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A Novel Robust Method for Estimating the Covariance Matrix of Financial Returns with Applications to Risk Management*

Arturo Leccadito[†,‡,§] Alessandro Staino[‡] Pietro Toscano[¶]

Abstract

In this paper we introduce the dynamic Gerber model (DGC) and compare its performance in the prediction of VaR and ES compared to alternative parametric, nonparametric and semiparametric methods to estimate the variance-covariance matrix of returns. Based on ES backtests, the DGC method produces, overall, accurate ES forecasts. Furthermore, we use the Model Confidence Set (MCS) procedure to identify the superior set of models (SSM). For all the portfolios and VaR/ES confidence levels we consider, the DGC is found to belong to the SSM.

Keywords: VaR; ES; Gerber statistic; parametric methods; nonparametric methods; semiparametric methods

*The views expressed here are those of the authors alone and do not necessarily reflect those of Wellington Management Company LLP. This article is intended to stimulate further research and is not a recommendation for adopting the proposed method.

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1 Introduction

In a globalized economy, turbulence in financial markets has become more frequent. Recent events like the COVID-19 pandemic, the surge in inflation, and the Russian-Ukrainian conflict have all triggered huge losses for investors worldwide. Providing methods that accurately measure financial market risk is therefore an increasingly crucial task. Over the past two decades, Value at Risk (VaR), a measure related to the quantile of the conditional portfolio return distribution, has become the standard measure of market risk. While VaR is a measure of risk easy to understand also for laymen, it has a number of shortcomings. First of all, it provides no information about returns exceeding the quantile. Secondly, VaR is not a coherent risk measure (see Artzner et al, 1999, for the properties of coherent risk measures). In particular, VaR is not subadditive, implying that the portfolio VaR could be larger than the sum of the VaRs of its components (see for instance Dhaene et al, 2006). Unlike VaR, Expected Shortfall (ES), defined as the conditional expectation of exceedances beyond the VaR, is a coherent risk measure (Acerbi and Tasche, 2002). Several methods to estimate risk measures are available, (see Nieto and Ruiz, 2016, for a survey of estimation methods concerning VaR). A well-known nonparametric method is historical simulation (HS), that estimates VaR and ES using the empirical counterparts under the assumption of i.i.d. portfolio returns. Parametric methods typically involve estimating GARCH-type of models, a few examples include Brooks and Persand (2003), Chu et al (2017), and Long et al (2020). Common alternatives are models based on extreme value theory, recent examples of which are represented by Bekiros et al (2019) and Echaust and Just (2020). A semiparametric approach to VaR/ES estimation combining a parametric model and HS is the filtered historical simulation (FHS) method of Barone-Adesi et al (2002). The FHS method exploits the idea of using the empirical quantile of random draws obtained with replacement from the standardized residuals of the parametric model. VaR is then obtained by rescaling this quantile using the predicted volatility from the parametric model. When the focus is only on forecasting VaR, a popular approach is to directly model the conditional quantile using quantile regression like in the conditional autoregressive VaR model of Engle and Manganelli (2004). Recently developed semiparametric methods that jointly estimate VaR and ES models include Patton et al (2019) and Taylor (2019).

As in Lopez and Walter (2000) and Skintzi and Xanthopoulos-Sisinis (2007), who discuss the importance of covariance matrix forecasting for risk management, in this paper we consider alternative methods to estimate the covariance matrix of returns. Indeed, besides HS, we compare, in terms of accuracy in forecasting VaR or jointly VaR and ES, a number of different methods (parametric and nonparametric) for obtaining the correlation and hence the covariance matrix. In addition to methods employing the standard Pearson correlation, we rely on methods based on a static or dynamic version of the robust correlation proposed by Gerber et al (2022). The measure is an extension of Kendall's Tau robust measure of pairwise movements of two series of returns. In particular, it is built based on the proportions of co-movements in the series of interest, i.e., on how many times the series simultaneous pierce some pre-specified thresholds. We contribute to the literature by introducing a dynamic version of the Gerber correlation matrix which we call the dynamic Gerber model (DGC). Like dynamic conditional correlation (DCC) models, the DGC model relies in the first stage of the estimation on univariate

GARCH models for the volatility of each asset return. In the second stage, based on the marginally standardized residuals, a dynamic Gerber matrix is established. The evolution of a such a matrix depends on a limited number of parameters (6, to be precise), even when the number of asset in the portfolio is large.

We consider three different portfolios consisting of developed equities (S&P 500 index), emerging equities (MXEF index), bonds (LBUSTRUU index) and gold. For different probability levels, we derive, in an out-of-sample exercise, VaR and ES for the three portfolios. We use a recently proposed procedure to backtest ES and the Model Confidence Set (MCS) procedure to identify the superior set of models (SSM). We find that models based on the DGC approach all in all give accurate ES forecasts and VaR/ES predictions more accurate than the competing methods.

The remainder of the paper is organized as follows. Section 2 presents the methods used in the estimations of VaR and ES, introduces the DGC model, and describes the MCS procedure. Section 3 presents the data used in this study and the results of the empirical analysis. Section 4 concludes.

2 Methodology

In this section we first present the static version of the robust measure of correlation introduced by Gerber et al (2022). Next, we discuss DCC models and introduce the novel DGC method. Finally, we explain how the risk measures of interest are derived and how to evaluate their predictions via the MCS procedure.

2.1 The Gerber Statistic

Denote by $r_{i,t}$ and $r_{j,t}$ the returns of asset i and asset j at time t. The Gerber statistic, introduced by Gerber et al (2022), is a robust measure of pairwise movements of the two series of returns defined as

$$g(i,j) = \frac{n_{ij}^c - n_{ij}^d}{n_{ij}^c + n_{ij}^d} \tag{1}$$

where

$$n_{ij}^c = \sum_{t=1}^{T} I\left(r_{i,t} \geq Q_i\right) I\left(r_{j,t} \geq Q_j\right) + \sum_{t=1}^{T} I\left(r_{i,t} \leq -Q_i\right) I\left(r_{j,t} \leq -Q_j\right) = n_{ij}^{UU} + n_{ij}^{DD}$$
$$n_{ij}^d = \sum_{t=1}^{T} I\left(r_{i,t} \geq Q_i\right) I\left(r_{j,t} \leq -Q_j\right) + \sum_{t=1}^{T} I\left(r_{i,t} \leq -Q_i\right) I\left(r_{j,t} \geq Q_j\right) = n_{ij}^{UD} + n_{ij}^{DU}.$$

Here T is the number of observations, Q_i and Q_j are thresholds, and $I(A)$ denotes the indicator function for the event A. Hence n_{ij}^c denotes the number of concordant pairs, i.e. the number of times both returns pierce their thresholds while moving in the same direction. Indeed, n_{ij}^c is equal to the sum of n_{ij}^{UU}, the number of pairs for which both returns are larger than their threshold, and n_{ij}^{DD}, the number of pairs for which both returns are smaller than their threshold times minus one. On the other hand, $n_{ij}^d = n_{ij}^{UD} + n_{ij}^{DU}$ represents the number of discordant pairs in the sample, i.e. the number of times both returns pierce their thresholds while moving in the opposite direction.

When several pairs of returns are involved, using (1) to construct a correlation matrix may lead to variance matrices that are not positive semidefinite. Therefore, Gerber et al (2022) define $n_{ij}^{NN} = \sum_{t=1}^{T} I\left(|r_{i,t}| \leq Q_i\right) I\left(|r_{j,t}| \leq Q_j\right)$ and propose replacing (1) with

$$g(i,j) = \frac{n_{ij}^{c} - n_{ij}^{d}}{T - n_{ij}^{NN}} \tag{2}$$

which instead yields positive semidefinite variance matrices.

Consider now the case of k different assets. Denote by $\boldsymbol{U}$ the $T \times k$ matrix with generic element $u_{t,j} = I\left(r_{j,t} \geq Q_j\right)$, $t = 1, \ldots, T$ and $j = 1, \ldots, k$. Similarly, denote by $\boldsymbol{D}$ the $T \times k$ matrix with generic element $d_{t,j} = I\left(r_{j,t} \leq -Q_j\right)$, $t = 1, \ldots, T$ and $j = 1, \ldots, k$. The $k \times k$ matrix $\boldsymbol{G}$ with element in position (i,j) given by the Gerber correlation (1) is then given by

$$\boldsymbol{G} = \left(\boldsymbol{U}'\boldsymbol{U} + \boldsymbol{D}'\boldsymbol{D} - \boldsymbol{U}'\boldsymbol{D} - \boldsymbol{D}'\boldsymbol{U}\right) \oslash \left(\boldsymbol{U}'\boldsymbol{U} + \boldsymbol{D}'\boldsymbol{D} + \boldsymbol{U}'\boldsymbol{D} + \boldsymbol{D}'\boldsymbol{U}\right) \tag{3}$$

where $\oslash$ means elementwise division.

With the further definition of the $T \times k$ matrix $\boldsymbol{N}$ with generic element $n_{t,j} = I\left(|r_{j,t}| \leq Q_j\right)$, $t = 1, \ldots, T$ and $j = 1, \ldots, k$, we can express the $k \times k$ matrix $\boldsymbol{G}$ with generic element eq. (2) as

$$\boldsymbol{G} = \left(\boldsymbol{\mathcal{C}} - \boldsymbol{\mathcal{D}}\right) \oslash \left(1 - \boldsymbol{\mathcal{N}}\right). \tag{4}$$

where $\boldsymbol{\mathcal{C}} = \frac{\boldsymbol{U}'\boldsymbol{U}}{T} + \frac{\boldsymbol{D}'\boldsymbol{D}}{T}$, $\boldsymbol{\mathcal{D}} = \frac{\boldsymbol{U}'\boldsymbol{D}}{T} + \frac{\boldsymbol{D}'\boldsymbol{U}}{T}$, $\boldsymbol{\mathcal{N}} = \frac{\boldsymbol{N}'\boldsymbol{N}}{T}$.

2.2 DCC Models

Assume that the multivariate time series of returns $\boldsymbol{r}_t = (r_{1,t}, \ldots, r_{k,t})'$ is described by

$$\boldsymbol{r}_t = \boldsymbol{\mu}_t + \boldsymbol{\Sigma}_t^{1/2} \boldsymbol{z}_t \tag{5}$$

where $\{\boldsymbol{z}_t\}_t$ is a sequence of independent and identically distributed random vectors such that $\mathbb{E}(\boldsymbol{z}_t) = 0$ and $\text{cov}(\boldsymbol{z}_t) = \boldsymbol{I}_k$ and $\boldsymbol{\Sigma}_t^{1/2}$ denotes the positive-definite square-root matrix of the conditional variance matrix of the returns, $\boldsymbol{\Sigma}_t$.

In DCC models (Engle, 2002), the $\boldsymbol{\Sigma}_t$ matrix, whose generic element is $\sigma_{ij,t}$, is decomposed as

$$\boldsymbol{\Sigma}_t = \text{diag}(\sigma_{11,t}^{1/2}, \ldots, \sigma_{kk,t}^{1/2})\boldsymbol{R}_t \,\text{diag}(\sigma_{11,t}^{1/2}, \ldots, \sigma_{kk,t}^{1/2}) \tag{6}$$

where $\boldsymbol{R}_t$ is the positive definite conditional correlation matrix. In this way, if $\rho_{ij,t}$ is the element of position (i, j) of the correlation matrix $\boldsymbol{R}_t$, then the corresponding element of $\boldsymbol{\Sigma}_t$ is found to be $\rho_{ij,t}\sqrt{\sigma_{ii,t}\sigma_{jj,t}}$.

Let $\hat{u}_{i,t} = r_{i,t} - \mu_{i,t}$ be the time t residual from the mean equation of asset i. We denote by $\boldsymbol{\eta}_t = (\eta_{1,t}, \ldots, \eta_{k,t})'$ the marginally standardized innovation vector:

$$\eta_{i,t} = \frac{\hat{u}_{i,t}}{\sqrt{\sigma_{ii,t}}} \quad i = 1, \ldots, k.$$

In this way, $\boldsymbol{R}_t$ is the covariance matrix of $\boldsymbol{\eta}_t$. Engle (2002) proposes modelling the correlation matrix as

$$\begin{aligned} \boldsymbol{S}_t =& (1 - a - b)\bar{\boldsymbol{S}} + a\boldsymbol{\eta}_{t-1}\boldsymbol{\eta}_{t-1}' + b\boldsymbol{S}_{t-1} \\ \boldsymbol{R}_t =& \boldsymbol{J}_t\boldsymbol{S}_t\boldsymbol{J}_t, \end{aligned} \tag{7}$$

where $\bar{\boldsymbol{S}}$ is the unconditional covariance matrix of $\boldsymbol{\eta}_t$, and $\boldsymbol{J}_t = \text{diag}(s_{11,t}^{-1/2}, \ldots, s_{kk,t}^{-1/2})$, where $s_{ii,t}$ denotes the element of position (i, i) of $\boldsymbol{S}_t$. In the first stage of the DCC model estimation, k univariate GARCH models are independently estimated for each one of the return series. In the second stage, the marginally standardized innovation vectors are derived and the parameters a and b in (7) are estimated.

2.3 A Dynamic Gerber Model

We propose a dynamic Gerber correlation (DGC) model based on a two-stage estimation like DCC models. In the first stage, the volatility of each asset is independently modelled via a GARCH model. Hence, like in DCC models, in the first stage k univariate GARCH models are independently estimated. In the second stage, based on the the marginally standardized innovation vectors, a dynamics is given to the Gerber correlation matrix, see eq.(4), rather than the Pearson correlation matrix. To be more precise, we identify the thresholds Q_i^η for each of the k time-series of marginally standardized innovation η_i $(i = 1, \ldots, k)$, and consider the following dynamics:

$$
\begin{aligned}
\boldsymbol{\mathcal{C}}_t =& (1 - a_C - b_C)\overline{\boldsymbol{\mathcal{C}}} + a_C \boldsymbol{\mathcal{I}}_{C,t-1} + b_C \boldsymbol{\mathcal{C}}_{t-1} \\
\boldsymbol{\mathcal{D}}_t =& (1 - a_D - b_D)\overline{\boldsymbol{\mathcal{D}}} + a_D \boldsymbol{\mathcal{I}}_{D,t-1} + b_D \boldsymbol{\mathcal{D}}_{t-1} \\
\boldsymbol{\mathcal{N}}_t =& (1 - a_N - b_N)\overline{\boldsymbol{\mathcal{N}}} + a_N \boldsymbol{\mathcal{I}}_{N,t-1} + b_N \boldsymbol{\mathcal{N}}_{t-1} \\
\boldsymbol{G}_t =& (\boldsymbol{\mathcal{C}}_t - \boldsymbol{\mathcal{D}}_t) \oslash (1 - \boldsymbol{\mathcal{N}}_t) \, .
\end{aligned}
\tag{8}
$$

The $\boldsymbol{\mathcal{I}}$ matrices appearing in (8) are obtained as follows:

$$
\begin{aligned}
\boldsymbol{\mathcal{I}}_{C,t} &= \boldsymbol{u}_t \boldsymbol{u}_t' + \boldsymbol{d}_t \boldsymbol{d}_t' \\
\boldsymbol{\mathcal{I}}_{D,t} &= \boldsymbol{u}_t \boldsymbol{d}_t' + \boldsymbol{d}_t \boldsymbol{u}_t' \\
\boldsymbol{\mathcal{I}}_{N,t} &= \boldsymbol{n}_t \boldsymbol{n}_t'
\end{aligned}
$$

where the element of position i of the $k \times 1$ vectors $\boldsymbol{u}_t$, $\boldsymbol{d}_t$ and $\boldsymbol{n}_t$ is $I\left(\eta_{i,t} \geq Q_i^\eta\right)$, $I\left(\eta_{i,t} \leq -Q_i^\eta\right)$, and $I\left(|\eta_{i,t}| \leq Q_i^\eta\right)$, respectively. The matrices $\overline{\boldsymbol{\mathcal{C}}}$, $\overline{\boldsymbol{\mathcal{D}}}$, and $\overline{\boldsymbol{\mathcal{N}}}$, are obtained as the unconditional expectations of $\boldsymbol{\mathcal{I}}_C$, $\boldsymbol{\mathcal{I}}_D$, and $\boldsymbol{\mathcal{I}}_N$, respectively. Contrary to the approach of Algieri et al (2021) that makes the Gerber correlation dynamic by assuming two parameters for each possible pair of assets, the proposed model employs only 6 parameters in total. To estimate the parameters appearing in (8), we assume a multivariate normal distribution and maximize the log-likelihood function

$$-\frac{1}{2}\sum_{t=1}^{T}\left(\log|\boldsymbol{G}_t| + \boldsymbol{\eta}_t'\boldsymbol{G}_t^{-1/2}\boldsymbol{\eta}_t\right), \tag{9}$$

under the constraints $\max\{|b_x|, |a_x|, |a_x + b_x|\} < 1$, $x \in \{C, D, N\}$. These conditions imply the stationarity of the three processes appearing in (8) (see Douc et al, 2013). The variance matrix is obtained as in (6), with the difference that $\boldsymbol{R}_t$ is replaced by $\boldsymbol{G}_t$.

2.4 Risk Measures

In this paper we make predictions for the portfolio VaR and ES using a Normal distribution assuming different forecasts for the variance matrix (and hence for portfolio volatility). Indeed, our main aim is to assess the impact of competing methods for estimating the variance matrix. Therefore we do not take into account alternative distributions for the innovation term possibly allowing for skewness and/or excess kurtosis. The VaR measure is defined implicitly as

$$\mathrm{P}\left(y_{T+1} \leq -\mathrm{VaR}_{T+1|T}(\tau)|\mathcal{F}_T\right) = \tau$$

where y_{T+1} is the portfolio return at time $T+1$ and $\mathcal{F}_T$ is the information available up to time T. Expected shortfall is instead defined as

$$\mathrm{ES}_{T+1|T}(\tau) = -\frac{1}{\tau}\int_{-\infty}^{-\mathrm{VaR}_{T+1|T}(\tau)} y f(y)\mathrm{d}y$$

where $f(y)$ is the predicted density for y_{T+1} conditional on $\mathcal{F}_T$. Given $\boldsymbol{\omega}$, the $k\times 1$ vector of portfolio weights, the forecasts for τ-VaR and τ-ES are given by

$$\mathrm{VaR}_{T+1|T}(\tau) = -\boldsymbol{\omega}'\boldsymbol{\mu}_{T+1|T} - \sqrt{\boldsymbol{\omega}'\boldsymbol{\Sigma}_{T+1|T}\boldsymbol{\omega}}\,\Phi^{-1}(\tau) \tag{10}$$

$$\mathrm{ES}_{T+1|T}(\tau) = -\boldsymbol{\omega}'\boldsymbol{\mu}_{T+1|T} + \sqrt{\boldsymbol{\omega}'\boldsymbol{\Sigma}_{T+1|T}\boldsymbol{\omega}}\,\frac{\phi(\Phi^{-1}(\tau))}{\tau}, \tag{11}$$

where $\boldsymbol{\mu}_{T+1|T}$ and $\boldsymbol{\Sigma}_{T+1|T}$ are the prediction we make based on the information up to time T to the mean vector and variance matrix at time $T+1$, respectively. In particular, both for static and dynamic models, the variance matrix $\boldsymbol{\Sigma}_{T+1|T}$ is derived based either on a Person or on a Gerber correlation matrix.

2.5 Evaluating VaR and ES Predictions

We use the MCS procedure of Hansen et al (2011) to classify the models based on their out-of-sample performance. The procedure is based on an optimality criterion so that the resulting superior set of models M^* will contain the best model with a given confidence level α.

It uses the idea of sequential testing, for which the generic set M^0, containing m_0 competing models, gets reduced in the number of elements by an elimination rule if the Equal Predictive Ability (EPA) null hypothesis is rejected. The procedure is iterated until the EPA hypothesis is not rejected for all the models left in the set, constituting the optimal model confidence set $M^*_{1-\alpha}$.

We use a loss function to compare forecast from different models. In particular, the smaller the value of the loss function for a given model, the more accurate are the predictions from the model. We denote by $l_{i,t}$ the loss associated with model i at time t. To evaluate VaR forecasts, we use the following loss function (see for instance González-Rivera et al, 2004):

$$l_{i,t}\left(y_t, \mathrm{VaR}^i_{t|t-1}(\tau)\right) = \rho_\tau\left(y_t + \mathrm{VaR}^i_{t|t-1}(\tau)\right) \tag{12}$$

where $\mathrm{VaR}^i_{t|t-1}(\tau)$ is the predicted τ-VaR at time t based on model i, y_t is the realized portfolio return and $\rho_\tau(u) = u(\tau - I(u < 0))$.

ES is not elicitable[1] on its own but it is jointly elicitable together with VaR using a suitable scoring function. Hence, we jointly assess VaR and ES forecasts considering the following functional form proposed by Fissler et al (2015):

$$\begin{aligned} l_{i,t}(y_t, \mathrm{VaR}^i_{t|t-1}(\tau), \mathrm{ES}^i_{t|t-1}(\tau)) &= \rho_\tau\left(y_t + \mathrm{VaR}^i_{t|t-1}(\tau)\right) - \tau y_t \\ -\frac{\mathrm{ES}^i_{t|t-1}(\tau)}{1+\exp\left(-\mathrm{ES}^i_{t|t-1}(\tau)\right)} &\times \Big(\mathrm{VaR}^i_{t|t-1}(\tau) - \mathrm{ES}^i_{t|t-1}(\tau) - I(y_t \leq -\mathrm{VaR}^i_{t|t-1}(\tau)) \\ &\times \frac{\mathrm{VaR}^i_{t|t-1}(\tau) + y_t}{\tau}\Big) + \log\left(\frac{2}{1+\exp\left(-\mathrm{ES}^i_{t|t-1}(\tau)\right)}\right), \end{aligned} \tag{13}$$

where $\mathrm{ES}^i_{t|t-1}(\tau)$ is the prediction model i makes for τ-ES at time t. The remaining details of the MCS procedure are given in Section A of the appendix.

[1] A measure is said to be elicitable if there exists at least one scoring function such that the correct forecast of the measure is the unique minimizer of the expectation of the scoring function.

Table 1: Data Description

Description	Ticker
S&P 500 index	SPX
MSCI Emerging Markets index	MXEF
Bloomberg Barclays U.S. Aggregate Bond index	LBUSTRUU
Gold	XAU

Table 2: Summary statistics for the returns of the four indices. The return of index j is calculated as $r_{j,t} = P_{j,t}/P_{j,t-1} - 1$, where $P_{j,t}$ is the index price at the end of week t.

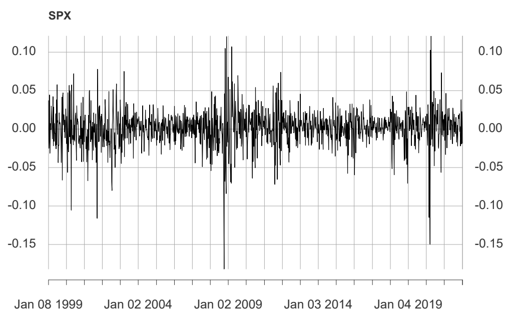
	Min	1st Qu.	Median	Mean	3rd Qu.	Max	Volatility
SPX	-0.1820	-0.0103	0.0014	0.0014	0.0143	0.1210	0.0248
MXEF	-0.2020	-0.0142	0.0016	0.0016	0.0178	0.2037	0.0296
LBUSTRUU	-0.0317	-0.0018	0.0009	0.0009	0.0042	0.0265	0.0051
XAU	-0.3179	-0.0276	0.0018	0.0018	0.0329	0.2359	0.0507

3 Empirical Analysis

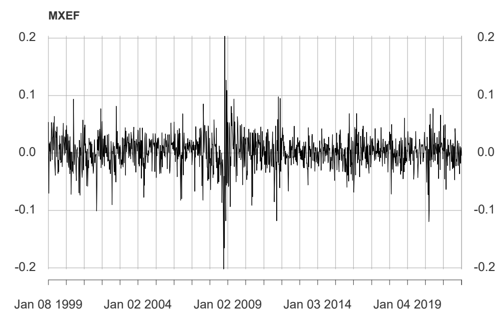
The dataset we use consists of weekly data for the indices described in Table 1. The data spans the period from January 22, 1999 to January 7, 2022 (1,199 observations). The motivation for choosing this four indices is that investors may build highly diversified portfolios by purchasing passive funds mimicking them.

For index j, we compute returns as $r_{j,t} = P_{j,t}/P_{j,t-1} - 1$, where $P_{j,t}$ is the index price at the end of week t. Table 2 presents descriptive statistics for the simple returns of the four indices and Figure 1 plots them. In Figure 2 we report the Pearson and Gerber correlation matrices calculated on the four time-series of returns. In the case of Gerber correlations, the thresholds are set to half the unconditional standard deviations.

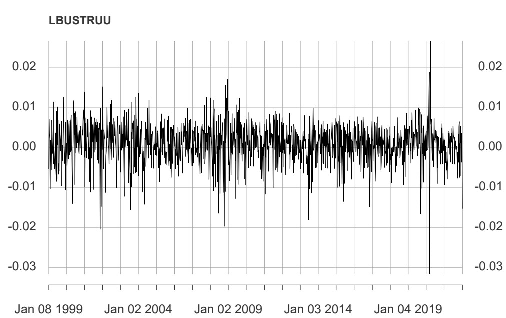
Figure 1: Time-series plots for the returns of the four indices. The return of index j is calculated as $r_{j,t} = P_{j,t}/P_{j,t-1} - 1$, where $P_{j,t}$ is the index price at the end of week t.



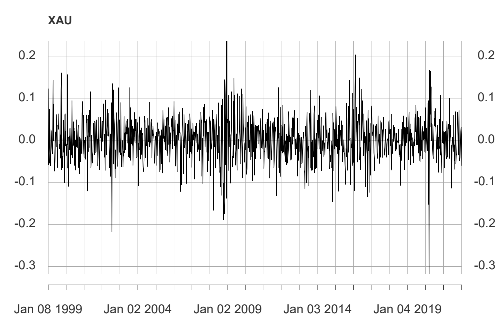
(a) SPX



(b) MXEF

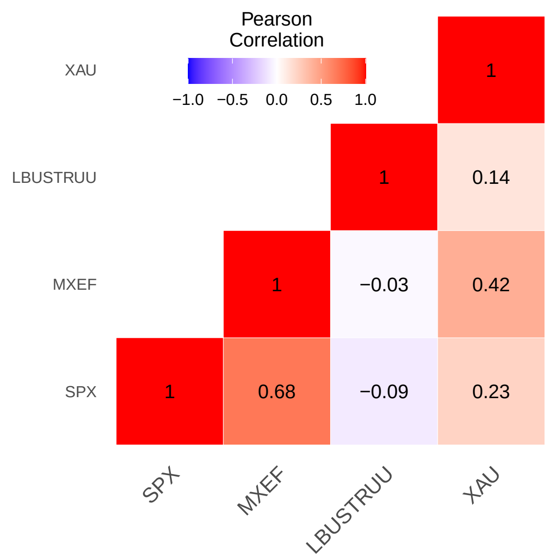


(c) LBUSTRUU

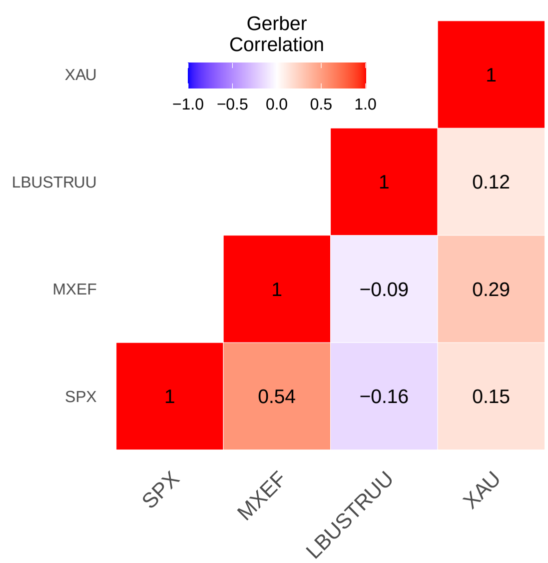


(d) XAU

Figure 2: Correlation plots for the returns of the four indices. The thresholds for the Gerber statistic are $Q_j = 0.5 \times \sigma_j$, $j = 1, \ldots, 4$, where σ_j is the standard deviation of the returns of index j.



(a) Pearson



(b) Gerber

3.1 In Sample Analysis

In this section, we use the whole sample to estimate the DGC model. For the first stage we use an ARMA(1,1)-GARCH(1,1) model for each of the return series. The thresholds for the Gerber statistic are assumed to be half the unconditional volatility. The estimated parameters are reported in Table 3. It is interesting to notice that the persistence, measured by the sum of the a and b parameters, is lower for the equation related to the dynamics of the $\boldsymbol{\mathcal{N}}$ matrix and higher for the dynamics of the $\boldsymbol{\mathcal{C}}$ and $\boldsymbol{\mathcal{D}}$ matrices. This result implies that, for each pair of assets, the probability of two consecutive extreme events is larger than the probability of observing returns for two consecutive weeks in the central part of the distribution. Figure 3 plots the Gerber correlations obtained from the estimated DGC model via the recursions (8) using the estimated parameters of Table 3. Some interesting features can be gleaned from the figure. For instance, it is possible to observe an increase in the correlation between equities and gold (panels c and d) after 2019. Furthermore, the correlation between bonds and gold became negative in the aftermath of the financial crisis of 2008 and became again positive around 2015.

3.2 Out-of-Sample Analysis

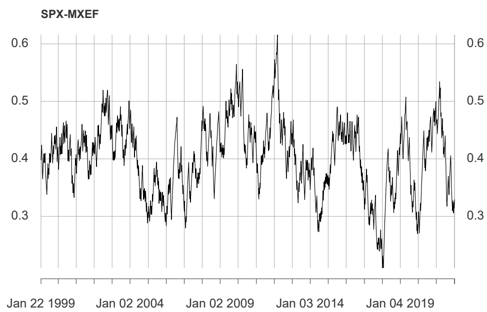
Here the focus is on estimating VaR and ES for the three portfolios of Table 4. The first one is the equally-weighted portfolio. The remaining two portfolios invest both 10% in gold However, they differ in the equity and bond allocation. Indeed, portfolio 2 invests approximately 60% of the allocation (excluding gold) in equities. On the contrary, portfolio 2 invests approximately 60% of the allocation

Table 3: Univariate GARCH Models Estimates (Panel A) and DGC Model Estimates (Panel B). We fit an ARMA(1,1)-GARCH(1,1) model for each of the return series. The thresholds for the Gerber statistic are assumed to be half the unconditional volatility. The table reports the estimated parameters (standard errors in parenthesis). '* * *', '* * ' and '*' denote significance at the 1%, 5% and 10%, respectively.

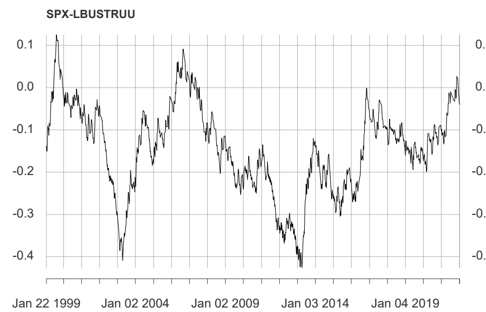
Panel A: Univariate GARCH Models				
	SPX	MXEF	LBUSTRUU	XAU
const. (mean eq.)	0.0025 * * *	0.0021 * *	0.0008 * * *	0.0002
	(0.0005)	(0.0009)	(0.0001)	(0.0010)
ar1	-0.7533 * * *	0.7130 * * *	-0.3248	0.9167 * * *
	(0.1338)	(0.2387)	(0.2619)	(0.0360)
ma1	0.6826 * * *	-0.6639 * * *	0.2237	-0.9387 * * *
	(0.1500)	(0.2553)	(0.2686)	(0.0327)
const. (variance eq.)	0.0000 * * *	0.0001 * * *	0.0000 * * *	0.0002 * * *
	(0.0000)	(0.0000)	(0.0000)	(0.0001)
ARCH	0.2635 * * *	0.1310 * * *	0.1103 * * *	0.0981 * * *
	(0.0355)	(0.0234)	(0.0107)	(0.0210)
GARCH	0.6996 * * *	0.7907 * * *	0.7861 * * *	0.8251 * * *
	(0.0341)	(0.0350)	(0.0168)	(0.0416)

Panel B: DGC Model	
a_C	0.0235 * * *
	(0.0075)
b_C	0.9706 * * *
	(0.0081)
a_D	0.0131 * * *
	(0.0046)
b_D	0.9869 * * *
	(0.0054)
a_N	0.0626 * * *
	(0.0185)
b_N	0.8373 * * *
	(0.0970)

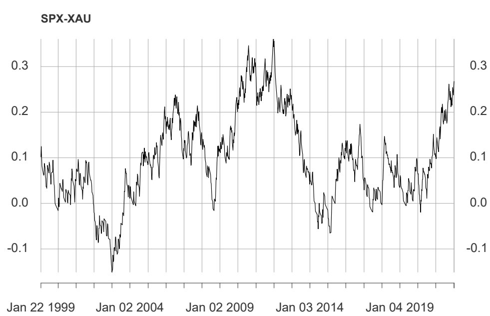
Figure 3: Time-Series Plots of the Gerber correlations based on the estimated GDC model.



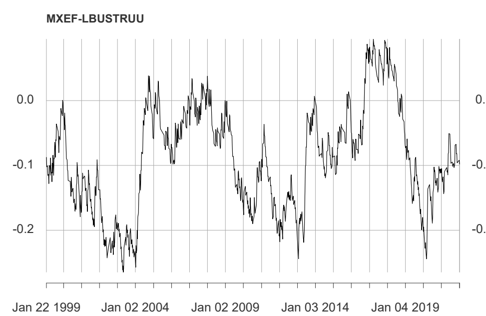
(a) SPX-MXEF



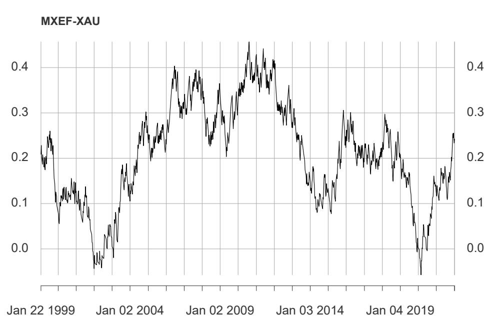
(b) SPX-LBUSTRUU



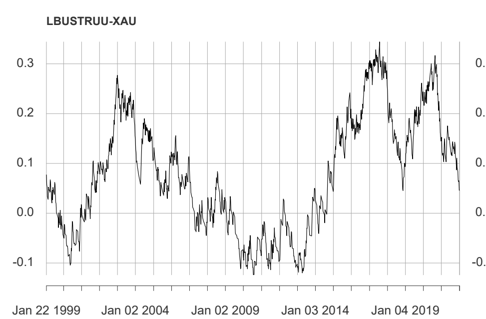
(c) SPX-XAU



(d) MXEF-LBUSTRUU



(e) MXEF-XAU



(f) LBUSTRUU-XAU

(excluding gold) in bonds.

Table 4: Weights of the three portfolios.

Asset	SPX	MXEF	LBUSTRUU	XAU
Portfolio 1	0.25	0.25	0.25	0.25
Portfolio 2	0.475	0.075	0.35	0.1
Portfolio 3	0.3	0.05	0.55	0.1

We assess how well the proposed DGC model predicts VaR and ES relative to a number of alternative methods. We consider non-parametric, parametric and semi-parametric methods. The first benchmark is a non-parametric method, historical simulation (HS). The method considers the last T portfolio returns and estimates τ-VaR as the negative of their τ-quantile. Instead, τ-ES is estimated under HS as minus one times the mean of portfolio returns that are less than the τ-quantile. The next two methods we consider, labelled PearsonHist and GerberHist, are for the 'static' models relying on the Pearson and Gerber correlation matrices, respectively. In particular, they use the last T observations of the k assets returns to estimate the (Person or Gerber) correlation matrix, $\boldsymbol{\Omega}_{T+1|T}$, and estimate the variance matrix as $\boldsymbol{\Sigma}_{T+1|T} = \boldsymbol{\Lambda}_{T+1|T}\boldsymbol{\Omega}_{T+1|T}\boldsymbol{\Lambda}_{T+1|T}$, where $\boldsymbol{\Lambda}_{T+1|T}$ is the matrix with the sample volatilities in the main diagonal and zero elsewhere. In the case of the GerberHist method, eq. (2) is used. VaR and ES are then estimated using (10)–(11) with $\boldsymbol{\mu}_{T+1|T}$ equal to the vector of sample means for the k assets returns. The next alternative method we consider is the DCC model (see Section 2.2). Note that we in the DCC and DGC models we use the same univariate specification, namely the ARMA(1,1)-GARCH(1,1) model for each of the return series. Hence, in both cases the vector $\boldsymbol{\mu}_{T+1|T}$ appearing in (10)–(11) is obtained using the mean prediction from each univariate ARMA(1,1) model.

Also, the predicted variance matrix is obtained as $\boldsymbol{\Sigma}_{T+1|T} = \boldsymbol{\Lambda}_{T+1|T}\boldsymbol{R}_{T+1|T}\boldsymbol{\Lambda}_{T+1|T}$ for the DCC model or as $\boldsymbol{\Sigma}_{T+1|T} = \boldsymbol{\Lambda}_{T+1|T}\boldsymbol{G}_{T+1|T}\boldsymbol{\Lambda}_{T+1|T}$ for the DGC model, where in both cases $\boldsymbol{\Lambda}_{T+1|T}$ is the diagonal matrix consisting of the predicted volatilities from each univariate GARCH(1,1) model. Finally, we consider the semi-parametric method filtered historical simulation (FHS) together with the DCC or DGC model. We implement the method as follows. We first estimate an ARMA(1,1)-GARCH(1,1) model on the last T portfolio returns. Denote by $\{z_t\}_{t=1,\ldots,T}$ the standardized portfolio returns, i.e. $z_t = y_t/\sigma_t$, where y_t is the portfolio return at time t and σ_t the time t volatility from the estimated GARCH(1,1) model. VaR and ES are then estimated by modifying (10)–(11) in the following way: i) $\Phi^{-1}(\tau)$ is replaced by the sample τ-quantile of the z series, say $q_z(\tau)$, and ii) $\frac{\phi(\Phi^{-1}(\tau))}{\tau}$ is replaced by minus one times the mean of the z_ts that are smaller than $q_z(\tau)$.

We move a window of length $T = 500$ to estimate the parameters (for parametric model) or to derive the empirical VaR or ES under the HS method. As a consequence, the first prediction we make is for August 22, 2008. We consider three different values of τ, 1%, 5%, and 10% when predicting VaR and ES.

3.3 Results

In this section, we first give the results of the ES-backtesting procedure[2] based on the results of Khalaf et al (2021), see Section B in the appendix. We opt for this recently proposed procedure because it allows to backtest ES only based on

[2]We do not report the results of VaR backtesting procedures (see for instance Christoffersen, 1998) since the statistical test we consider is based on cumulative violations associated with a sequence of quantiles in the left tail. Indeed some of tests of Khalaf et al (2021) are obtained as combinations of the p-values from VaR backtesting procedures.

a sequence of violations for an appropriately chosen sequence of VaRs. Therefore, contrary to the Du and Escanciano (2017) tests, it does not require knowing the entire conditional cumulative density function, which is difficult to derive for instance in the case of FHS methods. We consider conditional tests based on lags corresponding to one week, one month and two months, i.e. $m = 1$, $m = 5$, and $m = 10$. The results of the tests are reported, for all the alternative models we consider, in Table 5. Overall, models based on the dynamic Gerber correlation seem to have the best performance. Indeed, they perform well for portfolio 3 (Panel C) for all the three values of τ. For the first two portfolios (Panel A and B), instead, the null of 'accurate' ES predictions, see eq. (24), is rejected for $\tau = 1\%$, but not for the two remaining values of τ. We believe that the explanation of the result has to due with the distributional choice for the risk measures of Section 2.4. Like all the alternative models based on the normal distribution, the DGC model fails to accurately capture the tail expectation for small values of τ in the case of portfolios consisting mainly of highly volatile assets such as equities (like portfolio 1 and even more so portfolio 2).

Table 5: p-values (×100) for the backtesting procedure of Khalaf et al (2021). For each portfolio, for $m \in \{1, 5, 10\}$, and for $\tau \in \{1, 5, 10\}/100$, the table reports the p- values (multiplied by 100) for the test statistic $C_m(\tau)$, eq. (26). K is set to 100 and the p-values are obtained with the Monte Carlo method based on 50,000 simulations.

Panel A: Portfolio 1

	$\tau = 1\%$			$\tau = 5\%$			$\tau = 10\%$		
	$m = 1$	$m = 5$	$m = 10$	$m = 1$	$m = 5$	$m = 10$	$m = 1$	$m = 5$	$m = 10$
HS	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0006	0.0000	0.0000
PearsonHist	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0019	0.0000	0.0000
GerberHist	0.0000	0.0000	0.0000	0.0002	0.0000	0.0000	0.0059	0.0017	0.0001
GDC	14.7252	14.8256	1.2105	19.1657	18.6871	53.8092	19.2945	33.8882	63.6006
GDC FHS	11.5768	12.7558	2.6713	11.3130	12.0938	39.0368	19.4361	38.4679	75.5254
DCC	0.0913	0.0065	0.1168	13.8104	16.6517	52.9225	30.7109	35.0321	76.6289
DCC FHS	0.0007	0.0000	0.0000	3.1732	3.1596	15.7124	33.7484	35.1365	76.9704

Panel B: Portfolio 2

	$\tau = 1\%$			$\tau = 5\%$			$\tau = 10\%$		
	$m = 1$	$m = 5$	$m = 10$	$m = 1$	$m = 5$	$m = 10$	$m = 1$	$m = 5$	$m = 10$
HS	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
PearsonHist	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
GerberHist	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
GDC	0.0022	0.0069	0.1561	10.6607	12.5188	9.3389	10.5719	6.1867	13.1628
GDC FHS	1.2000	0.8010	2.0414	0.0383	0.4744	5.6654	5.2274	5.9787	20.0387
DCC	0.0489	0.0107	0.2566	0.5476	3.9250	18.0446	14.2625	11.0817	22.8662
DCC FHS	0.0000	0.0000	0.0000	0.0237	0.4184	3.9377	5.6148	7.2722	22.4237

Panel C: Portfolio 3

	$\tau = 1\%$			$\tau = 5\%$			$\tau = 10\%$		
	$m = 1$	$m = 5$	$m = 10$	$m = 1$	$m = 5$	$m = 10$	$m = 1$	$m = 5$	$m = 10$
HS	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.0000	0.0000
PearsonHist	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
GerberHist	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.0000	0.0000
GDC	24.6550	15.8000	18.5732	0.2709	6.4144	20.8786	26.8373	30.7581	63.6741
GDC FHS	17.8261	6.6865	17.5698	15.8550	22.8299	17.5160	2.4057	25.8272	66.3334
DCC	17.8016	14.3856	6.8976	0.2217	5.4016	24.5266	4.6571	41.9779	59.8495
DCC FHS	0.0000	0.0000	0.0000	0.0248	0.6288	7.3753	2.0414	27.7443	55.9721

Next, we use the MCS procedure using 5,000 bootstrap replications to derive the p-values associated with the test statistics involved. We report the SSMs, for the three portfolios we consider, in Tables 6–8 for the case of VaR predictions and in Tables 9–11 for the case of joint VaR/ES predictions. Each table reports, for each model i belonging to the identified SSM, the statistics $\max_j T_{ij}$ and $T_{i\cdot}$, p-values associated with the test statistics T_R and T_{max}, the ranking of the models in the SSM based on T_R and T_{max}, and the average of the loss function (12) when we forecast VaR or of (13) when we jointly forecast VaR and ES.

In the tables involving VaR predictions alone, there are just a few cases where the SSM does not include all the considered models. When that happens, just one or two models are excluded. For portfolio 1, we have that all the models belong to the SSM when the VaR confidence level, τ, is equal to 1% and 5% and that only model PearsonHist gets excluded when $\tau = 10\%$. For portfolio 2, all the models belong to the SSM when $\tau = 1\%$, only the HS model gets excluded when $\tau = 5\%$, and only the PearsonHist and GerberHist models get excluded when $\tau = 10\%$. Finally, for portfolio 3, all the models belong to the SSM when $\tau = 1\%$. The HS and PearsonHist models are respectively excluded when $\tau = 5\%$ and $\tau = 10\%$. Hence, the results about VaR predictions do not evidence much difference between static and dynamic models about the inclusion in the SSM, even though the few excluded ones are static. It is worth highlighting that the newly proposed models, DGC and DGC FHS, can produce VaR predictions that allow them to belong to the SSM. From the second and fifth columns in each panel of Tables 6–8 we see that dynamic models usually highly rank among the considered models. Based on the statistic $\max_j T_{ij}$, the models DCC, DCC FHS, DGC, and DGC FHS are always in the first four position. Based instead on the statistics $T_{i\cdot}$, there are more cases where some of the static models are ranked in the first four positions, but, in any case, dynamic models are always in the first two positions. However, at least one of the newly proposed models, DGC and DGC FHS, usually highly rank with the statistic $T_{i\cdot}$. To justify these results, we observe that the out-sample analysis uses a period, 22 August 2008 to 7 January 2022, characterized by moments of turmoil in the markets. Indeed, the period includes the final part of the global financial crisis in 2007-2008, the European sovereign debt crisis from 2009 until the mid to late 2010s, the Black Monday 2011, and the stock market crash in 2020 caused by the Covid-19 pandemic. Dynamic models highly rank in Tables 6–8 because they describe financial time series in the presence of turmoil better than static models. However, the rolling-window approach applied to make VaR predictions allows static models to stay in the SSM.

As far as the joint prediction of VaR and ES is concerned, from Tables 9–11 we see that all the SSM consist only of the newly proposed DGC model. The results are consistent across portfolios and confidence levels of the two risk measures we forecast. This means that, for the three portfolios consisting of the four assets considered in our study and for the period of investigation, the DGC model produces joint VaR/ES predictions that are more accurate than the competing models.

Table 6: Superior Set of Models, τ-VaR forecasting, Portfolio 1. For each model i in the SSM, the table reports the statistics $\max_j T_{ij}$ and $T_{i\cdot}$, the p-values associated with the test statistics T_R and T_{max}, the ranking of the models in the SSM based on T_R and T_{max}, and the average of the loss function (12) multiplied by 10^3.

Panel A: $\tau = 1\%$

Model i	Rank_{T_R}	$\max_j T_{ij}$	$p\text{-value}_{T_R}$	$\text{Rank}_{T_{max}}$	$T_{i\cdot}$	$p\text{-value}_{T_{max}}$	Loss$\times 10^3$
HS	7	1.3349	0.3726	6	1.5576	0.4966	1.1390
PearsonHist	6	1.2160	0.4414	5	1.3653	0.6130	1.1923
GerberHist	5	1.1382	0.4964	4	1.3076	0.6576	1.1972
DGC	4	0.3986	0.9474	7	2.6092	0.0284	1.1123
DGC FHS	1	-1.5257	1.0000	2	0.7316	1.0000	0.9992
DCC	3	-1.3588	1.0000	3	0.8890	1.0000	1.0330
DCC FHS	2	-1.3604	1.0000	1	-0.7316	1.0000	0.9772
Eliminated	-						

Panel B: $\tau = 5\%$

Model i	Rank_{T_R}	$\max_j T_{ij}$	$p\text{-value}_{T_R}$	$\text{Rank}_{T_{max}}$	$T_{i\cdot}$	$p\text{-value}_{T_{max}}$	Loss$\times 10^3$
HS	7	1.8419	0.0920	7	2.8842	0.0196	2.9443
PearsonHist	6	1.3249	0.3100	5	1.5595	0.5412	2.8978
GerberHist	5	1.1747	0.4086	3	1.4672	0.6122	2.8898
DGC	4	-1.0883	1.0000	4	1.4673	0.6118	2.7427
DGC FHS	2	-1.3231	1.0000	6	2.1562	0.5412	2.7370
DCC	1	-1.8161	1.0000	1	-1.0075	1.0000	2.6957
DCC FHS	3	-1.3055	1.0000	2	1.0075	1.0000	2.7192
Eliminated	-						

Panel C: $\tau = 10\%$

Model i	Rank_{T_R}	$\max_j T_{ij}$	$p\text{-value}_{T_R}$	$\text{Rank}_{T_{max}}$	$T_{i\cdot}$	$p\text{-value}_{T_{max}}$	Loss$\times 10^3$
HS	5	1.0792	0.4192	5	1.2831	0.6468	4.2724
GerberHist	6	1.8153	0.1116	6	2.2999	0.1132	4.3349
DGC	3	-1.3472	1.0000	3	0.7642	0.9976	4.1233
DGC FHS	4	-0.9968	1.0000	4	1.0000	0.9976	4.1361
DCC	2	-1.4086	1.0000	2	0.6361	1.0000	4.1108
DCC FHS	1	-1.6112	1.0000	1	-0.6361	1.0000	4.1017
Eliminated	PearsonHist						

Table 7: Superior Set of Models, τ-VaR forecasting, Portfolio 2. For each model i in the SSM, the table reports the statistics $\max_j T_{ij}$ and $T_{i\cdot}$, the p-values associated with the test statistics T_R and T_{max}, the ranking of the models in the SSM based on T_R and T_{max}, and the average of the loss function (12) multiplied by 10^3.

Panel A: $\tau = 1\%$

Model i	Rank$_{T_R}$	$\max_j T_{ij}$	p-value$_{T_R}$	Rank$_{T_{max}}$	$T_{i\cdot}$	p-value$_{T_{max}}$	Loss$\times 10^3$
HS	5	1.2236	0.3650	4	1.4753	0.4878	0.8991
PearsonHist	6	1.4226	0.2542	5	1.5087	0.4642	0.9449
GerberHist	7	1.4517	0.2418	6	1.5204	0.4538	0.9560
DGC	4	-0.3998	1.0000	7	2.2368	0.0846	0.8092
DGC FHS	1	-1.5921	1.0000	2	0.7357	1.0000	0.7165
DCC	3	-1.3414	1.0000	3	0.9579	1.0000	0.7528
DCC FHS	2	-1.4765	1.0000	1	-0.7357	1.0000	0.7039
Eliminated	-						

Panel B: $\tau = 5\%$

Model i	Rank$_{T_R}$	$\max_j T_{ij}$	p-value$_{T_R}$	Rank$_{T_{max}}$	$T_{i\cdot}$	p-value$_{T_{max}}$	Loss$\times 10^3$
PearsonHist	6	1.9959	0.0622	6	2.0739	0.1582	2.2252
GerberHist	5	1.8687	0.0802	5	1.9789	0.1948	2.2206
DGC	4	-1.1881	1.0000	4	1.8644	0.2430	2.0477
DGC FHS	3	-1.8190	1.0000	3	1.4437	0.8206	2.0252
DCC	1	-2.1129	1.0000	2	0.9656	0.8206	2.0089
DCC FHS	2	-2.0828	1.0000	1	-0.9656	1.0000	1.9939
Eliminated	HS						

Panel C: $\tau = 10\%$

Model i	Rank$_{T_R}$	$\max_j T_{ij}$	p-value$_{T_R}$	Rank$_{T_{max}}$	$T_{i\cdot}$	p-value$_{T_{max}}$	Loss$\times 10^3$
HS	5	1.8572	0.1096	3	1.9926	0.1590	3.2746
DGC	2	-1.7727	1.0000	2	0.4789	0.0031	3.0821
DGC FHS	4	-0.9693	1.0000	5	2.7562	0.0216 1	3.1030
DCC	1	-2.2889	1.0000	1	-0.4789	1.0000	3.0736
DCC FHS	3	-1.6894	1.0000	4	2.3547	0.0720	3.0934
Eliminated	PearsonHist,	GerberHist					

Table 8: Superior Set of Models, τ-VaR forecasting, Portfolio 3. For each model i in the SSM, the table reports the statistics $\max_j T_{ij}$ and $T_{i\cdot}$, the p-values associated with the test statistics T_R and T_{max}, the ranking of the models in the SSM based on T_R and T_{max}, and the average of the loss function (12) multiplied by 10^3.

Panel A: $\tau = 1\%$

Model i	Rank_{T_R}	$\max_j T_{ij}$	$p\text{-value}_{T_R}$	$\text{Rank}_{T_{max}}$	$T_{i\cdot}$	$p\text{-value}_{T_{max}}$	Loss$\times 10^3$
HS	5	1.0109	0.4916	5	1.2654	0.6282	0.6545
PearsonHist	7	1.1660	0.3986	6	1.2804	0.6144	0.6843
GerberHist	6	1.1384	0.4164	4	1.2576	0.7164	0.6873
DGC	4	-0.1536	1.0000	7	2.0570	0.1252	0.6124
DGC FHS	1	-1.3939	1.0000	2	0.3897	1.0000	0.5520
DCC	3	-0.9728	1.0000	3	1.1254	1.0000	0.5826
DCC FHS	2	-1.2656	1.0000	1	-0.3897	1.0000	0.5477
Eliminated	-						

Panel B: $\tau = 5\%$

Model i	Rank_{T_R}	$\max_j T_{ij}$	$p\text{-value}_{T_R}$	$\text{Rank}_{T_{max}}$	$T_{i\cdot}$	$p\text{-value}_{T_{max}}$	Loss$\times 10^3$
PearsonHist	6	1.9819	0.0648	6	1.9985	0.1868	1.6054
GerberHist	5	1.7295	0.1166	5	1.8254	0.2662	1.5969
DGC	4	-1.5486	1.0000	4	1.0687	0.7460	1.4846
DGC FHS	1	-2.0015	1.0000	3	0.5567	1.0000	1.4758
DCC	2	-1.8573	1.0000	1	-0.0662	1.0000	1.4707
DCC FHS	3	-1.6559	1.0000	2	0.0662	1.0000	1.4712
Eliminated	HS						

Panel C: $\tau = 10\%$

Model i	Rank_{T_R}	$\max_j T_{ij}$	$p\text{-value}_{T_R}$	$\text{Rank}_{T_{max}}$	$T_{i\cdot}$	$p\text{-value}_{T_{max}}$	Loss$\times 10^3$
HS	5	1.1866	0.3982	3	1.4942	0.5106	2.3411
GerberHist	6	1.6942	0.1584	5	1.8743	0.2694	2.3593
DGC	2	-1.7236	1.0000	2	0.2576	0.9994	2.2494
DGC FHS	4	-0.6579	1.0000	6	2.1185	0.1640	2.2722
DCC	1	-2.0043	1.0000	1	-0.2576	1.0000	2.2458
DCC FHS	3	-1.0946	1.0000	4	1.8724	0.2700	2.2665
Eliminated	PearsonHist						

Table 9: Superior Set of Models, joint τ-VaR and τ-ES forecasting, Portfolio 1. For each model i in the SSM, the table reports the statistics $\max_j T_{ij}$ and $T_{i\cdot}$, the p-values associated with the test statistics T_R and T_{max}, the ranking of the models in the SSM based on T_R and T_{max}, and the average of the loss function (13) multiplied by 10^3.

Panel A: $\tau = 1\%$

Model i	Rank$_{T_R}$	$\max_j T_{ij}$	p-value$_{T_R}$	Rank$_{T_{max}}$	$T_{i\cdot}$	p-value$_{T_{max}}$	Loss$\times 10^3$
DGC	1	-29.1663	1.0000	1	-29.1663	1.0000	22.8115
Eliminated	HS, DGC FHS,	PearsonHist, DCC,	GerberHist, DCC FHS				

Panel B: $\tau = 5\%$

Model i	Rank$_{T_R}$	$\max_j T_{ij}$	p-value$_{T_R}$	Rank$_{T_{max}}$	$T_{i\cdot}$	p-value$_{T_{max}}$	Loss$\times 10^3$
DGC	1	-28.6036	1.0000	1	-28.6036	1.0000	20.2943
Eliminated	HS, DGC FHS,	PearsonHist, DCC,	GerberHist, DCC FHS				

Panel C: $\tau = 10\%$

Model i	Rank$_{T_R}$	$\max_j T_{ij}$	p-value$_{T_R}$	Rank$_{T_{max}}$	$T_{i\cdot}$	p-value$_{T_{max}}$	Loss$\times 10^3$
DGC	1	-28.4352	1.0000	1	-28.4352	1.0000	19.0495
Eliminated	HS, DGC FHS,	PearsonHist, DCC,	GerberHist, DCC FHS				

Table 10: Superior Set of Models, joint τ-VaR and τ-ES forecasting, Portfolio 2. For each model i in the SSM, the table reports the statistics $\max_j T_{ij}$ and $T_{i\cdot}$, the p-values associated with the test statistics T_R and T_{max}, the ranking of the models in the SSM based on T_R and T_{max}, and the average of the loss function (13) multiplied by 10^3.

Panel A: $\tau = 1\%$

Model i	Rank_{T_R}	$\max_j T_{ij}$	$p\text{-value}_{T_R}$	$\text{Rank}_{T_{max}}$	$T_{i\cdot}$	$p\text{-value}_{T_{max}}$	$\text{Loss}\times10^3$
DGC	1	-15.3396	1.0000	1	-15.3396	1.0000	16.8861
Eliminated	HS, DGC FHS,	PearsonHist, DCC,	GerberHist, DCC FHS				

Panel B: $\tau = 5\%$

Model i	Rank_{T_R}	$\max_j T_{ij}$	$p\text{-value}_{T_R}$	$\text{Rank}_{T_{max}}$	$T_{i\cdot}$	$p\text{-value}_{T_{max}}$	$\text{Loss}\times10^3$
DGC	1	-15.1335	1.0000	1	-15.1335	1.0000	13.8046
Eliminated	HS, DGC FHS,	PearsonHist, DCC,	GerberHist, DCC FHS				

Panel C: $\tau = 10\%$

Model i	Rank_{T_R}	$\max_j T_{ij}$	$p\text{-value}_{T_R}$	$\text{Rank}_{T_{max}}$	$T_{i\cdot}$	$p\text{-value}_{T_{max}}$	$\text{Loss}\times10^3$
DGC	1	-28.4352	1.0000	1	-28.4352	1.0000	19.0495
Eliminated	HS, DGC FHS,	PearsonHist, DCC,	GerberHist, DCC FHS				

Table 11: Superior Set of Models, joint τ-VaR and τ-ES forecasting, Portfolio 3. For each model i in the SSM, the table reports the statistics $\max_j T_{ij}$ and $T_{i\cdot}$, the p-values associated with the test statistics T_R and T_{max}, the ranking of the models in the SSM based on T_R and T_{max}, and the average of the loss function (13) multiplied by 10^3.

Panel A: $\tau = 1\%$

Model i	Rank$_{T_R}$	$\max_j T_{ij}$	p-value$_{T_R}$	Rank$_{T_{max}}$	$T_{i\cdot}$	p-value$_{T_{max}}$	Loss$\times 10^3$
DGC	1	-16.2874	1	1	-16.2874	1	12.4309
Eliminated	HS, DGC FHS,	PearsonHist, DCC,	GerberHist, DCC FHS				

Panel B: $\tau = 5\%$

Model i	Rank$_{T_R}$	$\max_j T_{ij}$	p-value$_{T_R}$	Rank$_{T_{max}}$	$T_{i\cdot}$	p-value$_{T_{max}}$	Loss$\times 10^3$
DGC	1	-18.8147	1	1	-18.8147	1	10.7380
Eliminated	HS, DGC FHS,	PearsonHist, DCC,	GerberHist, DCC FHS				

Panel C: $\tau = 10\%$

Model i	Rank$_{T_R}$	$\max_j T_{ij}$	p-value$_{T_R}$	Rank$_{T_{max}}$	$T_{i\cdot}$	p-value$_{T_{max}}$	Loss$\times 10^3$
DGC	1	-18.0112	1	1	-18.0112	1	10.0040
Eliminated	HS, DGC FHS,	PearsonHist, DCC,	GerberHist, DCC FHS				

4 Conclusions

Methods that produce accurate forecasts of risk measures like VaR and ES are essential in an environment where market turmoils and substantial losses for investors are more and more frequent. In this study we have introduced a new method, DGC, to predict the two risk measures based on the dynamic version of the robust correlation proposed by Gerber et al (2022) that extends Kendall's Tau. As in DCC models, the proposed model is based, in a first stage of the estimation process, on univariate GARCH models. In an efficient way, the parameters in the recursions for the dynamic robust correlation matrix are estimated in the second stage. In an out-of-sample exercise, we have tested the performance of the proposed DGC method in accurately forecasting only VaR or VaR and ES jointly for portfolios consisting of four assets. For three different diversified portfolios realistically selected by many investors, we first backtest ES for the alternative models under scrutiny. With the exception of the portfolios consisting mainly of equities for the case $\tau = 1\%$, we do not reject the null of accurate ES predictions from models based on the DGC. Finally, we have derived, for VaR and ES corresponding to different probability levels, the superior set of models using the Model Confidence Set procedure. We have shown that for all the portfolios and VaR/ES confidence levels we consider, the DGC is part of the the superior set of models. A possible interesting development of the paper could be the use of the DGC method for portfolio selection, which we leave for future research.

Appendices

A The MCS Procedure

The relative performance between models i and j is obtained via the differential

$$d_{ij,t} = l_{i,t} - l_{j,t} \qquad \forall i, j \in M_0 \quad t = 1, \ldots, n \tag{14}$$

and the simple average loss of model i relative to the other models $j \in M$ at time t as

$$d_{i\cdot,t} = (m-1)^{-1} \sum_{j \in M \setminus i} d_{ij,t}. \tag{15}$$

For the elimination of inferior elements within the set M_0, two alternative sets of hypothesis are available to test the EPA:

$$\begin{cases} H_0 : \mathbb{E}(d_{ij}) = 0 \quad \forall i, j = 1, \ldots, m, \qquad \text{against} \\ H_1 : \mathbb{E}(d_{ij}) \neq 0 \end{cases} \tag{16}$$

or

$$\begin{cases} H_0 : \mathbb{E}(d_{i\cdot}) = 0 \quad \forall i = 1, \ldots, m, \qquad \text{against} \\ H_1 : \mathbb{E}(d_{i\cdot}) \neq 0 \end{cases} \tag{17}$$

Two statistics are then constructed to test the above hypotheses:

$$T_{ij} = \frac{\bar{d}_{ij}}{\sqrt{\widehat{\text{var}}(\bar{d}_{ij})}} \qquad\qquad T_{i\cdot} = \frac{\bar{d}_{i\cdot}}{\sqrt{\widehat{\text{var}}(\bar{d}_{i\cdot})}} \tag{18}$$

where $\bar{d}_{ij}$ constitute the relative average losses between i and j models and $\bar{d}_{i.}$ represents the average losses of the i^{th} model relative to the average losses across the models belonging to the set M:

$$\bar{d}_{ij} = n^{-1} \sum_{t=1}^{n} d_{ij,t} \qquad \bar{d}_{i.} = (m-1)^{-1} \sum_{j \in M \setminus i} \bar{d}_{ij} \tag{19}$$

The standard errors in the denominators of (18) are constructed by block bootstrap where p is the maximum number of significant parameters obtained by fitting an $AR(p)$ process to the d_{ij} terms.

The two hypotheses from (16) and (17) are mapped into two test statistics:

$$T_R = \max_{i,j \in M} T_{ij} \qquad T_{max} = \max_{i \in M} T_{i.}. \tag{20}$$

Because their distributions under the null are not known, they are simulated also by bootstrap. When the null hypothesis is rejected, the following elimination rules establish which of the models can be discarded:

$$e_R = \arg\max_{i} \left\{ \sup_{j \in M} T_{ij} \right\} \qquad e_{max} = \arg\max_{i} \left\{ T_{i.} \right\}. \tag{21}$$

Bernardi and Catania (2016) summarized the algorithm for the procedure as follows:

1. Set $M = M_0$.

2. Compute the test statistics under the null EPA hypothesis. If it is not rejected, set $M^*_{1-\alpha} = M$ and terminate the algorithm. If it is rejected use the elimination rule to determine the worst model.

3. Discard the model and repeat step 2.

The elimination rule defines a sequence of sets $M = M_0 \supset M_1 \cdots \supset M_m$, where $M_i = (e_{M_i}, \ldots, e_{M_m})$, each of which has a p-value associated with EPA test, so let P_{H_0,M_i} be the p-value associated with the null hypothesis H_{0,M_i}. The MCS p-value for model $e_{M_j} \supset M$ is defined as $\hat{p}_{e_{M_j}} = \max_{i \leq j} P_{H_0,M_i}$.

B Backtesting ES

In this appendix we provide the details of the procedure we implement to backtest ES.

Du and Escanciano (2017) proposed a procedure to backtest τ-ES based on the so-called cumulative violations (CV) process. The time-t value of this process is given by

$$H_t(\tau) = \frac{1}{\tau} [\tau - u_t] \, I \, (u_t \leq \tau) \tag{22}$$

where $u_t = G(r_t | \mathcal{F}_{t-1})$ is the Probability Integral (PIT) transform (Rosenblatt, 1952) and $G(\cdot | \mathcal{F}_{t-1}) = P(y_t \leq \cdot | \mathcal{F}_{t-1})$ is the conditional cumulative distribution function (cdf) of the portfolio return y_t.

To avoid estimating the portfolio cdf, Khalaf et al (2021) consider K equally spaced VaR levels with the larger one coinciding with τ, i.e. $\tau_j = (K - j + 1)\frac{\tau}{K}$ for $j = 1, \ldots, K$. Their ES backtesting procedures are based on the sum of VaR violations

$$N_t^K(\tau) = \sum_{j=1}^{K} I \left(y_t \leq -\text{VaR}_{t|t-1} \left(j \frac{\tau}{K} \right) \right), \tag{23}$$

since they show that, when returns are absolutely continuous, $\frac{N_t^K(\tau)}{K} \xrightarrow{D} H_t(\tau)$.

Khalaf et al (2021) hence consider the null

$$H_0 : \begin{cases} N_t^K(\tau)= j \text{ with probability } \theta_j \text{ for } j = 0, \ldots, K, \\ N_t^K(\tau) \perp\!\!\!\perp N_{t-h}^K(\tau), \forall h \neq 0, \end{cases} \tag{24}$$

where $\perp\!\!\!\perp$ denotes independence, and

$$\theta_0 = 1 - \tau, \qquad \theta_j = \frac{\tau}{K}, \quad j = 1, \ldots, K. \tag{25}$$

Note that $\mathbb{E}\left[N_t^K(\tau)\right] = \frac{K+1}{2}\tau$ under (24)–(25). As a consequence, in this paper we use a conditional backtest based on the idea that under the null, $\left\{N_t^K(\tau) - \frac{K+1}{2}\tau\right\}_t$ is a martingale difference sequence.

To this end, we use, for a sample of length n for $N_t^K(\tau)$, the Box–Pierce test statistic

$$C_m(\tau) = n \sum_{i=1}^{m} \rho_i^2(\tau) \tag{26}$$

where

$$\gamma_i(\tau) = \frac{1}{n-i} \sum_{t=i+1}^{n} \left(N_t^K(\tau) - \frac{K+1}{2}\tau\right)\left(N_{t-i}^K(\tau) - \frac{K+1}{2}\tau\right)$$

and $\rho_i(\tau) = \frac{\gamma_i(\tau)}{\gamma_0(\tau)}$. The p-vaues associated with the test statistic (26) can be obtained using the Monte Carlo test technique (Dufour, 2006) since, under the null (24)–(25), it is easy to simulate (23) and hence the test statistic (26), see Khalaf et al (2021) for further details.

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