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Modeling of a binomial decision tree for real options

Dissertation presented by
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Abstract

The conventional valuation methods for investment decisions are mainly based on simplistic methods but real world decisions are confronted more and more with uncertain future events which are not taken into account in the conventional calculations. Methods like the net present value and the decision tree analysis undervalue certain investments where some options, like an option to defer or to expand after a certain time, are present. With the real option analysis this additional flexibility is taken into account and can therefore better predict the value of the investment. These options can put a floor on the loss with the arrival of new information when they are exercised. The use of binomial option pricing gives a firm a better valuation of the return of a project in the presence of managerial flexibility in uncertain times. The thesis will cover the advantages and disadvantages as well as the new models, the Cox-Ross-Rubinstein and the Trigeorgis log-transformed methods, which will be tested and there will be a sensitivity analysis to further demonstrate the behavior of the implemented method.

Keywords : Binomial Tree Analysis, Real Option Valuation, Uncertainty, Stochastic Calculus

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Contents

List of Abbreviations

NPV	**N**et **P**resent **V**alue
DTA	**D**ecision **T**ree **A**nalysis
B&S	**B**lack & **S**choles model
CRR	**C**ox-**R**oss-**R**ubinstein
CCA	**C**ontingent **C**laim **A**nalysis

List of Symbols

Mathematical Background

t	Time
r	Discount rate
C_i	Cash flow at time i
N	Number of periods
B	Brownian motion
σ	Volatility
ϵ_t	Random normally distributed variable with mean 0 and standard variance of 1 and uncorrelated from each other for all t
Δ	Change over a time step or a step in the binomial tree
x	Ito process
α or $a(x,t)$	Drift of an Ito process
η	rate of return to the mean value
$\bar{x}$	Mean of the given asset
λ	Mean arrival rate of a Poisson process
p	probability of an up-movement in a binomial tree
$\mathbb{E}$	Expected value
$\mathbb{V}$	Variance

Introduction to Real Options

B	Bond value
S	Share value
n	Number of shares
$C(t,S_t)$	Call option price
$P(t,S_t)$	Put option price
K	Strike price
$N(x)$	Standard normal cumulative distribution function

Modeling of the Real Options

V	Value of the project
S	Price of a share with nearly perfect correlation to V
E	Equity value of the project
S_u or S_d	Return of the shares for an upward or downward movement respectively
r_f	Risk free interest rate
p	Risk neutral probability for up-movements inside the binomial tree
q	Real probability for up-movements inside the binomial tree
u	Multiplicative factor for up-movements inside the binomial tree
d	Multiplicative factor for down-movements inside the binomial tree
n	Number of shares

$R(i,j)$	Value of each node i for each level j
V_t	Geometric Brownian motion
α	Instantaneous expected return of the project
σ	Instantaneous standard derivation
Y_t	Arithmetic Brownian motion
H	Change of the arithmetic Brownian motion Y_t
C	External cash flows
I	Investment value
c	Multiplicative contracted value
e	Multiplicative expansion value
X	Salvage value for an investment
τ	Time step

Chapter 1

Introduction

An investment decision is a very important part of every company around the globe. The question, which everyone is asking, is if the investment is worth its cost. A project can be a change in production, a new facility, etc. in other words it can be a range of different things. For example, for a mobile phone company, the project can be the development of a new product. This would mean that there is the development cost of the new parts and the production of the said new phone. We can imagine a different project for every single company in the world. But we need to know if it creates value as the shareholders want to further increase their portfolio. And since the manager wants to keep its job he needs to consider the value creation. One can say that it is an easy decision while knowing all the cash flows, but in the real world we have at every moment an uncertainty about the future. So, we cannot say for sure what will be the cash flows of the future, the economy can go bad or maybe 'get better, the firm can lose or gain clients and so on. Without the uncertainty we could easily establish the value through the net present value of the project, however in the real world, this is not the case.

The following work will present a better valuation method than the widely used net present value (NPV). Due to more powerful computers we can perform more complex calculations at a faster speed and therefore, we can use methods which were not yet possible. The NPV method is indeed very easy to apply but with the uncertainty of the future it is not always the correct method to use. For bigger firms, small calculation errors can cost millions even though it is just an error of a thousandth. The tool that has to be created should be very easy to use and give better results without losing the simplicity of the NPV method. What is the biggest drawback of the NPV is its assumption of no uncertainty in the future cash flows and its constant discount rate. In real life both factors change with the market and with additional decisions of each firm which would imply that the initially calculated value is not correct anymore if the situations changes in the future. This is why it is needed to introduce a new way to calculate the value of a given project, so changing factors in the economy and inside the firm can be taken into account. Additionally, we need to be able to implement real options (see chapter 3) which would be a very difficult job with the NPV method. More on this topic will be discussed in chapter 2.

What is also important to notice is that the investment costs are sunk costs that means once the initial costs are paid it cannot be reimbursed so the investment becomes irreversible. Another point is the delay of the project which means that we pay the initial cost but wait before we execute the project which can change the decision to undergo the given project. It is like a financial American call option where you have the possibility but not the obligation to buy this asset and it can be exercised at any moment.

The aim of this thesis is the incorporation and the analysis of the existing methods for the real option valuation for binomial trees. As well as the decision why those methods are better than the NPV method in situations with an uncertain future. Finally, we need to show if we can put in place any investment decision we can imagine.

There are already 2 methods created, one by Cox Ross and Rubinstein and one by Trigeorgis. We will therefore base our valuation method on exactly those methods and we will further analyse their stability and their limitations. Additionally, the differences and the drawbacks are presented and compared. Another important question is why we would want a different method than the NPV method which is used in most of the situations. A step by step amelioration is provided in chapter 4. From here all the methods seen are implemented and the reason why each method is or is not appropriate for real options is elaborated with a numerical example. The last thing done in this work is the analysis for each component of the methods based on the binomial tree is analysed for their impact and their behaviour of the final value. Since the uncertainty of future events will impact also the risk-free interest rate and therefore also those models, Vasicek's interest rate model is used to see the change in value with the uncertain interest rates.

First the mathematical background needed will be revised so the model can be completely understood. Starting with a short explanation of the NPV method and how to calculate it. Then going over to the Ito calculus where the different stochastic differential equations are explained and solved. Finally, the tree structure, which is used for the valuation methods, is further explained and visualized.

Then, a short introduction of the real options as well as the Black & Scholes equations will be overseen. What kind of real options can be found in real life situations but keeping in mind that there are endless of different options imaginable. To better understand each possibility, it will be explained in terms of financial options since the models used later on are created for the financial market. The Black & Scholes equations simulate financial put and call options which are basic options in the the stock market. Since they are perceived as the correct value of the respective financial option, it will be the reference for our benchmark test. In order to make the link between the real options and the financial options the replicating portfolio theory is then introduced. And now we can use the methods initially created for the stock market for our real option valuation.

Finally, each step of how to find the two models, the Cox-Ross-Rubinstein and the Trigeorgis log-transformed method, and why we use any given method will be explained in chapter 4. Starting with the decision tree analysis which is based on the NPV method. Afterwards, the contingent claim analysis will be addressed and last but certainly not least, the final models for the binomial tree analysis for real options: the lattice tree methods. For the latter, the process on how to find the model will be elaborated, then the implementation of the real options. In real market conditions there are also volatile risk-free interest rates and that is why a simple model is introduced to see how the final value behaves to this change. The last point is about the stability of the two methods modelled before.

After finding those models, they have to be analysed numerically to see if they give indeed the correct values which is then done by using the Black & Scholes equations. Again, showing a simple numerical example to show the different advantages and disadvantages of each method presented in the previous chapter is also part of the chapter. The correctness of the final value calculated by the lattice tree methods will be analysed afterwards with the use of our benchmark. The sensitivity of each variable will be tested to see the behaviour of each component inside the model. Simple options are also implemented to show the possibilities of the real option valuation. To round this up, the stochastic interest rate will be used, and the stability will be tested, to see the explained behaviour in chapter 4.

Chapter 2

Mathematical Background

This chapter is dedicated to the necessary mathematical theory as well as one fundamental part of the finance industry: the net present value. Starting with the latter one, we will go over the theory for this evaluation method as well as the advantages and disadvantages which will clarify already why it is not always appropriate to use this finance tool. Then we will spend a lot of our time on the stochastic equations since they are the heart of the valuation tool with a stochastic payoff.

2.1 Net present value

2.1.1 Principle

The calculation of the net present value of a project is the most basic evaluation tool of an investment in the finance world of today. It is simply the difference between the present cash income and the present value of the cash outflow discounted to the current day:

$$NPV = \sum_{i=0}^{T} \frac{C_i}{(1+r)^i} \tag{2.1}$$

Where r is the discount rate, i the i^{th} time period and N the total number of time periods. C_i stands for a cash flow either positive or negative which means that money is incoming or outgoing respectively. A positive NPV value represents a profitable investment and on the other hand a negative value represents an investment with loss of money. The discount rate takes into account the various factors of devaluation of the money, e.g. inflation. For each firm, the future cash flows will be discounted by the weighted average cost of capital (WACC) which considers the level of debt and equity used as well as the cost of each one. The formula for the WACC is the following:

$$WACC = \frac{E}{E+D} r_{equity} + \frac{D}{E+D} r_{debt} \tag{2.2}$$

where E is the equity used, D the debt, r_{equity} is the cost of equity and r_{debt} is the cost of debt.

2.1.2 (Dis)advantages

The NPV method is indeed very easy to apply but it also has some drawbacks which are not always negligible in real world projects.
Managerial flexibility cannot be taken in account. For example, we have a project where we do not now when we will launch the new product and we want nevertheless calculate its respective value with the NPV. By holding on with the launch of the new product, we do not now from which date we have to discount the future cash flows. Imagine, we can have the possibility of deciding that it can be launched on every day for the next months. But with the use of the NPV and its constant discount rate we would always choose to launch right now since the discounted value will be worth less and less by pushing the launch further to the future. The only way it can be better to start the project in the future is by changing its discount rate. Then we would need to find a new discount rate for each future event. We can easily see that this approach is not as easy as it is supposed to be with a simple application of the NPV. Additionally, the uncertainty of future events that may occur produce also a change in the cash flows. This problem adds another complexity to the calculation of its value.

In other words if we have simple projects where we do not have a lot of choices the NPV method is the tool we need to use. However, if we can have multiple outcomes and we can change our mind about the conditions of the project, then we should look at another method which is easier to implement. On top of that the newly introduced method should also consider the ever changing market. That means that it should adapt to the trend of the market. So when the market goes down then the sales of a product will be affected and therefore the future cash flows. The same goes for a market increase. With the NPV method we cannot correctly assess the volatility of the market and this will be considered in the future methods.

2.2 Ito Calculus

In the following subsections we will talk about stochastic equations and how calculate them. We start with the Wiener Process which represents the noise of the stochastic equation. Then we attack the actual equations starting with a basic example. The generalized case is the Ito's Process which can be solved using Ito's lemma, a fundamental part of the stochastic theory. This section will be rounded up with different stochastic equations like the geometric Brownian motion, mean-reverting process and the jump process.

2.2.1 Wiener Process

A Wiener process can also be called Brownian motion and is a continuous time stochastic process which has 3 main properties: It is a Markov process which means that the future state only depends on the current state and not the past. In addition to that it has also independent increments, so the non-overlapping intervals are independent from each other and last but not least the changes of the process in any time interval are normally distributed, but the price described by the Wiener process should not be, because for example the price of an asset cannot be worth less than nothing. So, by using the logarithm of the price as a normally distributed variable we can get rid of this problem. Additionally, the prices of a stock grow at a compounded rate, so in order to simulate the price we need to find this rate which is a multiplication of different factors which are all represented as exponential values. Another reason is that with the log-normal distribution of the stock price the cheap stocks do not change with the same absolute values as the more expensive ones. In terms of percentage change, they will move with the more or less same value but not in absolute value.

Proposing a Wiener process, B_t, then the change of B_t, ΔB_t, is linked to the change of time:

$$\Delta B_t = \epsilon_t \sqrt{\Delta t} \tag{2.3}$$

where ϵ_t is a random normally distributed variable with mean 0 and standard variation 1 and each ϵ_t, where t signifies a certain time period, are completely uncorrelated. By dividing the time interval in a large number of intervals it is safe to say that the equation 2.3 can be easily expressed in continuous time with a nearly equal expression:

$$dB_t = \epsilon_t \sqrt{dt} \tag{2.4}$$

Again the distribution of ϵ_t has zero mean and unit standard deviation. Which gives dB_t a distribution of:

$$\mathbb{E}(dB_t) = 0$$
$$\mathbb{V}(dB_t) = (\mathbb{B}_{\approx})^{\not\models} = dt$$

2.2.2 Brownian Motion with Drift

In this case we have a Brownian Motion where the mean increases in time, e.g. a stock which is always changing from day to day but still after one year the mean of all possible outcomes of the stock price is gradually increasing instead of staying around 0. This phenomenon is also called the drift of the stochastic equation. This case can be expressed with a different stochastic equation:

$$dx = \alpha dt + \sigma dB_t \tag{2.5}$$

where dB_t is the change of a Wiener process and α is the drift parameter and σ the variance parameter also called volatility. The change of x is normally distributed and has a mean of $\mathbb{E}[\Delta x] = \alpha\Delta t$ and a variance of $\mathbb{V}[\Delta x] = \sigma^2\Delta t$.

2.2.3 Ito Process

The generalized equation of the previously given equation 2.5 can be called Ito process:

$$dx = a(x,t)dt + b(x,t)dB_t \tag{2.6}$$

There exist many different forms of such a process but there are three different types which are useful for the real options analysis: geometric Brownian motion, mean-reverting motion processes and the jump process.

Geometric Brownian Motion

An important case is the geometric Brownian motion with a drift where $a(x,t) = \alpha x$ and $b(x,t) = \sigma x$ and where α and σ are given constants:

$$dx = \alpha x dt + \sigma x dB_t \tag{2.7}$$

Knowing that $\frac{dx}{x}$ is normally distributed and those changes seems like logarithmic changes, therefor, the changes of x seems to be log-normally distributed. But since we are not working with a normal differential equation we need to do a change of variable to get the correct equation which will then be log-normally distributed.

For $F = log(x)$ we try to find the stochastic equation with this change of variable. Ito's Lemma is needed in order to find the wanted equation. Starting with a Taylor expansion of the proposed function F:

$$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial t}dt + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(dx)^2 + \frac{1}{6}\frac{\partial^3 F}{\partial x^3}(dx)^3 + \cdots \tag{2.8}$$

In normal calculus the higher order elements of the equation will vanish but not in this case. There will be still some terms staying contrary to the normal expansion. Considering again the general case 2.6 to find the final expression with their parameters.

$$(dx)^2 = a^2(x,t)(dt)^2 + a(x,t)b(x,t)(dt)^{\frac{3}{2}} + b^2(x,t)dt \tag{2.9}$$

Comparing the different powers of dt it can be derived that the $(dt)^2$ and $(dt)^{\frac{3}{2}}$ go faster to 0 as dt while dt becomes infinitesimal small. So the higher powers of dt are negligible to dt. The same goes for $(dx)^3$ but we only have terms which are negligible to dt itself. The final equation is then:

$$dF = \left[\frac{\partial F}{\partial t} + a(x,t)\frac{\partial F}{\partial x} + \frac{1}{2}b^2(x,t)\frac{\partial^2 F}{\partial x^2}\right]dt + b(x,t)\frac{\partial F}{\partial x}dB_t \tag{2.10}$$

Applying this equation to the geometric Brownian motion case the first and second derivative is needed: $\frac{\partial F}{\partial x} = \frac{1}{x}$, $\frac{\partial F}{\partial t} = 0$ and $\frac{\partial^2 F}{\partial x^2} = -\frac{1}{x^2}$. The equation of the geometric Brownian motion can be then rewritten as:

$$dF = (\alpha - \frac{1}{2}\sigma^2)dt + \sigma dB_t \tag{2.11}$$

Since F is normally distributed and the mean and variance of F are given in the previous point, the mean of $x(t)$ can be written as followed:

$$\mathbb{E}[x(t)] = x_0 e^{\alpha t} \tag{2.12}$$

and the variance is given by

$$\mathbb{V}[x(t)] = x_0^2 e^{2\alpha t}(e^{2\alpha t} - 1) \tag{2.13}$$

On figure 2.1 we can see some simulated trajectories. The volatility is fixed at 20% and the risk-free rate is at 5% and finally the initial value equals 10.

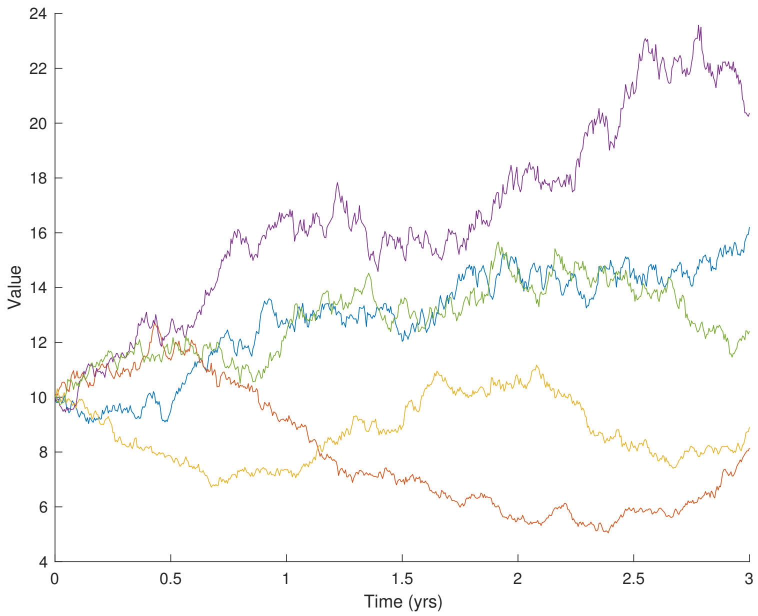


FIGURE 2.1: 5 simulated trajectories from the same brownian motion with initial value of 10, a drift of 5% and a volatility of 0.2

Mean-reverting Processes

On the other hand, the previously given stochastic equation is not always accurate while handling a different kind of assets like raw commodities. There might be some fluctuations of the price but in the long run they're rising in price due to the rising marginal cost of production. So, the fluctuations always come back to the mean of the given asset. So we introduce another term inside the equation 2.7:

$$dx = \eta(\bar{x} - x)dt + \sigma dB_t \tag{2.14}$$

where $\bar{x}$ is the mean of the given asset and η is the speed of the mean reversion. So the expected value and the variance are then given:

$$\mathbb{E}(x) = \bar{x} + (x_0 - \bar{x})e^{-\eta t} \qquad \mathbb{V}[x_t - \bar{x}] = \frac{\sigma^2}{2\eta}(1 - e^{-2\eta t}) \tag{2.15}$$

2.2.4 Jump Processes

A third process to consider is the jump process. If for example you hold a patent and after some time the competitors found another way to do the same which would then result in a big downward jump of the equity value. Unlike the other processes we do not have the continuity over the whole time frame. At a certain moment there can be a jump happening due to several reasons.

The jump itself will be also of random size as the arrival time will be modelled by a Poisson process.

We have then a typical equation of:

$$dx = a(x,t)dt + b(x,t)dB_t + c(x,t)dW_t \tag{2.16}$$

Where we have for the first two terms the same as for the initial Ito process, but the last term is due to the possible jump. We have for dW_t:

$$dW_t = \begin{cases} 0 & \text{with probability } \lambda dt \\ u & \text{with probability } 1 - \lambda dt \end{cases} \tag{2.17}$$

And λ represents the mean arrival rate of a Poisson process. This Process can be used in certain cases but for our methods we will only consider the geometric Brownian motion.

2.3 Tree Structures

We introduce 2 different types of tree structures. A tree structure is characterized by nodes and branches. We start with one single node where we add branches to this point and creating new nodes at the end of each branch. The initial point is their respective parent node. We can restart this procedure as often as we like to form a tree structure. In our case, we focus only on the possibility to add 2 outgoing branches for each created node.

Binomial tree

First of all we have a regular binomial tree. At each node we have a possibility to go up or to go down each is represented by a created branch. Every up movement has a probability p to be realized and each down movement a probability of (1-p). The drawback of this model is the fast increasing size of the tree. And so the computing time increases even further, since we have to visit every possible possibility inside that tree in order to find the initial value. This tree structure can be seen in figure 2.2.

Lattice model

Then we have the lattice model, where an upward then downward movement is the same as a downward then upward movement. As the horizon gets bigger it is faster to compute the needed values inside the tree. So, in the future we will

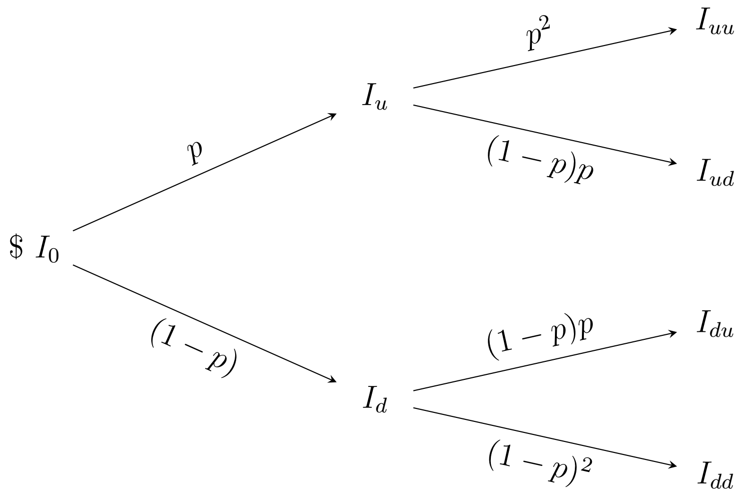


FIGURE 2.2: Binomial tree structure

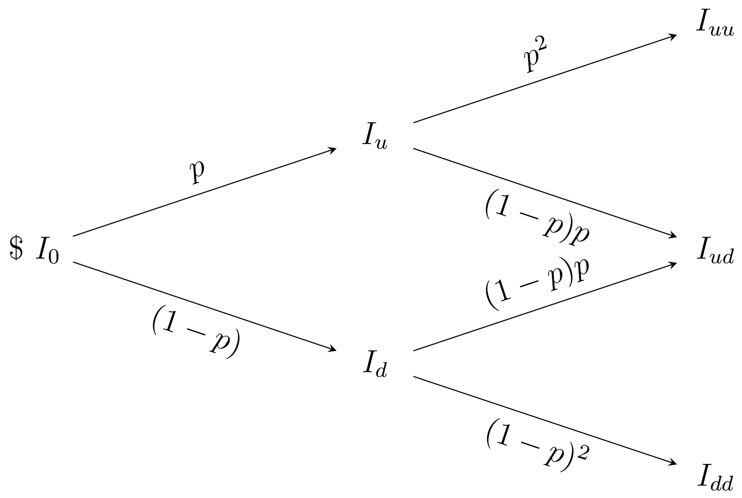


FIGURE 2.3: Lattice tree

use this model only in order to compute the real option valuation. Here on the other hand, we do not need to calculate all the possibilities since sometimes they are the the same for different scenarios. We can see an example in figure 2.3.

Chapter 3

Introduction to Real Options

This short chapter will go over the different possible real options, the replicating portfolio theory and the Black & Scholes Model. The first section is to visualize what kind of option we can actually incorporate and what it means in the industry, as well as a short comparison to the financial options. Then we will walk through the replicating portfolio theory which will make the link between a financial instrument and the latter options. Last but not least, the Black & Scholes model will be explained which is one of the best known equation in finance history.

3.1 Different Options

In this section, we walk through some options. There can be many more, but this is a list of the most obvious ones. There are no limitations of the complexity of the underlying options. Imagine a scenario and we can incorporate this into a different real option. First it will be explained what this option is all about with a short example to better understand each option and lastly, an equivalent to the financial option will be elaborated if it is possible. So, the most common options are listed below.

- Option to defer (learning option): We do not have an exact starting date for our project. That means it can be undergone in a certain time frame so that we maximize the profit made. For example, we are prepared to invest some money into a new product which is highly dependent on the market condition. In order to maximize the profit, you have to decide when the best moment to launch the new product is. Maybe if the market is very bad at the moment but in several months the markets recovers from the low, we can profit from waiting those months before launching the new product. It can be compared to a financial call option where you can exercise the option to buy the stock or in this case the launch of the product and the strike price can be then the investment needed for the start.

- Time to build option (can be shut down while constructing): It is very often used in research and development projects where you try to find a new way to do your business, but you can stop the project whenever you want. In most research projects the written code or new tests are not conclusive or helpful for the other branches of the firm and then you lose the totality of the invested capital. Especially research projects can be stopped in any case if the management deems the progress unpromising. This option on the other hand can be compared to a put option where you have the right to sell the infrastructure for a certain price like the put option.

- Option to shut down or abandon: This option is not the same as before since here you are able to sell your project for a certain price. After building a factory the head of operations can always decide to not use this facility and leaving the infrastructure untouched. In such a situation you can always sell the ground and the building to a third party. This can also be seen as a financial put option but with the special case where you can abandon the investment for further losses. Here it is possible to have losses and therefore having negative cash flows which is not exactly the case for the stock market where a stock cannot fall under 0 and so the shareholder can only loose his stake in the company.

- Option to alter operating scale: You can increase the return by scaling up the whole project. And on the opposite, if the market condition turns bad we are able to scale down the project without completely abandoning it. If a production facility gives promising results, then someone can replace the machines by more powerful ones to amplify the production output. In a way it can be also understood as a call option where you can be inclined to start the project on a negative cash flow and then exercise the option to grow. In this case the project has as a strike price the investment needed to grow. On the other hand, we have the option to decrease the scale of operation and there we can find an analogy to the put option where you can find an investment which might perform better than it supposed to for the next time and then you can exercise the option to sell parts of your equipment and produce less to cut losses.

- Growth option: Again, this is often used in research and development where one discovery can lead to new opportunities to profit. In cases where the new equipment for example brings us to new possibilities in the production line in other branches of the firm and so leading to new opportunities in the other department.
- Multiple interacting options: And of course, we can combine different options to correctly model the possible decisions a firm can make.

3.2 Replicating Portfolio

As we have already established that most of the real options can be interpreted like a financial option of the stock market we can now introduce the replicating portfolio theory. To correctly simulate the project this theory will make the link between the project, like a launch of a new mobile phone, and a financial instrument in the stock market. The main objective is to replicate the project's return and risk factors using bonds and shares. The bonds are risk free at a certain return rate. The shares deliver the volatility or the risk factor of the project. This factor depends on the firm for which we want to evaluate the project. Each firm can have a different structure and a different growth rate, etc. Also, the industry can give more stability in terms of payoff but there are many factors which should be analysed in order to correctly value the return rate and the volatility. So as said before, the portfolio should consist risk free bonds and shares. We can then model the tree as seen in the graph 3.1 where S is representing the shares, B the chosen bond, p the probability of an upward movement, r the return rate of the bond and n the number of shares. The shares should have the same risk as the project in order to correctly replicate its behaviour.

By using the replicating portfolio, we can use the existing theory on the financial options in the stock market for example. This will help us to evaluate the different projects and therefore the moment when to undergo the investment. But unfortunately, not in every case there can be found an equivalent portfolio which correctly imitates the project structure. But for the purpose of this thesis we will neglect the fact that the replicating portfolio cannot always be found in real life.

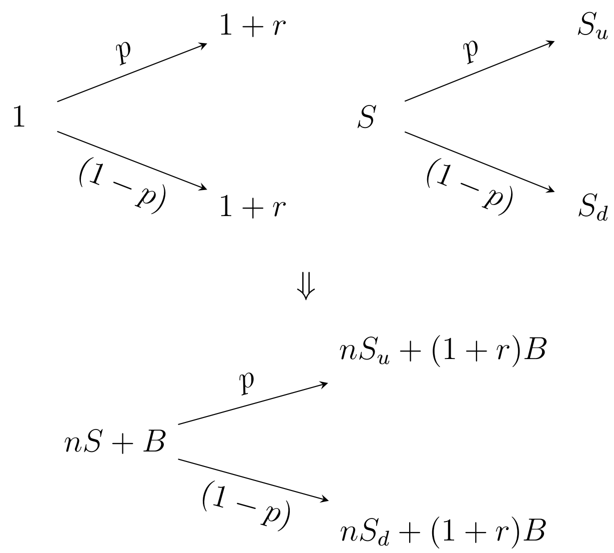


FIGURE 3.1: Principle of the replicating portfolio theory

3.3 Black and Scholes

The Black and Scholes formula, [2], was the first widely used formula for options pricing. Especially invented to find a theoretical price for the European options using their current share value, expected dividends, the option's strike price, expected interest rates, time to expiration and expected volatility. We can either have a call option or a put option. The call option is the right to buy a certain stock for a certain price. On the other hand, we have the put option which gives us the right but again not the obligation to sell a certain stock at a certain price. Of course, the value of the option changes with different factors. We can find the formula for the two options:

$$C(t, S_t) = S_t N(d_1) - N(d_2) K e^{-r(T-t)} \tag{3.1}$$

$$P(t, S_t) = -S_t N(-d_1) + N(d_2) K e^{-r(T-t)} \tag{3.2}$$

$$d_1 = \frac{ln\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right) t}{\sigma\sqrt{t}} \tag{3.3}$$

$$d_2 = d_1 - \sigma\sqrt{t} = \frac{ln\left(\frac{S_t}{K}\right) + \left(r - \frac{\sigma^2}{2}\right) t}{\sigma\sqrt{t}} \tag{3.4}$$

Where t is the time since the option was issued, S_t is the share price, K the strike price, r is the risk-free rate, σ is the respective volatility.
The first term of the calculation of the call is the expected profit from the right bought by the call. The second term however is everything that you pay over the course of time until the expiration date since the option can only be

exercised at the end of the time frame since we are dealing with the European version of the call option. Additionally, the N function is the standard normal cumulative distribution function:

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{\infty}^{x} e^{-\frac{z^2}{2}} dz \tag{3.5}$$

The American call or put option cannot be calculated by this formula. In the latter version you can exercise the option whenever you want to. That is why the American option should always be more expensive than the European one.

Chapter 4

Modelling of the Real Options

Since the net present value method is not the perfect way to evaluate our project, the tree structures will be combined with the NPV method in order to add already a certain complexity in decision making. Afterwards, the binomial tree structure is being used to accommodate already the first real options. Unfortunately, the stochastic nature of the economy is the reason why also this method cannot be taken as the correct way to evaluate the project decision. The first lattice method for real options with a basis of stochastic equations will be introduced with the Cox-Ross-Rubinstein or the Trigeorgis log-transformed method. With those models, the implementation of the real options can begin. To add a further complication, the constant risk-free interest can also become stochastic to get even closer to the real value of the project. The last point of the model is the stability analysis which is also important in order to know when we can use a certain method and when this method will not give the correct answer.

4.1 Decision Tree Analysis

Firstly, we consider the simple application of the net present value inside a tree structure. Each node will be calculated using the given formula:

$$\text{Net Present Value (= NPV)} \; = \sum_{i=0}^{N} \frac{C_i}{(1+r)^i}$$

where r is the discount rate at the given moment. It is a classic tool to evaluate the profitability of a certain project. We can find a formula for the expected value of the decision tree analysis by calculating recursively on each node the expected value of the values found of the outgoing nodes. Starting at the end and going back to the current position in time to find the expected value at our point of time.

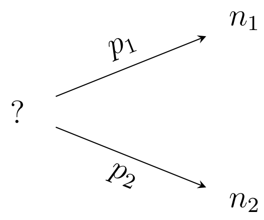


FIGURE 4.1: Example of a simple decision tree with two choices

Considering a simple example where we have a node in time with two outgoing branches with a node at the end where we know what the value of the ending node is. This example is represented on figure 4.1. Then we can find the expected value for our position by using the following formula:

$$\mathbb{E}(tree) = \frac{p_1 n_1 + p_2 n_2}{1+r} + n_0$$

where p_1 and p_2 are the real world probabilities of moving to node 1 or 2, n_1 and n_2 are the payoffs of node 1 and 2 and r is the discount rate. Additionally, we need to add the cash flow of the initial node n_0 since we might gain some money already at this point of time. Now we can find the expected value of every node with a generalized formula:

$$\mathbb{E}(node_j) = \frac{\sum_{i=1}^{N} p_i n_i}{1+r} + n_j \tag{4.1}$$

where p_i is the probability of moving to node i, n_i is the payoff of node i, r is the discount rate again, j is the node for which you would like to have the expected value and N is the number of outgoing nodes to the next step in the tree. n_j is the payoff at node j which is again positive for a cash income and negative for an investment paid. As said before we need to evaluate the tree recursively since the expected value of the very last node is always known since it is only the payoff at this step and we need always the nodes of the next step to calculate the value of our initial node.

It is already a better model than the simple application of the deterministic cash flow (=DCF) method. Since we can already analyze simple options with our model, for example, if we want to analyze the situation where you can change the start of the given project or when we get opportunities to realize another profitable project which would not be possible without the first project. Its downside, on the other hand, is the constant discount rate which normally changes over time as the markets change constantly and we should adapt our discount rate accordingly. With the time, the discount rate changes with the firm structure and many other factors as well. By realizing different projects, we might change different factors, who might influence the discount rate, which would implicate that we are not using the right one at the later stages of a project. Additionally, the income of each investment is also uncertain for most of the projects taken depending on the future economy. This cannot be taken into account correctly without complicating the tree structure even further and so it becomes unreadable. The factors of uncertainty are however considered in the next sections.

4.2 Contingent Claim Analysis

We now consider the contingent claim analysis where we account for the risk factor of each investment. The uncertainty of future events that change the gain from an investment is incorporated in the measure of the risk. Those quantifiable risk indicators help us to correctly predict if an investment should be considered. It is a forward-looking method rather than basing itself only on past occurrences. So, with the correct risk measurements we can better predict the value of the investment.

In the deterministic valuation of the decision tree analysis we used probabilities which are given based on the situation which is decided in this particular node. For example, at one node we put in the possibility of a defect which is provided by the manufacturer for example. In the latter case a certain production robot has the probability to fail of 5% in the first year of service. This probability will be fixed in the tree and therefore the probability that everything runs smoothly is of 95%. In the next model the risk of a project is inside the volatility which is fixed for a certain investment. So, the probability can be calculated and further transformed into the risk neutral probability. Additionally, we restrict the possible outcomes at each node at 2 which was not the case in the decision tree analysis where you can fix many different scenarios at each node.

The contingent claim analysis (=CCA) can be presented in form of a multiple decision tree, but we want to go even further and use a binomial tree. As presented before we can either go up or down. Each movement has a probability of p and 1-p respectively. The up movement means that the investment gains in value whereas the down movement represents a loss in value.

Let us define the different factors needed to create the analysis:

V	:=	total value of project
S	:=	price of the shares that are almost perfectly correlated with V
E	:=	equity value of the project for the shareholder
S_u	:=	return of the shares after an upward movement
S_d	:=	return of the shares after an downward movement
r_f	:=	risk-free interest rate
p	:=	risk-neutral probability for up-movements of V and S per period
q	:=	real probability for up-movements of V and S per period
u	:=	multiplicative factor for up-movements of V and S per period
d	:=	multiplicative factor for down-movements of V and S per period

We then try to replicate the investment by using shares and bonds to get the same risk and return structure as the given investment. The value can then be seen as $E = nS - B$ where n is the number of shares bought where its value is represented by S and B is the bond of maturity T. So, on the up movement we have $E_u = nS_u - (1 + r_f)B$ and on the down movement $E_d = nS_d - (1 + r_f)B$. The real world probability of happening to move up- or downwards are of the same likelihood. So we can find the last unknowns simply by substituting them:

$$n = \frac{E_u - E_d}{S_u - S_d} \qquad E = \frac{pE_u + (1-p)E_d}{1 + r_f} \tag{4.2}$$

$$B = \frac{E_u S_d - E_d S_u}{(S_u - S_d)(1 + r_f)} \qquad p = \frac{(1 + r_f) - d}{u - d} \tag{4.3}$$

In order to calculate when the project is undertaken we consider in this case the risk-adjusted probabilities, p, which differs to the previous case, with the DTA. The only value that has to be evaluated are E_d and E_u, the values of the equity value after one step and that means we have to find the multiplicators d and u. From there we can simply find $E_d = d\ E$ and $E_u = u\ E$. Now we can use the given method since we have found every unknown necessary for the implementation.

4.3 Lattice Methods for Real options

4.3.1 Modeling

Cox-Ross-Rubinstein binomial tree, [4]. The classical tool for the option pricing. We take the same parameters as before. We can see that the principle is equivalent to the previous point but will differ later in the analysis. In this case to make one upward movement then one downward is the same as one downward and one upward movement. So, we use a lattice tree as described earlier. With the replicating portfolio and the matching expected return of the tree and the stock we can find the following equality:

$$Se^{r_f \Delta t} = puS + (1-p)dS \tag{4.4}$$

$$\Rightarrow e^{r_f \Delta t} = pu + (1-p)d \tag{4.5}$$

From here on we can quickly find an expression for the risk neutral probability of going upwards:

$$p = \frac{e^{r_f \Delta t} - d}{u - d} \tag{4.6}$$

If the volatility of the given portfolio is σ we can find the standard derivation of its return in a short period of time of length Δt: $\sigma\sqrt{\Delta t}$. So, we can quickly find the variance which equals to $\sigma^2 \Delta t$. We want to examine the variance of the whole tree which can be done by starting with its definition and then substituting the corresponding values as well as replacing the probability by the latter found formula. This gives us an expression which we can develop and that should be equal to the variance of the portfolio:

$$\begin{aligned}
\mathbb{V} &= \mathbb{E}(X^2) - (\mathbb{E}(X))^2 \\
&= \left(pu^2 + (1-p)d^2\right) - (pu + (1-p)d)^2 \\
&= pu^2 - pd^2 + d^2 - p^2(u-d)^2 - d^2 - 2pd(u-d) \\
&= p\left(u^2 - d^2 - p(u-d)^2 - 2d(u-d)\right) \\
&= p\left(u^2 + d^2 - p(u-d)^2 - 2du\right) \\
&= p\left((u-d)^2 - p(u-d)^2\right) \\
&= p(1-p)(u-d)^2 \\
&= \frac{e^{r_f\Delta t} - d}{u-d}\left(1 - \frac{e^{r_f\Delta t} - d}{u-d}\right)(u-d)^2 \\
&= (e^{r_f\Delta t} - d)(u - d - e^{r_f\Delta t} + d) \\
&= ue^{r_f\Delta t} - ud + de^{r_f\Delta t} - e^{2r_f\Delta t} \\
&= \sigma^2\Delta t
\end{aligned}$$

Now we have imposed two conditions on p,u and d. A third one is added by Cox, Ross and Rubinstein via $u = \frac{1}{d}$. Then a solution is given by:

$$u = \exp\left(\sigma\sqrt{\frac{T}{N}}\right) = \frac{1}{d} \tag{4.7}$$

We replace the found values into our equality in order to check their viability:

$$\sigma^2\Delta t = e^{r_f\Delta t + \sigma\sqrt{\Delta t}} + e^{r_f\Delta t - \sigma\sqrt{\Delta t}} - 1 - e^{2r_f\Delta t} \tag{4.8}$$

We now use Taylors series expansion:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

We neglect the terms of Δt^2 and higher powers since we have a small step size and therefore, the higher powers are negligible in comparison to Δt.

$$\begin{aligned}
&= 1 + r_f\Delta t + \sigma\sqrt{\Delta t} + \frac{1}{2}(r_f\Delta t + \sigma\sqrt{\Delta t})^2 \\
&\quad + 1 + r_f\Delta t - \sigma\sqrt{\Delta t} + \frac{1}{2}(r_f\Delta t - \sigma\sqrt{\Delta t})^2 \\
&\quad - 1 \\
&\quad - (1 + 2r_f\Delta t + \frac{2}{2}(r_f\Delta t)^2 \\
&= \frac{1}{2}\left((r_f\Delta t)^2 + 2r_f\Delta t\sigma\sqrt{\Delta t} + \sigma^2\Delta t + (r_f\Delta t)^2 - 2r_f\Delta t\sigma\sqrt{\Delta t} + \sigma^2\Delta t\right) \\
&= \sigma^2\Delta t
\end{aligned}$$

We can see that equations 4.7 are indeed solutions to the previous system, 4.8. We can again find the last 3 unknowns as in the contingent claim analysis, 4.2:

$$n = \frac{E_u - E_d}{S_u - S_d} \qquad E = \frac{pE_u + (1-p)E_d}{e^{r_f\Delta t}} \tag{4.9}$$

$$B = \frac{E_uS_d - E_dS_u}{(S_u - S_d)e^{r_f\Delta t}} \tag{4.10}$$

We have now every unknown to implement a recursion formula. We introduce the opportunity value $R(i,j)$ where we denote j the j^{th} time step and i is the i^{th} node from the top at time j. Both index values start from 0. So, when we do an upward movement we will be at i and $j+1$ and for a downward movement we are one step further, $j+1$, and one node down, $i+1$. At each node we need to calculate the discounted value of the the latter nodes:

$$R(i,j) = e^{r_f\Delta t}\left[p \cdot R(i,j+1) + (1-p)\cdot R(i+1,j+1)\right] \tag{4.11}$$

This binomial tree can be visualized with the respective notations for each node on figure 4.2.

Trigeorgis log-transformed binomial tree. Model developed by Trigeorgis [9]. We now have to consider the stochastic processes of the different projects. We propose that the project follows a diffusion process (a Geometric Brownian Motion) given by the stochastic differential equation (=SDE):

$$dV_t = \alpha V_t dt + \sigma V_t dB_t, \qquad t \geq 0 \tag{4.12}$$

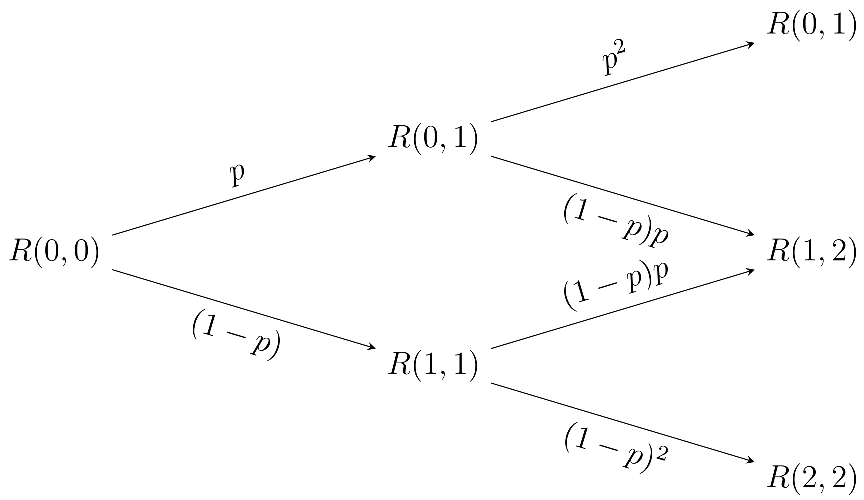


FIGURE 4.2: Visualisation of $R(i, j)$ for a tree with 3 levels

where $\alpha \in \mathbb{R}$ is the instantaneous expected return of the project, $\sigma \in \mathbb{R}^+$ the instantaneous standard derivation and finally $(B_t)_{t\geq 0}$ is a standard Brownian motion.
The log-transformation is represented by $Y_t := \ln(V_t), t \geq 0$ and gives this method his name. For every infinitesimal time interval dt the process Y follows an arithmetic Brownian motion. We can now transform it into the following equation under risk neutrality with $\alpha = r_f$:

$$dY = \ln\left(\frac{V_{t+dt}}{V_t}\right) = \left(r_f - \frac{1}{2}\sigma^2\right)dt + \sigma dB_t, \qquad t \geq 0 \tag{4.13}$$

We can prove that this process follows a normal distribution with mean, also known as the drift, equal to $\left(r_f - \frac{1}{2}\sigma^2\right)dt$ and standard variance of $\sigma^2 dt$. We then simplify it by replacing $\sigma^2 dt$ by K, defined as the time step:

$$dY \sim N(\mu K, K) \qquad \text{where } \mu = \frac{r_f}{\sigma^2} - \frac{1}{2} \tag{4.14}$$

We want to express everything in term of the variance. We now have a discrete Markov walk which goes up by ΔY with a probability of p and also down by the same amount but with a probability of $1 - p$. We also define $H := \Delta Y$, the state step. We can then calculate the expected value and the variance of one step:

$$\mathbb{E}(1\ step) = pH + (1-p)(-H) = 2PH - H \tag{4.15}$$

$$\mathbb{V}(1\ step) = H^2 - (E(\Delta X))^2 = H^2 - (2pH)H)^2 \tag{4.16}$$

We want that the discrete-time process is consistent with the continuous-time process, so their variances and their means should be equal which brings us to:

$$p = \frac{1}{2}\left(1 + \frac{\mu K}{H}\right) \tag{4.17}$$

$$H = \sqrt{K + (\mu K)^2} \tag{4.18}$$

Again we can calculate the opportunity value, R by discounting the future values to the node $R(i, j)$:

$$R(i,j) = \exp\left\{-r_f \frac{K}{\sigma^2}\right\}[p \cdot R(i, j+1) + (1-p) \cdot R(i+1, j+1)] \tag{4.19}$$

The model has to be adjusted for the occasional external cash flow that means that at some point of time there might be a cash flow like dividends or monthly payment of rent etc. Therefore, the usual formula for the expected value has to be adjusted to correctly value the cash flow for the next iterations. The value of the project is then denoted by V and the moment before and after by $^+$ and $^-$ respectively. C represents the external cash flow which is assumed to be a payment. Then, we have $V^+ = V^- - C$. The total opportunity value R' is:

$$R(V^+) = R(V^- + C) - C \tag{4.20}$$

The moment after the payment of the dividend for example is equal to the difference between the expected value of the value of the later levels of the tree regardless the payment and the paid sum of dividends. The given shift in the lattice is not everywhere the same. Since the shift is a local shift the earlier node structure has to be maintained. To correctly predict the value of each different real option we need check at each node if it is not better to apply a certain real option. Now we can implement the options of the firm into the recursion formula given in the following section.

4.3.2 Real Options Implementation

Option to defer until next period: Starting with the first option where we can wait a certain time period before starting the project and profiting from the expected return. So, we need to discount the future cash flows which would be generated if the project is undergone. But on the other hand, we need to compare it to the value of the cash flows if we actually do the project right now. We can then express this with our recurrence formula introduced earlier:

$$R(i,j) = \max\left\{e^{-r_f \frac{\sigma^2}{K}} E(R(i,j+1)), R(i,j)\right\}$$

Option to alter operating scale: The next on the list is where we can change the payoff of the project at any given time. In order to do so we need to invest capital, I_e, to scale up the project by a percental amount, e. We can then find the recursive formula:

$$R'(i,j) = R(i,j) + \max\{eV(i) - I_e, 0\}$$

On the other hand, if we want to scale down the project we should do so by a gain in capital, I_c due to the possible sell off of different machines, employees, ... In this case we gain money but loose on the percental contracted value c:

$$R'(i,j) = R(i,j) + \max\{I_c - cV(i), 0\}$$

Abandon an investment: When we abandon an investment, we can have three possible cases:

- The investment was into a research project and by abandoning the project we lose the whole value of the work done.
- Or the investment into a research project needs more money so that we the choice of either abandon the project or further invest into the research to realize the ultimate goal.
- The investment was linked with some infrastructure or needed raw material, etc. Then we can sell everything we have for the respective value, so we still get some money back from the investment.

For the first case we can either continue the project or just get a payoff of 0 so we do not make further losses:

$$R'(i,j) = \max\{R(i,j), 0\}$$

For the second case we only have to add the needed investment at $R(i,j)$:

$$R'(i,j) = \max\{R(i,j) - I, 0\}$$

Type of real option	Adjustment
Switch use for salvage X	$R'(i,j) = \max\{R(i,j), X\}$
Expand by e through investing an additional amount I_e	$R'(i,j) = R(i,j) + \max\{eV(i) - I_e, 0\}$
Contract by c, saving I_c	$R'(i,j) = R(i,j) + \max\{I_c - cV(i), 0\}$
Abandon by defaulting on an investment I	$R'(i,j) = \max\{R(i,j) - I, 0\}$
Defer until next period	$R(i,j) = \max\left\{e^{-r_f \frac{\sigma^2}{K}} E(R(i,j+1)), R(i,j)\right\}$

Table 4.1: Transformation to be applied to the binomial tree to correctly value the possible real options

The last case is another generalized example of the first one but this time instead of losing everything we gain some money X from the selloff of the goods:

$$R'(i,j) = \max\{R(i,j), X\}$$

4.3.3 Stochastic risk-free interest rates

Modification of the Cox-Ross-Rubenstein Binomial Tree. The constant risk-free interest rate is already described earlier. So now, the constant one is interchanged with an stochastic interest rate. The stochastic differential equation is slightly different than before:

$$dS_t = r_f(t) S_t dt + \sigma S_t dB_t, \qquad t \geq 0 \tag{4.21}$$

The binomial tree can still be constructed since its geometry does not depend on the variables used. An additional function is then needed to be defined:

$$a(t) = e^{r_f(t)\Delta t}, \qquad t \geq 0$$

The probabilities of each node are then:

$$p = \frac{a(t) - d}{u - d} \qquad \text{and} \qquad 1 - p = \frac{u - a(t)}{u - d} \tag{4.22}$$

In order to apply the ordinary Cox-Ross-Rubinstein method the intermediary interest rates have to be calculated. Afterwards, in order to apply them and correctly value the project, we have to recalculate the probabilities and the risk-free rate for each period.

$p^{(j)}$:= risk-neutral probability for up-movements after a node in time period j
$q^{(j)}$:= real probability for up-movements after a node in time period j

From here on it is exactly the same procedure as before but with different probabilities. We can apply the same procedure to the Trigeorgis log-transformed method without any additional information.

To simulate stochastic short term interest rates we will use Vasicek's model. This model is used for risk neutral measures which is indeed the case for our two methods. Vasicek based himself on a mean reverting process so that the interest rate will always come back to the mean eventually. So, we can find the stochastic differential equation:

$$dx = (\beta - \alpha_t x)dt + \sigma dB_t \tag{4.23}$$

where α_t, β and σ are all constant values. α is the speed of the mean reversion, β/α is the mean to which the process will come back and σ is, as always, the volatility of the process. We can generalize the solution by proposing a dependence of time to each constant. For the generalized case we can find the solution to the stochastic differential equation. We will start by finding a solution to the homogeneous case. First, we examine the generalized equation where α and β are dependent on time, t.

$$dx = \alpha_t x dt$$

We can find easily a solution of this process by separating the different variables and then integrating each side of the equation:

$$\begin{aligned} \frac{dx}{x} &= \alpha_t dt \\ \int_0^t \frac{1}{x} dx &= \int_0^t \alpha_u du \\ ln(x_t) - ln(x_0) &= \int_0^t \alpha_u du \\ ln\left(\frac{x_t}{x_0}\right) &= \\ x_t &= x_0 e^{\int_0^t \alpha_u du} \end{aligned}$$

Now we have the solution to the homogeneous part of the initial differential equation. With the use of the method of the variation of parameters we can propose a different process y_t to equalize the following process with the initial differential equation:

$$x = y_t e^{\int_0^t \alpha_u du} \tag{4.24}$$

So we compare equation 4.23 with the new one, 4.24, to find an expression for y_t:

$$dy_t = \beta_t exp\left(-\int_0^t \alpha_u du\right) dt + \sigma_t exp\left(-\int_0^t \alpha_u du\right) dB_t$$

We consider that $y_0 = x_0$ and we integrate as before from 0 to t:

$$y_t = x_0 + \int_0^t \beta_u exp\left(-\int_0^u \alpha_s ds\right) du + \int_0^t \sigma_u exp\left(-\int_0^u \alpha_s ds\right) dB_u$$

Now we can find the generalized solution of the mean reverting process:

$$x_t = \left(x_0 + \int_0^t \beta_u exp\left(-\int_0^u \alpha_s ds\right) du + \int_0^t \sigma_u exp\left(-\int_0^u \alpha_s ds\right) dB_u\right) exp\left(\int_0^t \alpha_u du\right) \tag{4.25}$$

Then, we come back to the model of Vasicek where σ, α and β are all independent of the time and we have to adjust α to $-\alpha$ to fulfill the model:

$$\begin{aligned} x_t &= \left(x_0 + \int_0^t \beta e^{\alpha u} du + \int_0^t \sigma e^{\alpha u} dB_u\right) e^{-\alpha t} \\ &= \left(x_0 - \frac{\beta}{\alpha}\left(1 - e^{\alpha t}\right) + \sigma \int_0^t e^{\alpha u} dB_u\right) e^{-\alpha t} \\ &= x_0 e^{-\alpha t} + \frac{\beta}{\alpha}\left(1 - e^{\alpha t}\right) + \sigma \int_0^t e^{-\alpha(t-u)} dB_u \end{aligned}$$

From here on we can directly find the expected value as well as the volatility of the interest rate:

$$\begin{aligned}
\mathbb{E}(x_t) &= x_0 e^{-\alpha t} + \frac{\beta}{\alpha}\left(1 - e^{\alpha t}\right) \\
\mathbb{V}(x_t) &= \mathbb{E}(x_t^2) - \mathbb{E}\left(\sigma \int_0^t e^{-\alpha(t-u)} dB_u\right)^2 \\
&= \sigma^2 \int_0^t e^{-\alpha(t-s)} ds \\
&= \frac{\sigma^2}{2\alpha}\left(1 - e^{\alpha t}\right)
\end{aligned}$$

To visualize Vasicek's model we will apply a numerical method. The chosen method is the Taylor 1.5 scheme found in the appendix A.1. For $\mu(\tau, Y) = \beta - \alpha Y$ and $\sigma(\tau, Y) = \sigma$ we can reduce the iterative equation to the following since $\sigma'' = \mu'' = \sigma' = 0$:

$$\begin{aligned}
Y_0 &= x_0 \\
Y_{i+1} &= Y_i + \mu(\tau_i, Y_i)\Delta_s + \sigma(\tau_i, Y_i)\Delta B_i^1 \\
&\quad + \mu'(\tau_i, Y_i)\sigma(\tau_i, Y_i)\Delta B_i^2 \\
&\quad + \frac{1}{2}\left(\mu(\tau_i, Y_i)\mu'(\tau_i, Y_i)\right)\Delta_s^2 \\
&\quad \text{for } i = 0, \cdots, N-1
\end{aligned}$$

With the use of the numerical approximation method we can see the Vasicek model on figure 4.3. We took the following parameters: $\alpha = 0.06$, $\beta = 0.12$, $r_0 = 0.06$, $\sigma = 0.02$, $T = 5$, and $N = 360$ where N is again the number of steps per year, T the maturity, r_0 the starting point, and the other parameters are the same as in the differential equation. We can clearly see that the risk-free rate is always returning to the same value as before since the mean is fixed at $\beta/\alpha = 0.06$.

4.3.4 Stability

Cox-Ross-Rubinstein: We can firstly observe that the logarithmic change of the process is equal to $ln(u)$ and since $u = \sigma\sqrt{\tau}$ where $\tau = T/N$ and by definition we have $K = \sigma^2\sqrt{\tau}$ we can identify that $ln(u) = \sqrt{K}$. To find a upper limit on the step size to make sure that the method is stable, we realize that if we tend N to infinity in the log-transformed method of Trigeorgis we can find the same results as in the Cox-Ross-Rubinstein method. First, we have to

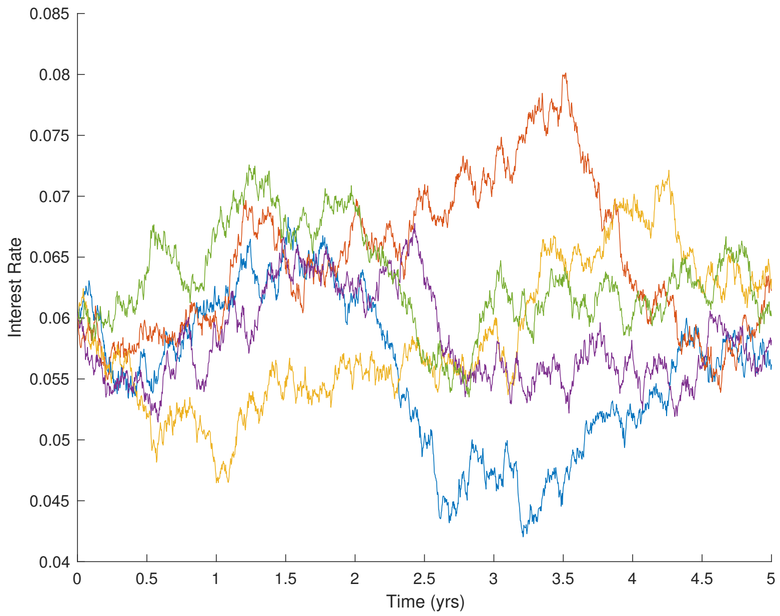


FIGURE 4.3: Interest Rate modeled with the Vasicek model with the parameters $\alpha = 0.06$, $\beta = 0.12$, $r_0 = 0.06$, $\sigma = 0.02$, $T = 5$, and $N = 360$

compare K with K^2 when $N \to \infty$:

$$K = \sigma^2 \sqrt{\frac{T}{N}} \quad K^2 = \sigma^4 \frac{T}{N} \tag{4.26}$$

We can quickly see that K^2 tends to 0 much faster than K. And then, we can assess that K^2 is negligible in the presence of K. Therefore, we can find for H in the model of Trigeorgis:

$$H = \sqrt{K + (\mu K)^2} \quad \overset{N \to \infty}{\Longrightarrow} \quad H \approx \sqrt{K} \tag{4.27}$$

So, we find the same value for H with both models. And now we have to analyse the approximation of the probability:

$$\begin{aligned} P &= \frac{1}{2}\left(1 + \frac{\mu K}{H}\right) \\ &= \frac{1}{2}\left(1 + \left(\frac{r_f}{\sigma^2} - \frac{1}{2}\right)\frac{K}{H}\right) \\ &\overset{N \to \infty}{\Longrightarrow} \frac{1}{2}\left(1 + \left(\frac{r_f}{\sigma^2} - \frac{1}{2}\right)\sqrt{K}\right) \qquad \text{with } K = \sigma^2 \tau \text{ and } m = r_f - \frac{1}{2}\sigma^2 \\ &= \frac{1}{2}\left(1 + \frac{m}{\sigma}\sqrt{\tau}\right) \\ &= q \end{aligned}$$

where q is the probability of the Cox-Ross-Rubinstein model. We will verify if q is indeed between 0 and 1.

$$0 \leq \frac{1}{2}\left(1 + \frac{m}{\sigma}\sqrt{\tau}\right) \Rightarrow -\frac{\sigma}{m} \leq \tau \tag{4.28}$$

$$\frac{1}{2}\left(1 + \frac{m}{\sigma}\sqrt{\tau}\right) \leq 1 \Rightarrow \sqrt{\tau} \leq \frac{\sigma}{m} \tag{4.29}$$

Putting those two conditions together we can find a single condition:

$$\tau \leq \frac{\sigma^2}{m^2} = \frac{\sigma^2}{\left(r_f - \frac{1}{2}\sigma^2\right)^2} \tag{4.30}$$

Now we have an upper bound on the step size depending on the volatility and the risk-free rate. For this model we encourage a higher volatility and less return if we want to increase the step size.

Trigeorgis log-transformed method: We have to verify that the probability is indeed between 0 and 1. We will start with the latter case:

$$\begin{aligned}
P &= \frac{1}{2}\left(1 + \frac{\mu K}{H}\right) \\
&\overset{(4.18)}{=} \frac{1}{2}\left(\frac{\mu K}{\sqrt{K + (\mu K)^2}}\right) \\
&\leq \frac{1}{2}\left(\frac{|\mu K|}{\sqrt{K + (\mu K)^2}}\right) \\
&\leq \frac{1}{2}\left(1 + \frac{|\mu K|}{|\mu K|}\right) \qquad \text{since } K \geq Var(\Delta Y) \geq 0 \\
&\leq 1
\end{aligned}$$

For the next step we need to express the Variance, 4.16, differently by substituting the probability, 4.17, inside it. From there we can find that $Var(\Delta Y) = H^2 - (\mu K)^2$. Since the Variance is always greater or equal to 0 we can deduce the following:

$$\begin{aligned}
&H^2 \geq (\mu K)^2 \\
&\Rightarrow |H| \geq |\mu K| \\
&\overset{H \geq 0}{\Rightarrow} -H \leq -|\mu K| \leq |\mu K| \leq H \\
&\Rightarrow -1 \leq \frac{\mu K}{H} \leq 1
\end{aligned}$$

With this knowledge we can find a lower limit on the probability:

$$P = \frac{1}{2}\left(1 + \frac{\mu K}{H}\right) \geq \frac{1}{2}(1 - 1) = 0 \tag{4.31}$$

And so since $P + (1 - P) = 1$ and $0 \leq P \leq 1$ we can assume that the method is unconditionally stable with no constraints on H and K.

Chapter 5

Analysis of the different methods

After building the models for the evaluation tools of the project, an analysis of the methods should be made to know how each method behaves. Starting with the basic valuation tool, the net present value. To add more complexity, we give an example of the decision tree analysis. As we have seen before, the contingent claim analysis is a better tool to evaluate real options. A basic example will be made to prove the point. Finally, we test our two main methods: the Cox-Ross-Rubinstein and the Trigeorgis log-transformed method. First, the correctness will be checked with the Black & Scholes formula by reproducing the European put option with our two methods. Then we will identify the correct number of time steps per year using the latter formula. From there, the sensitivity to different parameters, such as the volatility and the risk-free rate, will be checked and analyzed. Then we change the nature of the option to evaluate if we can use our models to calculate the value of the project. The last point will be about the stability of both methods.

5.1 Net Present Value

Firstly, we present an example of the net present value method which is very simple and easy to use. That is why it is widely used to valuate fixed projects without any flexibility. Considering now that we have a discount factor, r of 6% and we need an initial investment of 60 at the beginning and 40 after one year. This project will have a gain over 6 years of a constant value of 30 where the cash flows are visualized on figure 5.1.

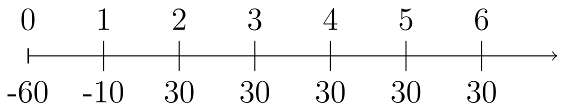


FIGURE 5.1: Cash flows of the project with an income of 30 per year

For this case we can observe a net present value of:

$$
\begin{aligned}
NPV &= -60 - \frac{10}{(1.06)} + \frac{30}{(1.06)^2} + \frac{30}{(1.06)^3} + \frac{30}{(1.06)^4} + \frac{30}{(1.06)^5} + \frac{30}{(1.06)^6} \\
&= 27.01
\end{aligned}
$$

So, we can say this project should be undergone, since it will be profitable. On the other hand if we now only get 20 per year which would imply that we pay 100 to get 100 where we can see the cash flows on figure 5.2.

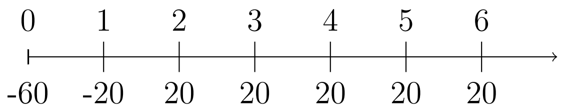


FIGURE 5.2: Cash flows of the project with an income of 20 per year

We calculate the NPV again:

$$
\begin{aligned}
NPV &= -60 - \frac{20}{(1.06)} + \frac{20}{(1.06)^2} + \frac{20}{(1.06)^3} + \frac{20}{(1.06)^4} + \frac{20}{(1.06)^5} + \frac{20}{(1.06)^6} \\
&= -12.73
\end{aligned}
$$

This time we can observe a negative NPV which implies that the project should not be done. Indeed, we have a payoff equal to the earlier invested capital, but since the money in the future is worth less than the money we pay right now we can confirm that we should not proceed with this investment. We can see that if the payoff changes the profitability of the project also changes. In order to predict correctly the future events, we should not follow a fixed payoff structure but rather have multiple possible payoffs and then take the expected value of the different scenarios. The first try to incorporate this possibility is with the decision tree analysis attacked in the next section. Most of the time the different payoffs are coming from the possibility to change the strategy taken previously and so it changes with the presence of real options.

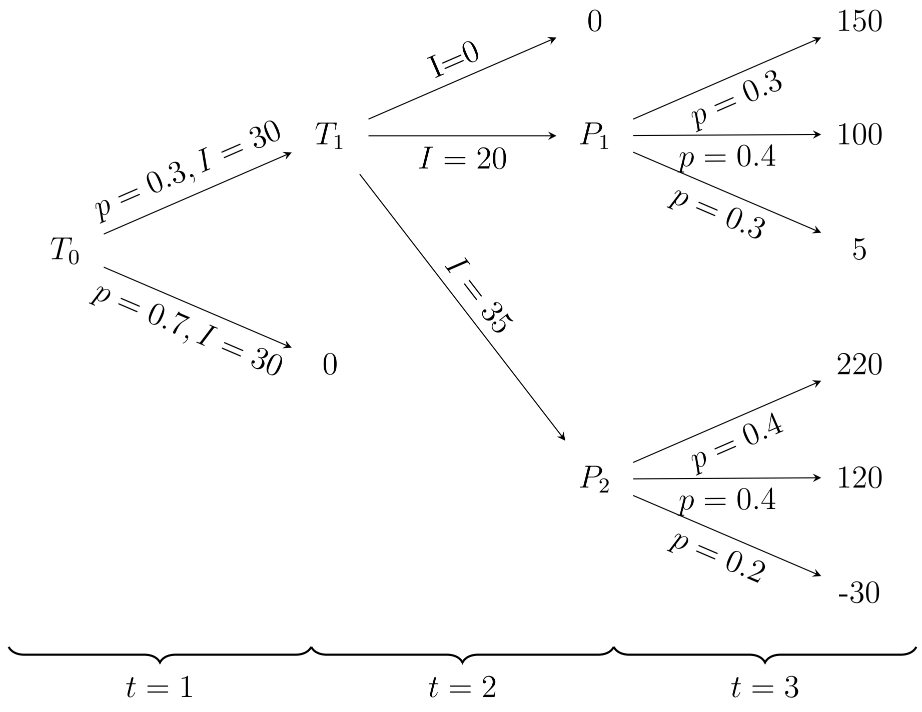


FIGURE 5.3: Example for the decision tree analysis

5.2 Decision Tree Analysis

For this section we will consider an example of a research and development project where we can always abandon the research at each given time step since we are not able to progress in the field of study. For the first time step we were successful with our studies with a probability of 0.3 and we gain nothing since we failed to find a new technology with a probability of 0.7. This already costs 28 in order to finance the research. At time 2 we can then construct 2 different plants: either costing 20 or 35. Each plant has different payoffs with respective probabilities which can be found in the next figure. The payoff is the net present values of the cash flows from year 3 to the end of the project.

In the figure 5.3 the probability of happening is denoted by p and the needed investment by I. The different choices to implement a different size of plant are mutually exclusive. So, we can calculate the value of the project at each node backwards and by discounting the investment at a rate of 6%:

$$\begin{aligned}
&Point \quad P_1: & 81.6 &= \frac{0.3\times150+0.4\times100+0.3\times5}{1.06}\\
&Point \quad P_2: & 122.64 &= \frac{0.4\times220+0.4\times120+0.2\times(-30)}{1.06}\\
&Point \quad T_1: & 87.64 &= max\left(0; \frac{100}{1.06}-15; \frac{138.68}{1.06}-25\right)\\
&Point \quad T_0: & -1.71 &= \frac{0.3\times87.64+0.7\times0}{1.06}-28
\end{aligned}$$

Where P_i is the plant i and T_i represents the stage of completion of the research.

So, we can see that the project would not be worth the cost since the NPV is negative in point T_0. We can now add a real option where we could sell the plant for a certain price X after year 3 for example. Considering now that the price also changes with the economic situation, so we assume the following prices:

$$\begin{array}{rclrcl} X_1(High) & = & 18 & X_2(High) & = & 30 \\ X_1(Medium) & = & 15 & X_2(Medium) & = & 25 \\ X_1(Low) & = & 12 & X_2(Low) & = & 20 \end{array}$$

Additionally, we can assume that the net present value of the payoff for the year 4 until the end can be assumed to be the following:

$$\begin{aligned} NPV(P_1(High))_4 &= 125 \\ NPV(P_1(Medium))_4 &= 80 \\ NPV(P_1(Low))_4 &= 2 \\ NPV(P_2(High))_4 &= 150 \\ NPV(P_2(Medium))_4 &= 70 \\ NPV(P_2(Low))_4 &= -30 \end{aligned}$$

We now have to check if the salvage value of the plant is higher than the NPV of the payoff of year 4 and later. This is only the case when we will be in the worst economic condition. For these two situations we need to recalculate the payoff. We need to keep in mind that at time 3 there is already a payoff generated by the respective plant which is simply the difference between the net present value at time 3 minus the one at time 4: $NPV(P_i)_3 - NPV(P_i)_4$.

$$\begin{aligned} NPV(P_1(Low))_3 &= \text{salvage value} + \text{cash flow in year 3} \\ &= 15 = 12 + (5 - 2) \\ NPV(P_2(Low))_3 &= 20 = 20 + (-30 - (-30)) \end{aligned}$$

This can now be put again inside the decision tree on figure 5.4

And we can now recalculate the net present value at each node recursively as we have done it before and we can find that the NPV_0 is positive this time and is equal to 1.123. This NPV is called a strategic NPV since the valued is

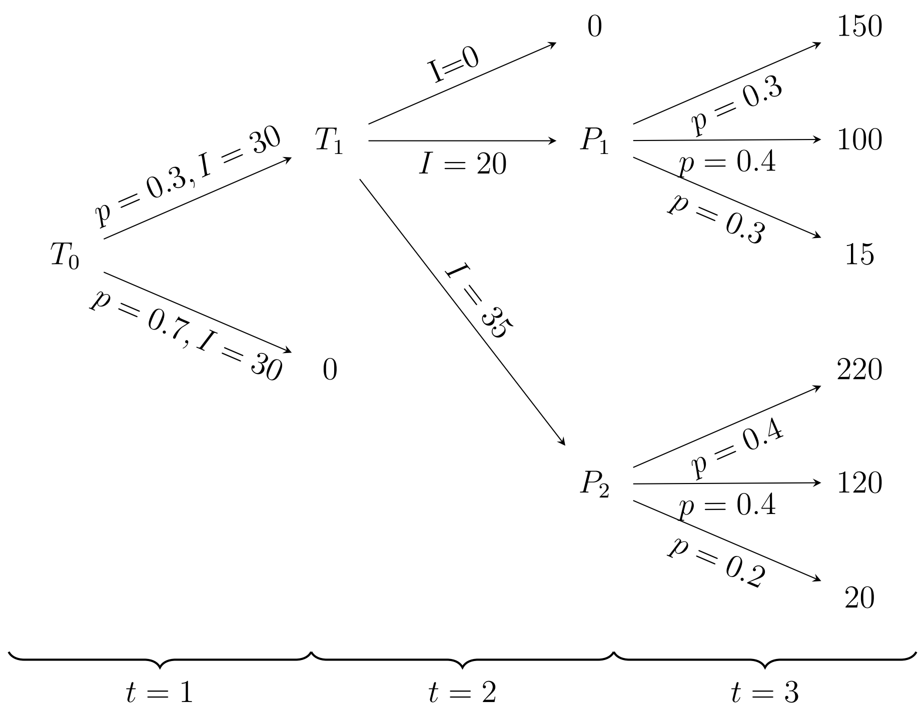


FIGURE 5.4: Example for the decision tree analysis with an option to sell the plant in year 3

embedded inside. The value of the NPV i absence of the real option is already known and equal to -1.708, often called static NPV. Now we can calculate the value of the option to abandon:

$$\begin{aligned}\text{Value of the option to abandon} &= \text{option premium}\\ &= \text{strategic NPV} - \text{static NPV}\\ &= 1.123 - (-1.708)\\ &= 2.831\end{aligned}$$

We can see for simple real options the decision tree analysis is well suited but if we shrink the step size the tree can become dense and so, very hard to calculate. Changing the discount factor for each node would even worsen the situation. Therefore, we will try to find a better solution for complex situations which need a near continuous decision making. The first step towards the solution will explained with the contingent claim analysis.

5.3 Contingent Claim Analysis

Now we consider stochastic interest rates which can be replicated by using the replicating portfolio method which replicates the payoff structure through financial transactions like risk free bonds and shares. We can use the formulas found earlier, 4.2:

$$n = \frac{E_u - E_d}{S_u - S_d} \qquad E = \frac{pE_u + (1-p)E_d}{1+r_f}$$

$$B = \frac{E_u S_d - E_d S_u}{(S_u - S_d)(1+r_f)} \qquad p = \frac{(1+r_f) - d}{u-d}$$

Let us assume a binomial tree. The upward movement gives us an increase of 50% and the downward movement gives us only 70% of the initial value, so that we have $u = 1.5$ and $d = 0.7$. Additionally, we have an initial portfolio value of 100. Each share is priced at 15. This time we consider a risk-free rate of 6%. We can now calculate each unknown with the given formulas:

$$\begin{aligned}
n &= \frac{E_u - E_d}{S_u - S_d} &&= \frac{150 - 70}{1.5 \times 15 - 0.7 \times 15} &&= 6.67 \\
p &= \frac{(1+r_f) - d}{u-d} &&= \frac{1.06 - 0.7}{1.5 - 0.7} &&= 0.45 \\
E &= \frac{pE_u + (1-p)E_d}{1+r_f} &&= \frac{0.45 \times 150 + 0.55 \times 70}{1.06} &&= 100 \\
B &= \frac{E_u S_d - E_d S_u}{(S_u - S_d)(1+r_f)} &&= \frac{150 \times 10.5 - 70 \times 22.5}{22.5 - 10.5} &&= 0
\end{aligned}$$

In this case we have a value of 100 which is the same as if we would do the calculation with the decision tree analysis. The difference with the DTA is that we use the risk-free probability in the contingent claim analysis. In order to check the equality for our example, we need to calculate the risk adjusted return of the shares which cannot always be calculated:

$$k = \frac{qS_u + (1-q)S_d}{S} - 1 = \frac{0.5 \times 22.5 + 0.5 \times 20.5}{15} - 1 = 10\%$$

We can now verify the value of our portfolio with the decision tree analysis.

$$\begin{aligned}
\text{Value with DTA} &= \frac{qE_u + (1-q)E_d}{1+k} \\
&= \frac{0.5 \times 150 + 0.5 \times 70}{1.1} \\
&= 100
\end{aligned}$$

This proves that the calculated value of the portfolio is the same with the two different methods. This changes in the presence of real options. We consider an option to abandon. We further assume that the initial investment cost, I_0 equals 104, so we would not undergo the investment since we have a negative $NPV = -4$. The investment cost at year 1 is $I_1 = I_0 \times (1 + r_f) = 110.24$. We abandon the project if the value is less than the investment cost:

$$E_u = max(V_u - I_1; 0) = 39.76$$
$$E_d = max(V_d - I_1; 0) = 0$$
$$E = \frac{pE_u + (1-p)E_d}{1 + r_f} = 17.892$$

where V_u and V_d are the values of the initial portfolio without the option to abandon after one upward and one downward movement respectively. We can again calculate the value of the option to abandon by taking the difference between the strategic and static NPV which equals to 21.892. On the other hand, we have the DTA method which can compute the value of the strategic NPV:

$$\text{Value with DTA} = \frac{qV_u + (1-q)V_d}{1 + k} = \frac{0.5 \times 39.76 + 0.5 \times 0}{1.1} = 18.073$$

And this time we have a difference between the two methods, but which one gives the correct value? We simply need to apply the replicating portfolio method to the newly found values:

$$n = \frac{E_u - E_d}{S_u - S_d} = \frac{39.76 - 0}{22.5 - 10.5} = 3.976$$
$$B = \frac{E_u S_d - E_d S_u}{(S_u - S_d)(1 + r_f)} = \frac{39.76 \times 10.5 - 0 \times 22.5}{22.5 - 10.5} = 41.748$$

And we can finally find the real value of the portfolio with

$$E = nS - B = 3.976 \times 15 - 41.748 = 17.892$$

So we can confirm that the correct value is indeed the one calculated with the CCA as already expected since we need to consider stochastic interest rates in presence of real options.

5.4 Benchmark with Black & Scholes

If we want to verify if the 2 methods, the Cox-Ross-Rubinstein and the Trigeorgis log-transformed method, give us correct values we need to set a benchmark to compare them to. We know that the Black & Scholes model is indeed used to correctly price an European put or call option. Since we transformed our project into a financial instrument through the replicating portfolio method we can use this formula to predict some prices of a certain real options. Since the Black & Scholes model is only verified for an European put and call option we need to make a change in our real option. Now we only consider the possibility to abandon the project for a salvage, X, at the last period of time. That means that we only have to check at the end:

$$R(i, j) = \max\{X - R(i, j), 0\}$$

We set certain parameters in order to standardize the tests and correctly interpret the results. To check if our methods are correct or if they converge by raising the number of intervals we take the following values:

$$\begin{aligned} T &= 0.25, 1, 3 \\ r_f &= 6\% \\ \sigma &= 0.2 \\ \frac{N}{T} &= 60, 120, 360, 720, 1080, 2160 \\ S &= 50, 60, \cdots, 150 \\ X &= 100 \end{aligned}$$

where T is the maturity of the option, r_f is the constant risk free rate, σ is the volatility of the underlying asset, N/T is the ratio between the number of steps and the maturity, so we can fix the time step more easily, S is the initial investment and finally X is the salvage for what we can resell our project. Now we have the necessary unknowns to evaluate the earlier modeled methods.

5.5 Time Step

So the first thing to do to correctly analyze the 2 given methods is to find out the size of the time step. But first we need to know that the complexity of those methods are both at $\mathcal{O}(n^2)$. That means if we increase the step number, n, the computation time will increase proportional to the square of the number

S	B&S	N/T					
		60	120	360	720	1080	2160
50	48.511	48.511	48.511	48.511	48.511	48.511	48.511
60	38.511	38.511	38.511	38.511	38.511	38.511	38.511
70	28.512	28.511	28.512	28.512	28.512	28.512	28.512
80	18.572	18.562	18.564	18.571	18.570	18.571	18.571
90	9.450	9.472	9.437	9.440	9.446	9.451	9.450
100	3.258	3.322	3.225	3.247	3.253	3.254	3.256
110	0.709	0.727	0.723	0.714	0.711	0.709	0.709
120	0.099	0.080	0.089	0.099	0.099	0.098	0.099
130	0.009	0.008	0.008	0.009	0.009	0.009	0.009
140	0.001	0.000	0.000	0.001	0.001	0.001	0.001
150	0.000	0.000	0.000	0.000	0.000	0.000	0.000

TABLE 5.1: Values of an European put option with the fixed values of $\sigma = 0.2$, $r_f = 6\%$, $T = 0.25$ and $X = 100$ computed by the Cox-Ross-Rubinstein method

of steps. That means we need to find an appropriate number of steps, so we can calculate the value in a reasonable time frame.

5.5.1 Cox-Ross-Rubinstein Binomial Tree

Starting with the Cox-Ross-Rubinstein binomial tree, we find in the table 5.1, 5.2 and 5.3 the computed values. We can compare them with the Black & Scholes model for the different maturities, 0.25, 1 and 3. We can fix the number of steps to the relative low ratio between the step number and the maturity, N/T of 360 which is nothing more than to take one step for each day if we assume that each month is approximately 30 days.

5.5.2 Trigeorgis log-transformed Binomial Tree

Continuing with the Trigeorgis log-transformed binomial tree, we can compute the same way as before the necessary values in figure 5.4, 5.5 and 5.6. And again, we find for the different maturities, 0.25, 1 and 3, that the ratio of 360 is the reasonable choice considering the computation time and the given accuracy. From now on we will only take the latter value to further analyze our methods.

5.5.3 Comparison

On table 5.7 we can compare directly the two methods given certain number of steps and with a maturity of 1. The values differ only very slightly so we can

S	B&S	N/T					
		60	120	360	720	1080	2160
50	44.179	44.179	44.179	44.179	44.179	44.179	44.179
60	34.239	34.233	34.236	34.238	34.239	34.239	34.239
70	24.669	24.653	24.664	24.666	24.667	24.668	24.669
80	16.200	16.199	16.190	16.201	16.197	16.200	16.199
90	9.611	9.581	9.604	9.604	9.609	9.612	9.611
100	5.166	5.133	5.149	5.160	5.163	5.164	5.165
110	2.542	2.543	2.553	2.546	2.544	2.542	2.542
120	1.161	1.175	1.151	1.163	1.161	1.160	1.161
130	0.498	0.490	0.497	0.499	0.499	0.499	0.498
140	0.204	0.205	0.202	0.202	0.204	0.204	0.203
150	0.080	0.078	0.078	0.080	0.080	0.080	0.080

Table 5.2: Values of an European put option with the fixed values of $\sigma = 0.2$, $r_f = 6\%$, $T = 1$ and $X = 100$ computed by the Cox-Ross-Rubinstein method

S	B&S	N/T					
		60	120	360	720	1080	2160
50	34.204	34.199	34.198	34.203	34.204	34.204	34.204
60	25.731	25.716	25.720	25.729	25.730	25.730	25.730
70	18.645	18.629	18.642	18.642	18.644	18.645	18.645
80	13.093	13.095	13.086	13.094	13.091	13.093	13.093
90	8.967	8.950	8.962	8.964	8.966	8.968	8.967
100	6.026	6.007	6.016	6.023	6.024	6.025	6.025
110	3.993	3.991	3.998	3.995	3.994	3.993	3.993
120	2.620	2.629	2.614	2.621	2.620	2.619	2.620
130	1.708	1.702	1.708	1.709	1.708	1.708	1.708
140	1.109	1.114	1.109	1.108	1.109	1.109	1.109
150	0.718	0.717	0.717	0.718	0.718	0.718	0.718

Table 5.3: Values of an European put option with the fixed values of $\sigma = 0.2$, $r_f = 6\%$, $T = 3$ and $X = 100$ computed by the Cox-Ross-Rubinstein method

S	B&S	N/T					
		60	120	360	720	1080	2160
50	48.511	48.511	48.511	48.511	48.511	48.511	48.511
60	38.511	38.511	38.511	38.511	38.511	38.511	38.511
70	28.512	28.512	28.512	28.512	28.512	28.512	28.512
80	18.572	18.562	18.564	18.571	18.570	18.571	18.571
90	9.450	9.473	9.438	9.440	9.446	9.451	9.450
100	3.258	3.324	3.226	3.247	3.253	3.254	3.256
110	0.709	0.728	0.724	0.714	0.711	0.709	0.709
120	0.099	0.081	0.089	0.099	0.099	0.098	0.099
130	0.009	0.008	0.008	0.009	0.009	0.009	0.009
140	0.001	0.000	0.000	0.001	0.001	0.001	0.001
150	0.000	0.000	0.000	0.000	0.000	0.000	0.000

TABLE 5.4: Values of an European put option with the fixed values of $\sigma = 0.2$, $r_f = 6\%$, $T = 0.25$ and $X = 100$ computed by the Trigeorgis log-transformed method

S	B&S	N/T					
		60	120	360	720	1080	2160
50	44.179	44.179	44.179	44.179	44.179	44.179	44.179
60	34.239	34.234	34.236	34.238	34.239	34.239	34.239
70	24.669	24.655	24.665	24.666	24.668	24.669	24.669
80	16.200	16.202	16.191	16.201	16.198	16.200	16.199
90	9.611	9.584	9.605	9.605	9.609	9.612	9.611
100	5.166	5.135	5.151	5.161	5.163	5.164	5.165
110	2.542	2.545	2.554	2.546	2.544	2.543	2.542
120	1.161	1.176	1.152	1.163	1.161	1.160	1.161
130	0.498	0.490	0.497	0.499	0.499	0.499	0.499
140	0.204	0.206	0.203	0.202	0.204	0.204	0.203
150	0.080	0.078	0.078	0.080	0.080	0.080	0.080

TABLE 5.5: Values of an European put option with the fixed values of $\sigma = 0.2$, $r_f = 6\%$, $T = 1$ and $X = 100$ computed by the Trigeorgis log-transformed method

		N/T					
S	B&S	60	120	360	720	1080	2160
50	34.204	34.201	34.199	34.203	34.204	34.204	34.204
60	25.731	25.720	25.722	25.730	25.730	25.730	25.731
70	18.645	18.634	18.644	18.643	18.644	18.645	18.646
80	13.093	13.099	13.089	13.095	13.092	13.093	13.093
90	8.967	8.954	8.965	8.964	8.967	8.968	8.967
100	6.026	6.011	6.018	6.023	6.025	6.025	6.026
110	3.993	3.995	4.000	3.995	3.994	3.993	3.993
120	2.620	2.632	2.615	2.622	2.620	2.620	2.620
130	1.708	1.704	1.709	1.709	1.709	1.708	1.708
140	1.109	1.116	1.110	1.108	1.109	1.109	1.109
150	0.718	0.718	0.717	0.719	0.718	0.718	0.718

TABLE 5.6: Values of an European put option with the fixed values of $\sigma = 0.2$, $r_f = 6\%$, $T = 3$ and $X = 100$ computed by the Trigeorgis log-transformed method

assume that the behavior will be the same for the sensitivity analysis. To avoid repetitive analysis, we will continue from now on with the Cox-Ross-Rubinstein method.

5.6 Volatility

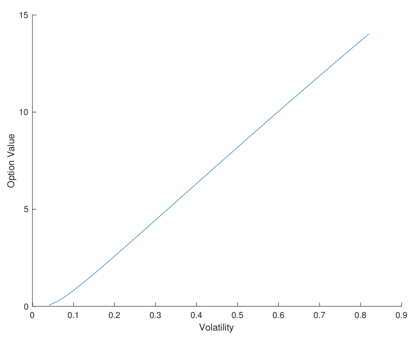
Now that we have fixed our environment we can start the analysis of the given methods for the different parameters, starting with the volatility. Intuitively we can say that the option should be worth more with higher volatilities, since the spread between the initial investment and the worst possible situation is greater with a high volatility and so the possibility to gain more money is greater too. We can analyze the change of the value of the put option graphically on figure 5.5a. As we have predicted the change in volatility depends linearly for reasonable volatilities but if we raise the volatility even further we can see that the curve converges to a limit which we can see on figure 5.5b.

5.7 Risk Free Rate

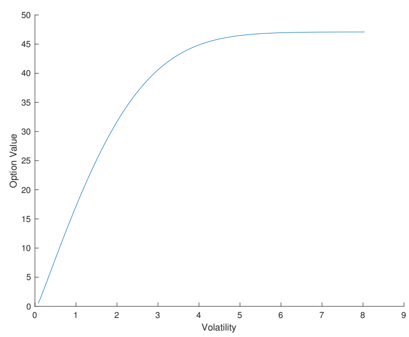
The next parameter to analyse is the risk-free rate. Logically, the price of the option will shrink over time to 0 as the risk-free rate rises. We can observe on figure 5.6a that the price is indeed lowering linearly with the behavior of the risk free rate.

		N/T					
		60		360		1080	
S	B&S	CRR	Trig	CRR	Trig	CRR	Trig
50	44.179	44.179	44.179	44.179	44.179	44.179	44.179
60	34.239	34.233	34.234	34.238	34.238	34.239	34.239
70	24.669	24.653	24.655	24.666	24.666	24.668	24.669
80	16.200	16.199	16.202	16.201	16.201	16.200	16.200
90	9.611	9.581	9.584	9.604	9.605	9.612	9.612
100	5.166	5.133	5.135	5.160	5.161	5.164	5.164
110	2.542	2.543	2.545	2.546	2.546	2.542	2.543
120	1.161	1.175	1.176	1.163	1.163	1.160	1.160
130	0.498	0.490	0.490	0.499	0.499	0.499	0.499
140	0.204	0.205	0.206	0.202	0.204	0.204	0.204
150	0.080	0.078	0.078	0.080	0.080	0.080	0.080

TABLE 5.7: Comparison between the values of an European put option with the fixed values of $\sigma = 0.2$, $r_f = 6\%$, $T = 1$ and $X = 100$ computed by the Cox-Ross-Rubinstein and the Trigeorgis log-transformed method

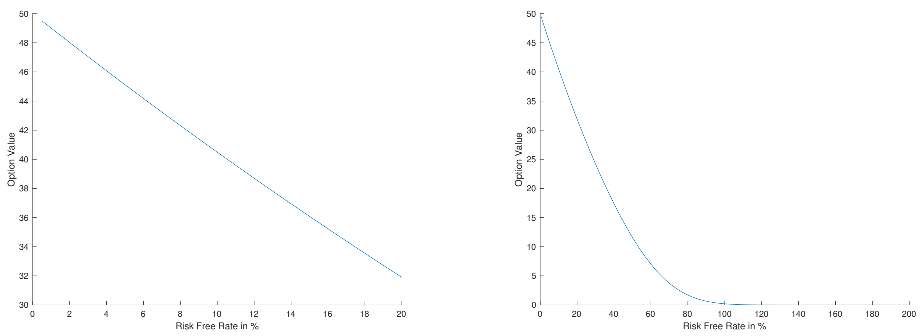


(A) Option value with lower levels of volatility



(B) Option value with higher levels of volatility

FIGURE 5.5: Sensibility of the option value to volatility calculated with the CRR method at the standard values and the initial investment value of 50 and the salvage value of 50 too



(A) Option value with lower levels of the risk free rate

(B) Option value with higher levels of the risk free rate

FIGURE 5.6: Sensibility of the option value to the risk free rate calculated with the CRR method at the standard values and the initial investment value of 50 and the salvage value of 50 too

But again, if we raise the level to unreasonable levels like a return rate of more than 60%, we can see again that it only converges to 0 and not continuing losing value linearly as it did before. This behavior can be observed on figure 5.6b.

5.8 Analysis of the Implementation of Real Options

For demonstration purposes we will only present two different real options. First, we start with the option to alter the operating scale after half a year for a project of 1 year. So, we can contract the project by a certain amount, c, or if the project is going well we can expand the production line for example and benefit from the good economic situation. The second real option is the possibility to abandon on an investment or proceed by paying it. And the second possibility is by abandoning the investment by getting paid a salvage. We will put us into a situation of a project of 1 year at the risk-free rate of 6%, a volatility of 0.2, an initial investment of 50 and a number of time steps of 360.

5.8.1 Altering Operating Scale Option

As mentioned before we can either contract or expand the investment after half a year. We will analyse the behaviour of the contracting or expanding value

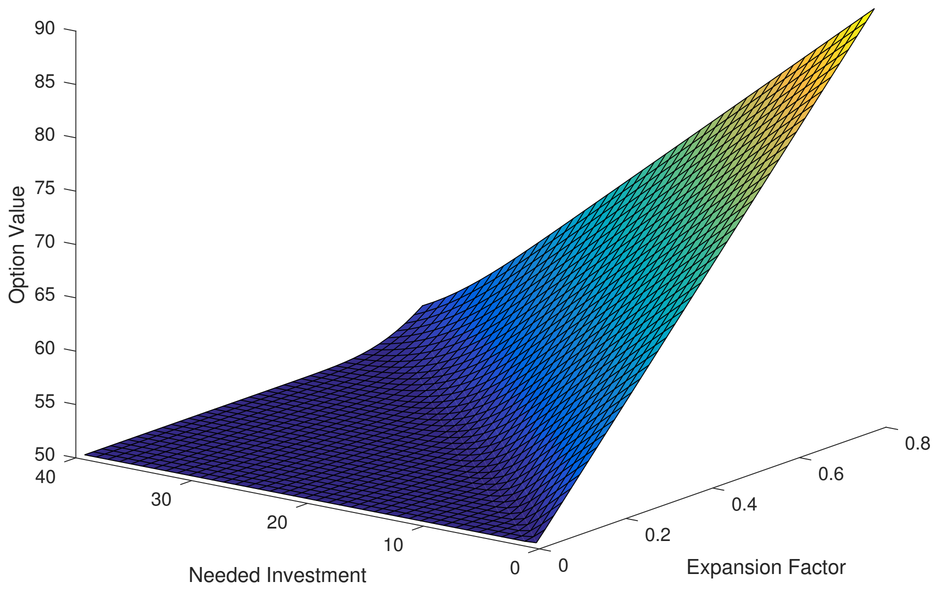


FIGURE 5.7: The expand option in terms of the different expand factors and the gain in money by expanding

and the additional investment needed.

Starting with the expansion option, we can see the behaviour of the percental expansion factor as well as the needed investment to expand on figure 5.7. We can see that the option is more and more profitable if we have the possibility to expand at a larger scale. The needed investment has the opposite effect as we could have anticipated. The higher the investment needed the lower is the value of the option.

On the other hand, we have the contraction value which behaves exactly the opposite way. We can see the behavior on figure 5.8. For higher gains, through the sell of equipment, the option value rises. At the same time if the contraction value rises the option value goes down since we produce less and therefore earn less money.

5.8.2 Abandoning Option

So, we will start by the option to abandon the project if we do not want to do the needed investment for example a quarterly payment. With greater

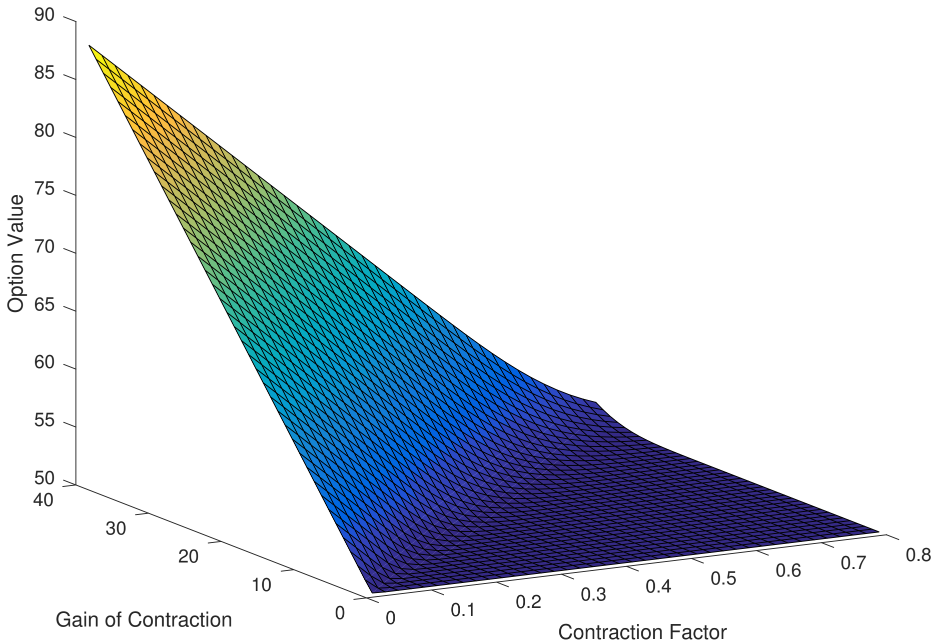


FIGURE 5.8: The contraction option in terms of the different contraction factors and the gain in money by contracting

quarterly pay the option converges to 0 as we could have anticipated which means that our method works. This is represented on figure 5.9.

Now we consider the option to abandon the project for a salvage X. This time we check at every time step if it is favorable to switch to the salvage value. Without any surprise the higher the salvage value is the better is the option. We can see the evolution of the relationship on figure 5.10.

5.9 Stochastic Interest Rates

To see the influence of stochastic interest rates, the model of Vasicek was used to simulate the evolution of the interest rates over time. We took the standard values for the table 5.8 to show the evolution of the calculated values in comparison to the fixed rate with different number of trials and with different investments at time 0. Additionally, the standard deviation of each trial set was calculated next to the mean. We took again the same parameters as we already did for the visualization of the Vasicek method: $\alpha = 0.06$, $\beta = 0.12$, $r_0 = 0.06$, $\sigma = 0.02$, $T = 5$, and $N = 360$. The other parameters are the standard ones

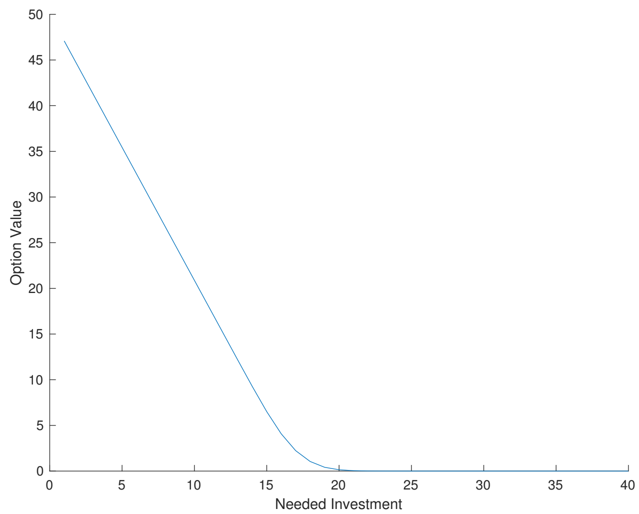


FIGURE 5.9: Option value of a quarterly investment needed to continue in terms of the needed investment

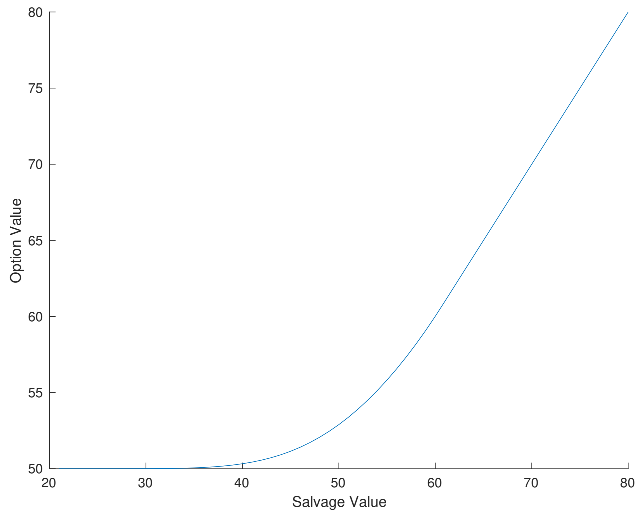


FIGURE 5.10: Abandon option for a salvage X in terms of the salvage

S	Fixed	50 trials		100 trials		400 trials	
		Mean	Sdt Dev	Mean	Sdt Dev	Mean	Sdt Dev
50	44.179	44.175	0.337	44.172	0.396	44.168	0.392
60	34.238	34.326	0.362	34.231	0.409	34.211	0.426
70	24.666	24.676	0.365	24.669	0.376	24.652	0.391
80	16.201	16.260	0.285	16.244	0.292	16.203	0.328
90	9.604	9.574	0.226	9.620	0.256	9.594	0.258
100	5.160	5.138	0.145	5.175	0.164	5.169	0.169
110	2.546	2.555	0.097	2.559	0.104	2.548	0.101
120	1.163	1.160	0.051	1.163	0.052	1.167	0.052
130	0.499	0.496	0.023	0.494	0.025	0.499	0.026
140	0.202	0.203	0.012	0.201	0.012	0.203	0.011
150	0.080	0.079	0.006	0.080	0.005	0.080	0.005

Table 5.8: Comparison of the mean and variance for the option value with stochastic interest rates computed with the Cox-Ross-Rubinstein method and with parameters: $\alpha = 0.06$, $\beta = 0.12$, $r_0 = 0.06$, $\sigma = 0.02$, $T = 5$, and $N = 360$

Number of trials	50	100	400
Time	1.3	2.6	10.5

Table 5.9: Computation time of the various set sizes for the stochastic interest rate computed with the Cox-Ross-Rubinstein method.

introduced earlier in this section.

As we can see the more trials we do per set the closer we get to the calculation with the fixed interest rate. So, we can say that the method works already quite well. But with the stochastic case we have a standard deviation which is something very important to calculate the real value of the project which might not follow the predicted way and so we can account for the latter factor with some precision. In addition to that, we can also change the target interest rate if we have knowledge about the change of some directives or if the project is taken much longer and so the yield curve should adapt accordingly to the change. As a project takes longer the return should also increase with the time. For continuous valuation of smaller projects, we would recommend to take 100 trials per set as the calculation time is at a reasonable range. We can see the average computing time for each set in table 5.9.

5.10 Stability Comparison

As analyzed earlier the Cox-Ross-Rubinstein method has its limitation. Those might not matter in real life situations but should be mentioned nevertheless. We have found a relation, 4.30, between the step size, the volatility and the risk-free rate:

$$\tau \leq \frac{\sigma^2}{m^2} = \frac{\sigma^2}{\left(r_f - \frac{1}{2}\sigma^2\right)^2}$$

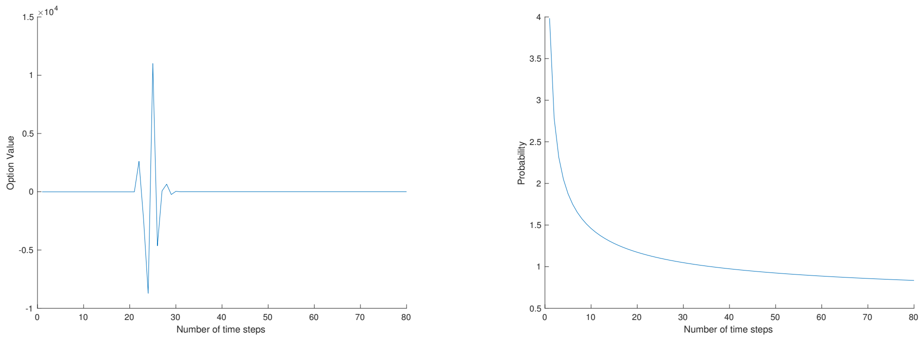
If we violate this inequality, we will find impossible probabilities. In other words, there will be a more than 100% chance that the tree will move upward which is indeed not possible.

To make our case we changed the risk-free rate to 30% and the volatility to 0.05, which would normally never happen in real life. Additionally, we put the initial investment at 50, the salvage cost at 100 and the maturity at 3 years for demonstration purposes. For those values We can see the different levels of the probability on figure 5.11b. At the very beginning we can find a probability of 400% which is indeed ridiculous. Only when the step number exceed 36 we get probabilities of less or equal to 1.

$$0.02778 = \frac{1}{36} < \frac{\sigma^2}{\left(r_f - \frac{1}{2}\sigma^2\right)^2} = \frac{0.05^2}{(0.3 - \frac{1}{2}0.05^2)^2} = 0.28 < \frac{1}{35} = 0.0286$$

Through this relation we can assume that this behavior is as expected. To see the reaction of the option value to the instability we need to look at figure 5.11a. At earlier levels the option value is around -9 and as the step number reaches the limit value the behavior gets even more unstable and we can see the reaction of the option value to impossible probabilities.

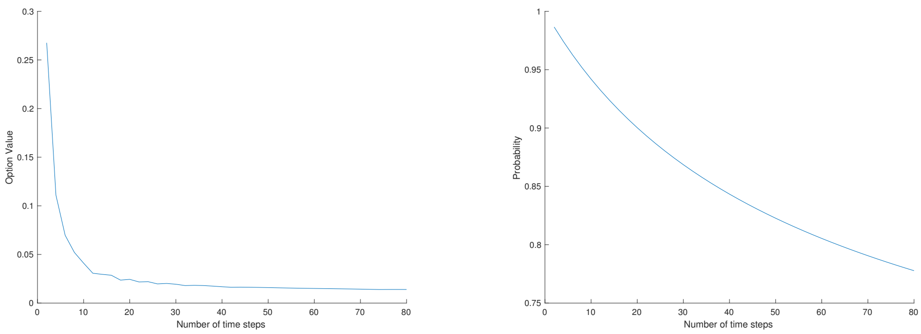
Now we can compare the reaction to the Trigeorgis log-transformed method which should always be stable even at very low number of time steps. The probabilities never exceed the 100% as we can see on figure 5.12b. At the beginning we start under the 100% mark and then we decrease further as the step count rises. Also, we do not have an unstable behavior for the option value. It starts quite high, but we can see a convergence towards the final value.



(A) Instability of the option value

(B) Calculated probabilities

FIGURE 5.11: Using the CRR method with lower numbers of the step size with a risk free rate of 30% and a volatility of 0.05



(A) Stability of the option value

(B) Calculated probabilities

FIGURE 5.12: Using the Trigeorgis log-transformed method with lower numbers of the step size with a risk free rate of 30% and a volatility of 0.05

Chapter 6

Conclusion

The purpose of the thesis is to find a more adequate method to implement a valuation tool for real life situations where we cannot predict exactly future events which is the case in the real world. The whole model should fit inside a lattice tree, which should also be binomial, as it can be seen in chapter 3. The simplistic method of the net present value can indeed not be used in more complex situations. Using a decision tree analysis to capture simple real options was the first approach but this time there cannot be captured stochastic elements of the real market. The contingent claim analysis tries to capture already more complex situations but here as well, it cannot correctly implement the stochastic nature of the future economy. But the final two methods, the Cox-Ross-Rubinstein method and the Trigeorgis log transformed method, will be based on the contingent claim analysis where the stochastic element will be introduced.

After the modeling of those 2 methods, the value can be correctly calculated as it can be seen with the benchmark taken from the Black & Scholes formula which is widely used to calculate European put and call options. After implementing such an option, the values can be compared and verified that those two methods give indeed the correct value as the number of time steps tend to infinity. The expected behavior of each component inside the model is indeed correctly captured by the given methods. An example of some simple real options are also given in the last chapter which represents only the beginning of the different possibilities which can be implemented but the implementation of more complex situations does not need much work to be put in place. The simple versions are used to show that the methods work indeed as expected. The Cox-Ross-Rubinstein method is not stable with any parameter but while using real life situations with more time steps the instability does not show itself in the different examples.

This work already shows that the valuation method of the real options is already a good start, but it is not yet used as "the" valuation tool. At the moment the most common used methods are still the net present value and sometimes the decision tree analysis. In the last decade the use of the proposed models is steadily rising, but more work still needs to be done about this topic before it will be used as the go to tool for nearly every investment decision. For the moment, the investors still prefer the simplicity of the NPV method. However, the two methods already proved themselves as a better valuation method which are unfortunately more difficult to implement.

Appendix A

Mathematical formulas

A.1 Taylor 1.5 Scheme

In order to visualize some complicated stochastic functions we need to use numerical methods to approximate the results. We introduce a common tool to do the task. Why did we choose the Taylor 1.5 scheme? It is the most accurate method which requires a reasonable amount of time to simulate. We will provide the iterative formula and its components for the general Ito process:

$$dx = \mu(x,t)dt + \sigma(x,t)dB_t \tag{A.1}$$

First of all we need two different Wiener processes, ΔB^1 and ΔB^2. Those two processes are correlated with a correlation factor of $\frac{\sqrt{3}}{2}$. So to find an expression we need to calculate them as followed:

$$\Delta B^1 = N(0,\sigma^2) \qquad \Delta B^2 = N(0,\frac{1}{3}\sigma^6) \qquad \text{with } Corr(\Delta B^1,\Delta B^2) = \frac{\sqrt{3}}{2}$$
$$\Longrightarrow \Delta B^1 = N_1^{(0,1)}\sigma \qquad \Delta B^2 = \frac{1}{2}\sigma^3\left(N_1^{(0,1)} + \frac{1}{\sqrt{3}}N_2^{(0,1)}\right)$$

The only thing left to define is$\sqrt{\Delta s} = \sigma$ and τ_i which is the i^{th} time point of the current simulation. From here on we can finally define our iterative method:

$$
\begin{aligned}
Y_0 &= x_0 \\
Y_{i+1} &= Y_i + \mu(\tau_i, Y_i)\Delta_s + \sigma(\tau_i, Y_i)\Delta B_i^1 \\
&= \frac{1}{2}\sigma(\tau_i, Y_i)\sigma'(\tau_i, Y_i)\left[\left(\Delta B^1\right)^2 - \Delta_s\right] \\
&+ \mu'(\tau_i, Y_i)\sigma(\tau_i, Y_i)\Delta B_i^2 \\
&+ \frac{1}{2}\left(\mu(\tau_i, Y_i)\mu'(\tau_i, Y_i) + \frac{1}{2}\sigma^2(\tau_i, Y_i)\mu''(\tau_i, Y_i)\right)\Delta_s^2 \\
&+ \left(\mu(\tau_i, Y_i)\sigma'(\tau_i, Y_i) + \frac{1}{2}\sigma^2(\tau_i, Y_i)\sigma''(\tau_i, Y_i)\right)\left[\Delta B_i^1\Delta_s - \Delta B_i^2\right] \\
&+ \frac{1}{2}\sigma(\tau_i, Y_i)\left(\sigma(\tau_i, Y_i)\sigma''(\tau_i, Y_i) + \sigma'(\tau_i, Y_i)\right)^2\left[\frac{1}{3}\left(\Delta B_i^1\right)^2 - \Delta_s\right]\Delta B_i^1 \\
&\text{for } i = 0, \cdots, N-1
\end{aligned}
$$

Where we have that:

$$
\begin{aligned}
\sigma'(\tau, Y) &= \frac{\partial \sigma}{\partial Y} & \sigma''(\tau, Y) &= \frac{\partial^2 \sigma}{\partial Y^2} \\
\mu'(\tau, Y) &= \frac{\partial \mu}{\partial Y} & \mu''(\tau, Y) &= \frac{\partial^2 \mu}{\partial Y^2}
\end{aligned}
$$

Appendix B

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