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Measures of Portfolio' Diversification

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Abstract

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Measures of Portfolio' Diversification

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Diversification is one the main and most important concept in the financial world. It is often said that diversification is the only free lunch in finance. From a qualitative point of view, the concept of diversification is quite clear: a portfolio is well-diversified if shocks in the individual components do not heavily impact on the overall portfolio. Relatively simple to understand then but profoundly difficult to define. Indeed, there is no broadly accepted precise and quantitative definition of diversification.

The first who proposed a mathematical formalization of diversification in a portfolio selection context was Markowitz in 1952 with his Modern Portfolio Theory. Even if it was path-breaking for that time, it doesn't clearly provide any proper definition of the term diversification or a specific measure of portfolio diversification.

Over the years, many different measures of diversification have been developed in the literature, each with its pros and cons. In the framework of this thesis, we have chosen to analyze six of them. Because we wanted to confront the weights concentration criterion with the risk minimization criterion, we decided to select measures that are based on the entropy of the weights and others that are based on the sources of risk. Those six different measures are the Shannon's Entropy, the Diversification Delta, the Diversification Ratio, the Marginal Risk Contributions, the Portfolio Diversification Index and the Effective Number of Bets.

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Contents

List of Tables

List of Figures

Chapter 1

Introduction

1.1 Context of the research

Diversification is one the main and most important concepts in the financial world. It is often said that diversification is the only free lunch in finance (Fernholz (2002)). Conceptually, the concept *diversification* represents the idea of bringing different elements together in order to introduce variety in a set of objects/things initially more uniform. The first who used diversification in an economic context and highlighted the benefits of the latter was maybe Bernoulli (1738) but he provided no mathematical foundation. To quote him: "*Another rule which may prove useful can be derived from our theory. This is the rule that it is advisable to divide goods which are exposed to some danger into several portions rather than to risk them all together*". We had to wait until Markowitz (1952) and the Modern Portfolio Theory in order to have the first mathematical formalization of diversification in a portfolio management context. The Mean-Variance Model of Markowitz is built around the idea that diversification is found by risk reduction. However, it doesn't clearly provide any proper definition of the term diversification or a specific measure of portfolio diversification.

Over the years, many different methods have been developed in order to measure the diversification of a portfolio and to create the most diversified portfolio but the concept itself seems to be hard to define. From a qualitative point of view, the concept of diversification is quite clear: a portfolio is well-diversified if shocks in the individual components do not heavily impact on the overall portfolio. Relatively simple to understand then but profoundly difficult to define. Indeed, as Meucci (2009) has noticed, there is no broadly accepted precise and quantitative definition of diversification.

The absence of precise definition or measure of diversification can represent an issue when we try to construct a diversified portfolio efficiently. And as the global financial crisis of 2008 dramatically demonstrated, there is an increasing need from both academic researchers and market practitioners for eligible ways of building more diversified portfolios.

1.2 Research Questions and Motivation

Many different measures of diversification have been developed in the literature, each with its pros and cons. In the framework of this thesis, we have chosen to analyze six of them. Because we wanted to confront the weights' concentration criterion with the risk minimization criterion, we decided to select measures that are based on the entropy of the weights and others that are based on the sources of risk. The six different measures are:

1. *The Shannon's Entropy.* Originally coming from the Information Theory and developed by Claude Shannon (1948) to solve communication problems, Shannon's Entropy has later been applied in finance to measure the amount of information given by observing the market.

2. *The Diversification Delta.* This measure introduced by Vermorken et al. (2012) is based on empirical entropy. However, due to several drawbacks, we will use an alternative version of it that has been developed by Salazar et al. (2014).

3. *The Diversification Ratio.* This measure proposed by Choueifaty (2006) is defined as the ratio of the portfolio's weighted average volatility to its overall volatility.

4. *The Marginal Risk Contributions.* This measure allows to decompose the total risk of a portfolio into the contributions of each individual assets.

5. *The Portfolio Diversification Index.* This measure indicates the number of unique investments in a portfolio and is useful to assess marginal and cumulative diversification benefits across asset classes.

6. *The Effective Number of Bets.* This measure uses the entropy of some factors (of risk) exposure distributions to indicate the diversification of a portfolio.

Having outlined the context, the criteria and the measures of diversification we will focus on in our analysis, we can define our main research question:

Among the six different measures of diversification we have decided to analyze, which one captures the most efficiently the diversification of a portfolio, from the risk minimization criterion as well as from the weights' concentration criterion?

Our motivation and ambition with this research question arises from the fact that the majority of analysis made in the literature are generally focused only on the performance of portfolios that have been created by maximizing an objective function which represents a measure of diversification. And the performance of a portfolio is depicted by its capacity to produce a maximum of returns while minimizing the risk. Few attention is given to weights' concentration even if it can have a serious impact on the overall risk profile of a portfolio. Moreover, it is not correct to say that a portfolio is diversified if its total weight is distributed among a small number of assets. Therefore, we wish to shed a light on the ability of the different measures under study to quantify a portfolio' diversification. We will focus our analysis only on historical data retrieved from Bloomberg. This way, the results that will come out of our analysis will not be distorted by estimation errors.

1.3 Structure of the Master's Thesis

Before getting into the heart of the study, let us look at the different chapters that structure the remainder of the master's thesis.

Chapter 2 We begin by reviewing the notion of risk and diversification. First, we define quickly the risk and present two of the most used measures of risk. Then we provide some definition that have been given to diversification in the literature. Some properties deemed desirable for a measure of diversification are also presented and we will try to answer the question "*Is a portfolio minimizing risk diversified*".

Chapter 3 This chapter covers the Modern Portfolio Theory developed by Markowitz as well as the CAPM. A section is also dedicated to the empirical results and shortcomings resulting from the model of Markowitz.

Chapter 4 In this chapter, we will review the six measures of diversification that are analyzed in this thesis. The chapter is divided into three parts: entropy measures, risk-based allocation and Diversification with regard to Risk Factors.

Chapter 5 This chapter is focused on the description of the methodology we followed to answer our main research question. First we describe the data we used and provide statistical descriptions. Then we turn to the structure of our analysis. We end up by explaining the computation of the Shannon's Entropy and the Risk Contribution.

Chapter 6 This chapter details our results as summarized here before and discuss the outcomes. The analyses are performed on Matlab. The Matlab scripts that we used can be found on the DIAL website, under the file linked to the electronic version of this thesis

Chapter 7 This final chapter concludes the thesis. We point out implications and underline the limitations of our work to finish with some suggestions for future research.

Chapter 2

Risk vs Diversification Measures

The concept of diversification has proved to be very difficult to define even if it is perhaps the most important of investment principles. After more than sixty years since the publication of the first mathematical formalization of diversification in portfolio selection analysis (Mean-Variance model of Markowitz (1952)), no formal unique definition or measure have been proposed for the concept.

The initial theoretical foundations of portfolio diversification was based on the principle that one investor could reduce the overall risk of his portfolio by investing in a broad range of assets. The theory developed around this idea was path-breaking for the time and provides a capital formalization of the link between diversification and risk, which can be summarized as follow; a portfolio is diversified if it reduces the overall risk. Risk can therefore be reduced but not eliminated. Indeed, as shown by the famous Capital Asset Pricing Model (CAPM)[1], there will always remain some non-diversifiable risk, or formally named systematic risk, in a portfolio and its return is related to the portion of that risk which cannot be eliminated.

The Modern Portfolio Theory developed by Markowitz (1952) and the CAPM have been largely adopted by academics and professionals of the financial sector. However, it has been

[1]This CAPM developed by Sharpe (1964), Lintner (1965) and Mossin (1966) will not be detailed here but a brief description of it can be found in the section 3.2

showed that their empirical results are not really adapted to the reality of markets. Moreover, the practical application of the Mean-Variance optimization leads most of the time to highly concentrated portfolios, which is clearly not in line with the idea of diversification and the well-known adage "don't put all you eggs in one basket".

Many other methods and measures of diversification have been proposed over the years, and lots of them are based on the minimization of risk. For example, an approach that have recently attracted a lot of attention from the practitioners is the *risk parity* which focuses on the allocation of risk rather than on the allocation of capital. The first author who used this term was Edward Qian (2005), even if the idea of risk parity goes back to the nineties, with the launch of the fund *All Weather asset allocation strategy* by Bridgewater Associates[2]. But can the diversification of a portfolio be attained only by minimizing the risk? And first, does it have a clear, unique and common definition of the concept of diversification? Even if no common definition of diversification has been obtained yet, it should however exist a way to properly define diversification as well as some common properties shared by all the different measures of diversification.

The following chapter will try to find a proper definition to the notion of diversification. Moreover, we will also try to answer the question "*Can a portfolio minimizing the risk be considered as truly diversified*". The methodology will consist to first defining the risk in section 2.1. Some example of the most known measures of risk will be also given in this section. In the section 2.2, we will then present some definitions about the concept of diversification found in the literature. Some common properties desirables for a measure of diversification will be also presented. After that, we will be able to answer the question that we have raised and also, we will be able to present the criteria of diversification we choose to focus on in our analysis part.

[2]See Schwartz, S. (2011). Risk Parity: Taking the Long View. *I&PE*, Retrieved April 2018 from: www.epe.com/risk-parity-taking-the-long-view/40025.article

2.1 Origins and Measures of Risk

The purpose of every investor is obviously to select and acquire the best investments possible, but they also have to manage the risk linked to these investments and consider the trade-off between risk and return. As Benjamin Graham, the "father of value investing", once said: *"The essence of investment management is the management of risks, not the management of returns"*. As we already know, the objective of diversification is to reduce the risk, but how can we define the risk and most significantly, how can we measure it? The following section will try to answer these questions by elaborating the concept of risk and by presenting some of the most popular risk measures.

2.1.1 Definition of Risk

In Finance, the risk can present itself under different forms such as the liquidity risk, the financial risk, the interest rate risk, the currency risk or also the operational risk. However, all these types of risk have something in common: the uncertainty. Indeed, the risk can be seen as the uncertainty of a future outcome (Mehra and Hemming (2003)) and if we limit our vision of the risk only to a financial point of view, we can define it as "*the uncertainty that an investment will earn its expected return*" (Reilly and Brown (2006)). In addition, as briefly mentioned above, the risk in finance is usually divided into two elements, which are the systematic risk and the unsystematic risk:

- The systematic risk is the risk that effect every asset and thus is a risk that has an effect in all financial markets. Therefore, it is the risk that we will never be able to eliminate and that cannot be diversified.
- The unsystematic or idiosyncratic risk is unique, specific to an asset and is thus the risk that can be diversified away within a portfolio.

2.1.2 Measures of Risk

Now that we have briefly defined the risk and the different forms that it can have, let us focus our attention on some of the most known and commonly used risk measures.

Volatility: The first one that we will present is the standard deviation, also called volatility. The volatility is surely the most common tool for measuring risk in financial markets. This measure simply indicates how much the returns fluctuate around their mean and is the square root of the variance. The more dispersion a return has around its mean, the more variable or volatile it is and thus, the more risky it is. Since the volatility is the square root of the variance, we will limit us to the equation of the latter, which is as follow:

$$Var = \frac{1}{N}\sum_{i=1}^{N}(R_i - \bar{R})^2 \tag{2.1}$$

Although the use of variance as a risk measure has been largely adopted in finance, especially due to the mean-variance model for portfolio selection developed by Markowitz (1952) (a detailed explanation of the MV model is shown in section 3.1), it implies that investors are indifferent between returns above and below the mean and this is far from true in reality. In addition, when investors think about risk, they usually have losses in mind, not swings towards higher than usual returns.

With relation to that, we also could mentioned the covariance. It simply represent the degree to which different assets move in relation to each other. More info on the covariance and its formula can be found in section 3.1.

Value-at-Risk: Another well-known risk measure is the Value-at-Risk (VaR), which was first introduced by the famous company JP Morgan in 1994. VaR basically measures the likelihood of losses, which makes it more straightforward to understand. It can be defined as follows:

$$F(Z(T) \leq VaR) = \zeta \tag{2.2}$$

Where

- F(.) is the cumulative probability distribution function,
- Z(T) is the loss and is defined by Z(T) = S(0) - S(T), with S(T) the Portfolio Value at time t,
- ζ is a cumulative probability associated with threshold value VaR, on the loss distribution of Z(t).

The explanation of the above formula could be summarized with the following sentence; "how much can one expect to lose, with a given cumulative probability ζ, for a given time horizon T?"

No other measures will be presented given that it is not strictly related to the topic of this thesis, but it is interesting to ask the question what properties a risk measure can have to be defined as such. Many different authors have written on the subject over the years, but a significant milestone was achieved when Artzner et al. (1999) proposed the first axioms of risk measurement. According to this paper, a risk measure has to obey the axioms in order to be called a coherent risk measure.

Let's consider X and Y, two random variables that denote the future loss of two portfolios. We say that a risk measure ψ is coherent if it adheres to the four axioms:

1. Monotonicity: if $X \leq Y$ then $\psi(X) \leq \psi(Y)$; the monotonicity axiom tells us that we associate higher risk with higher loss.
2. Positive homogeneity: $\psi(\lambda X) = \lambda\psi(X)$ for $\lambda > 0$; The homogeneity axiom ensures, since the risk of a stock comes from the stock itself, that we cannot increase or decrease risk by investing differing amounts in the same stock. The assumption of this axiom is that there is no liquidity risks, which is not true in the reality of the markets.
3. Translation invariance: $\psi(X + \chi) = \psi(X) - \chi$, where χ is a riskless bond. We will always receive back an investment in a riskless bond so such investment bears no loss

with probability 1. Consequently, since risk measures measure loss as a positive amount, a gain is negative and hence the riskless bond's investment must be subtracted.

4. Sub-additivity: $\psi(X+Y) \leq \psi(X) + \psi(Y)$. The subadditivity is the most important axiom because it ensures that a coherent risk measure considers portfolio diversification: the risk of an investment in both X and Y is lower than the sum of the risks of X and Y when taken separately.

By looking at the four axioms, we can notice the variance and the VaR that we have presented doesn't respect all of them and therefore can't be named coherent measures of risk. Indeed, the variance violates the axiom of monotonicity and the VaR does not obey the subadditivity axiom. The consequence of the latter is that the VaR of a portfolio can be greater than the sum of the individual risks. In addition, it has been proved that the use of VaR for portfolio optimization leads to more concentrate portfolios[4].

The incoherency of the VaR has raised a lot of questions and has led to the proposition of a wide variety of VaR-related risk measures. An well-known example is the Conditional Value-at-Risk (CVaR), also called Expected Shortfall (ES). ES has become a particularly popular risk measure due to its similarity to VaR: it gives an indication for the magnitude of the expected loss if the VaR threshold is breached, or in more simple words, it assesses "how bad things can get" if the VaR loss is exceeded. Briefly this can be defined by:

$$ES = \mathbb{E}\big[Z(T)|Z(T) > VaR\big] \tag{2.3}$$

[4]See McNeil (2002) and McNeil et al. (2005)

2.2 Notion of Diversification: Axioms and Definitions

2.2.1 Definitions of Diversification

The concept of diversification is more nuanced and complex than most people realize. The global financial crisis that we have experienced in 2008 highlights serious deficiencies in our true understanding of the subject. The fact that we still have difficulty understanding the concept of diversification maybe comes from the multitude of definitions that exist on the subject. Indeed, even if it is the cornerstone of all portfolio selection models, the term *diversification* can have many different definitions according to the literature. For some academics and practitioners, the definition of diversification depends on the method or model used.
For example, Fragkiskos (2014) distinguishes nine different definitions of diversification by gathering various approaches of portfolio diversification. These approaches are the market portfolio, the number of securities, mutual correlations, tail risk measures, returns, risk contributions, risk ratios, information theory and the principal portfolio respectively. It therefore doesn't present a unique definition and it also doesn't mention any common properties that a measure of portfolio diversification should have.

For others, the key characteristic of diversification is that it reduces the exposure to risk by combining different assets with low correlation of return (Reilly and Brown (2006)). However, if we define diversification only as a method that allow us to reduce the risk, we could find our self in some situation where we have indeed a portfolio with a minimal variance, composed only of not risky assets, but which is extremely concentrated in a few variety of assets and that does not represent a well-diversified portfolio. Therefore, it is not correct to limit the definition of diversification to the reduction of risk.

Another and maybe more interesting way to define diversification can be found in a choice theoretic framework. A recent paper on the subject has been published by De Giorgi and Mahmoud (2016). They provide the first comprehensive overview of the various existing formalizations of the notion of diversification from a choice theoretic perspective. In order to understand the definitions that will follow, it is important to clarify the theoretical setup that is used. We have to consider a decision maker who chooses from the vector space $\mathcal{X} = \mathbb{L}^{\infty}(\Omega, \mathcal{F}, \mathbb{P})$ composed of random variables bounded on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where Ω is the set of states of nature, $\mathcal{F}$ is a σ-algebra of events and $\mathbb{P}$ is a σ-additive probability measure on $(\Omega, \mathcal{F})$.

In such framework, and since diversification represents the idea of choosing variety over uniformity, the desirability of diversification can be axiomatized as the preference for a convex combination of choices[4] that are equivalently ranked. Axiomatically, the preference for diversification can be written as follows:

$$x_1 \sim \cdots \sim x_N \Rightarrow \sum_{i=1}^{N} \alpha_i x_i \succeq x_j \quad \text{for all } j = 1, \ldots, n \tag{2.4}$$

where $\mathcal{X}$ is a set of choices, $x_1, , x_N \in \mathcal{X}$ and $\alpha_1, , \alpha_N \in [0, 1]$ for which $\sum_{i=1}^{N} \alpha_i = 1$. This notion of diversification comes from Dekel (1989) and is equivalent to that of convexity of preferences, which states that $\alpha x + (1 - \alpha) y \succeq y$, for all $\alpha \in [0, 1]$, if $x \succeq y$.
Concretely, the definition states that *"an individual will want to diversify among a collection of choices all of which are ranked equivalently"*. To illustrate this, we could think of an investor evolving in the universe of asset markets and who has to make a choice between risky positions (equities, portfolios, bonds, derivatives,etc.).

Others similar definitions of diversification have been proposed by Chateauneuf and Tallon(2002). In their paper, they have introduced two notions of diversification: the first one is a stronger notion of *sure diversification* and the second one is a notion of *comonotone diversification.*

[4]In economy, the concept of convex preferences stipulates that if a decision maker is allowed to combine two (or several) choices, he is ensured under convexity that he is worse off than the least preferred of the two choices, that "the average is better than the extremes"

The notion of sure diversification stipulates that *"if the decision maker is indifferent between a collection of choices and can attain certainty by a convex combination of these choices, he should prefer that certain combination to any of the uncertain choices used in the combination"*. The formal definition is as follows:

Preference for sure diversification: *A preference relation $\succeq$ exhibits preference for sure diversification if for any $x_1, , x_N \in \mathcal{X}$ and $\alpha_1, , \alpha_N \geq 0$ satisfying $\sum_{i=1}^{N} \alpha_i = 1$ and c, $\beta \in \mathbb{R}$,*

$$\Big[_1 \sim \cdots \sim x_N \wedge \sum_{i=1}^{N} \alpha_i x_i = \beta\&_c\Big] \Rightarrow \beta\&_c \succeq x_i, \forall i = 1, \dots, N \tag{2.5}$$

The notion of *comonotone diversification* is basically the application of convexity of preferences to comonotonic random variables. Two random variables $w, y \in \mathcal{X}$ are said to be *comonotonic* if they yield the same ordering of the state space from best to worst, which can be written as $(x(\omega) - x(\omega'))(y(\omega) - y(\omega')) \geq 0$ for every $\omega, \omega' \in S$.

Chateauneuf and Lakhnati (2007) have proposed a weakening of the concept of preference for diversification which stipulates that the decision maker will want to diversify between two choices that are identically distributed. This weakened concept is referred to as preference for strong diversification:

Preference for strong diversification: *A preference relation $\succeq$ exhibits preference for strong diversification if for all x, $y \in \mathcal{X}$ with $F_x = F_y$ and $\alpha \in \big[0, 1\big]$, $\alpha x + (1 - \alpha) y \succeq y$.*

2.2.2 Properties of Diversification

The previous definitions provide a more formal framework to define the notion of diversification, but they still don't provide some properties that a diversification measure should have in order to be categorized as such. A proposition of such common properties can be found in Carmichael et al. (2015). They detailed five properties that are deemed desirable for a measure of portfolio diversification.

They can be summarized as follows:

- Property 1: A single asset portfolio must have the lowest diversification degree (degeneracy relative to portfolio size). This comes directly from the intuitive definition we have about diversification.

- Property 2: A portfolio formed solely with perfectly similar assets must have the lowest diversification degree (degeneracy relative to dissimilarity). This also is a direct result of the intuitive definition of diversification.

- Property 3: Consider a universe where an asset is duplicated. An unbiased portfolio construction process should produce the same portfolio, regardless of whether the asset was duplicated (duplication invariance). This property has been developed by Choueifaty et al. (2013) and stipulates that two portfolios, $U = A, B$ and $\mathbf{U} = A, A, B$, where asset A has been duplicated, must have the same diversification level.

- Property 4: An increase in portfolio size does not decrease the degree of portfolio diversification (non decreasing in portfolio size). This property comes from Markowitz (1952, p.89) where he states: "The adequacy of diversification is not thought by investors to depend solely on the number of different securities held".

- Property 5: A portfolio of less dissimilar assets is likely to offer less diversification than one of more dissimilar assets (non-decreasing in dissimilarity). Several authors see this property as desirable, like Markowitz (1952), Sharpe (1972) and Klemkosky and Martin (1975).

Those five properties form a useful tool to study diversification and give more structure to the evaluation of a measure of diversification and its characteristics.

2.3 Is a portfolio minimizing risk diversified?

As we have seen, the main idea we have on the diversification of a portfolio is closely linked to the one of overall risk reduction and minimization. But is it the only important criterion to create a diversified portfolio? If we limit the concept of diversification to risk reduction/minimization, then we only need to use a measure of risk to measure the diversification. Such reflexions haven been implied by several authors. For example, Markowitz (1952) emphasized the importance for a risk measure to encourage diversification and Föllmer and Schied (2010 and 2011) stated that *"a good measure of risk needs to promote diversification"*.

Therefore, we can see that the criterion of weights' concentration, as depicted by the well-known adage "*don't put all your eggs in one basket*", is not very considered when talking about diversification. In fact, lots of emblematic figures of the financial world doesn't seem to think that it is a bad idea to concentrate a portfolio in few investments. Indeed, if we look at the composition of Berkshire Hathaway's portfolio, the multinational conglomerate holding company created by the legendary investor Warren Buffet, in the late '80s, we can notice that it contained only a number of assets. To be precise, it was only composed of Washington Post, GEICO, ABC-Cap Cities and Coca-Cola. Moreover, the famous economist and accomplishing investor John Maynard Keynes once said about diversification: *"To carry ones eggs in a great number of baskets, without having time or opportunity to discover how many (baskets) have holes in the bottom, is the surest way of increasing risk and loss"*.

But does it mean that it is not important to not consider the weights' concentration when we want to diversify a portfolio? Of course, it is not. If we want to stay in line with the true concept of diversification, we have to consider the concentrations. For example, a portfolio showing the lowest risk level but where its investments are concentrated on a limited number of assets can't be considered as diversified. For that purpose, the measures of diversification we studied in the framework of this thesis will be analyzed on the basis of two criteria: first we will see if they can capture and express the risk present in a portfolio, as expressed by the variance/volatility and VaR described above. Second, we will see if they consider the weights' concentration when representing the diversification of a portfolio.

Chapter 3

Modern Portfolio Theory

The Modern Portfolio Theory ("MPT") is a portfolio selection approach proposed by Markowitz in his paper "Portfolio Selection", published in 1952 by the Journal of Finance[5]. Even though a lot of criticisms have been made on this theory, it is considered as path-breaking more than sixty years later after its publication and it is still used and studied today.
In this section, we will first discuss about the approach and then we will highlight the limitations and problems encountered by the MPT in the real world.

3.1 Markowitz's portfolios selection: model and assumptions

The major insight provided by Markowitz is that assets's risk and return profiles should not be viewed separately but should be evaluated by how these assets affect the overall portfolio's risk and return. Indeed, the approach developed by Markowitz considers the expected rate of return and risk of individual assets but also their interrelationship as measured by correlation.

[5] See Markowitz, H. (1952). Portfolio Selection. *Journal of Finance*, 7, 77-91.

We can see this analytically: we assume there are N risky assets composing the portfolio, such that each asset has a weight in the portfolio, noted ω_i, where each weight i represents the proportion of the i-th asset held in the portfolio. Hence, the weights can be represented by an N-vector $\omega = (\omega_1, \omega_2, ..., \omega_N)^T$ and $\sum_{i=1}^{N} \omega_i = 1$.

Now let's assume that the asset returns $R = (R_1, R_2, ..., R_N)^T$ have expected rates of return $\mu = (\mu_1, \mu_2, ..., \mu_N)^T$. The portfolio is composed of the set of assets thus its expected rate of return should depend on the expected rates of return of each asset included in it. So, the expected rate of return of the portfolio is the weighted average of the expected returns of the assets composing it. And because the expected rates of return of each assets are stochastic return, the overall return on the portfolio is given by the random variable R_P.

This give,

$$R_P = \sum_{i=1}^{N} \omega_i R_i \tag{3.1}$$

The expected value of a weighted sum is the weighted sum of the expected values so,

$$\mu_P = \sum_{i=1}^{N} \omega_i \mu_i \tag{3.2}$$

Expressed on the matrix form, it gives

$$R_P = \mathbf{w}^{\mathbf{T}} \mathbf{R} \tag{3.3}$$

$$\mu_P = \mathbf{w}^{\mathbf{T}} \boldsymbol{\mu} \tag{3.4}$$

The portfolio risk is measured by the portfolio variance σ_P^2, so the variance of a weighted sum. In order to express it we must define the "covariance". The covariance between R_i and R_j is defined as

$$\sigma_{ij} = \mathbb{E}\{[R_i - \mathbb{E}(R_i)][R_i - \mathbb{E}(R_j)]\} \tag{3.5}$$

This can also be expressed in terms of the correlation coefficient ρ_{ij},

$$\sigma_{ij} = \rho_{ij} \sigma_i \sigma_j \tag{3.6}$$

We thus have

$$\sigma_P^2 = \sum_{i=1}^{N} \omega_i^2 \sigma_i^2 + 2 \sum_{i=1}^{N} \sum_{i>1}^{N} \omega_i \omega_j \sigma_{ij} \tag{3.7}$$

And if we use the fact that the variance of r_i is σ_{ii} then

$$\sigma_P^2 = \sum_{i=1}^{N} \sum_{j=1}^{N} \omega_i \omega_j \sigma_{ij} \tag{3.8}$$

Expressed on the matrix form, the $N \times N$ covariance matrix between the returns is given by

$$\Sigma = \begin{pmatrix} \sigma_{11} & ... & \sigma_{1N} \\ .. & .. & .. \\ \sigma_{N1} & ... & \sigma_{NN} \end{pmatrix}$$

And the portfolio variance is given by

$$\sigma_P^2 = \mathbf{w}^{\mathbf{T}} \mathbf{\Sigma} \mathbf{w} \tag{3.9}$$

As a result, under the model, the portfolio return is the proportion-weighted combination of the constituent assets' returns and the portfolio volatility (risk) is a function of the correlations of the component assets.

We thus see that an investor get different combinations of μ and σ^2 depending on the choice of $\omega_1, \omega_2, ..., \omega_N$. Indeed, by choosing the portfolio weights, an investor effectively chooses between the available mean-variance pairs. He will thus choose a portfolio which give an efficient combination between mean and variance. So, in other words, a portfolio can be considered as efficient if it either minimizes the risk given a level of return or it maximizes returns given a level of risk.

The set of feasible portfolios, or the attainable set, is simply the set of all possible portfolios that can be formed by varying the portfolio weights such that $\sum_{i=1}^{N} \omega_i = 1$.

These two sets, the feasible one and the efficient one, are represented in figure 3.1. The attainable set is the interior of the ellipse and the efficient set is the upper left part of its boundary.

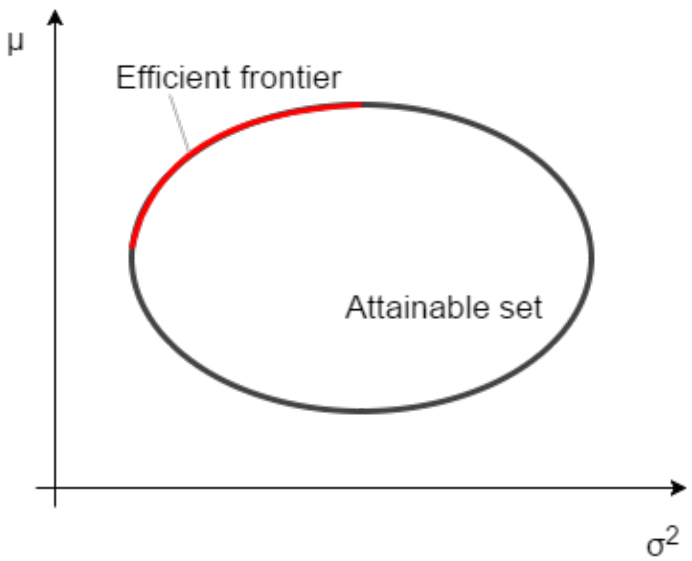


Figure 3.1: The efficient portfolios in the MV plane

The goal is then to choose the portfolio weighting factors optimally. However, even if both of the views about the efficiency of a portfolio are equivalent, the optimization does differ for these two cases: the portfolio minimizing the risk is a quadratic optimization with linear constraints, whereas in the portfolio maximizing the returns the objective function is linear, and the constraints are quadratic. In the framework of this thesis, I will only present the optimization minimizing the risk.

Model For the case of a minimal variance portfolio and a given portfolio return μ_p, the optimization problem consists in identifying optimal portfolio weights $\omega_1, \omega_2, ..., \omega_N$ which solve the following quadratic program:

$$\begin{aligned} MV = \min_{w} \quad & \sigma_P^2 = \omega^{\mathbf{T}} \mathbf{\Sigma} \omega \\ \text{s.t.} \quad & \mu_P = \omega^T \mu \\ & \omega^T \mathbf{i} = 1 \end{aligned} \tag{3.10}$$

where $\mathbf{i}$ is the $(N \times 1)$ vector of ones.

This program represents the situation where short sales are allowed. If we wanted to include the condition that only long positions are allowed in the model, we should add the constraint $\omega_i \geq 0$.

The solution of the constrained quadratic optimization problem presented above has been derived by Merton (1972). The latter has proved this problem can be solved using the method of Lagrange multipliers. A more detailed exposition of the results can be found on the paper written by Merton.

$$\omega^* = \boldsymbol{\mu_p}\mathbf{g} + \mathbf{f} \tag{3.11}$$

with

$$\mathbf{g} = \frac{1}{d}(c\Sigma^{-1}\boldsymbol{\mu} - b\Sigma^{-1}\mathbf{i}) \tag{3.12}$$

$$\mathbf{f} = \frac{1}{d}(b\Sigma^{-1}\boldsymbol{\mu} - a\Sigma^{-1}\mathbf{i}) \tag{3.13}$$

with $a = \boldsymbol{\mu}'\Sigma^{-1}\boldsymbol{\mu}$, $b = \boldsymbol{\mu}'\Sigma^{-1}\mathbf{i}$, $c = \mathbf{i}'\Sigma^{-1}\mathbf{i}$ and $d = ac - b^2$. The portfolio standard deviation is given by

$$\sigma = \sqrt{\frac{1}{d}(c\mu_P^2 - 2b\mu_p + a)} \tag{3.14}$$

The portfolio which minimizes the variance for a specified expected return is called a "frontier portfolio". It follows from this equation that all frontier portfolios ω^* are a linear combination of two portfolios **f** and **g**. The equation of the portfolio standard deviation represents the "efficient frontier", a hyperbola for mean-variance portfolios as represented in figure 3.1. Concretely, this efficient frontier consists of the set of all efficient portfolios that yield the highest return for each level of risk, as mentioned above, and provides an elegant way to achieve an efficient allocation: a higher expected return can only be achieved by taking on more risk. An interesting efficient allocation to focus on is the one at the bottom of the efficient frontier. This point constitutes the portfolio with the smallest variance among all feasible portfolios, or in other words, the portfolio with the minimum risk but also the minimum return. It is called the global minimum variance (GMV) portfolio.

Over the last several years, minimum variance portfolio has attracted more and more of investor' attention. This gain in popularity can be explained by two things. Firstly, the latest financial crisis has emphasized the importance of risk management and a minimum-variance strategy is obviously one of the perfect way to address it. And secondly, recent papers[8] support the long-standing critique of the CAPM (see section 3.2) that high-risk stocks don't always lead to high-return, as demonstrated by Fama and French (1992), meaning that a portfolio construction based on the minimum-variance principle could represent a highly efficient solution for portfolio selection.

The minimum variance portfolio is a specific portfolio on the mean-variance efficient frontier, very easy to compute and presenting a unique solution which is recognized to be robust since the portfolio ignore estimates of the expected returns[9]. The optimization function can simply be written as:

$$\begin{aligned} \text{GMV} = \min_{w} \quad & \omega^{\mathbf{T}} \mathbf{\Sigma} \omega \\ \text{s.t.} \quad & \omega^T \mathbf{i} \; = \; 1 \end{aligned} \tag{3.15}$$

Its weights are given by $\omega^*_{GMV} = \Sigma^{-1}\mathbf{i}/\mathbf{i}'\Sigma^{-1}\mathbf{i}$. We can observe that the weights can be thought of as a limiting case of the equation 3.10, if a mean-variance investor ignores the constraint on expected returns.

Assumptions The mean-variance analysis described above comes necessary with some assumptions. Their validity have been largely disused by the practitioners so I will only enumerate them here but the empirical results of Markowitz's MV model is presented in section 4.3:

- All investors are risk averse, meaning they prefer less risk to more for the same level of expected return

[8] As an example, Ang et al. (2006) published a paper where they document a low-risk and high-return empirical anomaly that they associate with idiosyncratic risk.

[9] The empirical problems of the MV model are discussed in section 3.3. However, it is important to mention that in this case, the expected returns do appear in the estimated covariance matrix, but it has been proved by Morrison (1990) that, under the assumption of Normal asset return, for any estimator of the covariance matrix, the MLE estimator of the mean is always the sample mean. This allows one to remove the dependence on expected returns for constructing the MLE estimator of the covariance matrix.

- Expected returns for all assets are known
- The variances and covariances of all asset returns are also known
- Investors need to know only the expected returns, variances, and covariances of returns to determine optimal portfolios. They can ignore skewness, kurtosis, and other attributes of a distribution
- There are no transaction costs or taxes

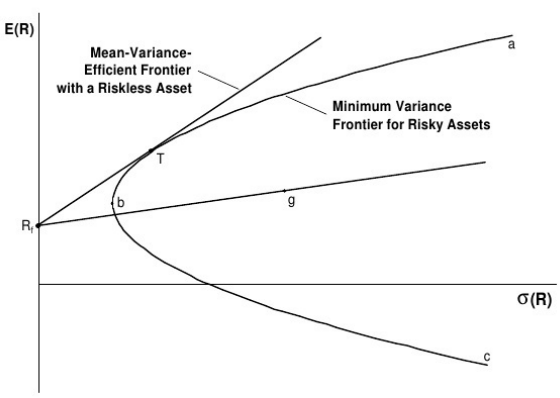
3.2 Capital Market Line and the CAPM Model

Following Markowitz's model, an investor is only interested in the mean and variance of the return on his portfolio and will therefore invest in one that is efficient in the mean-variance way. Based on this framework, Sharpe (1964) and Lintner (1965) developed an equilibrium model, called the Capital Asset Pricing Model or CAPM, by adding two key assumptions to the Markowitz model. The first one is that investors have identical views about risky assets mean returns, variances of returns, and correlations and the second one is that investors can borrow and lend at the risk-free rate without limit, and they can sell short any asset in any quantity.

The introduction of the second assumption, so allowing a portfolio to include risk-free security has an interesting effect: it turns the efficient frontier into a straight line, called the Capital Market Line, that is tangential to the risky efficient frontier and with a y-intercept equal to the risk-free rate. We can see from figure 3.2 that all efficient portfolios are a combination of the risk-free rate and a single risky tangency portfolio, T. This result comes from Tobin's (1958) "separation theorem" and the tangency portfolio T is the optimal portfolio only composed of risky assets.

As a summary, every investor that is a mean-variance optimizer will hold the same tangency portfolio of risky assets in conjunction with a position in the risk-free asset. So, because all investors hold the tangency portfolio and because the market must clear, the T portfolio is referred as the market portfolio.

Figure 3.2: Efficient frontier with a risk-free asset



Note: the figure comes from Fama and French (2003)

The equation of the capital market line (CML), and thus of the CAPM can be derived as

$$\mathbb{E}[R_i] = r_f + [\mathbb{E}[R_M] - r_f]\beta_{iM} \tag{3.16}$$

where $\mathbb{E}[R_i]$ is the expected return on asset i and β_{iM} is the market beta of asset i, defined by the ratio of the covariance of its return with the market return and the variance of the market return, $\beta_{iM} = \frac{cov(R_i,R_M)}{\sigma^2(R_M)}$. The interpretation of beta is that it measures the volatility of a security or portfolio in comparison to that of the market as a whole. In our case of portfolio selection, the beta represents the volatility of an asset composing the portfolio in comparison to the total volatility of the portfolio and is defined by $\beta_{iP} = \frac{cov(R_i,R_P)}{\sigma_P^2}$. In fact, beta is a measure of the systematic risk.

This equation can be interpreted as follows: the expected return on a security or a portfolio is equal to the risk-free rate (r_f) plus a risk premium. And the risk premium on a stock or portfolio varies directly with the level of systematic risk (beta). So concretely the security market line shows the risk or expected return tradeoff that exists with the CAPM. As stated earlier, since the idiosyncratic (unsystematic) risk can be eliminated through diversification by holding a large portfolio, more return means that an investor must take more systematic risk.

Another interesting element to look at is the slope of the CML, which is $\frac{\mathbb{E}[R_M]-r_f}{\sigma_{R_M}}$, also known as the Sharpe ratio. This ratio represents the expected excess return of a portfolio to the portfolio's volatility, or in other words the expected excess return per unit of risk. The Sharpe optimal portfolio is the one with maximum Sharpe ratio, which is nothing more than the tangency portfolio.

3.3 Empirical results of Markowitz's Mean-Variance Model

Despite its simplicity of comprehension and utilization, the traditional mean-variance model of Markowitz has many shortcomings when applied in practice. Theses shortcomings are mainly related to estimation errors. Indeed, we do not know in practice the true value of the parameters μ_i and σ_i. We therefore have to replace these unknown parameters by estimates, based on historical data and this obviously leads to estimation errors, which have a direct impact on the portfolio weights so that the desired properties of diversification are no longer met. The problems resulting from estimation errors are as follows:

1. Extreme allocations and sensitivity: the first problem we encounter with the MV analysis is that it tends to produce portfolios composed of combinations of extreme shorts and extreme longs, which are very sensitive to a change in the inputs. For example, Chopra (1993) has shown that small changes in the input mean-variance parameters lead to large variations in the composition of the optimal portfolio.

2. Estimations of the returns: weights of the optimal portfolio tend to be extremely sensitive to very small changes in the expected returns. For example, a small increase of the expected return of one asset can greatly alter the optimal portfolio' composition. Best and Grauer (1991) have demonstrated that and have also showed that the more assets there are in a feasible set, the more sensitive MV-efficient portfolio will be to changes in the means. Also, one of the result of Kallberg and Ziemba (1984) is that estimation errors in the means are about ten times as important as estimation errors in variances and covariances.

3. Estimation in variances and covariances: errors in estimated covariance matrices can also have considerable impact. Chopra and Ziemba (1993) demonstrated that, at a lower risk tolerance, the relative impact of errors in mean, variance and covariances is closer. However, the problems linked to variance and covariance' estimation errors can be mitigated to varying extents through the use of more robust estimation techniques, as we will see it later.

4. Amplification of large estimation errors: it has also been shown (Jobson and Korkie (1981), Michaud (1989)) that optimal portfolio tends to amplify large estimation errors in certain directions. Indeed, the risk of the estimated optimal portfolio is typically under-predicted and its return is over-predicted: if the variance of an asset appears to be small, or similarly the mean return of this asset appears to be large, as a result of being significantly underestimated, or overestimated respectively, the optimal portfolio will assign a large weight to it.

The major result of these estimation errors is that optimal MV portfolios are often extremely concentrated on a few asset, and this is clearly not in line with the notion of diversification. Beside from the estimation errors, the Mean-Variance approach allows extreme concentrations to occur by creating optimal portfolios composed of few assets which are presenting the highest (expected) returns (Bernstein, 2001). Many authors, such as Green and Hollifield (1992), Bera and Park (2008) or Yu, Lee and Chiou (2014) have published on the subject and proposed solutions to deal with this issue. The content of the following sections will thus be focused on the presentation of several diversification measures. Once the presentation will be made, we will analyze their results and see if they meet some criteria that any diversification measures must have to be named as such. These criteria and the reason of their selection will be developed in the chapter 5, as well as the description of the dataset and methodology we used to perform our analysis.

Chapter 4

From MV Selection to Alternative Methods

As we have seen, the use of Markowitz approach to create an optimal portfolio has proved to be problematic. Indeed, the method requires the expected return and variance of each security, and we have seen that the estimation of theses parameters leads to errors which result in the creation of optimal portfolios excessively concentrated in a limit number of assets. Therefore, the use of the MV method to construct a well-diversified portfolio doesn't seems to be the best choice. Moreover, the results shown by this method are clearly in contradiction with the notion of diversification that we have presented in chapter 2.

In the following chapter, we will present some measures of diversification that have been developed over the past few years. These measures can be grouped into the three following categories:

1. Entropy Measures
2. Risk-based allocation
3. Diversification with regard to Risk Factors

4.1 Entropy Measures

Entropy is a concept which was originally introduced in the information theory by Shannon (1948) and was defined in a statistical mechanics framework. Although it was not initially made to be applied in the portfolio management field, some interesting studies have been written on the subject and its use as an alternative measure of risk is now an established fact.

The first authors who have used entropy in a portfolio optimization framework were Philippatos and Wilson (1972). Since then, many other related works such as Hua and Xingsi (2003), Bera and Park (2008) or Meucci (2009) have been published. In the framework of this thesis aiming to review and analyze measures allowing to assess the diversification of investment portfolios, we will focus our attention on the Shannon entropy and the diversification delta.

4.1.1 Shannon Entropy and Equally Weighted Portfolio

As mentioned, many academic and practitioner's literature have proposed to use entropy approach to optimal portfolio selection. The main idea of this approach is to use entropy measure as the objective function for the optimization problem. In this case, weights of the portfolio are considered as the probability mass function of a random variable. We can obtain a pretty well-diversified portfolio by maximizing the entropy measure for such a random variable which are subject to some given constraints. Moreover, using entropy as the objective function guarantees non-negative weights for the assets in the portfolio, as they are seen as probabilities, which by definition take non-negative values.

Statistically, the concept of entropy can be resume as follows: it is a probabilistic measure of uncertainty or disorder. Basically, the entropy of a random variable is the "amount of information" contained in this variable: the more information it contains, the more order it has. Entropy is therefore a useful tool which is able to quantifies the amount of information in a random variable, which provides a good measure of the disorder present in a probability distribution. The most prominent entropy measure is surely the Shannon's entropy (H). In a portfolio construction framework, it allows to shrink the weight of the assets toward an equally weighted portfolio. It has the following form:

$$H(\omega) = -\sum_{i=1}^{N} \omega_i \log \omega_i, \tag{4.1}$$

where $\omega = (\omega_1, \omega_2, ..., \omega_N)$ is a portfolio allocation among N risky assets with properties that $\sum_{i=1}^{N} \omega_i = 1$, $\omega_i \geq 0$ and i = 1,2...,N.

The maximum value of $H(\omega)$ is reached when ω_i = 1/N, for all $i = 1, ..., N$ and the other extreme case occurs when $\omega_i = 1$ for one i, and = 0 for the rest, then $H(\omega) = 0$.

The Shannon's Entropy form that we used in our analysis is the exponential form:

$$H(\omega) = \exp\Big(-\sum_{i=1}^{N} \omega_i \log \omega_i\Big), \tag{4.2}$$

The measure reach a minimum equal to 1 if a portfolio is fully concentrated in a single component and a maximum equal to N, representing an Equally Weighted Portfolio.

A word on the Equally Weighted Portfolio The equally weighted portfolio that leads to a maximum value of H is a naive but well-known strategy in portfolio management. Indeed, investors are still using such simple allocation rules for allocating their wealth across assets even though other more sophisticated theoretical models have been developed in the last 60 years.

Besides the fact that it is computationally simple to implement, it has been proved that the $1/N$ asset allocation strategy can have a "*higher out-of-sample Sharpe ratio, a higher certainty-equivalent return, and a lower turnover than optimal asset allocation policies*" (DeMiguel et al. (2005)). Comparable results have been shown in Bloomfield et al. (1997), Jorion (1985) and Bera and Park (2008). Moreover, it does not involve any estimation or optimization and it also completely ignores the data. All it requires is to hold a portfolio with equal weights, so that $\omega_i = 1/N$ in each of the N risky assets.

A quick comparison can be made between this strategy and the one describes by the equation 3.10. Indeed, the 1/N portfolio can be seen as a strategy that does estimate the expected returns and covariance matrix, but imposes the restriction that expected returns are proportional to total risk rather than systematic risk, which can be mathematically written like $\mu_i \propto \Sigma_i \mathbf{1}_N$.

4.1.2 Diversification Delta

The diversification delta is an entropy-based ratio that has been originally developed by Vermorken et al. (2012). It can be defined as the ratio of the weighted average entropy of the assets minus the entropy of the portfolio divided by the weighted average entropy of the assets. Here, the exponential entropy is used. This idea, suggested by Campbell (1966) but also mentioned as a risk measure in Fabiozzi (2012), allows *"to avoid singularities of the entropy while still letting the uncertainty speak for itself"*.

Formally, for a given portfolio consisting of N risky assets $(X_1, ..., X_N)$ and weights $(\omega_1, ..., \omega_N)$, with $\sum_{i=1}^{N} \omega_i = 1$, the diversification delta developed by Vermorken et al. (2012) can be defined as the following ratio:

$$DD(P) = \frac{\exp(\sum_{i=1}^{N} \omega_i H(X_i)) - \exp(H((\sum_{i=1}^{N} \omega_i X_i))}{\exp((\sum_{i=1}^{N} \omega_i H(X_i))} \tag{4.3}$$

where f is the probability density function of X and $H(X) = -\int_x [f(x)] log(f(x)) dx$ is the differential entropy that is used as a measure of uncertainty.

The interpretation of this ratio is as follows: it is a ratio that compares the weighted individual assets and the portfolio. The diversification delta is *"designed to measure the diversification effect of a portfolio by considering the entropy of the assets and comparing it with the entropy of the portfolio"*. Therefore, the higher the level of entropy, the higher the uncertainty and inversely. The possible values of the ratio are ranged between 0 and 1.

The use of such ratio that compares the uncertainty of individual assets with the uncertainty of the portfolio is an interesting way to analyze a portfolio and quantify the effect of diversification. However, the diversification delta presents several issues as demonstrated by Salazar et al. (2014). By using the required properties of a risk measure as described in section 2.1.2, they have demonstrated that sub-additivity is not satisfied in the definition of the diversification delta. This entails the ratio can be negative but it also means it is not left bounded. Moreover, Salazar et al. (2014) have also shown that the left-hand side in the numerator of equation 4.2 is not homogeneous, meaning that *"changes in the size of the assets are not detected in the same way as changes in the portfolio"* and this leads to inconsistencies in the measurement of diversification.

Therefore, in order to address these issues, Salazar et al. (2014) have proposed to use the exponential entropy as a measure of uncertainty and formulated the following revised diversification delta DD*:

$$DD^*(P) = \frac{\sum_{i=1}^{N} \omega_i \exp(H(X_i)) - \exp(H((\sum_{i=1}^{N} \omega_i X_i))}{\sum_{i=1}^{N} \omega_i \exp(H(X_i))} \tag{4.4}$$

4.2 Risk-based Allocation: Diversification ratio and Marginal Risk Contributions

In this section, we will present two diversification measures based on a portfolio's risk assessment and management which have been quite popular over the last years. These measures are respectively named the Diversification ratio and the Marginal Risk Contributions.

4.2.1 Diversification Ratio

The Diversification ratio (DR) has been introduced in 2008 by Choueifaty and Coignard. This diversification measure can be defined as the ratio of a portfolio' weighted average of volatilities divided by its overall volatility. In essence, the DR measures "*the diversification gained from holding assets that are not perfectly correlated*" (Choueifaty et al. (2013)). Another way to represent the DR is to consider a portfolio which is exposed to F independent risk factors. Also, the portfolio's exposure to each risk factor is inversely proportional to the factor's volatility so that the risk budget of this portfolio is equally allocated across all risk factors and the $DR^2 = F$, meaning that the DR squared is equal to the number of independent risk factors represented in a portfolio.
Formally, given a portfolio composed of N assets $(X_1, ..., X_N)$ and weights $(\omega_1, ..., \omega_N)$, with $\sum_{i=1}^{N} \omega_i = 1$, we have:

$$DR = \frac{\sum_{i=1}^{N} \omega_i \sigma_i}{\sigma_P} \tag{4.5}$$

where σ_i is the volatility of asset i and σ_P the portfolio volatility. According to Choueifaty et al. (2013), the DR "*embodies the very nature of diversification*": the volatility of the overall (long-only) portfolio is less than or equal to the weighted sum of the assets volatilities.

In order to completely understand the intuition behind this ratio, an interesting decomposition of it has been formalized in Choueifaty et al. (2013). The decomposition has the following form:

$$DR(\mathbf{w}) = [\rho(\mathbf{w})(1 - CR(\mathbf{w})) + CR(\mathbf{w})]^{-1/2}, \tag{4.6}$$

where $\rho(\mathbf{w})$ is the volatility-weighted average correlation of the assets in the portfolio and $CR(\mathbf{w})$ is the volatility-weighted concentration ratio (CR) of the portfolio. These two are defined by

$$\rho(\mathbf{w}) = \frac{\sum_{i \neq j}^{N} (\omega_i \sigma_i \omega_j \sigma_j) \rho_{ij}}{\sum_{i \neq j}^{N} (\omega_i \sigma_i \omega_j \sigma_j)} \tag{4.7}$$

$$CR(\mathbf{w}) = \frac{\sum_{i=1}^{N} (\omega_i \sigma_i)^2}{(\sum_{i=1}^{N} \omega_i \sigma_i)^2} \tag{4.8}$$

From these equations, it can be seen that a portfolio which has a high concentration in few assets, or a portfolio consisting of highly correlated assets will have a low DR and therefore can be described as poorly diversified. In other words, the DR increases when $\rho(\mathbf{w})$ and, or $CR(\mathbf{w})$ decreases and inversely.
The extreme case of a DR equal to 1, representing a fully concentrated, mono-asset portfolio, is reached when correlations between assets or the CR is equal to 1.

4.2.2 Risk Parity and Marginal Risk Contributions

Risk Parity

The term "*marginal risk contributions*" comes from a portfolio construction strategy named the risk parity. The general idea behind the risk parity approach is that each asset, or asset class (such as bonds, stocks, commodities, interest rates, etc.), doesn't have the same contribution to the total risk of a portfolio. Therefore, to build a diversified portfolio, we need to focus on the amount of risk that each component i composing the portfolio represents rather than on the specific weight invested in each component i. In other words, instead of being focused on the allocation of capital like the "traditional" methods, risk parity is focused on the allocation of risk.

One could see risk parity as a mix between the minimum variance and the equally-weighted portfolios as described above. Indeed, the risk parity approach can be located between the minimum variance and equally-weighted portfolios because roughly speaking, building a risk parity portfolio is similar to create a minimum-variance portfolio subject to the constraint that each asset, or asset class, contributes equally to the total portfolio risk[8].

The approach has gained a lot of popularity in the last years and basically became standard practice for institutional investors. Some of them even claims it is the most efficient one. For example, Qian (2011) said that risk parity allocation results in better diversification and brings higher returns. One of the main advantage of the risk parity approach is that we do not need to use some estimation of expected returns. Indeed, as we have seen in the section 3.3, forecasting returns is a risky business and leads to estimation errors. On the other hand, the risk parity approach requires some accurate estimations of the variances and covariances, but robust estimation techniques exist (se for example Ledoit and Wolf (2004)) and have been proven to be relatively stable. Therefore, it is possible to estimate variances and covariances with a good deal of accuracy.

Marginal Risk Contributions

To measure the share of the total portfolio risk which is attributable to a component i, or simply the risk contribution of a component i, we have to compute the product of the allocation in i with its *marginal risk contributions*, which is given by *the change in the total risk of the portfolio induced by an infinitesimal increase in holdings of component i*. Therefore, in this context, a well-diversified portfolio is the one where all asset classes have the same marginal contribution to the total risk of the portfolio.

Based on the use of standard deviation as the measure of the total risk of a portfolio, we can calculate the contribution of each asset class to it. The marginal risk contributions of an asset class, noted $\partial_{xi}\sigma_P$, is given by the following expression:

$$\partial_{\omega i}\sigma_P = \frac{\partial \sigma_P}{\partial \omega_i} = \frac{\omega_i \sigma_i^2 + \sum_{j \neq i} \omega_j \sigma_{ij}}{\sigma_P} \tag{4.9}$$

[8]Proof of that can be found in Maillard, S., Roncalli, T., and Teiletche, J. (2009)

As mentioned, the total risk contribution from an asset i is computed as the product of the allocation in asset i with its marginal risk contribution. The risk contribution of i can therefore be written as:

$$RC_i = \sigma_i(\omega) = \omega_i \times \partial_{\omega i}\sigma_P = \omega_i \frac{\omega_i \sigma_i^2 + \sum_{j \neq i} \omega_j \sigma_{ij}}{\sigma_P} \tag{4.10}$$

And the total risk of the portfolio is thus nothing more than the sum of the total risk contributions,

$$TR = \sum_{i=1}^{N} RC_i = \sum_{i=1}^{N} \sigma_i(\omega) \tag{4.11}$$

The measure that we will use in our analysis is the risk contribution RC_i.

4.3 Diversification with regards to Risk Factors

The following measures of diversification are based on one of the best-known techniques in multivariate analysis, the Principal Components Analysis or PCA. The attractive characteristic of PCA is that it has the ability to decompose correlated variables into uncorrelated components, which can obviously be very interesting to use when we are trying to analyze the complex structure of financial markets. Indeed, PCA provides us with a way to identify uncorrelated risk sources in the market and pick stocks from those different risk sources. Therefore, after applying PCA for a stock picking purpose, the resulting portfolio size is more meaningful from the point of view of diversification.

Partovi and Caputo (2004) were the first to propose the idea of using PCA in the portfolio selection framework. Their basic idea was based on the fact that if there were no correlations among assets, the complexity in portfolio selection dramatically decreased. Since the publication of their paper, Partovi and Caputo (2004) have inspired many academic researcher but also many portfolio managers to construct principal portfolios (uncorrelated portfolios), especially when risk reduction and management became the priority, after the financial crisis of 2008.
Here we will present two interesting diversification indexes which are based on the theoretical framework of PCA: the Portfolio Diversification Index, developed by Rudin and Morgan (2006), and the Effective Number of Bets, developed by Meucci (2009).

4.3.1 Brief Review of the Principal Component Analysis

Principal Component Analysis (PCA) is a statistical method of dimension reduction that is used to reduce the complexity of a data set while minimizing information loss. It transforms a data set in which there are a large number of correlated variables into a new set of uncorrelated variables, called the principal components. The first principal component account for as much of the variability as possible, and each succeeding component accounts for as much of the remaining variability as possible. Each principal component is a linear combination of the original variables in which the coefficients indicate the relative importance of the variable in the component.

Consider the linear combinations

$$
\begin{aligned}
Y_1 &= e_{11}X_1 + e_{12}X_2 + e_{1p}X_p \\
Y_2 &= e_{11}X_1 + e_{22}X_2 + e_{2p}X_p \\
&\vdots \\
Y_1 &= e_{p1}X_1 + e_{p2}X_2 + e_{pp}X_p
\end{aligned}
$$

This can be seen as a linear regression where we are predicting Y_i from $X_1, X_2, ..., X_p$. There is no intercept but $e_{i1}, e_{i2}, ..., e_{ip}$ can be viewed as regression coefficients.
Formally, the variance and covariance are written as

$$var(Y_i) = \mathbf{e}_i^{'}\Sigma\mathbf{e}_i = \lambda_i \tag{4.12}$$

$$cov(Y_i, Y_j) = \mathbf{e}_i^{'}\Sigma\mathbf{e}_j \tag{4.13}$$

where $\lambda_1, \lambda_2, ..., \lambda_p$ are the eigenvalues of the variance-covariance matrix Σ and the vectors $e_1, e_2, ..., e_p$ are the corresponding eigenvectors.
The goal of the PCA is, for the i^{th} principal component Y_i, to select $e_{i1}, e_{i2}, ..., e_{ip}$ such that $var(Y_i)$ is maximized with the constraints that the sum of squared coefficients add up to one and each new component defined is uncorrelated with all the previously defined components.
Formally, we have to maximize $var(Y_i) = \mathbf{e}_i^{'}\Sigma\mathbf{e}_i$ subject to the constraints:

$$\mathbf{e}_i^{'}\Sigma\mathbf{e}_i = 1 \tag{4.14}$$

$$
\begin{aligned}
&cov(Y_1, Y_i) = \mathbf{e}_1^{'}\Sigma\mathbf{e}_i = 0, \\
&cov(Y_2, Y_i) = \mathbf{e}_2^{'}\Sigma\mathbf{e}_i = 0, \\
&\vdots \\
&cov(Y_{i-1}, Y_i) = \mathbf{e}_{i-1}^{'}\Sigma\mathbf{e}_i = 0
\end{aligned}
$$

4.3.2 Portfolio Diversification Index

The Portfolio Diversification Index is a measure developed by Rudin and Morgan (2006) which quantify the diversification distinctly related to correlations. Concretely, the PDI indicates the diversification' level of a given portfolio and can assess whether the addition of new assets to the portfolio improves its diversification and by how much (Smith (2006)). Therefore, this index can be compared to the Diversification Ratio, as described above. Indeed, these two measures both aim to quantify the diversification distinctly related to correlations and provide frameworks for constructing diversified portfolio (Randy O'Toole (2014)).

Given a portfolio consisting of N risky assets, the equation for the PDI is as follows:

$$PDI = 2\sum_{i=1}^{N} i\mathrm{RS}_i - 1, \tag{4.15}$$

where $RS_i = \frac{\lambda_i}{\sum_{j=1}^{N} \lambda_j}$ is the relative strength of the ith principal component, λ_i being the eigenvalue associated with the ith principal component. From this equation, we can see that the diversification properties of a portfolio are *"conveyed by means of the relative strength of these factors"* (Smith, 2006).

The PDI can have values comprised between 1 and N and the interpretation of these extreme cases is as follows:

- A PDI = 1 represents a portfolio dominated by a single factor and thus completely undiversified. In that case, $\mathrm{RS}_1 \approx 1$ and the others are null.
- a PDI = N represents a portfolio totally diversified. In that case, $\mathrm{RS}_i \approx 1/\mathrm{N}$ for all i.

It is thus straightforward that a high PDI value indicates a well-diversified portfolio and inversely, a low PDI value indicates a less diversified portfolio.

4.3.3 Effective Number of Bets

Another use of PCA in a portfolio construction framework can be found in Meucci (2009). In his paper, Meucci proposed to transform a data set composed of a large number of correlated assets into a new set of principal portfolios representing uncorrelated risk sources related to the original assets. He then introduced a "diversification distribution", a tool to analyze the structure of a portfolios concentration profile. The diversification distribution is expressed as the ratio of each principal portfolios variance to its total variance. In this principal portfolio environment, all the principal portfolios are uncorrelated and therefore the total variance of the portfolio can be written as a sum of the contributions of each factor's variance. The ratio of individual principal portfolio variance to the total variance is then in the range of 0 to 1 and sum to 1. Here the maximum diversification is reached when the diversification distribution is close to one. Therefore, based on this method, a well-diversified portfolio is the one in which all the risk sources composing it are invested equally.
Moreover, Meucci (2009) introduced a diversification index that represented the effective number of uncorrelated bets (ENB) in a portfolio. The idea is that if the number of uncorrelated bets were small, the risks were rather concentrated in few sources and less diversified.

The construction of a portfolio following such method could be compared to the one following the risk parity approach, as described above. These two allocations strategy present a similarity even if they are conceptually different. Indeed, they both construct portfolios by allocating risks. But the way these two strategies allocate risk is completely distinct. The risk parity strategy allocates investments in such a way that they equally contribute to the risk of the total portfolio whereas the ENB strategy allocates risk based on uncorrelated principal portfolios. The consequence is that risk parity allocation strategy can have rather concentrated risk if most assets have high correlations with each other. As an example, consider an extreme case with all stocks perfectly positive correlated. The result is that allocating equal risk budget to all stocks is actually the same as holding one stock. The ENB strategy does not have such a problem since it is allocating its risk budget based on uncorrelated risk sources. Though in the case of perfect correlations all variation would be explained by the first principal component.

Formally, the development of Meucci (2009) began with the decomposition of the the variance-covariance matrix Σ of the N assets:

$$\mathbf{E}'\mathbf{\Sigma}\mathbf{E} = \Lambda \tag{4.16}$$

where $\Lambda \equiv diag(\lambda_1, \lambda_2, ..., \lambda_N)$ is a diagonal matrix. The columns of Λ are the respective eigenvectors $\mathbf{E} \equiv (e_1, e_2, ..., e_N)$ which define a set of N uncorrelated portfolios (principal portfolios). The returns of the principal portfolios can be expressed as $r_F = \mathbf{E}'\mathbf{R}$, where R denotes the random returns on the N assets, and the variances are the eigenvalues $(\lambda_1, \lambda_2, ..., \lambda_N)$. Following that, the original portfolio ω can be expressed as a combination of the uncorrelated principal portfolios with weights $\boldsymbol{\omega}_F = \mathbf{E}^{-1}\boldsymbol{\omega}$.

Also, the introduction of uncorrelated factors allows to write the total variance of the portfolio as the sum of the contributions of each factor's variance:

$$\boldsymbol{\omega}'\mathbf{\Sigma}\boldsymbol{\omega} = \sum_{i=1}^{N} \omega_{Fi}^2 \lambda_i^2 \tag{4.17}$$

The contribution of the i^{th} factor to the total portfolio variance is defined by what Meucci calls the diversification distribution: $p_i(\omega) = \frac{\omega_{Fi}^2 \lambda_i^2}{\sum_{i=1}^{N} \omega_{Fi}^2 \lambda_i^2}$. Each $p_i(\omega)$ is positive and their sum equal to 1.

The interpretation given by Meucci (2009) to the diversification distribution is that it represents a set of probability masses associated with the principal portfolios and therefore, a portfolio ω is well diversified when *"the probability masses p_i are approximately equal and thus the diversification distribution is close to uniform"*. Moreover, by measuring the dispersion of the probability distribution, which can be done thanks to its entropy, we can thus measure the diversification of the portfolio.

$$ENB(\omega) = \exp\Big(- \sum_{i=1}^{N} p_i(\omega) ln(p_i(\omega)) \Big) \tag{4.18}$$

A maximal value of $ENB(\omega)$ equal to N would mean that the risk is evenly spread amongst the different factors and the portfolio is thus well diversified portfolio, whereas a value of $ENB(\omega)$ equal to 1 would mean that the portfolio is concentrated in a single risk factor.

Chapter 5

Data and Methodology

In this chapter, we describe the methodology that has been followed in order to conduct our studies on the diversification measures presented in chapter 4. The goal of the analysis is obviously to provide answers to the research questions raised in the chapter 1.
To realize that, we will first describe the data that have been collected. Then we will present the procedure we followed in order to perform our analysis. The implementation of Shannon's Entropy will be also described in this section. Finally, we will analyze the results obtained and try to answer the research questions that we put these measures in relation with the criteria selected to see if they meet the requirements that define a diversification measure. The computer program chooses to complete our analysis is Matlab. The scripts that we used can be found on the DIAL website, under the file linked to the electronic version of this thesis.

5.1 Description of the Data

Database

The data used in our empirical study were obtained on Bloomberg and are based on a weekly basis for the period from 29 December 1989 to 4 May 2018, which provides 1,480 weekly returns. The prices showed correspond to the closing prices of each Friday. Also, we consider the following five asset classes:

- US large capitalization stocks: we used the S&P 500 equity index as a proxy. This index represent a market value weighted index of the 500 larger cap companies in the United States. It is constructed by adjusting the closing S&P 500 closing prices to corporate actions (S&P Global (2016)). The index is denoted by the ticker SPXT Index.

- US Treasury bonds: we used the S&P US Treasury Bond Index as a proxy. This index is a market-value weighted index that seeks to measure the performance of the United States Treasury Bond Market. Treasury Bond, or Treasury-bill (T-bill), is a *short-term debt obligation backed by the U.S. government with a maturity of less than one year* (definition from Investopedia). It is seen as the safest investment that can be made, which explain why a T-bill investment is usually referred as risk-free. It is denoted by the ticker SPBDUSBT Index.

- Commodities: we used the S&P Goldman Sachs Commodity Index as a proxy. This index is the first major commodity index which was originally developed by Goldman Sachs. It is designed to be investable by including the most liquid commodity futures[9]. The index currently comprises 24 commodities from all commodity sectors: energy products, industrial metals, agricultural products, livestock products and precious metals. It is denoted by the ticker SPGSCITR Index.

- US corporate bonds: we used the SPDR Bloomberg Barclays U.S. Corporate Bond Index (Bloomberg Barclays U.S. Corporate Total Return Value Unhedged USD) as a proxy. This index "*measures the performance of the investment grade U.S. corporate bond market*" (State Street Global Advisors (2018)). Each security that are included into the index must be fixed rate, in U.S. dollar, taxable and rated investment grade as defined by the Index methodology. It is denoted by the ticker LUACTRUU Index.

- Real Estate: we used the Dow Jones Equity All REIT Index as a proxy. This index is designed to "*measure all publicly traded real estate investment trusts in the Dow Jones*

[9] A commodity is a basic good used in commerce such as oil, sugar, rice, etc.

U.S. stock universe classified as equity REITs according to the S&P Dow Jones Indices REIT Industry Classification Hierarchy" (from Dow Jones Equity All REIT Index).

Various Statistics

In Table 5.1, the main descriptive statistics of each asset class over the entire sample period are presented. The average return and volatility are annualized but both Value-at-Risk computations are done with weekly returns. We can observe that the lowest volatility is achieved by the Treasury Bonds with 3.62%, which is logical given the nature of this asset class. The second lowest volatility asset class is the corporate bonds with a 5.12% volatility. The remaining asset classes are much more volatile. Indeed, we can see that the bonds are at least four times less volatile than the three other asset classes of our dataset. The S&P 500 index present a volatility equal to 16.16% while Commodities and Real Estate show volatilities above 20% (20.79% and 20.02% respectively). The interpretation of the risk linked to each asset classes provided by the VaR is similar to that of the volatilities in the sense that for both probability level (1-α), the most volatile asset class is Commodities, followed by Real Estate, S&P 500, Corporate Bonds and finally Treasury Bonds respectively. However, given that the VaR are calculated on a weekly returns basis, their values represent the probability that an asset class will lose at least α-percent of its value each week.

Table 5.1: Descriptive statistics of the returns of the asset classes

	S&P 500	T. Bonds	Commod.	Corp. Bonds	Real Estate
Average Return (%)	10.54	5.06	3.52	6.53	11.98
Volatility (%)	16.16	3.62	20.79	5.12	20.02
Sharpe Ratio[10]	0.39	1.38	0.06	1.25	0.50
VaR_5 (%)	3.71	0.83	4.79	1.17	4.57
VaR_1 (%)	5.24	1.17	6.77	1.65	6.47

It is interesting to notice that the asset classes which present the lowest volatilities, and thus which are considered the less risky, have the highest Sharpe ratio (SR) that we mentioned in section 3.2. Indeed, the Treasury Bonds have a SR of 1.38, closely followed by the Corporate

Bonds with a SR of 1.25. In comparison the latter asset classes, the S&P 500 and Real Estate index have much higher average returns, but their respective high volatilities lead to a much lower SR (0.39 and 0.50 respectively).

Table 5.2 presents the correlations between de different asset classes. We can see that the Treasury and Corporate bonds are negatively correlated with all the other asset classes except with each other, presenting a highly positive correlation (93.07%). Commodities are also negatively correlated with all the other classes and doesn't seem to present any correlation with the equity stocks (S&P 500). Finally, Real Estate appear to be highly positively correlated with the S&P 500 Index.

Table 5.2: Correlation Matrix in %

	S&P 500	T. Bonds	Commod.	Corp. Bonds	Real Estate
S&P 500	100	-80.11	3.08	-58.24	80.46
Treasury Bond	-80.11	100	-60.31	93.07	-66.97
Commodities	3.08	-60.31	100	-75.62	-4.67
Corporate Bonds	-58.24	93.07	-75.62	100	-42.53
Real Estate	80.46	-66.97	-4.67	-42.53	100

To apply the ENB and PDI measures, we have also performed a Principal Component Analysis (PCA) on the variance-covariance matrix estimated over the entire database. This allows us to identify the factors that affect the returns of each asset classes. The results are given in the Table 5.3 below. Concretely it shows the effects, or exposures, of each of the factors on the different asset classes[11]. The variance of each factor represents the eigenvalues that are obtained after performing the PCA, as explained in section 4.3.1. Moreover, all the factors are obviously uncorrelated and explain the entire variability of the asset classes' returns. The interpretation of the factors resulting from the PCA is as follows:

- With an exposure equal to 82.99%, the first factor F1 seems to be related to the real estate risk. Also, we can notice that F1 is significantly exposed to S&P 500 (53.68%).

[11] The exposures is ranged between -100% and 100% but total position, or factor exposures can be greater than 100%, as it can be observed

This can be explained by the fact that the Real Estate and S&P 500 are highly correlated, as we can see in Table 5.2. Therefore, we can interpret this factor as a risk coming mostly from Real Estate, but also from the equities composing the S&P 500 Index.

- The second factor F2 is surely related to the commodity risk, with an exposure of 98.63%. In addition, all the other weights have an absolute value of maximum 11.50%, confirming this interpretation.
- The third factor F3 represents the remaining risk related to S&P 500 (83.16%) that is not explained by F1. This can be confirmed by the fact that F3 is negatively and significantly exposed to Real Estate (-54.99%).
- The fourth and last factor F4 is heavily loaded on Corporate Bonds (94.97%) but also, on a smaller scale, on the Treasury Bonds (29.97%). Therefore, we could interpret F4 as an interest rate risk factor.

Table 5.3: PCA Factors' Exposures (%)

	F1	F2	F3	F4
S&P 500	53.68	11.50	83.16	7.01
Treasury Bond	-4.11	-4.82	-4.48	29.97
Commodities	-14.48	98.63	-4.91	5.68
Corporate Bonds	-2.32	-5.28	-4.09	94.97
Real Estate	82.99	9.38	-54.99	0.59
Total Position	115.76	109.41	14.71	138.23
Variance	0.0404	0.0352	0.0037	0.0000
Percent Explained	50.95	44.39	4.67	0.00
Cumulative	50.95	95.33	100.00	100.00

Another thing that we can notice by looking at Table 5.3 is that half of the total variance is explained by F1, which we have deduced related to Real Estate and equities (S&P 500) too. This seems logical since Real Estate and equities are the second and third most volatiles asset classes (see Table 5.1).

5.2 Methodology

Concretely, the analysis we provide on the measures of diversification consists of two parts:

- Firstly, we analyze the results given by the different measures separately. In order to do that, we generate randomly the weights of 10 portfolios and then apply the measures of diversification on them. The creation of these 10 portfolios can be done thanks to the Matlab function *rand*. We thus create 10 portfolios with weights assigned to the 5 different asset classes we are considering in our study. Once the portfolios created, we can apply the diversification measures considered to them and to analyze the results delivered.

- Secondly, we compute the maximum diversification portfolios by maximizing the function we have implemented for each measures of diversification. Therefore, we obtain the optimal weights of 6 portfolios, each of them representing the optimal portfolio associated to a measure of diversification. The risk and concentration profiles of these portfolios is then analyzed, and we try to answer the research questions of the thesis. The maximum diversification portfolios are implemented thanks to the Matlab function *fmincon*.
  Since the function needs a vector of weights as an input, we have decided to assign a vector composed of equal weights. The consequence of this initial vector is that the optimal portfolio constructed by maximization of the Shannon Entropy function is an Equally-Weighted portfolio. Therefore, this allows us to analyze the risk profile of a portfolio which is completely diversified from the concentration's point of view and to compare its results with the other optimal portfolios.

It is important to mention that for the construction of the portfolios, the Random ones or the Maximum Diversification ones, we use no short-selling constrain, meaning the weights can't be negative. We back up this decision for two reasons. The first reason is that *"since the constructed portfolio weights obtain through the maximum entropy approach are in the form of probabilities, the weights are certainly non-negative"* (Bera and Park (2008)). The second reason is that the no short-selling constraint allows to reduce the estimation error. Jagannathan and Ma (2003) concluded, that no short selling constraint helps to reduce estimation error in the covariance matrix, when one wants to use the returns of a higher frequency, which is our case, as we use weekly returns in our analysis. Moreover, according to the financial industry standards, most asset managers are not allowed to sell short.

Computation of Shannon's Entropy and the Risk Contribution

Before going any further into the analysis, we have to describe the method we used in order to implement the Shannon's entropy but also explain how we compute the Risk Contribution measure.

Calculation of Shannon's Entropy from histograms We used an expression that allows to calculate the entropy from histograms as developed by Harris (2006) for a discrete distribution and by Rich and Tracy (2006) for a histogram.

To express the Shannon's entropy from the situation where the density is represented as a histogram, the range of the variable must be divided into n intervals (l_k, u_k), with $k = 1, ..., n$ so that

$$H(X) = -\sum_{k=1}^{n} \int_{l_k}^{u_k} f(x) \log f(x) dx \tag{5.1}$$

After that, the kth term in this summation is related to the kth bin of a histogram, with width $x_k = u_k - l_k$. Therefore, the bin probabilities p_k, for $k = 1, ..., n$, defined as

$$p_k = \int_{l_k}^{u_k} f(x) dx \tag{5.2}$$

can be approximated by $x_k f(y_k)$, the area of a rectangle of height $f(y_k)$ where y_k is a representative value within the interval (l_k, u_k) (Wallis (2006)). The kth integral in the sum above can also be approximated by $x_k f(y_k) \log f(y_k)$ and after rewriting the latter in terms of the bin probabilities, we ended up with the new expression of the entropy:

$$H(X) = -\sum_{k=1}^{n} p_k \log \left(\frac{p_k}{x_k} \right) \tag{5.3}$$

only for the case of $x_k = 1$. When x_k is constant but not equal to 1, then

$$H(X) = -\sum_{k=1}^{n} p_k \log(p_k) + \log x \tag{5.4}$$

If x_k is not constant, equation 5.3 "*calls for bin-by-bin adjustments before comparisons of entropy between histograms with different bin configurations can be made*" (Wallis (2006)).

Implementation of the RC measure When we apply the Risk Contribution measure to a portfolio, the results obtained represent a vector composed of values equal to the RC of each portfolio' assets, which is in our case, a vector of 5 RC. Since all the other measures analyzed delivered a unique value, we use Shannon's entropy to assess the results provided by the RC measure. This allows us then to compare the values obtained with those of the other measures considered. Such implementation was suggested by Riccardo Brignone (2015). Concretely, instead of using the weight, we incorporate the RC into Shannon's entropy, leading to the following expression:

$$RCH(\omega) = \exp\Big(-\sum_{i=1}^{N} \sigma_i(\omega) \log \sigma_i(\omega)\Big) \tag{5.5}$$

Here also, the measure reaches a minimum equal to 1 if a portfolio is fully concentrated in a single component and a maximum equal to N.

Chapter 6

Analysis

Based on the methodology detailed in chapter 5, we now report and analyze our results to answer our research questions. First, we describe the results obtain from the Random Portfolios and provide a short analysis. Then we describe the findings related to the computation of the Maximum Diversification Portfolios. After that, we provide the main analysis of our thesis and we try to assess the different measures of diversification according to the selected criteria.

6.1 Random Portfolios: Results and Analysis

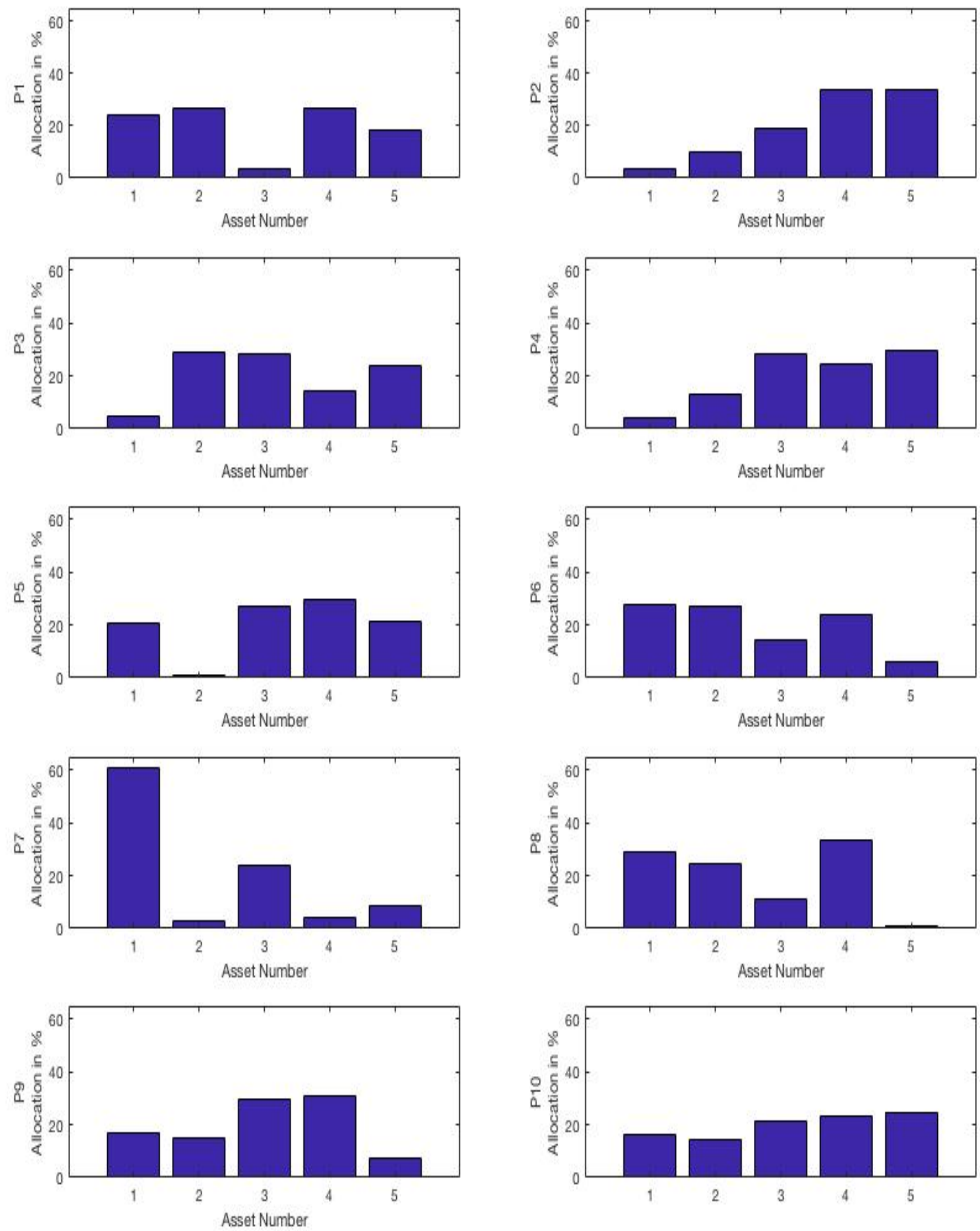
The allocations generated for each of the 10 portfolios can be found in Table B.1 of Appendix A. The graphic representations of these allocations have also been provided in Figure A.1 of Appendix A. As a first step, we analyze the diversification level of the different portfolios in terms of distribution of their weights, so basically in terms of their concentrations. The measures used to do that are obviously the weights-based measures which are the Shannon's Entropy and Diversification Delta. After that we analyze the diversification level with regards to the other measures that are more risk-based. Then we compare the results obtained with the more traditional measures of portfolio risk before moving to the other part of our analysis related to the Maximum Diversification Portfolios.

6.1.1 Weight-based Analysis

According to the Shannon's Entropy (SE), most of the random portfolios seem to be well-diversified. Indeed, the SE's values are mostly comprised between 4.03 and 4.9 and knowing that the maximum attainable is 5, we can say that most of portfolios have a high level of diversification from the SE's perspective. The two only exceptions are P7 and P8 which have an entropy on 2.94 and 3.94 respectively. When looking at the compositions of these portfolios, we can see that P7 is mostly loaded on the S&P 500 index (60,97% of the total allocations) and on the commodities (23,91%), while the three other asset classes remain below 8.39%. The allocations of P8 are approximatively well distributed between the equities, the Treasury and Corporate bonds (29.20%, 24.64% and 33.70% respectively) while the remaining weights are mainly allocated to the commodities (11.24% while the real estate only represent 1.22%).
The portfolio having the maximum SE is P10. Its different allocations are more or less similar to the one of an equally-weighted portfolio, with allocation varying between 14.63% and 24.78%. No surprise then than it nearly has a SE value close to the maximum of 5.

The results obtained from the Diversification Delta (DD) are clearly more mixed. Indeed, bearing in mind that the values of DD are ranged between 0 and 1, the latter representing a perfectly diversified portfolio, we can see that the portfolios don't seem to be well-diversified since the majority of them are presenting a DD comprised between 0.33 and 0.38. According to these numbers, all the portfolios are more or less on the same level of diversification. It is also interesting to notice that P8 is considered as one of the most diversified portfolios according to DD. It is at the same level of diversification as P6, with a DD also equal to 0.38. Notice that P6 has a high degree of diversification according to SE (4.51). P10 is therefore not the most diversified portfolio anymore based on the DD. However, P7 is still considered as the least diversified portfolio both with the SE and with the DD.

Figure 6.1: Random Portfolios Generated



6.1.2 Risk-based Analysis

Among the four remaining measures of diversification, three of them are directly related to the diverse sources of risk present in a portfolio. Recalling what we mentioned in section 4.2.1, DR^2 represents the effective number of independent risk factors represented in a portfolio. The term effective means that the the risk factors are never truly independent since the different sectors and styles of a market often move in the same direction, albeit imperfectly.

Therefore, when looking at the values obtained in Table B.1 for DR^2, we can say that the majority of portfolios are effectively exposed to two or three (in fact the maximum is 2.57) independent risk factors (let us remember that we identified 4 different risk factors according to the decomposition of the variance-covariance matrix given by the PCA). And if we compare these results to the different values of the PDI obtained, we can see that they are closely related, which is totally normal since both measures can be compared with each other, as explained earlier.
Moreover, we can notice that for both measures, the portfolio with the lowest degree of diversification is P7 (PDI of 1.91 and DR^2 equal to 1.69). Here also, the measures seems to designate P7 as the least diversified portfolio, like the SE and the DD.
A last observation that can be made about the values of both measures is that the PDI provided slightly less volatile results than the DR^2. PDI is ranged between 2.19 and 2.40 (without taking account of the value of P7 of course) while DR^2 varies between 2.17 and 2.57 (also without P7).

The results produced by the ENB measures are not really different than those previously analyzed but some differences do exist between them. First most portfolios (7 out of 10 to be precise) seems to be exposed to less than 2 risk factors while the DR and the PDI show an exposure to two or three risk factors. But one result that seems quite surprising is that P10 is designated as the least diversified portfolio by ENB. So far, the measures we have reviewed all considered P10 as the most or one of the most diversified portfolios of the group.

A more detailed explanation of the ENB values can be found on Table B.3. It shows the diversification distribution $p_i(\omega)$ calculated. We can see that each portfolio is more than 70% exposed to the first risk factor F1 (factor related to real estate and equities). Given that, it is easy to understand why ENB values don't exceed two for most of the cases. And we see also that P10 has the highest exposure to F1 with a $p_1(\omega)$ of 96.08%, explaining its status of least diversified portfolio. The portfolio with the highest degree of diversification according to the ENB measure is P8, with a $p_1(\omega)$ of "only" 68.62%. This is more consistent with the results of other measures.

Now if we look at the RC entropy, the results don't seems to differ much from those of previous measures. Recall that we first calculated the risk contributions for each portfolio, then we implemented the Shannon's entropy of the RC, resulting in the values shown in Table B.1. Also, by construction, a portfolio which would have its risk contributions equally allocated between the five asset classes would show an RC entropy of 5.
The values obtained thanks to the RC entropy are mainly located between 2.21 and 2.93, with some outliers which present entropies equal to 3.17, 3.19 and 3.23, representing P10, P6 and P5 respectively. The most diversified portfolio of the group (P5) is therefore different from the other candidates designated by the other measures, and the least diversified is P7, which is here similar to previous conclusions.

Now if we look at Table B.2 of Appendix A, it is interesting to notice that some Risk Contributions are negatives. Indeed, for P5 and P7 the Treasury Bonds asset classes have a negative value equal to -0.01% and -0.12% respectively. The explanation of such results can be found in the definition of the risk contribution and marginal risk contribution: as explained in section 4.2.2, the risk contribution is computed as the product of the allocation in asset class i with its marginal risk contribution and the latter is computed from the co-variance matrix, which, by definition, can have negative values. In fact, the marginal risk contribution (MRC) could more intuitively be expressed using a concept we have presented previously, Beta, and thus the interpretation of a negative MRC is that it suggests we increase the weight of the asset class concerned in the portfolio[12]. Also, when looking at the Table B.2, we can understand why P5 presents the highest RC entropy given the distribution of its different risk contributions.

So, as a quick conclusion, we see that all the measures don't come to the same conclusion about the diversification of portfolios, even if some trends are noticeable. Indeed, the different measures are showing some level of diversification that are similar for some portfolios, but it seems that there is no collective agreement that has been reached on one unique portfolio that is the most diversified, even if P8 is designated as a portfolio having one of the highest diversification degree by all the measures, except for the SE.

[12]This interpretation has been found on *Risk Decomposed: Marginal Versus Risk Contributions*

6.1.3 Comparison with Overall Risk Performances of Portfolios

Now that we have analyzed the results provided by the different measures of diversification, the logical following step is thus to link them with the portfolio risk measures and see if the indications about the level of diversification in the portfolios given by the six measures analyzed are well aligned with the level of risk. But before we begin, we want to mention that since the volatility and VaR evolve in the same way, we choose to refer to the volatility when describing the portfolio' risk level.

First, if we begin with the analysis that has been made from the weights-based measures SE and DD, we verify that P7 is indeed the portfolio with the lowest level of diversification and by far. Its volatility is equal to 13.01% while the second least diversified portfolio has a volatility only reaching 9.89%. Moreover, as indicated by its Sharpe Ratio, we can see that this portfolio has a performance that is not very risk-adjusted. This conclusion about P7 is also shared by the DR, the PDI and the RC entropy so it seems that almost all measures of diversification, weights-based as well as risk-based, have reached the same conclusion about the portfolio having the lowest diversification degree.
According to the ENB measure, it is P10 the least diversified but giving its volatility equal to 8.99%, it is not the case. Compared to other volatilities, P10 is indeed not the riskiest portfolio, but it is also not the safest as indicated by the SE. The portfolio showing the smallest volatility is P6 (6.36%) and it is also the one having the highest Sharpe Ratio (equal to 1.00).

It is interesting to notice that the portfolio which presents the most diversification according to almost all the measures is a portfolio presenting an important level of concentration given its SE (3.94 while the others have an entropy of over 4) compared to the other portfolios. A portfolio that seems to be diversified according to the weights allocation criterion and the risk minimization criterion is P6 which has a low 6.99% volatility, a high SE as well as a high DD (4.51 and 0.38 respectively), and also high risk-based measures levels (RC entropy = 3.19, ENB = 1.85, DR = 1.60 and PDI = 2.38).

The review of the results showed us that the diversification concept of a portfolio is not interpreted in the same way by the different measures we have analyzed. Indeed, the interpretation given by each measure on the portfolio diversification can vary significantly depending on the same portfolio. The following section is focused on portfolios that have been computed thanks to the maximization of the measures of diversification analyzed. We will see if this optimization allows us to obtain similar portfolios sharing some common characteristics.

6.2 Maximum Diversification Portfolios: Results and Analysis

The following step of our analysis is to create optimal portfolios constructed by the maximization of the different measures of diversification. Concretely, the optimizations problems are created with an objective function, which varies with the diversification's measure optimized, subject to the following two constraints: $\omega \geq 0$ and $\sum_{i=1}^{N} = 1$, meaning that short-selling is not allowed. In addition, the initial weights vector which is used as an input of the optimization problems is a vector of equal weights, meaning that the weight allocated in each asset class is equal to 20%.

6.2.1 Results Analysis: Portfolios Compositions

The optimal allocations obtained are presented numerically in Table 6.1 below but a graphic representation can be found in Figure 6.2.

We can observe that the composition of the portfolio obtained by maximizing the Shannon's Entropy doesn't change from the composition of the initial vector used. The resulting portfolio is therefore an Equally Weighted Portfolio. This is not surprising since a portfolio of equal weights already represents a portfolio with maximum diversification following the SE characteristics. The portfolio obtained by maximization of the Diversification Delta is also more or less equally allocated, which can be explained by the entropy-related nature of the DD measure.

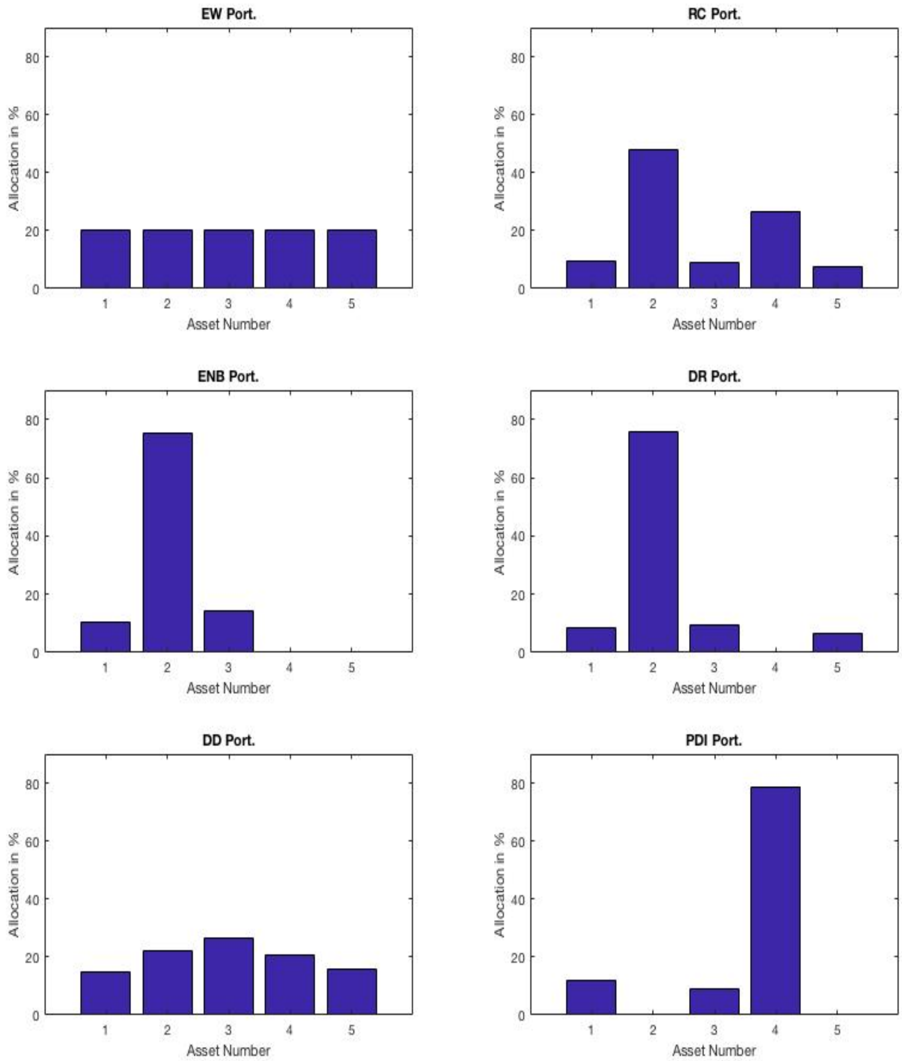
The compositions of the ENB, DR and PDI Portfolios are quite difficult to understand if we are strictly referring to the basic notion of diversification: the three of them are massively loaded on a single asset class. ENB and DR Portfolios have more than 75% (75.31% and 75.54% respectively) of the total portfolio weights allocated to the Treasury Bonds and the PDI Portfolio is composed at 78.68% of Corporate Bonds. Knowing the usual qualitative definition of portfolio diversification as expressed by the famous adage *"don't put all your eggs in one basket"*, such compositions of portfolio are therefore a bit strange.

Finally, the RC Portfolio is more understandable compared to the three previously reviewed. Indeed, the majority of the portfolio is invested in bonds, with 47.74% in Treasury Bonds and 26.61% in Corporate Bonds. The remaining part of the portfolio is invested at an equivalent level in the equities, commodities and real estate.

Table 6.1: Maximum Diversification Portfolios Allocations (%)

	SE Port.	RC Port.	ENB Port.	DR Port.	DD Port.	PDI Port.
S&P 500	20.00	9.35	10.18	8.56	14.91	12.13
T. Bonds	20.00	47.74	75.31	75.54	22.21	0.01
Commod.	20.00	8.80	14.51	9.54	26.69	9.18
Corp. Bonds	20.00	26.61	0.00	0.00	20.46	78.68
Real Estate	20.00	7.49	0.00	6.37	15.74	0.00

Figure 6.2: Maximum Diversification Portfolios



6.2.2 Results Analysis: Risk and Scores of Measures

As we have seen, some composition of portfolios can be difficult to understand. However, those allocations have been obtained by maximizing the different measures of diversification so normally the portfolios should present high scores of measures and thus high degree of diversification.

If we look at Table 6.2 and at the results of the maximum SE Portfolio, which is basically an Equally Weighted Portfolio (EWP), we can see it has indeed the maximum SE value of 5, meaning that it represents a highly diversified portfolio from the perspective of the weights concentration. But it doesn't seem to be the case for the other measures. According to the ENB and DR measures, the EWP is the least diversified portfolio of the group. And if we look at the VaR and volatility level, it is clearly the riskiest portfolio of all. Moreover, the corresponding Sharpe Ratio of the EWP is very low (0.4), which indicates that the portfolio is not well risk-adjusted.

Table 6.2: Risk and Performance Statistics of the Maximum Diversification Portfolios

	SE Port.	RC Port.	ENB Port.	DR Port.	DD Port.	PDI Port.
Aver. Ret. (%)	6.38	5.81	4.89	5.32	5.71	6.27
Variance (%)	0.72	0.20	0.19	0.17	0.69	0.25
Volatility (%)	8.53	4.52	4.43	4.15	8.31	5.02
VaR_5	1.95	1.03	1.01	0.95	1.90	1.15
VaR_1	2.75	1.46	1.43	1.34	2.68	1.62
Sharpe Ratio	0.40	1.28	1.10	1.28	0.69	1.25
SE	5.00	3.80	2.07	2.27	4.88	1.94
RC Entropy	3.31	5.00	2.74	3.81	3.06	2.46
ENB	1.23	2.11	3.06	2.58	1.58	2.47
DR	1.56	1.80	1.75	1.86	1.57	1.58
DR^2	2.43	3.24	3.06	3.46	2.46	2.50
DD	0.37	0.43	0.42	0.45	0.37	0.37
PDI	2.31	2.48	2.75	2.77	2.29	3.06

Another portfolio that presents a very high value for SE is the Diversification Delta Portfolio (4.88). This is not very surprising since the DD is also an entropy-based measure, but it is interesting to notice that the DD Portfolio has also an important level of risk with a volatility equal to 8.31%. The two riskiest portfolios are thus those that are have been constructed by minimizing the weights concentration (the maximization the entropy implies the minimization of weights concentration). Therefore, following the observation of our results, it can safely be said that the concentration criterion alone doesn't seem to be adequate to construct diversified portfolios.

Interesting also to notice the SE and DD portfolios have not a high diversification delta. In fact, they have the lowest DD score with the PDI portfolio (0.37). This is quite surprising, especially for DD Portfolio because its construction is from the beginning based on the maximization of the Diversification Delta. The portfolio with the highest DD score is the one maximizing the Diversification Ratio of Choueifaty and Coignard (2008). Such portfolio has actually be named *"Most Diversified Portfolio"* by Choueifaty and Coignard and we can see that it presents indeed some characteristics that we are looking for in a well-diversified portfolio. First it is the less risky of the group with a volatility of 4.15 and it also shows the highest Sharpe ratio (1.28). In addition, it has the highest score for most measures. The only measure that doesn't consider it as a well-diversified portfolio is the Shannon's Entropy (2.27), which is explained by the high percentage of its weights allocated to the Treasury Bonds (75.54%). This extreme allocation is also traduced by a high percentage of risk contributions in T-bills and a strong exposure to F4 as shown in Table B.4 and B.5 of Appendix A. However, the portfolio seems to be efficiently exposed to the different risk factors, as we can see from the high scores of ENB, DR^2 and PDI (2.58, 3.46 and 2.77 respectively).

The only portfolio having high scores in both entropy-based and risk-based measures is the Risk Contribution Portfolio. The portfolio is constructed in a way that all asset classes contributes to approximatively 20% of the total portfolio risk (this is not surprising given that the portfolio maximizes Shannon's entropy which uses the risk contributions instead of the weights as inputs). The result is that the portfolio has a RC Entropy of 5 and shows a good distribution of weights as depicted by a score of 3.80 in SE. Moreover, its risk profile is very low with a volatility of 4.52% and it shares the highest Sharpe Ratio of 1.28 with the DR Portfolio. It also shows an efficient exposure to all the risk factors (ENB=2.11, DR^2=3.24 and PDI=2.48). Therefore, as the latter, the Risk Contribution Portfolio seems to present lots of characteristics of a well-diversified portfolio, from the concentration perspective as well as from the risk-minimization perspective.

We have seen it, the portfolios constructed with PCA-based measures (ENB and PDI Port) are extremely concentrated, especially in the bonds, Treasury or Corporates. As a result, they show a poor score in SE, as well as in RC entropy which implies that the risk is not well distributed between the asset classes. However, given the construction of the portfolios, their expositions to the various sources of risk is efficient according to their scores for ENB, DR^2 and PDI. The volatilities of ENB and PDI portfolios are also quite low (4.43% and 5.02% respectively) and the Sharpe Ratios high (1.10 and 1.25 respectively). We are thus here in the case of a portfolio construction which minimizes the risk but has an allocation of weights that is extremely concentrated, which doesn't really reflect the basic idea of diversification.

6.2.3 Criteria of Diversification: Discussion

The results of our analysis have emphasized the point we have developed in the section 2.3: some measures of diversification are only based on the risk minimization criterion and doesn't consider the weights concentration of a given portfolio. However, the only use of a measure based on the entropy, like SE and DD, in order to quantify the diversification of a portfolio doesn't seem to be very efficient. Indeed, our results have shown that a portfolio can have weights that are well distributed between the different asset classes but have high volatility level and thus a high risk profile. The Equally-Weighted portfolio created by maximizing the Shannon's entropy is a good example of that.

Moreover, following our analysis, the results we have obtained seem to indicate that the Risk Contributions and the Diversification Ratio are more adequate measures in order to quantify the diversification of a portfolio and also to create a well-diversified one. They are indeed embodying the characteristics of diversification we have defined early on in Chapter 2 by providing high scores to portfolios which show a minimal risk profile but also small concentrations in each asset classes.

Also, both measures verify some of the desirable properties that a measure of diversification should have (section 2.2.2) like property 1, 2 and 3 as shown by Carmichael et al. (2015) and Maillard, Roncalli and Teiletche (2009). However, since they don't verify properties 4 and 5, we can be totally satisfied with the results they are providing. Also, the DR has been critized by Lee (2011) and Taliaferro (2012) because the measure is not associated with a clear objective function. Therefore, a portfolio constructed by maximizing the DR *"only has desirable properties by accident"* (Taliaferro (2012)).

The ENB and PDI have provided mitigates results because it doesn't seem the measures takes into account the concentration of the weights. As we have seen, the measures give high scores, and thus indicate a high degree of diversification to portfolios that have weights concentrated on few asset classes. Carmichael et al. (2015) have also shown the measures are not efficient to quantify diversification. Based on Embrechts et al. (1999), they have demonstrated that ENB has two major shortcomings: *"it does not distinguish between negative and positive correlation (and thus cannot incorporate the benefits arising from negative correlation) and it can only be computed if portfolio risk is measured by its variance or volatility, which are adequate measures of risk only in the unlikely case that asset returns are normally distributed"*.

Chapter 7

Conclusion and Further Research

This thesis wished to provide a thorough study of six measures of portfolio diversification: the Shannon's Entropy, the Diversification Delta, the Diversification Ratio, the Risk Contributions, the Portfolio Diversification Index and the Effective Number of Bets. The results were obtained on a database composed of five asset classes. The data were collected from Bloomberg and are based on a weekly basis for the period from 29 December 1989 to 4 May 2018.

The introduction of the thesis presented the research question that we wanted to tackle. In short, we wanted to study the ability of the different measures to quantify the diversification level of a portfolio from the perspective of the risk minimization criterion and from the weights concentration criterion. After reviewing the relevant theoretical content in chapters 2, 3 and 4, chapter 5 detailed the methodology followed and chapter 6 finished with the presentation and discussion of the results.

In this conclusion, we will be focus on the implications of the results and we will answer to the question that has been raised at the beginning of the thesis. We will end up with the limitations of our works as well as suggestions for future research.

7.1 Implications of the Results

The main implication arising from our research is that it seems very difficult to find a portfolio which is well-diversified from the weights concentration perspective and also from the risk minimization perspective at the same time. Indeed, the results we have obtained in our analysis clearly show that if we want to obtain a portfolio with the minimum level of risk, we have to do it at the cost of more concentration of the weights.
For example, based on our dataset, if we wanted to minimize the risk of a portfolio, it involved more weights on the Treasury or Corporate Bonds. The Maximum Diversification Portfolios constructed by maximization of the ENB, DR and PDI measures are good examples of that. They concentrate more than 75% of their total weights in one of the two bonds asset class to obtain a low risk profile. But such extreme allocations are not really in line with the qualitative definition of diversification.

Another implication arising from our research is that the portfolios which have weights that are (relatively) well distributed between the different asset classes are showing a riskiest profile than the others. The SE and DD Portfolios reflect that pretty well. Moreover, it is interesting to notice that their respective performances are not well risk-adjusted as represented by their Sharpe Ratio. This implication is quite surprising since lots of studies (we can mention Bera and Park (2008), DeMiguel et al. (2005), Jorion (1985) as examples) have been made on the subject of portfolios constructed with equal weights contribution and have proven that such strategy (the $1/N$ asset allocation strategy to name it) delivers strong performances. Therefore, we have decided to attribute the reason of such findings to the choice of our database. Indeed, on one side we have the bonds which are characterized by a low volatility and on the other side, we have the commodities, the equities and real estate which are riskier asset classes.

Finally, the results of our study allow us to provide an answer to the research question we have raised in chapter 1. Indeed, among the six different measures of portfolio diversification we have analyzed, the one which seems to capture the most efficiently the diversification from the two criteria perspective is the risk contributions measure. It is the only one which seems to consider the risk minimization criterion as well as of the weights concentration criterion when judging the diversification level of a portfolio. Moreover, the Risk Contribution Portfolio formed by the maximization of the measure represents a portfolio where the risk is equally distributed between the different asset classes and where the weights are not too concentrated, as depicted by the score of its Shannons Entropy.

7.2 Limitations and Suggestions for Further research

The results of our thesis need to be put into perspective. Indeed, as we have seen, the concept of diversification is complex and hard to define in a financial context. In the framework of our thesis, we have decided to compare the measures of diversification based on two criteria but the choice of the latter can be challenged. Indeed, the absence of a common definition for the concept of diversification in the literature makes it difficult to perform a comparison of different measures of diversification based on such imprecise definition. Moreover, the reality of the market has proven that a large majority of academics and practitioners only consider the risk minimization criterion when trying to diversify a portfolio. A clarification of the concept is therefore something further research should put their attention on.

In addition, we choose to use the Shannon's Entropy to analyze and quantify the concentrations of a portfolio. However, such choice may not be the most appropriate. As a matter of fact, several critics have been made about the use of Shannon's Entropy as a measure of portfolio diversification. For example, Carmichael et al. (2015) showed that the measure doesn't verify Property 2, 3, 4 and 5 that have been described in section 2.2.2. Therefore, the use of another entropy in order to measure the diversification of a portfolio might be necessary. The Rao's Quadratic Entropy (Rao (1982)) has proved to be a strong candidate by Carmichael et al. (2015). Such measure indeed verifies the five properties that are deemed desirable for diversification measure and should be considered for further researches.

A last point would be about the choice of the variance/volatility and the VaR to measure the risk of a portfolio. We have seen in section 2.1.2 that both variance and VaR can't be considered as coherent measures of risk. The variance violates the axiom of monotonicity and the VaR does not obey the sub-additivity axiom. Therefore, it could be interesting to analyze some measures of diversification which are considered as coherent measures of risk, like the Conditional VaR (also called Expected Shortfall).

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Appendix A

Summary Statistics and Results

Table A.1: Randomly Generated Portfolios Allocations, Risk and Performance Statistics(%)

	P1	P2	P3	P4	P5	P6	P7	P8	P9	P10
S&P 500	24.02	3.42	4.67	4.39	20.79	27.86	60.96	29.20	17.08	16.08
T. Bonds	26.69	9.78	28.79	13,05	1.13	27.32	2.74	24.63	14.85	14.63
Commod.	3.74	19.22	28.39	28.34	26.92	14.42	23.91	11.24	29.81	21.22
Corp. Bonds	26.91	33.65	14.39	24.51	29.62	24.10	3.98	33.69	30.96	23.29
Real Estate	18.63	33.91	23.74	29.69	21.52	6.29	8.38	1.22	7.27	24.77
Aver. Ret. (%)	7.18	6.59	5.53	5.96	6.37	6.29	7.17	6.35	5.42	6.46
Variance (%)	0.55	0.81	0.74	0.91	0.98	0.49	1.69	0.40	0.67	0.81
Volatility (%)	7.40	9.00	8.62	9.54	9.89	6.99	13.01	6.36	8.18	8.99
VaR_5	1.69	2.05	1.97	2.18	2.25	1.60	2.97	1.45	1.87	2.05
VaR_1	2.39	2.90	2.78	3.08	3.19	2.26	4.20	2.05	2.64	2.90
Sharpe Ratio	0.57	0.73	0.64	0.62	0.64	0.90	0.55	1.00	0.66	0.72
SE	4.41	4.03	4.39	4.33	4.14	4.51	2.94	3.94	4.48	4.90
RC Entropy	2.93	2.48	2.56	2.56	3.23	3.19	2.21	2.82	2.69	3.17
ENB	2.03	1.49	1.73	1.55	1.27	1.85	1.51	2.33	2.11	1.20
DR	1.47	1.50	1.54	1.50	1.51	1.60	1.30	1.60	1.55	1.54
DR^2	2.17	2.24	2.38	2.26	2.28	2.57	1.69	2.55	2.40	2.36
DD	0.33	0.35	0.35	0.34	0.35	0.38	0.25	0.38	0.36	0.36
PDI	2.32	2.19	2.24	2.23	2.30	2.36	1.91	2.40	2.27	2.28

Table A.2: Risk Contribution for Random Portfolios (%)

	P1	P2	P3	P4	P5	P6	P7	P8	P9	P10
S&P 500	45.40	3.79	4.84	4.35	23.87	55.38	70.03	64.31	20.50	21.10
T. Bonds	1.91	0.24	0.48	0.01	-0.01	1.76	-0.12	2.99	0.44	0.06
Commod.	2.91	25.38	53.92	45.19	40.18	25.39	21.60	19.26	65.27	31.27
Corp. Bonds	6.43	4.83	1.67	2.47	2.60	5.62	0.06	11.24	4.68	2.71
Real Estate	43.35	65.76	39.08	47.98	33.36	11.85	8.43	2.19	9.12	44.86
Cumulative (%)	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00

Table A.3: Diversification Distribution for Random Portfolios(%)

	P1	P2	P3	P4	P5	P6	P7	P8	P9	P10
F1	78.87	89.85	86.03	89.99	95.05	81.95	86.03	68.62	74.89	96.08
F2	10.17	0.01	7.45	3.58	2.29	0.32	0.06	0.11	17.49	0.29
F3	1.40	5.12	2.54	3.34	0.09	7.76	13.83	14.42	0.95	0.13
F4	9.55	5.02	3.98	3.09	2.56	9.97	0.09	16.86	6.67	3.51
Cumulative (%)	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00

Table A.4: Risk Contribution for Maximum Diversification Portfolios (%)

	SE Port.	RC Port.	ENB Port.	DR Port.	DD Port.	PDI Port.
S&P 500	29.17	19.99	16.94	18.77	19.04	19.41
T. Bonds	0.17	20.01	31.72	36.95	0.37	0.00
Commod.	31.22	20.00	51.34	27.04	52.72	16.56
Corp. Bonds	2.56	20.01	0.00	0.00	2.69	64.03
Real Estate	36.88	19.99	0.00	17.24	25.19	0.00
Cumulative (%)	100.00	100.00	100.00	100.00	100.00	100.00

Table A.5: Diversification Distribution for Maximum Diversification Portfolios (%)

	SE Port.	RC Port.	ENB Port.	DR Port.	DD Port.	PDI Port.
F1	95.45	54.89	38.56	52.31	87.74	30.34
F2	0.37	0.19	18.85	1.71	7.48	1.70
F3	0.15	0.09	2.01	0.05	0.01	3.28
F4	4.03	44.82	40.58	45.93	4.76	64.68
Cumulative (%)	100.00	100.00	100.00	100.00	100.00	100.00

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