. 107–111 (1947)
42. Zambelli, G.: Colorings of k-balanced matrices and integer decomposition property of related polyhedra. Operations Research Letters **35**(3), 353–356 (2007)

Appendix

Proof (of Theorem 11). We prove the result for nonbipartite graphs, as for bipartite graphs it holds thanks to the total unimodularity of the incidence matrix [29]. Let $\mathcal{F} \in \mathfrak{F}_G$. By Theorem 10, it is sufficient to prove that $P(G) = \{x\colon M_G^{\mathcal{F}} x = \mathbf{1}, x \geqslant \mathbf{0}\}$ if and only if G is solid.

($\Rightarrow$) Suppose that $G = (V, E)$ is not solid. Then, there exists a separating cut C which is not tight, and hence a perfect matching M of G such that $|M \cap C| > 1$. In [16, Lemma 2.19], de Carvalho et al. proved that a cut is separating if and only if for each edge, there exists a perfect matching that contains this edge and exactly one edge in the cut. Hence, since C is separating, for every edge $e \in M$, there exists a perfect matching M_e of G including e for which $|M_e \cap C| = 1$. Then, let $p = \frac{1}{|M|-1}((\sum_{e \in M} \chi^{M_e}) - \chi^M)$. By construction, $p \in \{x\colon M_G^{\mathcal{F}} x = \mathbf{1}, x \geqslant \mathbf{0}\}$, since M_e, $e \in E$, and M intersect each tight cut exactly once by definition. But $p^\top \chi^C < 1$, hence $p \notin P(G)$.

($\Leftarrow$) Let $P = \{x\colon M_G^{\mathcal{F}} x = \mathbf{1}, x \geqslant \mathbf{0}\}$, suppose that $P \neq P(G)$, and let us prove that G is not solid. Since $P(G) \subseteq P$ and every integer point of P is the characteristic vector of a perfect matching of G, P has a fractional vertex $p \notin P(G)$. Then, let $\mathcal{C}$ be the family of cuts associated with inequalities (2a) that are not satisfied by p. By Theorem 10, every point of P satisfies to equality the inequalities (2a) associated with tight cuts. Hence, no cut in $\mathcal{C}$ is tight. Let $\mathcal{M}$ denote the family of all perfect matchings of G. Let $\delta(U)$ be a cut in $\mathcal{C}$ such that there exists no cut $C \in \mathcal{C}$ for which $|C \cap M| \leqslant |\delta(U) \cap M|$ for all M in $\mathcal{M}$ and $|C \cap M'| < |\delta(U) \cap M'|$ for some $M' \in \mathcal{M}$.

Let us prove that $\delta(U)$ is separating, that is, G/U and $G/\overline{U}$ are both matching covered. This will contradict the solidity of G. Let u be the contraction of U in G/U and suppose that G/U is not matching covered. Then, by Theorems 5 and 6, there exists a node set S such that $|\mathcal{O}((G/U)\backslash S)| > |S|$ or S is a barrier with adjacent nodes. In both cases, u belongs to S, as otherwise G is not matching covered since $|U|$ is odd. Since $\bigcup_{K \in \mathcal{O}((G/U)\backslash S)} \delta(K)$ is contained in $\delta(S)$, we have

$$\begin{aligned}
\sum_{K \in \mathcal{O}((G/U)\backslash S)} p^\top \chi^{\delta(K)} &\leqslant p^\top \chi^{\delta(S)} \\
&= p^\top \chi^{\delta(u)\backslash E(S)} + \sum_{s \in S\backslash\{u\}} p^\top \chi^{\delta(s)\backslash E(S)} \\
&\leqslant p^\top \chi^{\delta(U)} + \sum_{s \in S\backslash\{u\}} p^\top \chi^{\delta(s)} \\
&< 1 + (|S| - 1),
\end{aligned}$$

where the last inequality holds since p violates the inequality (2a) associated with U but satisfies (2b). Thus, there is an odd component L of $(G/U)\backslash S$ such that $p^\top \chi^{\delta(L)} < 1$. Equalities (2b) imply that L is nontrivial so $L \in \mathcal{C}$. Note that every odd component of $(G/U)\backslash S$ is an odd component of $G\backslash(S\backslash\{u\} \cup U)$. For every perfect matching M of G, we have

$$\begin{aligned}|\delta(U)\cap M| + |\mathcal{O}((G/U)\backslash S)| - 1 &\geqslant |\delta(U)\cap M| + |S| - 1\\ &= |\delta(U)\cap M| + \sum_{s\in S\backslash\{u\}} |\delta(s)\cap M|\\ &\geqslant |\delta(S\backslash\{u\}\cup U)\cap M|\\ &\geqslant |(\bigcup_{K\in\mathcal{O}((G/U)\backslash S)} \delta(K))\cap M|\\ &= \sum_{K\in\mathcal{O}((G/U)\backslash S)} |\delta(K)\cap M|\\ &\geqslant |\delta(L)\cap M| + |\mathcal{O}((G/U)\backslash S)| - 1,\end{aligned}$$

where the last inequality holds because $|\delta(K)\cap M| \geqslant 1$ for all $K \in \mathcal{O}((G/U)\backslash S)$. This implies that $|\delta(L)\cap M| \leqslant |\delta(U)\cap M|$ for every perfect matching M.

Let us show that this is impossible. First, if $|\mathcal{O}((G/U)\backslash S)| > |S|$, then:

$$\begin{aligned}|\mathcal{O}((G/U)\backslash S)| - 1 + |M\cap\delta(U)| &> |S| - 1 + |M\cap\delta(U)|\\ &\geqslant |M\cap\delta(\mathcal{O}((G/U)\backslash S))|\\ &\geqslant |\mathcal{O}((G/U)\backslash S)| - 1 + |M\cap\delta(L)|,\end{aligned}$$

which contradicts the choice of $\delta(U)$ in $\mathcal{C}$.

Otherwise, we have $|\mathcal{O}((G/U)\backslash S)| = |S|$ and, by Theorem 6, there exists an edge $e \in S$, and since G is matching covered, there exists a perfect matching M_e of G including e. Then, we have:

$$\begin{aligned}|\mathcal{O}((G/U)\backslash S)| + |M_e\cap\delta(U)| &= |S| + |M_e\cap\delta(U)|\\ &\geqslant |M_e\cap\delta(S)| + 2 + |M_e\cap\delta(U)|\\ &\geqslant |M_e\cap\delta(\mathcal{O}((G/U)\backslash S))| + 2\\ &\geqslant |\mathcal{O}((G/U)\backslash S)| + |M_e\cap\delta(L)| + 2,\end{aligned}$$

and M_e contradicts the choice of $\delta(U)$ in $\mathcal{C}$.

Therefore, G/U is matching covered. Similarly, $G/\overline{U}$ is matching covered. Hence, $\delta(U)$ is separating and G is not solid. □

Proof (of Theorem 12). (1. ⇒ 2.) Every face of a box-TDI polyhedron is also box-TDI.

(2. ⇔ 3.) Since an affine space has a single face, this follows from Theorems 3 and 10.

(2.&3. ⇒ 4.) By Theorem 10, $\mathrm{aff}(P(G)) = \{x\colon M_G^{\mathcal{F}} x = \mathbf{1}\}$. Since $\mathrm{aff}(P(G))$ is integer and box-TDI, so is $\mathrm{aff}(P(G)) \cap \mathbb{R}^E_{\geqslant 0}$. In particular, the latter is $P(G)$. Let B be a basis of $M_G^{\mathcal{F}}$, $M = B^{-1}M_G^{\mathcal{F}}$, and $b = B^{-1}\mathbf{1}$. Since $M_G^{\mathcal{F}}$ is equimodular, M is totally unimodular by Theorem 1, and $P(G) = \{x\colon Mx = b, x \geqslant \mathbf{0}\}$. Note that b is integer since $\mathbf{1} \in \mathrm{lattice}(M_G^{\mathcal{F}})$.

(4. ⇒ 1.) When M is totally unimodular, $\{x\colon Mx = b, x \geqslant \mathbf{0}\}$ is box-TDI [29]. □

Proof (of Lemma 14). Since $\delta(U)$ is tight, $P(G/U)$ is the orthogonal projection of $P(G)$ onto the coordinates indexed by $\delta(U) \cup E(\bar{U})$. Such projections preserve box-TDIness [37, Page 323], and the result follows.

Proof (of Lemma 15). Let G be a near-brick and B its brick. If $G = B$ there is nothing to prove. Thus, we assume that there exists a nontrivial tight cut of G. Let $\delta(U)$ be a nontrivial tight cut of G such that $U \cap V(B) \neq \varnothing$. Suppose by contradiction that B contains two vertex-disjoint odd circuits C_1 and C_2, and let $\tilde{C}_1$ and $\tilde{C}_2$ be two circuits of G such that $C_1 \subseteq (V(\tilde{C}_1), \tilde{C}_1)/\overline{U}$ and $C_2 \subseteq (V(\tilde{C}_2), \tilde{C}_2)/\overline{U}$. Since G is a near-brick, $\mathfrak{F}_G = \varnothing$. That is, G/U is bipartite, and the lengths of $\tilde{C}_1$ and $\tilde{C}_2$ have the same parity as those of C_1 and C_2. Therefore, both $\tilde{C}_1$ and $\tilde{C}_2$ share some edges with $\delta(U)$, and, hence, $V(C_1) \cap V(C_2) \neq \varnothing$, a contradiction. ◻

Proof (of Theorem 16). Let $G = (V, E)$ be a near-brick.

($\Rightarrow$) We equivalently prove that if G has two vertex-disjoint odd circuits, then $P(G)$ is not box-TDI. Suppose that G contains two vertex-disjoint odd circuits. Since G is a near-brick, $\mathfrak{F}_G = \varnothing$. Thus, A_G is face-defining for $\operatorname{aff}(P(G))$, by Theorem 10. By Theorem 2, A_G is not equimodular. Thus, $P_{\mathrm{PM}}(G)$ is not box-TDI by Theorem 3.

($\Leftarrow$) Suppose that G is odd intercyclic. By Lemma 15, its brick is odd intercyclic, hence, is a solid graph by [16, Lemma 2.29]. Thus, by Theorem 11, $P(G) = \operatorname{aff}(P(G)) \cap \{x \geqslant \mathbf{0}\}$. By Theorem 2 and Theorem 10, A_G is an equimodular face-defining matrix for $\operatorname{aff}(P(G))$. Thus, $P(G) = \{x \colon A_G x = \mathbf{1}\} \cap \{x \colon x \geqslant \mathbf{0}\}$ is box-TDI by Theorem 3 and the definition of box-TDIness. ◻

Observation 26. Let G be a matching covered graph and $\delta(U) \in \mathcal{F}$ for some $\mathcal{F} \in \mathfrak{F}_G$. Then, $\begin{bmatrix} M_{G/U}^{\mathcal{F}_{G/U}} & \mathbf{0} \end{bmatrix}$ is composed of rows of $M_G^{\mathcal{F}}$.

Proof (of Theorem 17). The "only if" part comes from Lemma 14, hence let us prove the "if" part.

Suppose that $G \backslash \{u, w\}$ has precisely two even connected components $G[U]$ and $G[W]$, and let $X = U \cup \{u\}$ form a 2-separation cut $\delta(X)$ such that $P(G/X)$ and $P(G/\overline{X})$ are box-TDI.

Since $\delta(X)$ is tight, there exists $\mathcal{F} \in \mathfrak{F}_G$ such that $\{\delta(X)\} \cup \mathcal{F}$ is laminar. By laminarity of $\mathcal{F}$, the tight cuts of $\mathcal{F}$ have either a shore $Y \subseteq X$ or a shore $Z \supseteq X$. The former type of cuts correspond to rows $M_{G/\overline{X}}^{\mathcal{F}_{G/\overline{X}}}$, the latter don't.

Let e be an edge of G. Up to replacing $U \cup \{u\}$ by $W \cup \{w\}$, we may assume that e belongs to $E(U \cup \{u, w\})$. Let B be a basis of $M_G^{\mathcal{F}}$ not containing e, and denote by μ^e the e-th column of $M_G^{\mathcal{F}}$. Let us prove that there exists a $0, \pm 1$ vector x such that $Bx = \mu^e$. By Theorem 1, this will imply the equimodularity of $M_G^{\mathcal{F}}$, and hence the box-TDIness of $P(G)$ by statement 3 of Theorem 12.

Suppose $uw \in E(G)$ and let K and H be the graphs respectively obtained from G/X and $G/\overline{X}$ by removing edges parallel to uw. Then, e is in $E(H)$, and $P(H)$ is box-TDI as the removal of duplicate edges does not impact box-TDIness.

Let B_H (resp. B_K) be the submatrix of B whose rows are indexed by $\delta(Y)$ for $Y \subseteq X$ (resp. $Y \supseteq X$) and whose columns are indexed by the edges of H (resp. K) corresponding to columns of B. Let ν^e denote the e-th column of $M_{G/\overline{X}}^{\mathcal{F}_{G/\overline{X}}}$. Note that ν^e is the restriction of μ^e to the rows of $M_{G/\overline{X}}^{\mathcal{F}_{G/\overline{X}}}$.

There are two cases.

First, suppose that H is bipartite. Then, $\delta(X) \notin \mathcal{F}$ and no cut of $\mathcal{F}$ has a shore contained in X. In particular, note that $[B_H, \nu^e]$ is a submatrix of A_H, and the rows of B_H are indexed by $\delta(h)$ for all $h \in V(H)$. Since H is 2-connected, any matrix M obtained from B_H by removing a single row is face-defining for $P(H)$. Let M be such a matrix and C a basis of M. Denote by ρ^e the column obtained from ν^e by removing the coordinate associated with the removed row. By the total unimodularity of M, there exists a 0,±1 vector v such that $Cv = \rho^e$. Let y be the 0,±1 vector obtained by completing y with 0 coordinates on the columns of B_H that are not columns of C. We have $My = \rho^e$, and since the row removed from $[B_H, \nu^e]$ is a linear combination of the rows of $[M, \rho^e]$, we also have $B_H y = \nu^e$.

Second, suppose that H is nonbipartite. Let M be the matrix obtained from $M_{G/\overline{X}}^{\mathcal{F}_{G/\overline{X}}}$ by removing the duplicates of column uw. Then, B_H is the submatrix of M whose columns are indexed by those of B contained in M. Hence, a basis C of B_H is also a basis of M. By Theorem 10, M is face-defining for $P(H)$, and is a submatrix of $M_G^{\mathcal{F}}$ by Observation 26. By Theorem 1 and by statement 3 of Theorem 12, since $P(H)$ is box-TDI, there exists a 0,±1 vector v such that $Cv = \nu^e$. Completing v with 0 coordinates on the columns of B_H that are not columns of C yields a 0,±1 vector y such that $B_H y = \nu^e$.

Suppose that $y_{uw} = 0$ or uw is a column of B. Let x be the vector of $\{0,\pm 1\}^{E(G)}$ obtained by completing y with 0 coordinates on the edges $E(G)\backslash E(H)$. Note that nonzero coordinates of x involve only edges of $E(U \cup \{u, w\})$.

Suppose now that $y_{uw} \neq 0$ and uw is not a column of B. Due to the structure of the matrix, one can check that if every basis of B_K contains uw, then there exists a basis C_H not containing uw. Considering this basis instead yields a y such that $y_{uw} = 0$. This case has been treated just above. Then, let C_K be a basis of B_K not containing uw. Since $P(K)$ is box-TDI, there exists a 0,±1 vector z such that $C_K z = \chi^{uw}$. Now, define the vector $x \in \{0,\pm 1\}^{E(G)}$ as follows:

- if $e \in E(H)$ and $e \neq uw$, then let $x_e = y_e$,
- if $e \in E(K)$, then let $x_e = y_{uw} z_e$.

Now, let us prove that $Bx = \mu^e$. Let B_r be a row of B, and let us treat the different possibilities for $B_r x$. First, there are the rows of B associated with $\delta(s)$ for $s \in V$.

- If B_r is associated with $\delta(s)$ for some $s \in U$, then $x(\delta(s)) = y(\delta(s)) = 1$ if $s \in e$ and 0 otherwise. That is, $B_{\delta(s)} x = \mu^e_{\delta(s)}$.
- If B_r is associated with $\delta(s)$ for some $s \in W$, then $x(\delta(s)) = z(\delta(s)) = 0$. That is, $B_{\delta(s)} x = \mu^e_{\delta(s)}$.

- If B_r is associated with $\delta(u)$, we have either $x(\delta(u)) = y(\delta(u, U)) + y_{uw} = \nu^e_{\delta(u)}$, or we have $x(\delta(u)) = y(\delta(u, U)) + y_{uw} z(\delta(u, W)) = y(\delta(u, U)) + y_{uw}\chi^{uw}_u = y(\delta(u)) = \nu^e_{\delta(u)}$.
- Similarly, $x(\delta(w)) = \nu^e_{\delta(w)}$.

Second, there are the rows of B associated with tights cuts of $\mathcal{F}$ which are tights cuts of H or K.

- If B_r corresponds to a tight cut $\delta(Y)$ of H with $u \notin Y$, then $B_r x = y(\delta(Y, \{w\} \cup X \backslash Y)) = \nu^e_{\delta(Y)}$.
- If B_r corresponds to a tight cut $\delta(Y)$ of H with $u \in Y$, then $B_r x = x(\delta(Y, (w \cup U) \backslash Y) \backslash \{uw\}) + x_{uw} + x(\delta(u, W))$. Hence, we have either $B_r x = y(\delta(Y, (w \cup U) \backslash Y) \backslash \{uw\}) + y_{uw} = y(\delta(Y)) = \nu^e_{\delta(Y)}$. Or we have $B_r x = y(\delta(Y, (w \cup U) \backslash Y) \backslash \{uw\}) + y_{uw} z(\delta(u, W)) = y(\delta(Y, (w \cup U) \backslash Y) \backslash \{uw\}) + y_{uw} = y(\delta(Y)) = \nu^e_{\delta(Y)}$.
- The cases where B_r corresponds to a tight cut of K are similar.

Since ν^e is a restriction of μ^e, overall we have $Bx = \mu^e$, which concludes the case in which $uw \in E(G)$.

Suppose $uw \notin E(G)$. Recall that adding or removing edge duplicates maintains box-TDIness. Since $G[U]$ and $G[W]$ are even and connected, both G/X and $G/\overline{X}$ contain the edge uw. Since their perfect matching polytope is box-TDI, so is that of $(G/X) + uw$ and $(G/\overline{X}) + uw$. Those are the two $\delta(X)$-contractions of $G+uw$, in which uw is an edge, hence $P(G+uw)$ is box-TDI as we have shown above. Since $P(G) = P(G + uw) \cap \{x : x_{uw} = 0\}$, $P(G)$ is also box-TDI. □

Proof (of Corollary 18). Let H be the graph obtained by replacing the edge uv of G with the path $u, u_1, \ldots, u_{2k}, v$, with $k \in \mathbb{Z}_{>0}$. Note that $\delta(\{u, u_1, \ldots, u_{2k}\})$ is a 2-separation cut of G, and the $\delta(\{u, u_1, \ldots, u_{2k}\})$-contractions are G and C_{2k+2}. The latter is bipartite, hence its perfect matching polytope is box-TDI. Therefore, by Theorem 17, $P(H)$ is box-TDI if and only if $P(G)$ is. □

Proof (of Corollary 21). By Lemma 14, if $P(G)$ is box-TDI, then so is the perfect matching polytope of any near-brick that arises from a sequence of contractions of tight cuts of G. By Theorem 16, all these near-bricks is odd intercyclic. □

Proof (of Lemma 22). Let $k \in \mathbb{Z}_{\geqslant 0}$ and $x \in kP(G)$. Denote by $x_{|G/U}$ the restriction of x to G/U. Since $\delta(U)$ is tight, note that $x_{|G/U} \in kP(G/U)$ and $x_{|G/\overline{U}} \in kP(G/\overline{U})$. Then, if both $P(G/U)$ and $P(G/\overline{U})$ have the IDP, $x_{|G/U}$ and $x_{|G/\overline{U}}$ are respectively the sum of k perfect matchings of G/U and $G/\overline{U}$. Since the contributions of $x_{|G/U}$ and $x_{|G/\overline{U}}$ to the edges of $\delta(U) = \delta(\overline{U})$ are identical, pairing appropriately these matchings of G/U and $G/\overline{U}$ decomposes x into k perfect matchings of G. □

A graph whose PMP is box-TDI on the contrary to one of its subgraph

We provide an example highlighting that, unlike the elegant characterization of box-TDI matching polytopes given in Theorem 9, it is not possible to characterize box-TDI perfect matching polytopes solely in terms of forbidden subgraphs. This limitation arises because, in the context of perfect matchings, using a subgraph approach requires that if an edge is deleted, all edges that appear exclusively in perfect matchings containing the deleted edge must also be removed.

Let G be the graph illustrated in Figure 5. By Theorem 8 and [16, Lemma 2.29] G is solid, since the bricks obtained with the tight cut decomposition associated with the family

$$\mathcal{H} = \{\delta(\{1,2,3\}, \delta(\{5,6,7\}, \delta(\{8,9,10\}), \delta(\{11,12,13\}), \delta(\{14,15,16\})\}$$

are all K_4. A maximal subfamily of laminar nontrivial tight cuts with the odd-cycle property of $\mathcal{H}$ is

$$\mathcal{F} = \{\delta(\{1,2,3\}, \delta(\{5,6,7\}, \delta(\{8,9,10\}), \delta(\{11,12,13\})\}.$$

The family $F \in \mathfrak{F}_G$ gives the matrix $M_G^{\mathcal{F}}$ which is equimodular[4]. By Theorem 12, $P(G)$ is box-TDI.

The PMP of the brick $\overline{C}_6$ is not box-TDI by Theorem 16. Hence, none is any of its fully odd subdivision by Corollary 18. Since the subgraph in red in Figure 5 is a fully odd subdivision of $\overline{C}_6$, it implies that G contains a subgraph whose PMP is not box-TDI whereas the PMP of G is.

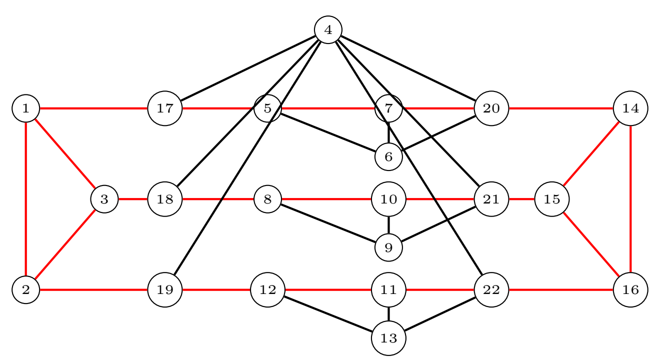


Fig. 5: The smallest matching covered graph containing a fully odd subdivision of $\overline{C}_6$ (in red) as a subgraph and whose perfect matching polytope is box-TDI.

[4] The equimodularity has been checked by enumerating all the nonzero maximal minors and verifying that they equal up to absolute value.