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A general firm-value model under partial information

Cheikh MBAYE [*] Abass SAGNA [†] Frédéric VRINS [‡]

Abstract

We introduce a new structural default model which purpose is to combine enhanced economic relevance and affordable computational complexity. Our approach exploits the information conveyed by a noisy observation of the firm value combined with the firm's actual default state. Moreover, it is rather general since any diffusion can be used to depict the firm's dynamics. However, this realistic setup comes at the expense of important computational challenges. To mitigate them, we propose an implementation based on recursive quantization. A thorough analysis of the approximation error resulting from our numerical procedure is provided. The power of our method is illustrated on the pricing of CDS options. This analysis reveals that the observation noise has a significant impact on the credit spreads' implied volatility.

Keywords: finance, credit risk, structural model, noisy information, non-linear filtering.

1 Introduction

In recent years, credit risk has received more attention from the industry, regulators and academia. The 2008 financial crisis, for instance, stressed the relevance of relying on sound modeling frameworks to better assess and manage the default likelihood of firms, and of trades' counterparties in particular. Alongside the reduced-form framework, the structural approach counts among the most popular setups. It consists of modeling the evolution of a firm's credit worthiness by using a stochastic representation of its economic fundamentals: the firm value (the value of its assets) and its debt. The first model of this class is built upon the option pricing theory derived by Black, Scholes and Merton; a firm will default prior to a given time T if the firm's asset value falls below the debt level at that time [25]. In order to better model the actual time of the default event, first passage time models were introduced. In these approaches, the default time of the firm is modeled as the first passage time of the firm value process below the debt threshold; see, e.g., the celebrated Black-Cox model [3]. More complex structural models could be thought of by considering a time-dependent or a stochastic random barrier, but this makes harder the task of finding the law of the corresponding first passage time models. Furthermore, the analytical complexity of the model increases when considering a stochastic interest rates [22]. This theoretical complexity makes difficult to obtain closed form expressions of the firm's capital structure components or for the survival probability, leading to the use of numerical procedures. This might be a serious issue because model calibration typically requires to call the price function many times.

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Most structural models and specially the Black and Cox model suffer from the drawback that they fail to reproduce spread consistent with empirical observations. They reproduce lower spreads that decrease to zero for short maturities. This limitation comes from the fact that the valuation takes place in the filtration of the firm's assets, where the default event is a predictable stopping time. Being predictable, short-term default risk is not priced by the models. The predictability issue is, among others, addressed in [7] and [9], who considered random default barriers and time dependent volatilities. The same problem is handled differently in [34, 37] by introducing unpredictable jumps into the dynamics of the firm's value process. In the case of option pricing, [12] introduced a structural default model by incorporating a two-sided jump process in the firm dynamics leading to a variety of shapes for the implied volatility of equity options with a decent calibration performance to CDS spreads. It follows that, in these models, the default time is no longer a predictable stopping time. Recently, [33] applied the default model introduced by [12] to the valuation of CDS options and related optionalities such as extension risk using an efficient Monte Carlo algorithm based on Brownian bridges. In the context of counterparty risk valuation, structural models with jumps as been devised for example by [21] and [1]. In the first paper [21], the authors developed a methodology for valuing the counterparty credit risk inherent in the credit default swaps using a multidimensional structural model. The paper [1] presented a multivariate version of a structural default model with jumps using Lévy processes in order to evaluate the bilateral credit value adjustment in the presence of wrong-way risk and the bilateral debt value adjustment for equity contracts. In a more recent paper, the authors proposed an integrated pricing framework for credit value adjustment for equity and commodity products by taking into account the inclusion of risk mitigation clauses such as netting, collateral and initial margin provisions [2]. Another class of models rely on modified structural models with partial information. For example, Duffie and Lando proposed in [14] a model where the investors only have partial information about the actual financial health of the firm: only a noisy accounting report is accessible, and merely at some discrete time intervals. By doing so, the default time becomes totally inaccessible and, as a result, the corresponding short-term spreads do not vanish any longer. Similar extensions have been developed in a continuous-time setup, see, e.g., [13] or more recently, [16] and [20]. Notice that most of these models aim to be analytically tractable. As mentioned above, analytical tractability is appealing, but sets stringent limits regarding the dynamics that one can postulate for the firm value process.

In this paper, we generalize and improve earlier models in the sense that we consider a structural approach that is realistic, both in terms of filtration and dynamics. More explicitly, [11] and [31] can deal with arbitrary firm value diffusions. This comes at the expenses of a heavier computational complexity. Furthermore, the default state of the firm is not accessible to the investor. This does not comply with market practice since, in reality, bankruptcy is a legal – hence public– statement. To circumvent this issue, we consider the same information set as [13] but, whereas the latter restrict the firm value dynamics to be a continuous and invertible function of a Gaussian martingale, our framework can cope with arbitrary diffusions. To mitigate the loss of analytical tractability and deal with the resulting computational cost, we propose a numerical scheme based on fast quantization recently introduced in [28]. This technique is faster compared to [11] and [31] as Monte Carlo simulations can be avoided. Hence, the proposed method leads to a tractable structural default model under partial information, supporting general diffusions.

One of the main results of this paper is that, even in a simple GBM setup, the noise impacting the observed firm value has a major impact on the estimated credit worthiness. In particular, we show on a CDS option example that the noise power has a significant impact on the implied volatility of credit spreads.

The remainder of the paper is organized as follows. We first introduce the model in Section 2; various information flows are considered and the corresponding survival probabilities are derived.

Section 3 presents the computation of the survival probabilities using the recursive quantization method and the stochastic filtering theory. We then briefly review the quantization method. Eventually, in Section 4, we apply our method on a real financial application: the pricing of options on credit default swaps (CDS).

2 The model

As usual in quantitative finance, we represent the uncertainty of our economy through a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We consider a first passage time model in a structural framework. More precisely, let X be the stochastic process associated with the actual value of the firm and $\mathbf{a} \in \mathbb{R}$ stands for the default barrier, assumed constant. The default time of the firm is then represented by the random time $\tau_X > 0$, defined as:

$$\tau_X := \inf \{u \geq 0 : X_u \leq \mathbf{a}\}, \quad \mathbf{a} < X_0 \tag{1}$$

where $\inf \emptyset := +\infty$, as usual. We restrict ourselves to work within a finite time horizon, T.

We consider a partial information model where the true firm value process X (called *signal process* hereafter) is not directly observable; the investor can only access Y (*observation process*), a stochastic process correlated to X. We suppose a general diffusion framework whereby the dynamics of X and Y are governed by the following stochastic differential equations (SDEs):

$$\begin{cases} dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t, & X_0 = x_0, \\ dY_t = h(t, Y_t, X_t)dt + \nu(t, Y_t)dW_t + \delta(t, Y_t)d\widetilde{W}_t, & Y_0 = y_0, \end{cases} \tag{2}$$

where $(W, \widetilde{W})$ is a two-dimensional Brownian motion. We suppose that the functions $b, \sigma, \nu, \delta : [0, +\infty) \times \mathbb{R} \to \mathbb{R}$ are Lipschitz in x uniformly in t and that $\sigma(t, x) > 0$ for every $(t, x) \in [0, +\infty) \times \mathbb{R}$. These conditions ensure that the above SDEs admit a unique strong solution. Moreover we assume that h is locally bounded and Lipschitz in (y, x), uniformly in t and that $\nu(t, y) > 0$ and $\sigma(t, y) > 0$ for every $(t, y) \in [0, +\infty) \times \mathbb{R}$.

Remark 2.1. The results in the paper can be straightforwardly extended in the case when the default barrier $\mathbf{a}$ is a piecewise constant function of time $\mathbf{a} : [0, \infty) \to [0, \infty)$, with $0 < \mathbf{a}(0) < x_0$. The other extensions are beyond the scope of this paper. Nevertheless, note that crossing probabilities for the Brownian motion are available even when the barrier is a piecewise linear function on $[0, T]$ (see, e.g., [30]), while approximations can be found for the general case of nonlinear boundaries or satisfying the so-called Chercassov condition (see [20] for details).

2.1 Information flows

As recalled above, the common investor can only have an imperfect – or noisy – information regarding the actual value of the firm's assets. Economically, X would represent the value of the firm, which cannot be observed by the common investors, while Y might be the market price of an asset issued by the firm, accessible to all market participants. The threshold $\mathbf{a}$ would stand for the solvency capital requirement imposed by regulators. Mathematically, we take as the investors' information the natural filtration of the observed signal, noted $\mathbb{F}^Y := (\mathcal{F}_t^Y)_{t \geq 0}$ where $\mathcal{F}_t^Y := \sigma(Y_u, 0 \leq u \leq t)$. However, one might argue that this way of modeling the information is not realistic either, as the investor would then be unable to monitor the survival/default state of the reference entity. Indeed, it is clear from (1) that the default indicator process $H = (H_t)_{t \geq 0}$, $H_t := \mathbb{1}_{\{t \geq \tau_X\}}, t \geq 0$ is adapted to $\mathbb{F}^X$ but not to $\mathbb{F}^Y$. Hence, τ_X is not an $\mathbb{F}^Y$-stopping time and, consequently, it is not possible to determine whether $\tau_X \leq t$ given $\mathcal{F}_t^Y$.

We address this point by considering a more realistic information flow, defined as the progressive enlargement of $\mathbb{F}^Y$ with $\mathbb{F}^H$, the natural filtration of the default indicator process,

$$\mathcal{F}_t^H := \sigma(H_u, 0 \leq u \leq t), \quad t \geq 0.$$

In other words, we analyze two investors' information flows. On the one hand, we have the information available to the common investor, defined as the progressive enlargement of the natural filtration of the default indicator process with that of the noisy firm value process, noted $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ where

$$\mathcal{F}_t := \mathcal{F}_t^Y \vee \mathcal{F}_t^H, \quad t \geq 0.$$

In this setup, the following relationships hold:

$$\mathbb{F}^H \subsetneq \mathbb{F}^X = \mathbb{F}^W \subsetneq \mathbb{G} \quad \text{and} \quad \mathbb{F}^Y \subsetneq \mathbb{G},$$

where $\mathbb{F}^W$ is the natural filtration of the Brownian motion driving X and $\mathbb{G} := (\mathcal{G}_t)_{t\geq 0}$ is the full information, i.e., the information available for example to a small number of stockholders of the company, who have access to Y and X.

On the other hand, the natural filtration of the actual (i.e., noise-free) firm value process, $\mathbb{F}^X$, could be seen as the information available to insiders. Notice that in this case, there is no need to enlarge $\mathbb{F}^X$ with $\mathbb{F}^H$ to make the default time observable since τ_X is an $\mathbb{F}^X$-stopping time.

2.2 Survival probability

A fundamental output of a credit model is the survival probability of the firm up to time t conditional upon $\mathcal{F}_s$, $s \leq t \leq T$:

$$\mathbb{P}(\tau_X > t | \mathcal{F}_s) = \mathbb{E}\left[\mathbb{1}_{\{\tau_X > t\}} \middle| \mathcal{F}_s\right]. \tag{3}$$

Note that this probability collapses to zero whenever $\{\tau_X \leq s\}$. Recall that the specificity of our approach is that the actual value of the firm X is not revealed in $\mathcal{F}_s$; only a noisy version Y is accessible.

Using the Markov property of X, the independence between W and $\widetilde{W}$ as well as the chain rule of conditional expectations, the following result holds.

Proposition 2.1. *We have, for $s \leq t$,*

$$\mathbb{P}\left(\tau_X > t \middle| \mathcal{F}_s\right) = \mathbb{1}_{\{\tau_X > s\}} \frac{\mathbb{E}\left[\mathbb{1}_{\{\tau_X > s\}} F(s,t,X_s) | \mathcal{F}_s^Y\right]}{\mathbb{P}\left(\tau_X > s \middle| \mathcal{F}_s^Y\right)} \tag{4}$$

where, for every $x \in \mathbb{R}$,

$$F(s,t,x) := \mathbb{P}\left(\inf_{s<u\leq t} X_u > \mathbf{a} \middle| X_s = x\right) \tag{5}$$

is the conditional survival probability under full information. Furthermore, on the set $\{\tau_X > s\}$, it holds that

$$\mathbb{P}(\tau_X > t | \mathcal{F}_s^Y) \leq \mathbb{P}(\tau_X > t | \mathcal{F}_s). \tag{6}$$

Proof. Using the *Key lemma* (see e.g. Lemma 3.1 in [15]), a key result from the theory of conditional expectations, we have

$$\begin{aligned}
\mathbb{P}\left(\tau_X > t \middle| \mathcal{F}_s\right) &= \mathbb{1}_{\{\tau_X > s\}} \mathbb{E}\left[\mathbb{1}_{\{\tau_X > t\}} \middle| \mathcal{F}_s^Y \vee \mathcal{F}_s^H\right] \\
&= \mathbb{1}_{\{\tau_X > s\}} \frac{\mathbb{E}\left[\mathbb{1}_{\{\tau_X > t\}} \middle| \mathcal{F}_s^Y\right]}{\mathbb{P}\left(\tau_X > s \middle| \mathcal{F}_s^Y\right)} \\
&= \mathbb{1}_{\{\tau_X > s\}} \frac{\mathbb{P}\left(\tau_X > t | \mathcal{F}_s^Y\right)}{\mathbb{P}\left(\tau_X > s \middle| \mathcal{F}_s^Y\right)}. \qquad (7)
\end{aligned}$$

The proof is completed by noting that $\mathbb{P}\left(\tau_X > t|\mathcal{F}_s^Y\right) = \mathbb{E}\left[\mathbb{1}_{\{\tau_X > s\}} F(s,t,X_s)|\mathcal{F}_s^Y\right]$ (see, e.g., [31]). Inequality (6) follows from (7) since on the event $\{\tau_X > s\}$,

$$\mathbb{P}\big(\tau_X > t|\mathcal{F}_s^Y\big) = \mathbb{P}(\tau_X > t|\mathcal{F}_s)\, \mathbb{P}\big(\tau_X > s|\mathcal{F}_s^Y\big) \leq \mathbb{P}(\tau_X > t|\mathcal{F}_s)$$

or, equivalently, $\mathbb{P}\big(\tau_X \leq t|\mathcal{F}_s\big) \leq \mathbb{P}\big(\tau_X \leq t|\mathcal{F}_s^Y\big)$.

□

Remark 2.2. The inequality (6) will be illustrated numerically in Section 4. Loosely speaking, it means that the lower the information available regarding the state of the system ($\mathcal{F}_s^Y \subset \mathcal{F}_s^Y \vee \mathcal{F}_s^H = \mathcal{F}_s$), the higher the conditional default probability.

2.3 The problem

Having clearly stated the expression of interest, namely the survival probability of the reference entity up to time t conditional upon the investor's information up to time s, we need to actually compute it. To comply even more with real market practice, we further consider that we can only access Y at some *discrete times* up to s. To that end, let us start by fixing a time discretization grid over $[0,t]$:

$$0 = t_0 < \cdots < t_m = s < t_{m+1} < \cdots < t_n = t.$$

Our aim is to approximate the right hand side of (4) by recursive quantization. In some specific models (those for which (2) admits an explicit solution (X,Y), like in the Black-Scholes framework), we will consider the discrete trajectories $(X_{t_k}, Y_{t_k})_{k=0,\ldots,n}$. In more general models, we need to make a discrete time approximation of the quantity of interest. To this end, we suppose that we have access to a trajectory of Y sampled at m times: $(\bar{Y}_{t_0}, \ldots, \bar{Y}_{t_m})$, with $t_0 = 0$ and $t_m = s$ (which in practice will be approximated from the paths of the Euler-Maruyama scheme associated to the stochastic process Y) and will estimate (3) by

$$\frac{\mathbb{P}\left(\tau_X > t \middle| \mathcal{F}_s^{\bar{Y}}\right)}{\mathbb{P}\left(\tau_X > s \middle| \mathcal{F}_s^{\bar{Y}}\right)}$$

on the event $\{\tau_X > s\}$, where $\mathcal{F}_s^{\bar{Y}} := \sigma(\bar{Y}_{t_k},\ t_k \leq s) = \sigma(\bar{Y}_{t_0}, \ldots, \bar{Y}_{t_m})$.

2.4 Discrete time approximation

We denote by $\bar{X}$ the Euler-Maruyama scheme associated to the process X in Equation (2), namely:

$$\bar{X}_s = \bar{X}_{\underline{s}} + b(\underline{s}, \bar{X}_{\underline{s}})(s - \underline{s}) + \sigma(\underline{s}, \bar{X}_{\underline{s}})(W_s - W_{\underline{s}}), \quad \bar{X}_0 = x_0,$$

with $\underline{s} = t_k$ if $s \in [t_k, t_{k+1})$, for $k = 0, \dots, n$. Based on the Euler-Maruyama scheme, we introduce the discretized version of our state-observation processes $(\bar{X}, \bar{Y})$

$$\begin{cases} \bar{X}_{t_{k+1}} = \bar{X}_{t_k} + b(t_k, \bar{X}_{t_k})\Delta_k + \sigma(t_k, \bar{X}_{t_k})(W_{t_{k+1}} - W_{t_k}) \\ \bar{Y}_{t_{k+1}} = \bar{Y}_{t_k} + h(t_k, \bar{Y}_{t_k}, \bar{X}_{t_k})\Delta_k + \nu(t_k, \bar{Y}_{t_k})(W_{t_{k+1}} - W_{t_k}) + \delta(t_k, \bar{Y}_{t_k})(\widetilde{W}_{t_{k+1}} - \widetilde{W}_{t_k}) \end{cases} \tag{8}$$

where $k \in \{0, \dots, n-1\}$ for the signal process and $k \in \{0, \dots, m-1\}$ for the observation process and where $\Delta_k := t_{k+1} - t_k$.

Assuming that we have access to a discrete trajectory of Y, $(\bar{Y}_{t_0}, \dots, \bar{Y}_{t_m})$, our first goal is to approximate

$$\frac{\mathbb{P}\big(\tau_X > t |\, \mathcal{F}_s^Y\big)}{\mathbb{P}\big(\tau_X > s |\, \mathcal{F}_s^Y\big)} \quad \text{by} \quad \frac{\mathbb{P}\big(\tau_{\bar{X}} > t | \mathcal{F}_s^{\bar{Y}}\big)}{\mathbb{P}\big(\tau_{\bar{X}} > s | \mathcal{F}_s^{\bar{Y}}\big)}, \tag{9}$$

where, from (1),

$$\tau_{\bar{X}} := \inf\{u \geq 0, \bar{X}_u \leq \mathbf{a}\}.$$

Using the Brownian Bridge method and the Markov property of $(\bar{X}_{t_k}, \bar{Y}_{t_k})_k$, we show that (9) can be written in a closed formula. To this end, we will use the Theorem 2.5. in [31], where the probability $\mathbb{P}\big(\tau_{\bar{X}} > t | \mathcal{F}_s^{\bar{Y}}\big)$ has been computed for every $s \leq t$. The challenging question in our situation is not only to find a closed formula to compute (9) but, to be also able to find a tractable numerical recursive algorithm to compute numerically the resulting formula.

Theorem 2.2. *We have:*

$$\frac{\mathbb{P}\big(\tau_{\bar{X}} > t | \mathcal{F}_s^{\bar{Y}}\big)}{\mathbb{P}\big(\tau_{\bar{X}} > s | \mathcal{F}_s^{\bar{Y}}\big)} = \Psi(\bar{Y}_{t_0}, \dots, \bar{Y}_{t_m}), \tag{10}$$

where for $y = (y_0, \dots, y_m) \in \mathbb{R}^{m+1}$,

$$\Psi(y) := \frac{\mathbb{E}\big[\bar{F}(t_m, t_n, \bar{X}_{t_m}) K_{\mathbf{a}}^m L_y^m\big]}{\mathbb{E}[K_{\mathbf{a}}^m L_y^m]}, \tag{11}$$

with

$$K_{\mathbf{a}}^m := \prod_{k=0}^{m-1} G_{\Delta_k \sigma_k^2}^{\bar{X}_{t_k}, \bar{X}_{t_{k+1}}}(\mathbf{a}), \qquad L_y^m = \prod_{k=0}^{m-1} g_k(\bar{X}_{t_k}, y_k; \bar{X}_{t_{k+1}}, y_{k+1})$$

and where for every $x \in \mathbb{R}$,

$$\bar{F}(t_m, t_n, x) := \mathbb{E}\Big[\prod_{k=m}^{n-1} G_{\Delta_k \sigma_k^2}^{\bar{X}_{t_k}, \bar{X}_{t_{k+1}}}(\mathbf{a}) \big|\, \bar{X}_{t_m} = x\Big]. \tag{12}$$

The function g_k *is defined by*

$$\begin{aligned} g_k(x_k, y_k; x_{k+1}, y_{k+1}) &:= \frac{\mathbb{P}\big((\bar{X}_{t_{k+1}}, \bar{Y}_{t_{k+1}}) = (x_{k+1}, y_{k+1}) | (\bar{X}_{t_k}, \bar{Y}_{t_k}) = (x_k, y_k)\big)}{\mathbb{P}\big(\bar{X}_{t_{k+1}} = x_{k+1} | \bar{X}_{t_k} = x_k\big)} \\ &= \frac{1}{(2\pi\Delta_k)^{1/2}\delta_k} \exp\left(- \frac{\nu_k^2}{2\delta_k^2 \Delta_k} \Big(\frac{x_{k+1} - m_k^1}{\sigma_k} - \frac{y_{k+1} - m_k^2}{\nu_k} \Big)^2 \right) \end{aligned} \tag{13}$$

with $m_k^1 := x_k + b_k\Delta_k$ *and* $m_k^2 := y_k + h_k\Delta_k$. *Finally,*

$$\begin{aligned} G_{\Delta_k \sigma_k^2}^{x_k, x_{k+1}}(\mathbf{a}) &= \mathbb{P}\big(\inf_{u \in [t_k, t_{k+1}]} \bar{X}_u \geq \mathbf{a} | \bar{X}_{t_k} = x_k\big) \\ &= \left(1 - \exp\left(- \frac{2(x_k - \mathbf{a})(x_{k+1} - \mathbf{a})}{\Delta_k \sigma^2(t_k, x_k)} \right) \right) \mathbb{1}_{\{x_k \geq \mathbf{a};\, x_{k+1} \geq \mathbf{a}\}}. \end{aligned} \tag{14}$$

Proof. Following Theorem 2.5. in [31], we have,

$$\mathbb{P}\big(\tau_{\bar X} > t|\mathcal{F}_s^{\bar Y}\big) = \frac{\mathbb{E}\big[\bar F(t_m,t_n,\bar X_{t_m})K_{\mathbf{a}}^m L_y^m\big]}{\mathbb{E}[L_y^m]} \quad \text{and} \quad \mathbb{P}\big(\tau_{\bar X} > s|\mathcal{F}_s^{\bar Y}\big) = \frac{\mathbb{E}\big[K_{\mathbf{a}}^m L_y^m\big]}{\mathbb{E}[L_y^m]}.$$

□

Therefore, we are left with the estimation of $\Psi(y)$ in (11), for $y = (\bar Y_{t_0},\ldots,\bar Y_{t_m})$. Owing to the form of the random vector $K_{\mathbf{a}}^m$, we may put it together with L_y^m to get a similar formula as the filter estimate in a standard nonlinear filtering problem. In other words, we may write for $y := (y_0,\ldots,y_m)$,

$$\Psi(y) = \frac{\mathbb{E}\big[\bar F(t_m,t_n,\bar X_{t_m})\, L_{y,\mathbf{a}}^m\big]}{\mathbb{E}[L_{y,\mathbf{a}}^m]} \quad \text{where} \quad L_{y,\mathbf{a}}^m = \prod_{k=0}^{m-1} g_k(\bar X_{t_k},y_k;\bar X_{t_{k+1}},y_{k+1})\times G_{\Delta_k\sigma_k^2}^{\bar X_{t_k},\bar X_{t_{k+1}}}(\mathbf{a}).$$

In the next section, we show how this function Ψ can be estimated efficiently by using recursive quantization.

3 Implementation by recursive quantization

Notice that defining the operator $\pi_{y,m}$, for every bounded measurable function f, by

$$\pi_{y,m} f := \mathbb{E}\big[f(\bar X_{t_m})L_{y,\mathbf{a}}^m\big],$$

we have

$$\Psi(y) = \frac{\pi_{y,m}\bar F(t_m,t_n,\cdot)}{\pi_{y,m}\mathbf{1}} =: \Pi_{y,m}\bar F(t_m,t_n,\cdot), \tag{15}$$

where $\mathbf{1}(x) = 1$, for every real x. Then it is enough to tell how to compute the numerator

$$\pi_{y,m}\bar F(t_m,t_n,\cdot) = \mathbb{E}\left[\bar F(t_m,t_n,\bar X_{t_m})\prod_{k=0}^{m-1} g_k^{\mathbf{a}}(\bar X_{t_k},y_k;\bar X_{t_{k+1}},y_{k+1})\right]$$

where

$$g_k^{\mathbf{a}}(\bar X_{t_k},y_k;\bar X_{t_{k+1}},y_{k+1}) = g_k(\bar X_{t_k},y_k;\bar X_{t_{k+1}},y_{k+1}) \times G_{\Delta_k\sigma_k^2}^{\bar X_{t_k},\bar X_{t_{k+1}}}(\mathbf{a}).$$

At this stage, several methods involving Monte Carlo simulations such as the particle method can be used to approximate $\Pi_{y,m}$. Optimal quantization is an alternative and some times as a substitute to the Monte Carlo method to approximate such a quantity (we refer to [35] for a comparison of particle like methods and optimal quantization methods).

To use the optimal quantization methods we have to quantize the marginals of the process $(\bar X_{t_k})_k$, means, to represent every marginal $\bar X_{t_k}$, $k = 0,\ldots,n$, by a discrete random variable $\widehat X_{t_k}^{\Gamma_k}$ (we will simply denote it $\widehat X_{t_k}$ when there is no ambiguity) taking N_k values $\Gamma_k = \{x_1^k,\ldots,x_{N_k}^k\}$. As we will see later, we have also need in our context to compute the transition probabilities $\hat p_k^{ij} = \mathbb{P}(\widehat X_{t_k} = x_j^k|\widehat X_{t_{k-1}} = x_i^{k-1})$, for $i = 1,\ldots,N_{k-1}$; $j = 1,\ldots,N_k$. To this end, we may use stochastic algorithms to get the optimal grids and the associated transition probabilities (see e.g. [27, 35]). This method works well but may be very time consuming. The so-called marginal functional quantization method (see [11, 31, 32]) is used as an alternative to the previous method. It consists of constructing the marginal quantizations by considering the ordinary differential equation (ODE) resulting from the substitution of the Brownian motion appearing in the dynamics of X in (2) by a quadratic quantization of the Brownian motion (see [23]). This procedure performs the marginal quantizations quite instantaneous and works well enough from the numerical point of view even if the rate of convergence (which has not been computed yet from the theoretical point of view) seems to be poor. As an alternative to the two previous methods, we propose the recursive marginal quantization (also called fast quantization) method introduced in [28]. It consists of quantizing the process $(\bar{X}_{t_k})_{k=0,\dots,n}$, based on a recursive method involving the conditional distributions $\bar{X}_{t_{k+1}}|\bar{X}_{t_k}$, $k = 0, \dots, n-1$. For the problem of interest, this last method is more performing than the previous ones due to its computation speed and to its robustness.

On the other hand, the function $\bar{F}$ has been estimated by Monte Carlo method in [11, 31]. For competitiveness reasons of the recursive quantization w.r.t. the previously raised methods, we propose here to approximate both quantities Π and $\bar{F}$ by the recursive quantization method.

3.1 Approximation of $\Pi_{y,m}$ by recursive quantization

Let us focus on the computation of the numerator appearing in the right hand side of (15). Indeed, the denominator is just a particular case of the latter. Notice first that $\pi_{y,m}$ can be computed from the following recursive formula:

$$\pi_{y,k} = \pi_{y,k-1} H_{y,k}, \qquad k = 1, \dots, m, \tag{16}$$

where, for every $k = 1, \dots, m$ and every bounded and measurable function f, the transition kernel $H_{y,k}$ is defined by

$$H_{y,k} f(z) = \mathbb{E}\left[f(\bar{X}_{t_k}) g^{\mathbf{a}}_{k-1}(\bar{X}_{t_{k-1}}, y_{k-1}; \bar{X}_{t_k}, y_k) | \bar{X}_{t_{k-1}} = z\right] \quad \text{with} \quad H_{y,0} f := \mathbb{E}[f(\bar{X}_0)].$$

In fact, for any bounded Borel function f we have

$$\begin{aligned}
\pi_{y,k} f &= \mathbb{E}\left[f(\bar{X}_{t_k}) \prod_{\ell=0}^{k-1} g^{\mathbf{a}}_{\ell}(\bar{X}_{t_\ell}, y_\ell; \bar{X}_{\ell+1}, y_{\ell+1})\right] \\
&= \mathbb{E}\left[\mathbb{E}\Big(f(\bar{X}_{t_k}) \prod_{\ell=0}^{k-1} g^{\mathbf{a}}_{\ell}(\bar{X}_{t_\ell}, y_\ell; \bar{X}_{\ell+1}, y_{\ell+1}) \big| \mathcal{F}^{\bar{X}}_{k-1}\Big)\right].
\end{aligned}$$

Because $\bar{X}$ remains Markov, one obtains

$$\begin{aligned}
\pi_{y,k} f &= \mathbb{E}\left[\mathbb{E}\Big(f(\bar{X}_{t_k}) g^{\mathbf{a}}_{k-1}(\bar{X}_{t_{k-1}}, y_{k-1}; \bar{X}_k, y_k) \big| \mathcal{F}^{\bar{X}}_{k-1}\Big) \prod_{\ell=0}^{k-2} g^{\mathbf{a}}_{\ell}(\bar{X}_{t_\ell}, y_\ell; \bar{X}_{\ell+1}, y_{\ell+1})\right] \\
&= \mathbb{E}\left[H_{y,k} f(\bar{X}_{t_{k-1}}) \prod_{\ell=0}^{k-2} g^{\mathbf{a}}_{\ell}(\bar{X}_{t_\ell}, y_\ell; \bar{X}_{\ell+1}, y_{\ell+1})\right] \\
&= \pi_{y,k-1} H_{y,k} f.
\end{aligned}$$

Equipped with the quantization of the marginals of the process $\bar{X}$, the functional $\pi_{y,k}$ can be approximated recursively by optimal quantization : $\hat{\pi}_{y,k} = \hat{\pi}_{y,k-1} \widehat{H}_{y,k}$ where $k \geq 1$. The (i,j) component of the $N_k \times N_{k-1}$ matrix $\widehat{H}_{y,k}$ is given by

$$\widehat{H}^{ij}_{y,k} = g^{\mathbf{a}}_{k-1}(x^i_{k-1}, y_{k-1}; x^j_k, y_k)\, \hat{p}^{ij}_k\, \delta_{x^j_k} \qquad \text{where} \qquad p^{ij}_k = \mathbb{P}(\widehat{X}_{t_k} = x^j_k | \widehat{X}_{t_{k-1}} = x^i_{k-1}).$$

In this expression, $(\widehat{X}_{t_k})_k$ is the quantization of the process $(\bar{X}_t)_{t\geq 0}$ over the time steps $t_k, k = 1,\ldots,m$ on the grids Γ_k. As a consequence, $\Pi_{y,m}\bar{F}(t_m,t_n,\cdot)$ can be estimated by

$$\widehat{\Pi}_{y,m}\bar{F}(t_m,t_n,\cdot) = \sum_{i=1}^{N_m} \widehat{\Pi}^i_{y,m}\bar{F}(t_m,t_n,x^i_m) \tag{17}$$

where

$$\widehat{\Pi}^i_{y,m} := \frac{\widehat{\pi}^i_{y,m}}{\sum_{j=1}^{N_m}\widehat{\pi}^i_{y,m}}, \quad i=1,\ldots,N_m$$

with $\widehat{\pi}_{y,m}$ the (optimal quantization) estimator of $\pi_{y,m}$. The latter is defined in a recursive way as follows:

$$\begin{cases} \widehat{\pi}_{y,0} = \widehat{H}_{y,0} \\ \widehat{\pi}_{y,k} = \widehat{\pi}_{y,k-1}\widehat{H}_{y,k} := \Big[\sum_{i=1}^{N_{k-1}} \widehat{H}^{i,j}_{y,k}\widehat{\pi}^i_{y,k-1}\Big]_{j=1,\ldots,N_k}, \quad k=1,\ldots,m \end{cases} \tag{18}$$

with

$$\widehat{H}^{ij}_{y,k} = g^{\mathbf{a}}_{k-1}(x^i_{k-1},y_{k-1};x^j_k,y_k)\,\hat{p}^{ij}_k\,\delta_{x^j_k}\ . \tag{19}$$

Our aim is now to use the (marginal) recursive quantization method to estimate the $\bar{F}(t_m,t_n,x^i_m)'$s.

3.2 Approximation of $\bar{F}(t_m,t_n,\cdot)$ by recursive quantization

Recall that for every x, we have

$$\bar{F}(t_m,t_n,x) = \mathbb{E}\left[\prod_{k=m}^{n-1} G^{\bar{X}_{t_k},\bar{X}_{t_{k+1}}}_{\Delta_k\sigma_k^2}(\mathbf{a})\,\middle|\, \bar{X}_{t_m}=x\right].$$

As previously, define $\pi_{n,m}$ by

$$\big(\pi_{n,m}f\big)(x) := \mathbb{E}\left[f(\bar{X}_{t_n})\prod_{k=m}^{n-1} G^{\bar{X}_{t_k},\bar{X}_{t_{k+1}}}_{\Delta_k\sigma_k^2}(\mathbf{a})\,\middle|\, \bar{X}_{t_m}=x\right],$$

for every bounded and measurable function f. Hence, $\bar{F}(t_m,t_n,x)$ takes the simple form

$$\bar{F}(t_m,t_n,x) = \big(\pi_{n,m}\mathbf{1}\big)(x).$$

For every such function f, define the transition kernel H_k as,

$$\big(H_kf\big)(z) := \mathbb{E}\left[f(\bar{X}_{t_k})G^{\bar{X}_{t_{k-1}},\bar{X}_{t_k}}_{\Delta_{k-1}\sigma^2_{k-1}}(\mathbf{a})\,\middle|\,\bar{X}_{t_{k-1}}=z\right],$$

for every $k=m+1,\ldots,n$ and set

$$H_mf := \mathbb{E}\left[f(\bar{X}_{t_m})\right]. \tag{20}$$

This yields, for every $k=m+1,\ldots,n$,

$$\begin{aligned}(\pi_{k,m}f)(x) &= \mathbb{E}\Big[\mathbb{E}\Big(f(\bar{X}_{t_k})\prod_{i=m}^{k-1} G^{\bar{X}_{t_k},\bar{X}_{t_{k+1}}}_{\Delta_k\sigma_k^2}(\mathbf{a})\big|(\bar{X}_{t_\ell})_{\ell=m,\ldots,k-1}\Big)\big|\bar{X}_{t_m}=x\Big] \\ &= \mathbb{E}\Big[\mathbb{E}\Big(f(\bar{X}_{t_k})G^{\bar{X}_{t_{k-1}},\bar{X}_{t_k}}_{\Delta_{k-1}\sigma^2_{k-1}}(\mathbf{a})|(\bar{X}_{t_\ell})_{\ell=m,\ldots,k-1}\Big)\prod_{\ell=m}^{k-2}G^{\bar{X}_{t_\ell},\bar{X}_{t_{\ell+1}}}_{\Delta_\ell\sigma_\ell^2}(\mathbf{a})|\bar{X}_{t_m}=x\Big] \\ &= \mathbb{E}\Big[\mathbb{E}\Big(f(\bar{X}_{t_k})G^{\bar{X}_{t_{k-1}},\bar{X}_{t_k}}_{\Delta_{k-1}\sigma^2_{k-1}}(\mathbf{a})|\bar{X}_{t_{k-1}}\Big)\prod_{\ell=m}^{k-2}G^{\bar{X}_{t_\ell},\bar{X}_{t_{\ell+1}}}_{\Delta_\ell\sigma_\ell^2}(\mathbf{a})|\bar{X}_{t_m}=x\Big] \\ &= (\pi_{k-1,m}H_kf)(x).\end{aligned}$$

Consequently, if one has access to the recursive quantizations $(\widehat{X}_{t_k})_{k=m,\dots,n}$ as well as to the transition probabilities $\{\hat{p}_k^{ij}, k = m+1,\dots,n\}$ of the process $(\bar{X}_{t_k})_{k=m,\dots,n}$, the quantity $\bar{F}(t_m, t_n, x)$ can be estimated by

$$\widehat{F}(t_m, t_n, x) := \sum_{j=1}^{N_n} \widehat{\pi}_{n,m}\, \delta_{\{x_m^j = x\}}, \tag{21}$$

where the $\widehat{\pi}_{n,m}$'s are defined from the recursive formula

$$\begin{cases} \widehat{\pi}_{m,m} = \widehat{H}_m \\ \widehat{\pi}_{k,m} = \widehat{\pi}_{k-1,m}\widehat{H}_k := \Big[\sum_{i=1}^{N_{k-1}} \widehat{H}_k^{i,j}\widehat{\pi}_{k-1,m}\Big]_{j=1,\dots,N_k}, \quad k = m+1,\dots,n \end{cases} \tag{22}$$

with

$$\widehat{H}_k^{ij} = G_{\Delta_{k-1}\sigma_{k-1}^2}^{x_{k-1}^i, x_k^j}(\mathbf{a})\, \hat{p}_k^{ij}\, \delta_{x_k^j}, \quad i = 1,\dots,N_{k-1} \text{ and } j = 1,\dots,N_k.$$

3.3 Approximation of $\Pi_{y,m}\bar{F}(t_m, t_n, \cdot)$ by recursive quantization

Combining equations (17) and (21), the conditional survival probability $\Pi_{y,m}F(t_m, t_n, \cdot)$ will be estimated (for a fixed trajectory $y = (y_0,\dots,y_m)$ of the observation process $(Y_{t_0},\dots,Y_{t_m})$) by

$$\widehat{\Pi}_{y,m}\widehat{F}(t_m, t_n, \cdot) = \sum_{i=1}^{N_m}\sum_{j=1}^{N_n} \widehat{\Pi}_{y,m}^i \widehat{\pi}_{n,m}\, \delta_{x_m^j}. \tag{23}$$

We now proceed with the quantization of the signal process X using the recursive quantization method.

3.4 The recursive quantization method

Recall that for a given $\mathbb{R}^d$-valued random vector X with distribution $\mathbb{P}_X$, the $L^r(\mathbb{P}_X)$-optimal quantization problem of size N for X (or for the distribution $\mathbb{P}_X$) aims to approximate X by a Borel function of X taking at most N values. If $X \in L^r(\mathbb{P})$ and defining $\|X\|_r := (\mathbb{E}|X|^r)^{1/r}$ where $|\cdot|$ denotes an arbitrary norm on $\mathbb{R}^d$, this turns out to solve the following optimization problem (see, e.g., [18]):

$$e_{N,r}(X) = \inf\{\|X - \widehat{X}^\Gamma\|_r, \Gamma \subset \mathbb{R}^d, \text{card}(\Gamma) \le N\} = \inf_{\substack{\Gamma\subset\mathbb{R}^d \\ \text{card}(\Gamma)\le N}} \left(\int_{\mathbb{R}^d} d(x,\Gamma)^r d\mathbb{P}_X(x)\right)^{1/r} \tag{24}$$

where $\widehat{X}^\Gamma$, the quantization of X on the subset $\Gamma = \{x_1,\dots,x_N\} \subset \mathbb{R}^d$ (called a codebook, an N-quantizer or a grid), is defined by

$$\widehat{X}^\Gamma = \text{Proj}_\Gamma(X) := \sum_{i=1}^{N} x_i \mathbb{1}_{\{X \in C_i(\Gamma)\}}$$

and where $(C_i(\Gamma))_{i=1,\dots,N}$ is a Voronoï partition of $\mathbb{R}^d$ satisfying, for every $i \in \{1,\dots,N\}$,

$$C_i(\Gamma) \subset \{x \in \mathbb{R}^d : |x - x_i| = \min_{j=1,\dots,N} |x - x_j|\}.$$

Keep in mind that for every $N \geq 1$, the infimum in (24) is reached at one grid at least. Any N-quantizer realizing this infimum is called a L^r-optimal N-quantizer. Moreover, if $\text{card}(\text{supp}(\mathbb{P}_X)) \geq N$ then the optimal N-quantizer is of size N (see [18] or [26]). On the other hand, the quantization error $e_{N,r}(X)$ decreases to zero as the grid size N goes to infinity, with rate $N^{-1/d}$. This convergence rate (known as Zador Theorem) has been investigated in [10] and [36] for absolutely continuous probability measures under the quadratic norm on $\mathbb{R}^d$. A detailed study of the convergence rate under an arbitrary norm on $\mathbb{R}^d$ and for both absolutely continuous and singular measures is available in [18].

The recursive quantization of the Euler-Maruyama scheme of an $\mathbb{R}^d$-valued diffusion process has been introduced in [28]. The method consists of a sequence of quantizations $(\widehat{X}_{t_k}^{\Gamma_k})_{k=0,\ldots,n}$ of the Euler-Maruyama scheme $(\bar{X}_{t_k})_{k=0,\ldots,n}$ defined recursively as

$$\widetilde{X}_0 = \bar{X}_0, \quad \widehat{X}_{t_k}^{\Gamma_k} = \text{Proj}_{\Gamma_k}(\widetilde{X}_{t_k}) \quad \text{and} \quad \widetilde{X}_{t_{k+1}} = \mathcal{E}_k(\widehat{X}_{t_k}^{\Gamma_k}, Z_{k+1}),\ k = 0,\ldots,n-1, \tag{25}$$

where $(Z_k)_{k=1,\ldots,n}$ is an i.i.d. sequence of $\mathcal{N}(0; I_q)$-distributed random vectors, independent from $\bar{X}_0$ and

$$\mathcal{E}_k(y,z) = y + \Delta b(t_k,y) + \sqrt{\Delta}\sigma(t_k,y)z, \quad y \in \mathbb{R}^d,\ z \in \mathbb{R}^q,\ k = 0,\ldots,n-1.$$

The sequence of quantizers satisfies for every $k \in \{0,\ldots,n\}$,

$$\Gamma_k \in \text{argmin}\{\widetilde{D}_k(\Gamma),\ \Gamma \subset \mathbb{R}^d,\ \text{card}(\Gamma) \leq N_k\},$$

where for every grid $\Gamma \subset \mathbb{R}^d$, $\widetilde{D}_{k+1}(\Gamma) := \mathbb{E}\big[d(\widetilde{X}_{t_{k+1}}, \Gamma)^2\big]$.

This recursive quantization method raises some problems among which the computation of the quadratic error bound $\|\bar{X}_{t_k} - \widehat{X}_{t_k}^{\Gamma_k}\|_2 := \big(\mathbb{E}|\bar{X}_{t_k} - \widehat{X}_{t_k}^{\Gamma_k}|_2\big)^{1/2}$, for every $k = 0,\ldots,n$. It has been shown in [28, 29] that for any sequences of (quadratic) optimal quantizers Γ_k for $\widetilde{X}_{t_k}^{\Gamma_k}$, for every $k = 0,\ldots,n-1$, the quantization error $\|\bar{X}_{t_k} - \widehat{X}_{t_k}^{\Gamma_k}\|_2$ is bounded by the cumulative quantization errors $\|\widetilde{X}_{t_i} - \widehat{X}_{t_i}^{\Gamma_i}\|_2$, for $i = 0,\ldots,k$. This result, formally stated in Proposition 3.1 below, is obtained under the following assumptions:

1. *L^2-Lipschitz assumption.* The mappings $x \mapsto \mathcal{E}_k(x, Z_{k+1})$ from $\mathbb{R}^d$ to $L^2(\Omega, \mathcal{A}, \mathbb{P})$, $k = 1 : n$ are Lipschitz continuous i.e.

$$(\text{Lip}) \quad \equiv \quad \forall\, x,\, x' \in \mathbb{R}^d, \quad \big\|\mathcal{E}_k(x, Z_{k+1}) - \mathcal{E}_k(x', Z_{k+1})\big\|_2 \leq [\mathcal{E}_k]_{\text{Lip}}|x - x'|,\ k = 1 : n.$$

2. *L^p-linear growth assumption.* Let $p \in (2,3]$.

$$(\text{SL})_p \ \equiv\ \forall\, k \in \{1,\ldots,n\},\ \forall\, x \in \mathbb{R}^d, \quad \mathbb{E}|\mathcal{E}_k(x, Z_{k+1})|^p \leq \alpha_{p,k} + \beta_{p,k}|x|^p.$$

We state the recursive quantization error bound in the following proposition.

Proposition 3.1. *Let $\widehat{X} = (\widehat{X}_{t_k})_{k=0:n}$ be defined by* (25) *and suppose that all the grids Γ_k are quadratic optimal. Assume that both assumptions* (Lip) *and* $(\text{SL})_p$ *(for some $p \in (2,3]$) hold and that $X_0 \in L^p(\mathbb{P})$. Then,*

$$\big\|\bar{X}_{t_k} - \widehat{X}_{t_k}\big\|_2 \leq C_{d,p} \sum_{i=0}^{k} [\mathcal{E}_{i+1:k}]_{\text{Lip}} \left[\sum_{\ell=0}^{i} \alpha_{p,\ell}\beta_{p,\ell+1:i}\right]^{\frac{1}{p}} N_i^{-\frac{1}{d}} \tag{26}$$

where $C_{d,p} > 0$ and $\alpha_{p,0} = \mathbb{E}\,|X_0|^p = \|X_0\|_p^p$, $\beta_{p,\ell:i} = \prod_{m=\ell}^{i} \beta_{p,m}$ (with $\prod_{\emptyset} = 1$) and

$$[\mathcal{E}_{i:k}]_{\text{Lip}} := \prod_{\ell=i}^{k} [\mathcal{E}_\ell]_{\text{Lip}},\ 1 \leq \ell \leq k \leq n \qquad \textit{and} \qquad [\mathcal{E}_{k+1:k}]_{\text{Lip}} = 1.$$

The following result gives the explicit expressions of the probability weights and transition probabilities.

Proposition 3.2. *Let Γ_{k+1} be a quadratic optimal quantizer for the marginal random variable $\widetilde{X}_{t_{k+1}}$. Suppose that the quadratic optimal quantizer Γ_k for $\widetilde{X}_{t_k}$ is already computed and that we have access to its associated weights $\mathbb{P}(\widetilde{X}_{t_k} \in C_i(\Gamma_k))$, $i = 1, \ldots, N_k$. The transition probability $\hat{p}_k^{ij} = \mathbb{P}(\widetilde{X}_{t_{k+1}} \in C_j(\Gamma_{k+1}) | \widetilde{X}_{t_k} \in C_i(\Gamma_k)) = \mathbb{P}(\widehat{X}_{t_{k+1}} = x_{k+1}^j | \widehat{X}_{t_k} = x_k^i)$ is given by*

$$\hat{p}_k^{ij} = \Phi\big(x_{k+1,j+}(x_i^k)\big) - \Phi\big(x_{k+1,j-}(x_i^k)\big), \tag{27}$$

where $\Phi(\cdot)$ is the cumulative distribution function of the standard Gaussian distribution,

$$x_{k+1,j-}(x) := \frac{x_{k+1}^{j-1/2} - m_k(x)}{v_k(x)} \quad \text{and} \quad x_{k+1,j+}(x) := \frac{x_{k+1}^{j+1/2} - m_k(x)}{v_k(x)},$$

with $m_k(x) = x + \Delta b(t_k, x)$, $v_k(x) = \sqrt{\Delta}\sigma(t_k, x)$ and, for $k = 0, \ldots, n-1$ and for $j = 1, \ldots, N_{k+1}$,

$$x_{k+1}^{j-1/2} = \frac{x_{k+1}^j + x_{k+1}^{j-1}}{2},\ x_{k+1}^{j+1/2} = \frac{x_{k+1}^j + x_{k+1}^{j+1}}{2},\ \text{with } x_{k+1}^{1/2} = -\infty, x_{k+1}^{N_{k+1}+1/2} = +\infty.$$

Once we have access to the marginal quantizations and to their associated transition probabilities, the right hand side of (23) can be computed explicitly. The error analysis allowing to compute the convergence rate of our numerical schemes is detailed in the appendix. The next section is devoted to a numerical illustration of our approach.

4 Numerical results

In the remaining part of the paper, we analyze the behavior of our approach from various standpoints by relying on a Black-Scholes[1] type of model. More specifically, the dynamics of the signal process X and observation process Y take the form

$$\begin{cases} dX_t = X_t(\mu dt + \sigma dW_t), & X_0 = x_0, \\ dY_t = Y_t(\mu dt + \sigma dW_t + \delta d\widetilde{W}_t), & Y_0 = y_0, \end{cases} \tag{28}$$

meaning that

$$\frac{dY_t}{Y_t} = \frac{dX_t}{X_t} + \delta d\widetilde{W}_t.$$

In other words, the yield associated with the observation process Y is a noisy version of the yield of the signal process X. This section deals with the pricing of CDS option, and $\mathbb{P}$ refers to the risk-neutral probability measure. Because X is positive, one must choose $\mathbf{a} > 0$.

We start by analyzing the model's behavior in terms of conditional survival probabilities in Section 4.1. Then, in Section 4.2, the impact of the noise parameter δ on the implied volatilities of CDS options is discussed.

[1]Our approach encompasses the estimation of general diffusions of the form (2), but the Black-Scholes type of dynamics allows exact expressions of the quantities we would like to estimate and thus can be used as a simple illustration the convergence of our numerical schemes.

4.1 Comparison of conditional survival probabilities

First, we compare the function $F(s,t,x) = \mathbb{P}(\inf_{s\le u\le t} X_u > \mathbf{a} | X_s = x)$ to its quantized version $\widehat{F}(s,t,x)$ defined in (21). This is possible because in the Black-Scholes setup (28), the exact expression of $F(s,t,x)$ is known analytically (see chapter 3 in [19]):

$$F(s,t,x) = \Phi(h_1(x,t-s)) - \left(\frac{\mathbf{a}}{x}\right)^{2\sigma^{-2}(\mu-\sigma^2/2)} \Phi(h_2(x,t-s)), \tag{29}$$

where

$$h_1(x,u) = \frac{1}{\sigma\sqrt{u}}\left(\log\left(\frac{x}{\mathbf{a}}\right) + \left(\mu - \frac{1}{2}\sigma^2\right)u\right),$$
$$h_2(x,u) = \frac{1}{\sigma\sqrt{u}}\left(\log\left(\frac{\mathbf{a}}{x}\right) + \left(\mu - \frac{1}{2}\sigma^2\right)u\right).$$

Second, we compare the conditional default probabilities $\mathbb{P}(\tau_{\bar{X}} \le t | \mathcal{F}_s^{\bar{Y}} \vee \mathcal{F}_s^{\bar{H}})$ to $\mathbb{P}(\tau_{\bar{X}} \le t \mid \mathcal{F}_s^{\bar{Y}})$. The first conditional default probability is estimated by (23). The second one is estimated, using our procedure, by

$$\widehat{\varpi}_{y,m}\widehat{F}(t_m,t_n,\cdot) = \sum_{i=1}^{N_m}\sum_{j=1}^{N_n} \widehat{\varpi}^i_{y,m}\widehat{\pi}_{n,m}\,\delta_{x_m^j} \tag{30}$$

where the $\widehat{\varpi}^i_{y,m}$'s are obtained from (18) by replacing the function $g_k^{\mathbf{a}}$ by g_k of equation (13).

The purpose here is to understand what is the specific impact of the information regarding the default state, i.e., $\mathcal{F}_s^{\bar{H}}$ on the conditional survival probability. More explicitly, we compare the conditional probability of $\tau_{\bar{X}} \le t$ given $\mathcal{F}_s^{\bar{Y}}$ or $\mathcal{F}_s^{\bar{Y}} \vee \mathcal{F}_s^{\bar{H}}$.

Remark 4.1. Notice that the working example (28) can lead to explicit solutions. In such a simple case, one could perfectly consider a formula similar to (11), but relying on the exact solutions on the SDEs of X and Y; the only change will come from the function g_k which involves the conditional density of $(X_{t_{k+1}}, Y_{t_{k+1}})$ given (X_{t_k}, Y_{t_k}). However, recall that our purpose here is to provide a general framework, in which the signal and the observation processes can be arbitrary diffusions, with no explicit solutions. Therefore, we do not make use of the specific explicit solutions (X,Y) but, instead, restrict ourselves to work with their discretized versions, that is, with $(\bar{X},\bar{Y})$.

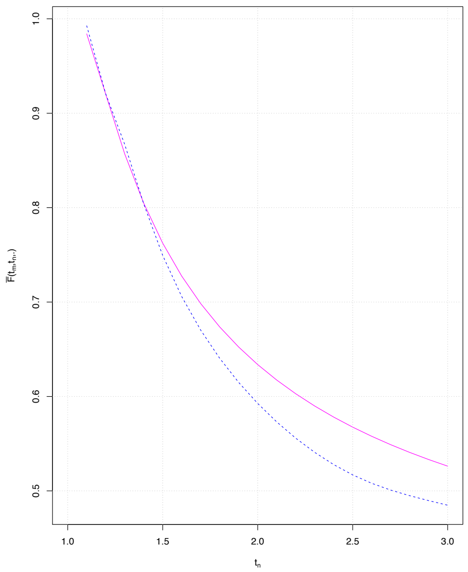
4.1.1 Convergence of the quantization

To compare the functions $F(s,t,\cdot)$ and $\widehat{F}(s,t,\cdot)$, we choose the following set of parameters (like those of [31]): $\mu = r = 0.03$, $\sigma = 0.09$, $\delta = 0.5$, $x_0 = y_0 = 86.3$ and $\mathbf{a} = 76$. Figure 1 shows the convergence of the quantized function $\widehat{F}(t_m,t_n,x)$ (dotted blue) towards the exact one, $F(t_m,t_n,x)$ (solid magenta). We set $t_m = 1$, $t_n \in [1.1, 3]$ and x to be one particular point on the grid $\{x_m^i, i = 1,\ldots,N_m\}$ (see (21)). Having fixed t_m, $F(t_m,t_n,x)$ depends on both t_n and x. Therefore, to show the convergence, we take a fixed x and plot both $F(t_m,t_n,x)$ and $\widehat{F}(t_m,t_n,x)$ with respect to t_n. The number of discretization points m is set to 50 and the convergence is achieved by increasing the number of quantization points N_n. The theoretical convergence rate is of the form $C \times m/N$, where the size of the quantization grids N is constant here along the time discretization steps: $N = N_m = N_n$. The constant C is an upper bound of $B_k(G)$ in Equation (36) and involves the Lipschitz coefficients (with respect to x_k and x_{k+1}, see Proposition 6.2 or [31, 27] for similar results) of the (discretized) survival probability under full information (the function $G^{x_k,x_{k+1}}_{\Delta_k\sigma_k^2}(a)$ in (14)). So, the constant C is in general difficult to compute. Putting for

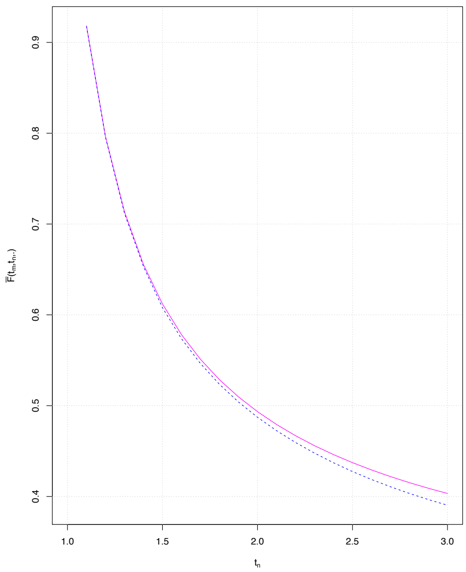
example $N = 100$ gives an error rate $0.5 \times C$. However, remark that the numerical results usually overperform the theoretical results in quantization methods (see [26, 28], for example).
Recall that in [31], the function $F(s,t,\cdot)$ were estimated by means of Monte Carlo simulations. So it will be interesting to compare the performances of our quantization scheme with the ones of Monte Carlo simulations when computing the survival function $F(s,t,\cdot)$. To do so, let's fix $x = 83.19103$ a given point in the grid $\{x_m^i, i = 1,\ldots,N_m\}$, $t_m = 1$ and $t_n = 1.5$. In this case, the exact value of $F(s,t,x)$ is 0.88518 using Equation (29). In order to achieve the desired level of convergence to the exact value, we need $6 \cdot 10^5$ paths when using Monte Carlo simulations while only 30 quantization points are required for the quantization scheme. The corresponding computation time is 0.08695 seconds for the quantization method and 2.42706 seconds with Monte Carlo simulations. We then reduce the computational cost for about 30 times by using quantization techniques.

4.1.2 Impact of the default information

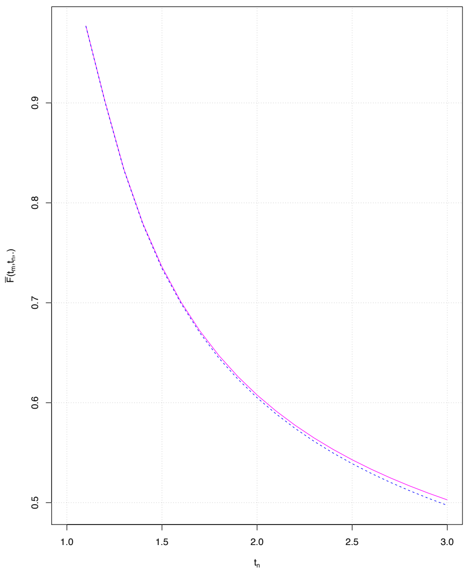
In order to check the statements in Remark 2.2, we proceed with the numerical comparison between the conditional default probabilities $\mathbb{P}(\tau_{\bar{X}} \leq t|\mathcal{F}_s^{\bar{Y}} \vee \mathcal{F}_s^{\bar{H}})$ and $\mathbb{P}(\tau_{\bar{X}} \leq t|\mathcal{F}_s^{\bar{Y}})$, respectively estimated by (23) and (30). Setting $s = t_m = 1$, $t = t_n$ and considering the same parameter set as in the previous figure but with $t_n \in [1.1, 11]$, Figure 2 depicts the trajectories of the observation process $\bar{Y}$ from 0 to t_m and the associated conditional default probabilities as a function of t_n. First, we notice that inequality (6) holds since, for the two trajectories of $\bar{Y}$ represented in Fig. 2(a) and Fig. 2(c), the curve $\mathbb{P}(\tau_{\bar{X}} \leq t|\mathcal{F}_s^{\bar{Y}})$ (dotted blue) dominates the curve $\mathbb{P}(\tau_{\bar{X}} \leq t|\mathcal{F}_s^{\bar{Y}} \vee \mathcal{F}_s^{\bar{H}})$ (solid magenta) in both Fig. 2(b) and Fig. 2(d). Second, it is clear by comparing these last two figures that the gap between these two curves is larger for a downward trajectory of $\bar{Y}$ than for an upward trend. This can be understood from the fact that the firm is less exposed to default risk in the former case than in the latter, and that the actual information about the default state of a firm is less relevant for a very creditworthy firm than for company with a higher default likelihood. In Figure 3, the left side graph (Fig. 3(a)) represents two trajectories of the observation process $\bar{Y}$ for the same $\delta = 0.5$, while the right side graph (Fig. 3(b)) gives the corresponding cumulative distribution functions $\mathbb{P}(\tau_{\bar{X}} \leq t|\mathcal{F}_s^{\bar{Y}} \vee \mathcal{F}_s^{\bar{H}})$. We observe that the corresponding default probability of the downward trajectory of $\bar{Y}$ (dashed magenta) is above the one of the upward trajectory (dotted blue). This is due to the fact that the less creditworthy a firm is, the more its default probability of probability. Figure 4 shows the dependence of default probabilities on the noise parameter δ. Note that, for different δ, have also the effect of a bad (downward) or good (upward) direction of the observation process $\bar{Y}$ on the default probabilities (see Figure 3). In order to limit this effect and only show the dependence on δ, we consider three trajectories of $\bar{Y}$ with different δ which have neither a downtrend nor an uptrend by keeping the same source of noise for each path. In Figure 4, the left side graph (Fig. 4(a)) represents two trajectories of the observation process $\bar{Y}$ for $\delta \in \{0.1, 0.3, 0.5\}$, while the right side graph (Fig. 4(b)) gives the corresponding cumulative distribution functions $\mathbb{P}(\tau_{\bar{X}} \leq t|\mathcal{F}_s^{\bar{Y}} \vee \mathcal{F}_s^{\bar{H}})$. Clearly, we notice that the noisier the observations are, the higher the default probability.



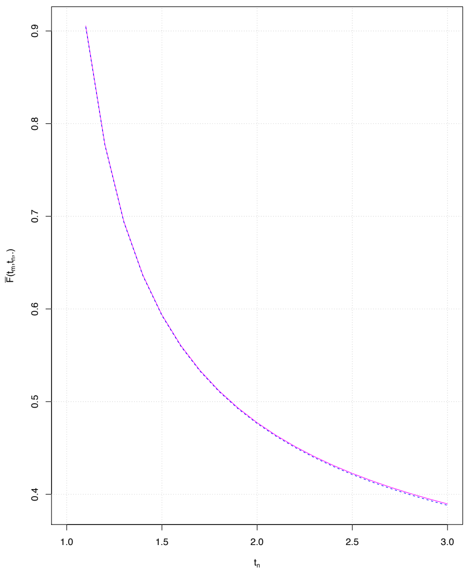
(a) $N_n = 10$



(b) $N_n = 30$

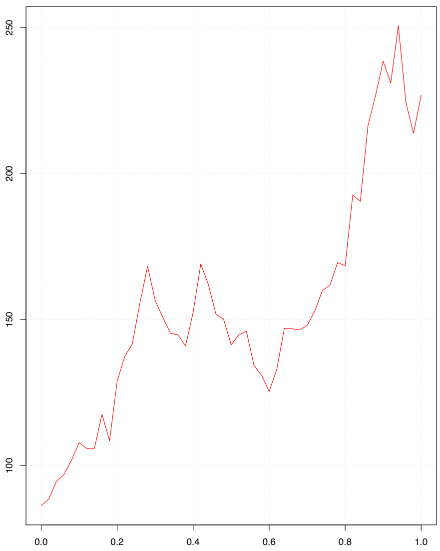


(c) $N_n = 50$

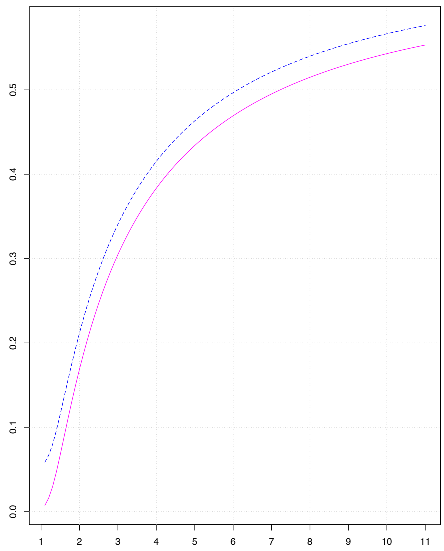


(d) $N_n = 100$

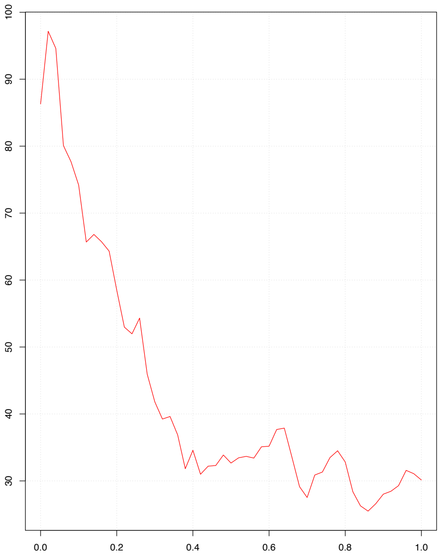
Figure 1: Convergence of $\bar{F}(t_m, t_n, \cdot)$ estimated by $\widehat{F}(t_m, t_n, \cdot)$ (dotted blue) towards $F(t_m, t_n, \cdot)$ (solid magenta). The chosen quantization point x_m can differ when moving N_n since $\bar{F}(t_m, t_n, \cdot)$ is a random variable. Hence, the corresponding figures can take different shapes but, we have only to make sure that the convergence is achieved when increasing N_n.



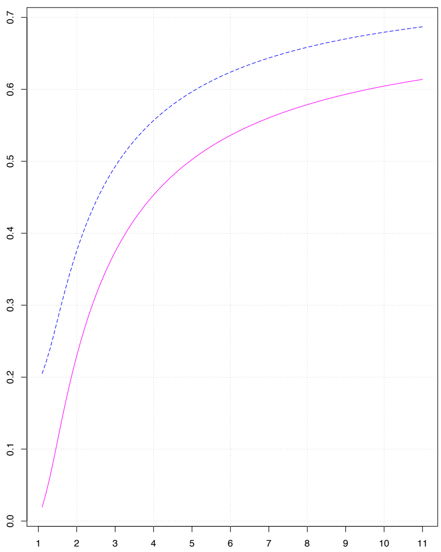
(a) $\bar{Y}$; Upward trajectory



(b) Default probability

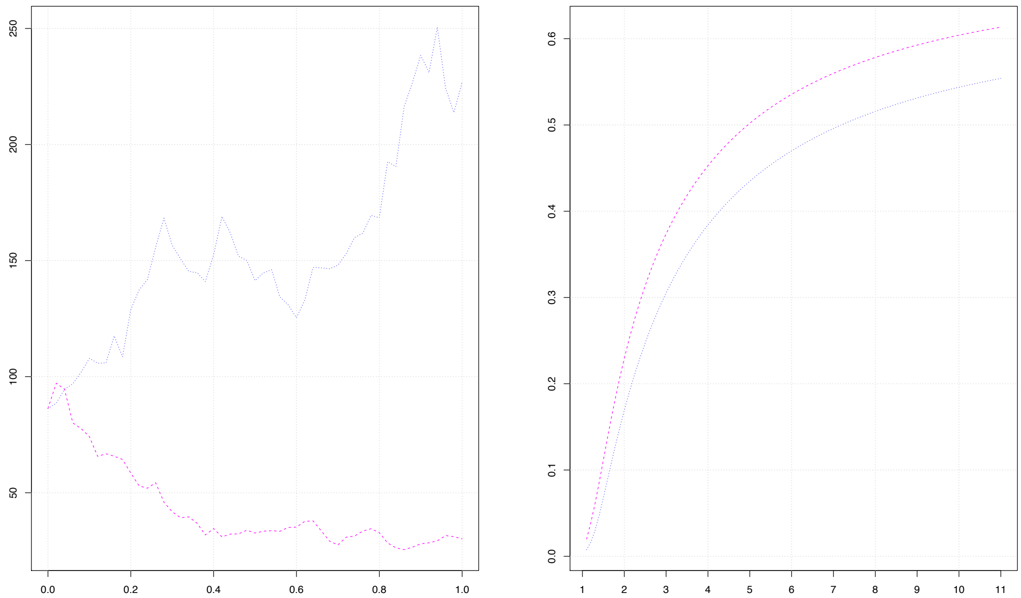


(c) $\bar{Y}$; Downward trajectory



(d) Default probability

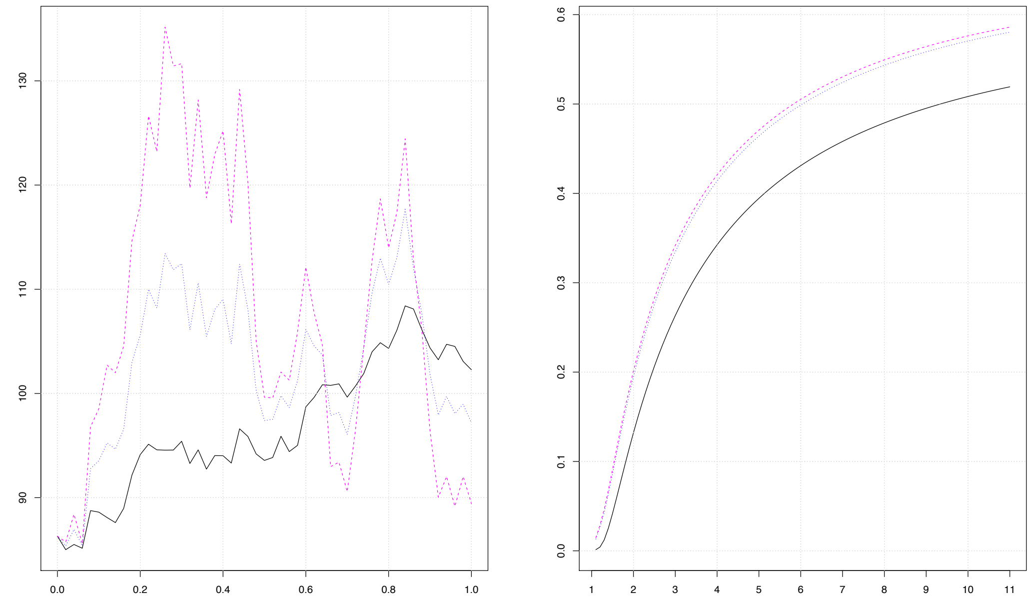
Figure 2: Upward (panel a) and downward (panel c) trajectories of $\bar{Y}$ and the associated conditional default probabilities functions $\mathbb{P}(\tau_{\bar{X}} \leq t | \mathcal{F}_s^{\bar{Y}} \vee \mathcal{F}_s^{\bar{H}})$ (solid magenta) and $\mathbb{P}(\tau_{\bar{X}} \leq t | \mathcal{F}_s^{\bar{Y}})$ (dashed blue) (panels b and d), with $t_m = 1$ and $t_n \in [1.1, 11]$ and number of quantization points $N_n = N_m = 30$.



(a) Trajectories of $\bar{Y}$

(b) Default probability

Figure 3: Two trajectories of $\bar{Y}$ (panel a) and the associated conditional default probabilities $\mathbb{P}(\tau_{\bar{X}} \leq t|\mathcal{F}_s^{\bar{Y}} \vee \mathcal{F}_s^{\bar{H}})$ (panel b). Upward (dotted blue) and downward (dashed magenta) trajectories of $\bar{Y}$ and the associated conditional default probabilities functions, with $\delta = 0.5$, $t_m = 1$ and $t_n \in [1.1, 11]$ and number of quantization points $N_n = N_m = 30$.



(a) Trajectories of $\bar{Y}$

(b) Default probability

Figure 4: Three trajectories of $\bar{Y}$ (panel a) and the associated conditional default probabilities $\mathbb{P}(\tau_{\bar{X}} \leq t | \mathcal{F}_s^{\bar{Y}} \vee \mathcal{F}_s^{\bar{H}})$ (panel b) in function of the noise parameter $\delta = 0.1$ (solid black), $\delta = 0.3$ (dotted blue), $\delta = 0.5$ (dashed magenta), with $t_m = 1$ and $t_n \in [1.1, 11]$ and number of quantization points $N_n = N_m = 30$.

4.2 Application to CDS option pricing

In this section, we apply our model to the pricing of CDS options. We start with a brief reminder about CDS and options on the latter (a.k.a. credit swaptions). The specific pricing formulae will be given in a firm-value approach using partial information theory and optimal quantization.

A CDS is an agreement between two counterparties to buy or sell protection against the default risk of a third party called *reference entity*. We note τ_X the default time of the latter. In this case, if the contract is signed at time s, starts at time T_a and has maturity T_b, the protection buyer pays a coupon (or spread) k at payments dates $T_{a+1}, \ldots, T_b$ as long as the reference entity does not default or until τ_X (whichever comes first). If the default occurs at time τ_X with $T_a < \tau_X \leq T_b$, the protection seller will make a single payment LGD (that we assume to be a known constant) to the protection buyer. A CDS option (CDSO) is an over-the-counter European option written on a CDS; it gives the right (or the obligation) to enter into a specific CDS contract at some future point in time, with a given counterparty. From this perspective, it is worth recalling the no-arbitrage pricing equations of those products. For more details about the mechanics of this product, we refer to [4] and [5].

The time-s price of a general buyer CDS with unit notional starting at time T_a with maturity T_b, $s \leq T_a < T_b$, a credit spread k and loss given default LGD is noted $CDS_s(a, b, k)$. It is given by the difference of the conditional risk-neutral expectations of the protection and the premium discounted cashflows (see [4]):

$$\begin{aligned} CDS_s(a,b,k) =& \,\mathbb{E}\left[LGD\mathbb{1}_{\{T_a<\tau_X\le T_b\}}D_s(\tau_X)|\mathcal{F}_s\right] \\ & -k\mathbb{E}\left[\sum_{i=a+1}^{b}\left(\mathbb{1}_{\{\tau_X\ge T_i\}}\alpha_i D_t(T_i)+\mathbb{1}_{\{T_{i-1}\le\tau_X<T_i\}}\alpha_i\frac{\tau_X-T_{i-1}}{T_i-T_{i-1}}D_t(\tau_X)\right)\middle|\,\mathcal{F}_s\right] \end{aligned}$$

with α_i the day count fraction between dates T_{i-1} and T_i which, in a standard CDS, is around 0.25 (quarterly payment dates) and $D_s(t) = e^{-r(t-s)}$ is a time-s discount factor with maturity t and constant interest rates r. In an enlargement of filtration setup, where $\mathcal{F}_s := \mathcal{F}_s^Y \vee \mathcal{F}_s^H$, this expression can be developed explicitly thanks to the Key lemma:

$$CDS_s(a,b,k) = \mathbb{1}_{\{\tau_X>s\}}\left(-LGD\int_{T_a}^{T_b}D_s(u)\partial_u P_s(u)du - k\,C_s(a,b)\right), \tag{31}$$

where

$$P_s(t) := \frac{S_s(t)}{S_s(s)} \tag{32}$$

and $S_s(t) = \mathbb{P}\left(\tau_X \ge t\middle|\mathcal{F}_s^Y\right)$ is known as the *Azéma supermartingale* and $C_s(a,b)$ is the risky duration, i.e., the time-s value of the CDS premia paid during the life of the contract for a unit spread $k = 1$:

$$C_s(a,b) := \sum_{i=a+1}^{b}\alpha_i D_s(T_i)P_s(T_i) - \int_{T_{i-1}}^{T_i}\frac{u-T_{i-1}}{T_i-T_{i-1}}\alpha_i D_s(u)\partial_u P_s(u)du.$$

The spread which, at time s, sets the forward start CDS at 0, called *par spread*, is given by:

$$\mathbb{1}_{\{\tau_X>s\}}k_s^\star(a,b) := \mathbb{1}_{\{\tau_X>s\}}\frac{-LGD\int_{T_a}^{T_b}D_s(u)\partial_u P_s(u)du}{C_s(a,b)}. \tag{33}$$

The no-arbitrage price of a call option on such a contract at time $s = 0$ becomes

$$\begin{aligned} PSO(a,b,k) =& \,\mathbb{E}\left[(CDS_{T_a}(a,b,k))^+ D_0(T_a)\right] \\ =& \,D_0(T_a)\mathbb{E}\left[S_{T_a}(T_a)\left(LGD - \sum_{i=a+1}^{b}\int_{T_{i-1}}^{T_i}g_i(u)D_{T_a}(u)P_{T_a}(u)du\right)^+\right] \end{aligned} \tag{34}$$

where $g_i(u) := LGD(r + \delta_{T_b}(u)) + k\frac{\alpha_i}{T_i-T_{i-1}}(1-(u-T_{i-1})r(u))$, with $\delta_s(u)$ the Dirac delta function centered at s.
The survival processes $S_{T_a}(\cdot)$ and the future prices of discount bond $P_{T_a}(\cdot)$ can be computed using our optimal quantization approach. To do so, using equations (23) and (30), one only needs to set

$$P_{T_a}(u) = \widehat{\Pi}_{y,a}\widehat{F}(T_a,u,\cdot) \quad \text{and} \quad S_{T_a}(T_a) = \widehat{\varpi}_{y,a}\mathbf{1}. \tag{35}$$

Hence the randomness that enters the expectation (34) comes solely from the trajectory of the observation process Y on $[0, T_a)$. Thus, after computing the aforementioned survival processes using optimal quantization, the expectation (34) can be estimated using Monte Carlo simulation. This justifies the fact that (34) can be computed using a hybrid Monte Carlo-optimal quantization procedure.

Just like usual equity options, it is the standard market practice to quote them in term of its Black implied volatility $\bar{\sigma}$, based on the assumption that the credit spread follows a geometric Brownian motion.[2] Using the results based on [6], the Black formula for (payer) credit swaptions at time 0 with maturity T_a is

$$PSO^{Black}(a, b, k, \bar{\sigma}) = C_0(a, b) \left[k_0^\star(a, b)\Phi(d_1) - k\Phi(d_2)\right]$$

where

$$d_1 = \frac{\ln \frac{k_0^\star(a,b)}{k} + \frac{1}{2}\bar{\sigma}^2 T_a}{\bar{\sigma}\sqrt{T_a}}, \qquad d_2 = d_1 - \bar{\sigma}\sqrt{T_a}.$$

Hence, the implied volatility $\bar{\sigma}$ associated with the CDS option price $PSO(a, b, k)$ is given by finding the value $\bar{\sigma}$ solving

$$PSO(a, b, k) = PSO^{Black}(a, b, k, \bar{\sigma}).$$

We now discuss the figures implied by our model in the context of CDS options pricing. The set of parameters is the same as before, except that here we take $\sigma = 5\%$ and δ is varying.[3] Table 1 shows the estimated premiums of a payer CDS option and the corresponding Black volatilities with various strikes and noise powers δ. First, we notice that the value of the payer CDS option is decreasing with respect to the strike for a fixed δ. Second, we observe that both CDS option prices and implied volatilities are increasing with δ. This can be explained by the fact that the higher the δ, the noisier the observations, the higher the uncertainty, and the higher the perceived default probability. It follows an increase of the prices of the payer CDS option and the corresponding implied volatilities. By contrast, while the option prices are always decreasing with the strike, this is not the case with the implied volatilities except for $\delta = 2\%$ and $\delta = 3\%$. In the case where $\delta = 1\%$, the implied volatility is increasing with respect to the strike. Hence, depending on the parameter δ, one observes different skews, as shown in Figure 5 using a three-dimensional representation of the implied volatility.

k (bps)	Payer			Implied vol (%)		
	$\delta = 1\%$	$\delta = 2\%$	$\delta = 3\%$	$\delta = 1\%$	$\delta = 2\%$	$\delta = 3\%$
52.9	0.004655	0.007214	0.009276	69.44	133.84	196.80
66.2	0.003739	0.006077	0.008032	73.36	124.16	173.70
79.4	0.003107	0.005298	0.007081	76.85	121.08	161.57

Table 1: CDS option prices and corresponding Black volatilities (with spread $k_0^\star(a, b) = 66.20$ pbs, $T_a = 1$ and $T_b = 3$) implied by the structural model using Monte Carlo simulations ($1.5 \cdot 10^5$ paths) for $\sigma = 5\%$, various noise powers δ and different strikes (80%, 100%, 120%)$k_0^\star(a, b)$. The loss given default is $LGD = 60\%$ and the number of quantization points is $N_n = N_m = 30$

.

[2]Recall that this does not mean in any way that the market naively believes that credit spreads exhibit log-normal dynamics. Market participants simply rely on the Black-Scholes machinery to convert a price into a quantity that is more intuitive to traders, namely implied volatilities.

[3]Notice that the purpose of this example is to illustrate the behavior of the model. The calibration of the model to actual market data is beyond the scope of the paper and is left for future work. In particular, we fixe the parameter σ to 5% in order to reproduce realistic volatilities in the CDS options since we increase the noise parameter δ.

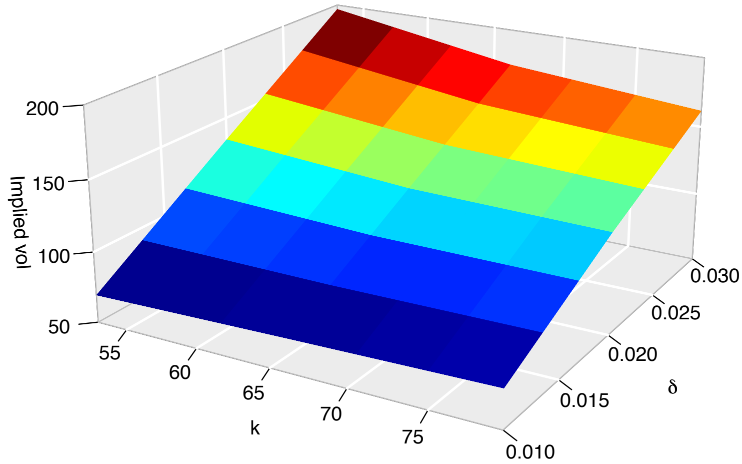


Figure 5: CDS option implied volatility surface with respect to different strikes (80%, 100%, $120\%)k_0^\star(a,b)$ and various noise powers δ (1%, 2%, 3%) using Monte Carlo simulations ($1.5 \cdot 10^5$ paths). We consider a par spread $k_0^\star(a,b) = 66.20$ pbs, $T_a = 1$, $T_b = 3$ and $\sigma = 5\%$. The loss given default is $LGD = 60\%$ and the number of quantization points is $N_n = N_m = 30$.

Note that with $N_n = N_m = 30$, the computational time to generate the quantization grid is about 0.16779 seconds. Once the grid is generated, it takes 0.26455 seconds to compute the Azéma supermartingale $S_{T_a}(T_a)$ in Equation (35). The corresponding computation time for the survival probability $P_{T_a}(u)$ (see (35)) for a fixed u is 0.32017 seconds when using the exact formula of the function $F(T_a, u, \cdot)$ and 18.39194 seconds is required when using the quantization of the function $F(T_a, u, \cdot)$. Indeed the CPU time of the CDS option computation using Equation (34) takes about 111 hours [4] with the parameters used in Table 1. In the computation of the CDS option, the exact formula of $F(T_a, u, \cdot)$ were used to compute $P_{T_a}(\cdot)$ in (35).

[4]The code was executed with a DELL CORE i5 vPro and the coding language is Rcpp.

5 Conclusion

In this paper, a new structural model for credit risk has been proposed. Our model deals with incomplete information, where the default state and a noisy observation of the firm's value are accessible to the investor. It can be seen as an extension of [13], as the firm value process triggering the default is no longer restricted to be a continuous and invertible function of a Gaussian martingale, but can be any diffusion. This more general framework benefits however from limited analytical tractability. Therefore, we propose a numerical method that relies on nonlinear filtering theory associated with recursive quantization. Compared to earlier works such as [11] or [31], our numerical procedure is based on the fast quantization method recently introduced in [28], which avoids using Monte Carlo simulations to compute the conditional survival probabilities. A rigorous analysis of the global error impacting the estimation of the survival processes is performed. We analyze the shapes of the default probabilities which are characterized by a path-dependent feature keeping the memory of the whole path of the observed process. Eventually, we quantify the impact of the volatility of the noise impacting the firm value process on the pricing of CDS options and the corresponding implied volatilities using a hybrid Monte Carlo-optimal quantization method.

Future research will deal with the calibration of the model to actual market prices. Exact calibration to a given term-structure of credit spreads should be possible by adopting a time-dependent barrier (as in [8]) or using time change techniques (similar to [24]). Another possible research area is to extend the applications to other credit-sensitive securities. A full quantization scheme was proposed to estimate the conditional default probabilities, but not for the pricing of CDS option; the latter involved Monte Carlo simulations. One possible route towards a full quantization scheme would be to exploit functional quantization methods. Finally, the proposed model can also be useful to deal with counterparty risk with possibly an extension to the multi-dimensional case by following the work of [21] or [1]. The implementation of the latter may be challenging because of its complexity intrinsically linked to the generality of the model we have considered in this paper.

6 Appendix

In this section, we investigate the analysis of the error resulting from the approximation of

$$\Pi_{y,m}\bar{F}(t_m,t_n,\cdot) = \big[\Pi_{y,m}\big(\pi_{n,m}\mathbf{1}\big)\big](\cdot)$$

by $\big[\widehat{\Pi}_{y,m}\big(\hat{\pi}_{n,m}\mathbf{1}\big)\big](\cdot)$. This error is an aggregation of three terms (see the proof of Theorem 6.4) involving the approximation errors $|\Pi_{y,m}\bar{F}(t_m,t_n,\cdot) - \widehat{\Pi}_{y,m}\bar{F}(t_m,t_n,\cdot)|$ and $|\big(\pi_{n,m}\mathbf{1}\big)(x) - \big(\hat{\pi}_{n,m}\mathbf{1}\big)(x)|$, $x \in \mathbb{R}$. The two following results give the bounds associated with the former approximation errors. They are variants of Theorem 4.1. and Lemma 4.1. in [27] tailored to our context. Hence, we refer to the above reference for more detailed proofs.

Theorem 6.1. *Suppose that Assumption (*Lip*) holds true. Then, for any bounded Lipschitz continuous function f on $\mathbb{R}^d$ we have,*

$$|\Pi_{y,m}f - \widehat{\Pi}_{y,m}f| \leq \frac{K_g^m}{\phi_m \vee \hat{\phi}_m} \sum_{k=0}^{m} A_k^m(f,y)\, \|\bar{X}_{t_k} - \widehat{X}_{t_k}^{\Gamma_k}\|_2,$$

where

$$\phi_m := \pi_{y,m}\mathbf{1}, \quad \widehat{\phi}_m := \widehat{\pi}_{y,m}\mathbf{1},$$

$$A_k^m(f,y) := 2\frac{\|f\|_\infty}{K_g^m}[g_k^2]_{\rm Lip}(y_{k-1},y_k) + 2\frac{\|f\|_\infty}{K_g^m}\sum_{j=k+1}^{m}[\mathcal{E}]_{Lip}^{j-k-1}\Big([g_j^1]_{\rm Lip}(y_{j-1},y_j) + [\mathcal{E}]_{\rm Lip}[g_j^2]_{Lip}(y_{j-1},y_j)\Big),$$

and, for every $k \in \{1,\ldots,m\}$, $[g_k^1]_{\rm Lip}(y,y')$ and $[g_k^2]_{\rm Lip}(y,y')$ are such that for every $x, x', \hat{x}, \hat{x}' \in \mathbb{R}^d$,

$$|g_k^a(x,y;x',y') - g_k^a(\widehat{x},y;\widehat{x}',y')| \le [g_k^1]_{\rm Lip}(y,y')\,|x-\widehat{x}| + [g_k^2]_{\rm Lip}(y,y')\,|x'-\widehat{x}'|.$$

The quantities K_g and $[\mathcal{E}]_{\rm Lip}$ are defined as

$$K_g = \max_{k=1,\cdots,m}\|g_k^a\|_\infty \qquad \textit{and} \qquad [\mathcal{E}]_{\rm Lip} = \max_{k=1,\ldots,m}[\mathcal{E}_k]_{\rm Lip}.$$

Remark that the existence of $[g_k^1]_{\rm Lip}(y_{k-1},y_k)$ and $[g_k^2]_{\rm Lip}(y_{k-1},y_k)$ is guaranteed by the fact that the function $g_k^a(x,y;x',y')$ is Lipschitz with respect to (x,x').

Let us give now the error bound associated to the approximation of $\pi_{n,m}\mathbf{1}$.

Proposition 6.2. *Let $y = (y_0,\ldots,y_m) \in (\mathbb{R}^q)^{m+1}$. Then, we have for any $x \in \mathbb{R}$*

$$\big|\big(\pi_{n,m}\mathbf{1}\big)(x) - \big(\hat{\pi}_{n,m}\mathbf{1}\big)(x)\big| \le \sum_{k=m+1}^{n} B_k(G)\,\|\bar{X}_{t_k} - \widehat{X}_{t_k}^{\Gamma_k}\|_2 \tag{36}$$

where

$$B_k(G) = \Lambda^{n-m-1}\Big([G_m^1]_{\rm Lip}\,\delta_{\{k=m\}} + \big([G_k^1]_{\rm Lip} \vee [G_{k-1}^2]_{\rm Lip}\big)\,\delta_{\{k\in\{m+1,\ldots,n-1\}\}} + [G_n^2]_{\rm Lip}\,\delta_{\{k=n\}}\Big)$$

with

$$\Lambda = \max_{k=m+1,\ldots,n}\big\|G_{\Delta_k\sigma_k^2}^{(\bullet,\bullet)}(\mathbf{a})\big\|_\infty.$$

Proof. Recall that

$$\big(\pi_{n,m}\mathbf{1}\big)(x) = \mathbb{E}\Big(\Lambda_m(\bar{X}_{t_{m:n}})\big|\,\bar{X}_{t_m} = x\Big)$$

where for every $k \ge m$,

$$\Lambda_m(\bar{X}_{t_{m:k}}) := \prod_{\ell=m}^{k-1} G_{\Delta_k\sigma_k^2}^{\bar{X}_{t_k},\bar{X}_{t_{k+1}}}(\mathbf{a}),$$

with the convention that $\Lambda_m(\bar{X}_{t_{m:m}}) = 1$. Now, we have for any $k \ge m$,

$$\Lambda_m(\bar{X}_{t_{m:k}}) - \Lambda_m(\hat{X}_{t_{m:k}}) = \Big(G_{\Delta_{k-1}\sigma_{k-1}^2}^{\bar{X}_{t_{k-1}},\bar{X}_{t_k}}(\mathbf{a}) - G_{\Delta_{k-1}\sigma_{k-1}^2}^{\hat{X}_{t_{k-1}},\hat{X}_{t_k}}(\mathbf{a})\Big)\Lambda_m(\bar{X}_{t_{m:k-1}}) + G_{\Delta_{k-1}\sigma_{k-1}^2}^{\hat{X}_{t_{k-1}},\hat{X}_{t_k}}(\mathbf{a})\big(\Lambda_m(\bar{X}_{t_{m:k-1}}) - \Lambda_m(\hat{X}_{t_{m:k-1}})\big).$$

Since the function $G_{\Delta_k\sigma_k^2}^{(\bullet,\bullet)}(\mathbf{a})$ is Lipschitz and bounded and that for any $k \ge m+1$, $\Lambda_m(\bar{X}_{t_{m:k-1}}) \le \Lambda^{k-m-1}$, we have

$$|\Lambda_m(\bar{X}_{t_{m:k}}) - \Lambda_m(\hat{X}_{t_{m:k}})| \le \Big([G_k^1]_{\rm Lip}|\bar{X}_{t_{k-1}} - \hat{X}_{t_{k-1}}| + [G_k^2]_{\rm Lip}|\bar{X}_{t_k} - \hat{X}_{t_k}|\Big)\Lambda^{k-m-1} + \Lambda|\Lambda_m(\bar{X}_{t_{m:k-1}}) - \Lambda_m(\hat{X}_{t_{m:k-1}})|.$$

Keeping in mind that $\Lambda_m(\bar{X}_{t_{m:m}}) = \Lambda_m(\hat{X}_{t_{m:m}}) = 1$, we deduce from an induction on k that

$$|\Lambda_m(\bar{X}_{t_{m:n}}) - \Lambda_m(\hat{X}_{t_{m:n}})| \leq \Lambda^{n-m-1} \sum_{k=m+1}^{n} [G_k^1]_{\rm Lip} |\bar{X}_{t_{k-1}} - \hat{X}_{t_{k-1}}| + [G_k^2]_{\rm Lip} |\bar{X}_{t_k} - \hat{X}_{t_k}|.$$

The result follows by noting that

$$\begin{aligned} \big|\big(\pi_{n,m}\mathbf{1}\big)(x) - \big(\hat{\pi}_{n,m}\mathbf{1}\big)(x)\big| &\leq \mathbb{E}\big(|\Lambda_m(\bar{X}_{t_{m:n}}) - \Lambda_m(\hat{X}_{t_{m:n}})|\,\big|\bar{X}_{t_m} = x\big) \\ &\leq \Lambda^{n-m-1} \sum_{k=m+1}^{n} [G_k^1]_{\rm Lip} |\bar{X}_{t_{k-1}} - \hat{X}_{t_{k-1}}| + [G_k^2]_{\rm Lip} |\bar{X}_{t_k} - \hat{X}_{t_k}| \end{aligned}$$

and by using the non-decreasing property of the L^p-norm. □

The global error induced by our numerical procedure is obtained by analyzing the error resulting from the estimation of

$$\Pi_{y,m} F(t_m, t_n, \cdot) = \mathbb{P}\big(\tau_X > t_n |\, (Y_{t_0}, \ldots, Y_{t_m}) = y\big)$$

using (23). To compute the convergence rate of the quantity $\mathbb{E}\big|\mathbb{1}_{\{\tau_{\bar{X}} > t\}} - \mathbb{1}_{\{\tau_X > t\}}\big|$ towards 0, we need the following additional assumptions (see [17]). We suppose that the diffusion X is homogeneous and

(**H1**) b is a $\mathcal{C}_b^\infty(\mathbb{R})$ function and σ is in $\mathcal{C}_b^\infty(\mathbb{R})$.

(**H2**) There exists $\sigma_0 > 0$ such that $\forall x \in \mathbb{R}, \sigma(x)^2 \geq \sigma_0^2$ (*uniform ellipticity*).

We have the following result.

Proposition 6.3 (See [17])**.** *Let $t > 0$. Suppose that Assumptions (**H1**) and (**H2**) are fulfilled. Then, for every $\eta \in (0, \frac{1}{2}[$ there exists an increasing function $K(T)$ such that for every $t \in [0, T]$ and for every $x \in \mathbb{R}$,*

$$\mathbb{E}_x\left[\left|\mathbb{1}_{\{\tau_X > t\}} - \mathbb{1}_{\{\tau_{\bar{X}} > t\}}\right|\right] \leq \frac{1}{n^{\frac{1}{2}-\eta}} \frac{K(T)}{\sqrt{t}},$$

where n is the number of discretization time steps over $[0, t]$.

Theorem 6.4. *Suppose that the coefficients b and σ of the continuous signal process X are such that Assumptions (**H1**) and (**H2**) are satisfied and let $\eta \in (0, \frac{1}{2}]$. We also suppose that Assumption (Lip) holds. Then, for any bounded Lipschitz continuous function $f : \mathbb{R}^d \mapsto \mathbb{R}$ and for any fixed observation $y = (y_0, \ldots, y_m)$ we have*

$$\begin{aligned} \Big|\mathbb{P}\big(\tau_X > t_n | (\bar{Y}_{t_0}, \ldots, \bar{Y}_{t_m}) = y\big) - \sum_{i=1}^{N_n} \sum_{j=1}^{N_m} \widehat{\Pi}_{y,m}^i \widehat{\pi}_{n,m}\, \delta_{x_m^j}\Big| &\leq \mathcal{O}\Big(n^{-\frac{1}{2}+\eta}\Big) \\ &\quad + \sum_{k=0}^{n} C_k^n(\bar{F}(s,t,\cdot), y)\, \|\bar{X}_{t_k} - \widehat{X}_{t_k}^{\Gamma_k}\|_2, \end{aligned}$$

where

$$C_k^n(\bar{F}(s,t,\cdot), y) = \frac{K_g^m}{\phi_m \vee \hat{\phi}_m} A_k^m(f, y)\, \delta_{\{k \leq m\}} + B_k(G)\, \delta_{\{k \geq m+1\}}$$

and where K_g^m, ϕ_m, $\hat{\phi}_m$, $A_k^m(f, y)$ and $B_k(G)$ are defined in Theorem 6.1 and in Proposition 6.2.

Proof. We have

$$\begin{aligned}
\Big|\mathbb{P}(\tau_X > t_n|(\bar{Y}_{t_0},\ldots,\bar{Y}_{t_m}) = y) - \sum_{i=1}^{N_n}\sum_{j=1}^{N_m}\widehat{\Pi}^i_{y,m}\widehat{\pi}_{n,m}\,\delta_{x_m^j}\Big| \\
\le \big|\mathbb{P}(\tau_X > t_n|(\bar{Y}_{t_0},\ldots,\bar{Y}_{t_m}) = y) - \mathbb{P}(\tau_{\bar{X}} > t_n|(\bar{Y}_{t_0},\ldots,\bar{Y}_{t_m}) = y)\big| \\
+ \big|\Pi_{y,m}\bar{F}(t_m,t_n,\cdot) - \widehat{\Pi}_{y,m}\bar{F}(t_m,t_n,\cdot)\big| \\
+ \big|\widehat{\Pi}_{y,m}\bar{F}(t_m,t_n,\cdot) - \widehat{\Pi}_{y,m}\hat{F}(t_m,t_n,\cdot)\big|. \qquad (37)
\end{aligned}$$

Moreover,

$$\begin{aligned}
\big|\widehat{\Pi}_{y,m}\bar{F}(t_m,t_n,\cdot) - \widehat{\Pi}_{y,m}\hat{F}(t_m,t_n,\cdot)\big| &= \Big|\sum_{i=1}^{N_m}\Big(\widehat{\Pi}^i_{y,m}\bar{F}(t_m,t_n,x_m^i) - \widehat{\Pi}^i_{y,m}\hat{F}(t_m,t_n,x_m^i)\Big)\Big| \\
&\le \sup_{x\in\mathbb{R}}\big|\bar{F}(t_m,t_n,x) - \hat{F}(t_m,t_n,x)\big|\sum_{i=1}^{N_m}\widehat{\Pi}^i_{y,m} \\
&= \sup_{x\in\mathbb{R}}\big|\bar{F}(t_m,t_n,x) - \hat{F}(t_m,t_n,x)\big| \\
&= \sup_{x\in\mathbb{R}}\big|(\pi_{n,m}\mathbf{1})(x) - (\hat{\pi}_{n,m}\mathbf{1})(x)\big|.
\end{aligned}$$

The error bound for $|(\pi_{n,m}\mathbf{1})(x) - (\hat{\pi}_{n,m}\mathbf{1})(x)|$ is independent from x and is given by (36). On the other hand we also have

$$\begin{aligned}
\big|\mathbb{P}(\tau_X > t_n|(\bar{Y}_{t_0},\ldots,\bar{Y}_{t_m}) = y) - \mathbb{P}(\tau_{\bar{X}} > t_n|(\bar{Y}_{t_0},\ldots,\bar{Y}_{t_m}) = y)\big| \\
\le \mathbb{E}\big(|\mathbb{1}_{\{\tau_X > t_n\}} - \mathbb{1}_{\{\tau_{\bar{X}} > t_n\}}|\big|(\bar{Y}_{t_0},\ldots,\bar{Y}_{t_m}) = y\big) \\
\le \frac{1}{\mathbb{P}((\bar{Y}_{t_0},\ldots,\bar{Y}_{t_m}) = y)}\,\mathbb{E}\big|\mathbb{1}_{\{\tau_X > t_n\}} - \mathbb{1}_{\{\tau_{\bar{X}} > t_n\}}\big|.
\end{aligned}$$

Proposition 6.3 and Theorem 6.1 conclude the proof. □

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