"Three essays in social choice and mechanism design"

Athanasiou, Efthymios

ABSTRACT

This thesis consists of three distinct chapters. In the first chapter I consider a production economy a la Mirrlees in which the earning capability of individuals is endogenous. Individuals are heterogeneous with respect to their preferences and their propensity to benefit from a given investment in human capital. I look for allocation rules satisfying properties that capture the objective of equalizing opportunities. I characterize four allocation rules that both encompass different perspectives of equality of opportunity and justify different levels of public intervention. In the second chapter, along with coauthors, we put forward a model of private goods with externalities. Agents derive benefit from communicating with each other. In order to communicate they need to have a language in common. Learning languages is costly. In this setting no individually rational and feasible Groves mechanism exists. We characterize the best-in-class feasible Groves mechanism and the best-in-clas...

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THREE ESSAYS ON SOCIAL CHOICE AND MECHANISM DESIGN

Efthymios ATHANASIOU

THESE PRESENTEE EN VUE DE L’OBTENTION DU GRADE DE DOCTEUR EN SCIENCES ECONOMIQUES ET DE GESTION

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The reader will find that throughout this monograph I rarely use ‘I’ to refer to the author. I don’t know whether I suffer from delusions of grandeur. I guess the jury is still out on this one. Nonetheless, there is a more subtle reason that compelled me to use ‘we’ rather than ‘I’. This thesis has been a collaborative effort in more ways than meet the eye.

First and foremost my deepest gratitude goes to Francois Maniquet. It takes a kind and patient thesis supervisor to suffer me as a student, and Francois has been way more than that. He is an insightful guide who supervises closely and attentively. And he also became a friend on whose advice I rely heavily to this day. Unfortunately, I can blame him for none of the deficiencies of my thesis, yet I have to credit him for much of whatever may be good in it.

Giacomo Valletta, or as I affectionately call him, Giac, has been an immense source of support and, I have to concede, inspiration. The reader will recognize him as a coauthor in one of the chapters, yet his contribution extends beyond that. His academic acumen is greater than what he gives himself credit for and I have benefited from it on several occasions. And he does not snore, a quality I look for in a coauthor.

I have had the fortune and the privilege of being a PhD student at CORE. Against all odds, CORE has maintained over a period spanning 40 years a stimulating and lively intellectual environment from which I have profited immensely. For someone like me, who aspires to become a theorist, CORE constitutes a rare academic haven. I have profited from a long list of individuals with whom CORE gave me a chance to interact. They are too numerous to name. The administrative staff, the professors, the other students, the visitors, each one has pledged a contribution to this thesis. Santanu Dey, given his background in operations research is an unlikely coauthor. It has been a rare privilege to engage him in my work. I have learned a lot from him. It was not chance, but, rather, CORE that made this collaboration come to fruition.

Claude d’Aspremont has been a teacher and a thesis advisor whose work on social choice and mechanism design provided me with the inspiration and the foundations to develop my thesis. Throughout the years he has been generously accessible and kindly supportive. Yves Sprumont has made incisive comments on my work and took great care to suggest improvements. His work on mechanism design has served me in my own pursuits. Eve Ramaekers has been impressively meticulous in her feedback over the years. I have benefited from her perspective often and I have relied on her support in many ways.

Pierre Pestieau, Herve Moulin, Jim Schummer, Marc Fleurbaey, Shlomo Weber, Juan Moreno-Ternero, Chris Chambers, William Thomson, Rauf Boucekkine and countless others, all took time to make helpful comments and suggestions. Without question, memory serves me poorly and this list is too short. Apologies.

During my time in Belgium I had to the good fortune to make precious fiends. Their support and encouragement cannot be overstated. Marie Louize and Helene you are my favorite french people. I know you think the competition is poor, but the truth is you have made my day countless times. Matteo, Matias, Maria, Bastien, Celine, Sylvette thank you for your support and kindness.

Finally, as much as I need to acknowledge the help of numerous people, the usual disclaimer applies. Faults and deficiencies remain my own.
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To my parents and my grandfather,
for their unwavering generosity
Well, he hands you a nickel
He hands you a dime
He asks you with a grin
If you’re havin’ a good time
Then he fines you every time you slam the door
I ain’t gonna work for Maggie’s brother no more

*Bob Dylan* on incentives

Most economists would agree, economics is about incentives. I subscribe to this view, albeit with some reservation. The first chapter on my thesis indulges this reservation. Much of what boils down to policy recommendations is shaped by concerns about efficiency and incentives. A more subtle factor, which nonetheless has surprising bite, is the measure of social welfare one elects to base his prescriptions on. Too much is swept under the carpet if one fails to give adequate considerations to this matter. Yet too little of consequence can ever be said if one suppresses altogether the incentive issue. Chapters 2 and 3 are mostly about incentives.

I regret this dichotomy. Ideally, the discussion of most matters of economic interest should cover all three aspects. A careful discussion of the criterion that ranks economic outcomes and the assumptions that found it, a notion of efficiency and the trade-offs it makes necessary and, finally, an explicit mention to incentives and the challenges the implementation exercise presents. But we all need to make compromises.

Each chapter is written so as to be read independently of the rest. Moreover, each contains a detailed introduction that outlines the results it contains and the model it is based on. In what follows I make some general comments on the broad theme that underlies the thesis and binds the chapters together. My intention is merely to whet the appetite of the reader and to encourage him to delve deeper.

The first chapter brings up the issue of equality of opportunity. The notion is opaque and elusive. So considerable effort is expended in defining it. The chapter aims at building rules that embody it and, that, moreover can be readily applied to economic contexts where ethical concerns rightfully become factors. The model alludes to education. How should educational policies be shaped if one strives to bring about equality of opportunity? That is the underlying question I pursue.

The analysis is first-best. The incentive issue is suppressed. I deemed this necessary in order to focus on issues that mainstream economics often neglects. Yet I acknowledge that the discussion is incomplete. There remains the issue of implementation, a question that I intend to take up on separate paper. As far as this thesis is concerned I may only alert the reader to the omission.

Chapters 2 and 3 are related. They both concern strategy-proof implementation. The context, though, is somewhat different. Chapter 2 deals with a private good in the presence...
of an externality, while chapter 3 deals with a pure private indivisible good. There is, nonetheless, a common underlying theme. The aim of both papers is to isolate those mechanisms, within a broader class defined by a set of desirable properties, that perform better on welfare terms.

In Chapter 2, along with my coauthors Santanu Dey and Giacomo Valletta, we put forward a model of agent collaboration. Agents derive benefit from communicating with each other. In order to communicate they need to have a language in common. Learning languages is costly. In this setting no individually rational and feasible Groves mechanism exists. We characterize the best-in-class feasible Groves mechanism and the best-in-class individually rational Groves mechanism.

In Chapter 3 a benevolent Planner wishes to assign an indivisible private good to a number of claimants, each valuing the object differently. A Second-Best Efficient mechanism is a Strategy-Proof mechanism that is not Pareto dominated by another Strategy-Proof mechanism. In this context, I characterize the set of mechanisms that are Second-Best Efficient. It turns out that this set contains mechanisms that destroy the good for some profiles.

Read together, chapters 2 and 3 reveal something surprising about the interplay between incentives and efficiency considerations. To destroy resources is to destroy welfare. Yet sometimes, under incentive constraints, welfare considerations may compel you to destroy the resource you are charged to allocate. This has been a gratifying discovery. And it reinforced my conviction that economic insights become more powerful when issues are raised against a backdrop that accounts comprehensively for incentives, efficiency and welfare concerns.
Chapter 1

Educational Outcomes and Equality of Opportunity

1.1 Introduction

Equality of opportunity does not a priori preclude the prevalence of an unequal distribution of social outcomes. Roemer [63] invokes the ‘non-discrimination’ principle which asserts that only relevant attributes should matter in competing for a position in the social hierarchy. Social outcomes should not be determined by supposedly irrelevant characteristics, such as race, sex and heritage. Yet pinning down the characteristics that ‘should’ matter is far from straightforward. Indeed, in order to define Equality of Opportunity, one needs to understand the nature of individual differences.

The model we put forward is nothing more than an extension of Mirrlees [37]. In contrast to Mirrlees, individuals differ in their preferences and productivities are endogenous. We incorporate into the model an earnings function that transforms investment in human capital into earnings capabilities. Individuals are characterized by different innate abilities and thus have different earnings possibilities. However, a priori, these initial discrepancies that may reflect family background and genetic predisposition, among other things, need not translate into an unequal distribution of marginal productivities. The actual productivity with which an agent may transform labour into the consumption good will be determined, inter alia, by the amount he chooses to ‘invest in himself’.

At first glance, this presents the policy maker with a range of new possibilities. For instance, one may opt for equalizing productivities or strive to guarantee everyone a certain amount of investment in human capital. Although, throughout the paper, we insist on a first best approach, we find that such solutions are excluded if one is to respect Pareto Efficiency. In figure 1.1, we introduce a rough version of the model. On the left hand side we have the standard Mirrlees consumption space, comprising labour and a consumption good, such as money. On the right hand side we plot two earnings functions, reflecting a supposition that individual A is more talented than individual B. If we were to impose a human capital investment profile \((m_A, m_B)\) and then bill each individual for the expenditure \(m_A + m_B\) equally, we would have A and B facing the same budget line (assuming a constant marginal productivity technology). We have equalized consumption opportunities. Suppose, then, that we let individuals maximize their preferences. The resulting allocation \((z_A, z_B)\) is not Pareto efficient: The allocation \((z_A', z_B)\) Pareto dominates \((z_A, z_B)\).\(^1\)

\(^1\)To obtain this allocation we invest \((0, m_B')\) and charge individuals A and B zero and \(m_B' = m_A + m_B\) respectively. Individual A’s budget line is in effect the labour axis.
Figure 1.1: An investment profile that equalizes opportunities.

In the public finance literature similar models have been investigated in an effort to isolate features of optimal taxation when individuals react to taxes by simultaneously adjusting their labour and educational choices. Bruno [8], Ulph and Hare [29] and Ulph [64] discuss models that are somewhat more restrictive than the one we put forward. Crucially, in their work individuals differ only with respect to their ability to gain from an investment in education, while all individuals share the same preferences. The setting is one of informational asymmetries. Individuals have the option of concealing certain pieces of information that are important for the design of the optimal tax scheme. These papers conclude that educational expenses should be distributed regressively. Lately, Maldonado [33] and Bovenberg and Jacobs [7] discuss the degree of ‘ regressivity’ and its determinants. These results are driven by the combination of heterogeneity among individuals and informational asymmetries. For instance, the degree of regressivity depends on the role that educational subsidies play in alleviating the incentive problem. Our point of view is first-best: all relevant information is known. Thus, we intend to isolate the effect that heterogeneity has on its own on the optimal distribution of educational expenses.

Recently, Cremer et al. [13] showed that even in the absence of any difference across individuals, in a first-best environment, wage equalization may be suboptimal. We can reproduce their finding with the help of figure 1.2. Suppose that there are two individuals in the economy. They share the same preferences defined over consumption and labour time. Both face a simple linear earnings function: their marginal productivity equals an amount of money times some positive parameter \( \beta \). The social planner needs to decide how to invest a fixed and exogenous amount \( M \) in the education of those two individuals. Suppose that he decides to equalize wages. He invests an amount \( M/2 \) in each. Hence, both individuals face the same budget line, the one through points \((0,0)\) and \((1, M/2)\). They maximize their preferences subject to this constraint. In figure 1.2 we depict an indifference curve that represents individual preferences that is, in addition, supported by the budget line generated by the policy of equalizing marginal productivities. This serves as our reference. Consider, now, an alternative policy. The entire amount \( M \) is invested in individual 1. The corresponding budget line in figure 1.2 is the one through points \((0,0)\) and \((1, M)\). Individual 2’s budget line is the segment of the labour axis up to point \((1,0)\). Suppose, then, that we equalize social welfare by transferring resources lump-sum
from individual 1 to individual 2. In the figure the magnitude of the transfer equals \( T \). If individuals maximize their preferences subject to this educational and tax policy, they end up better-off compared to the previous scenario. This counterexample cannot be replicated in our context because the actual cumulative investment in human capital is endogenous and not bounded explicitly. As we shall see, there always exists a Pareto efficient allocation involving no taxes such that all individuals that are identical in all respects both enjoy the same level of welfare and have the same marginal productivity.

What is striking about the example in figure 1.1 is that it illustrates how the absence of intervention (individuals fully bear the cost of their education) may be welfare superior over a policy that strives to equalize the distribution of marginal productivities. Therefore, we have to abandon the most intuitive interpretation of equality of opportunity, that of ‘leveling the playing field’, and examine more subtle factors. It is insights such as this that make the first-best analysis an exercise worth a paper of its own. It is more than a prerequisite for introducing second-best considerations. It serves to clarify and fix ideas regarding equality of opportunity.

Our departing premise is that two broad categories of individual differences, preferences and innate abilities, warrant differential treatment in the context of equality of opportunity. This can be best illustrated by an example. Suppose Sarah and Matt are equally hardworking lawyers. However, Sarah has secured a better job than Matt, because her mother, a well established lawyer, intervened on her behalf. Sarah enjoys a higher income than Matt thanks to her family background. Matt is not to blame for falling short of Sarah. He is not accountable for the fact that his parents are, say, doctors and therefore could not help him in the job market. The principle of compensation dictates that the difference in welfare between Sarah and Matt should be eliminated. Let us now consider a different story, that of Maria and Nelson. Both are alike in all respects except that Nelson is not as hard a worker as Maria is. Naturally Maria earns more than Nelson. Both Maria and Nelson are responsible for their disposition towards labour. In recognition of this fact, the principle of responsibility dictates that the difference in income between Maria and Nelson be preserved\(^2\). In brief, the differences in social outcomes that we should strive to

\[^2\text{See Arneson [3] for a discussion on whether individual preferences develop under circumstances that}\]
correct are those solely due to differences in innate abilities and social backgrounds. On the contrary, there is no reason to amend differences in social outcomes that are solely the result of differences in preferences. The two objectives, Compensation and Responsibility, will play a crucial role in the definition of Equality of Opportunity. This distinction, following Fleurbaey [17], can be viewed as a particular manifestation of Roemer’s non-discrimination principle.

A series of papers (see Fleurbaey and Maniquet [22] for a review), in different contexts, demonstrate that Compensation, Responsibility and Pareto efficiency clash with one another. One cannot fully fulfill all these separate requirements. In the framework of our model we reiterate this conclusion, perhaps more emphatically, as we find that the incompatibility holds, independently of the earnings function. What ought to be an appropriate compromise is far from apparent. One of the two principles needs to be abandoned in order to accommodate the other. Any point of view regarding this trade-off is treated as a moral stance in the face of an ethical difficulty.

Fleurbaey and Maniquet [19], [20] are the two contributions which come closest to our approach. Their model involves a production economy with exogenous productivities. They are the first to recognize the incompatibility between Compensation and Responsibility in the context of a production economy. Roemer [63] attempts a similar exercise in which he employs conceptual tools that appear to derive from the ideas of Compensation and Responsibility. He adopts, however, a different model. The aim of this paper is to revisit the conclusions of these works, drawing new insights and recasting certain solutions built on new foundations.

The paper proceeds as follows. Section 1 presents the model. In section 2 we develop and discuss an object, akin to a budget set, which is instrumental in the derivation of our results. In section 3 we elaborate on the consequences of Pareto efficiency. Section 4 presents the axioms that encapsulate the main ideas we have addressed in this brief introduction. Section 5 proposes solutions and discusses their implications. Section 6 sums up and discusses the potential extension to second best. In the appendix we prove propositions 3, 4 and 5.

1.2 The model

Let \( \mathbb{N}_+ \) be the set of positive integers and \( \mathcal{N} \) denote the set of all finite and non-empty subsets of \( \mathbb{N}_+ \). A population of agents is some \( N \in \mathcal{N} \). Let \( \mathcal{R} \) denote the set of complete, transitive, convex, monotonic\(^3\) and continuous preference relations over bundles \( z \equiv (l, c) \in Z \equiv [0, 1] \times \mathbb{R}_+, \) where \( l \) stands for labour time, \( c \) denotes consumption and \( \mathbb{R}_+ \) is the non-negative real line. Individuals are endowed with a preference \( R_i \in \mathcal{R} \). The strict preference and indifference counterparts of \( R_i \) are \( P_i \) and \( I_i \) respectively.

For each \( N \in \mathcal{N} \) and all \( i \in N \), let \( a_i \in \mathbb{R}_+ \) determine the individual's innate ability or talent. If for some \( i, j \in N \) we have that \( a_i > a_j \), we infer that individual \( i \) is more talented than individual \( j \). Each individual \( i \in N \) may forgo consumption to invest an amount \( k_i \in \mathbb{R}_+ \) in order to build up his productivity. The earnings function is captured by \( g : \mathbb{R}_+^2 \to \mathbb{R}_+ \) that transforms pairs of \( (k_i, a_i) \) into a constant individual marginal productivity. We require that the earnings function satisfies the following properties.

---

\(^3\)With respect to leisure and consumption.
**Assumption 1** The function $g$ is such that:

(a) for each $a > 0$, it is twice continuously differentiable in $k$,

(b) for each $a > 0$ and $k \geq 0$,
\[
\frac{\partial g(k,a)}{\partial k} > 0, \quad \frac{\partial^2 g(k,a)}{\partial k^2} < 0, \quad g(0,a) = 0,
\]

(c) for each $a > 0$ there exist $k' > 0$ such that $\frac{\partial g(k',a)}{\partial k} \leq 1$,

(d) for each pair $a, a' > 0$, $a > a'$ implies that for each $k > 0$ we have $g(k,a) > g(k,a')$.

Loury [32] and Becker et al. [6] develop theoretical models that involve earnings functions similar to the one we propose. Assumption 1 is more general than what is typically found in the public finance literature. Bovenberg and Jacobs [29] and Maldonado [33], for instance, make additional assumptions concerning cross-derivatives. Finally, Assumption 1 conforms with the empirical evidence. Psaharopoulos and Patrinos [50] conclude that “investment in education behaves in a more or less similar manner as investment in physical capital.”

Let $G$ be the family of functions satisfying Assumption 1. For each $N \in \mathbb{N}$ define $R_N \equiv (R_i)_{i \in N}, a_N \equiv (a_i)_{i \in N}$. An economy $e_N$ is described by the following profile:

\[
e_N = (R_N, a_N, g) \in \mathcal{E} \equiv \bigcup_{N \in \mathbb{N}} (\mathbb{R}^{|N|} \times \mathbb{R}_+^{|N|} \times G).
\]

Slightly abusing the notation we will sometimes treat $R_N, a_N$ as sets of elements, writing for instance $a_i \in a_N$ and $\min a_N \equiv \{a_i \in a_N : a_i \leq a_j, \text{ for each } a_j \in a_N\}$.

Agents are endowed with one unit of leisure and zero units of the consumption good. Any amount of investment in human capital will be paid out of labour income. The consumption and investment goods are perfect substitutes. We impose no physical constraint on the amount that can be invested. Imagine perfect credit markets that lend amounts of the investment good at no interest. Returns to scale are constant\footnote{The story we are unraveling is consistent with an OLG model in which individuals live two periods, investment in human capital occurs when young, consumption when old, and the distribution of innate abilities remains constant across different generations.}. In order to facilitate comprehension we may raise the following question: what constitutes an attainable bundle $(l_i, c_i) \in Z$ for some individual $i \in N$, in some economy $e_N \in \mathcal{E}$? In the standard setup with exogenous productivities and constant returns to scale one would have to require that $c \leq w_i l$, where $w_i \geq 0$ stand for the marginal productivity and the product $w_i l$ bears the interpretation of earnings or gross income. In our setting marginal productivities are endogenous, their formation being dictated by some earnings function. Individual $i$ needs to invest in himself, so to speak, in order to build a capacity to produce the consumption good. Therefore, the fruit of his labour, gross income, needs to be sufficient to sustain both the investment in his productive capabilities, as well as consumption.

We will say that a bundle $z_i \in Z$ belongs to the **Outcome Set** available to some individual $i$ facing a transfer $t_i \in \mathbb{R}$, and write $z_i \in OS(g, a_i, t_i)$, if and only if there exists $k_i \in \mathbb{R}_+$ such that $c_i + k_i \leq g(k_i, a_i)l_i + t_i$.  

\[1.2.1\]
An allocation $z_N \in Z[N]$ is feasible in economy $e_N \in \mathcal{E}$ if and only if there exists a vector of transfers $t_N \in \mathbb{R}^{|N|}$, such that $\sum_{i \in N} t_i \leq 0$, and for each $i \in N$ we have $z_i \in OS(g, a_i, t_i)$. The set of feasible allocations for economy $e_N$ is denoted $Z(e_N)$.5

Our model relies on a number of assumptions. First, education has no intrinsic value. Investment in human capital is introduced as a means to an end, the end being consumption. One could rectify this by introducing a third dimension to the consumption set and defining preferences over bundles $(l, c, k) \in [0, 1] \times \mathbb{R}_+^2$. Investment in human capital becomes a means to an end and an end in itself. Without doubt, under such an extension, the parameter $k$ connotes the role of education better. We do not maintain that such a qualification to our model will make the problem we are investigating intractable. However, it does not serve our purpose. It would not be surprising to find that in a world where individuals have different preferences over education, equalizing marginal productivities clashes with Pareto efficiency. Contrary to that, such a claim carries some weight in a world where education is perceived by all individuals in the same way.

A second key assumption is that the earnings function is deterministic. Alternatively, $g(k, a)$ could be a distribution over marginal productivities. Along these lines one could use stochastic dominance criteria to capture the effect of the inputs $k, a$. Naturally, preferences should also be qualified to account for uncertainty. Working under this assumption, introduces an extra parameter that varies across individuals, namely, attitude towards risk. The argument against introducing uncertainty resembles the one we employed to defend our choice to assign no intrinsic value to education. We want to model the minimum degree of diversity across individuals that is sufficient to establish our main result. More diversity works in favor of the clash between equality of opportunity and Pareto efficiency.

A third assumption that we make involves the absence of credit constraints. We could imagine that each individual $i \in N$ faces a constraint $\bar{c}_i$ on the amount he may borrow and subsequently invest to build earning capabilities. If credit constraints were present we would need to have a story regarding the circumstances that bring them about and, more importantly, an opinion on whether these constraints are morally justifiable. We choose to suppress this matter. Besides, the mechanics of the model are not significantly altered with the introduction of credit constraints. Most of our results, and certainly the key ones, can be reitered in a more general set up that accounts for credit constraints. In addition, Cremer et al. suggest that credit constraints work against the equalization of marginal productivities.

Finally, we assume no education externalities. Indeed, it could be the case that individual marginal productivity is a function of the entire profile of investment in human capital, $k_N$ (and/or the entire profile of innate abilities, $a_N$). In fact, equalization of marginal productivities, which our model excludes as it clashes with Pareto efficiency, could be rationalized in a model that incorporates educational externalities. Whether in reality educational externalities are prevalent remains an open question. There is, however, empirical evidence that suggests that they are not (see Pritchett [51] for a recent reference).

1.3 The Outcome Frontier

The objective of this section is to define and develop a concept that will enable us to represent both sources of heterogeneity in the consumption space. Practically speaking, we will demonstrate that differences in innate abilities correspond to differences in the

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5An equivalent definition is $Z(e_N) = \{z_N \in Z[N] : \text{there exists } k_N \in \mathbb{R}_+^{|N|} \text{ s.t. } \sum_{i \in N} (c_i + k_i) \leq \sum_{i \in N} g(k_i, a_i, t_i)\}$. It will prove helpful, however, to offer a definition of feasibility that makes explicit the role of lump-sum transfers.
outcome sets available to each individual. In order to be more precise, we need to explore the properties of the outcome set. We accomplish this task in two steps. First, we show that, for all parameter values, the set $OS(a_i, t_i)^6$ is closed and bounded, and we provide a characterization of its bound. Second, we determine the effect that changes in the value of $t_i$ and $a_i$ have on the bound of the outcome set$^7$.

The following definition states that a feasible bundle lies on the bound of $OS(a_i, t_i)$ if and only if in any closed ball around it there exists another bundle which is not feasible. Formally, for each $a_i \in \mathbb{R}_+$ and $t_i \in \mathbb{R}$, a bundle $\tilde{z}_i \in OS(a_i, t_i)$ belongs to the outcome frontier, denoted $OF(a_i, t_i)$, if and only if for each $\epsilon > 0$, there exists $z'_i \in Z$ such that

$$z'_i \in \{ z_i \in Z : ||z_i - \tilde{z}_i|| < \epsilon \} \text{ and } z'_i \notin OS(a_i, t_i).$$

The definition above helps to fix ideas, although it remains silent as far as the properties of the outcome frontier are concerned. Lemma 1 below provides a characterization of the outcome sets available to each individual. In order to be more precise, we need to explore two functions that are inherently connected to the outcome frontier.

Consider the problem

$$\max_{k \geq 0} g(k, a_i)l_i - k + t_i \text{ subject to } l_i = l'_i.$$  

We will show that for each $a_i \geq 0$ and $t_i \in \mathbb{R}$ there exists a finite value $k^*(l'_i; a_i, t_i) \in \mathbb{R}_+$ such that for each $l'_i \in [0, 1],$

$$k^*(l'_i; a_i, t_i) = \max_{k \geq 0} \{ g(k, a_i)l'_i - k + t_i \}.$$  

In other words, we will establish that the solution to the maximization problem defined above always exists and, moreover, it is finite and unique. The case $l'_i = 0$ is trivial. For each $k > 0$ we have $g(k, a_i)l'_i - k + t_i < t_i$, while for $k = 0$ we have $g(0, a_i)l'_i - k + t_i = t_i$. Hence, $k^*(0; a_i, t_i) = 0$. Let us focus on values of $l'_i$ belonging to the interval $[0, 1]$. According to the first order condition associated with the problem, $k^*(l'_i; a_i, t_i) \in \mathbb{R}_+$ is a maximum only if it solves

$$(\partial g(k, a_i) \frac{\partial}{\partial k} l'_i - 1)k = 0. \quad (1.3.1)$$

If for each $k > 0$ we have $\frac{\partial g(k, a_i)}{\partial k} l'_i < 1$, then the constraint $k \geq 0$ binds and, consequently, $k^*(l'_i; a_i, t_i) = 0$. Suppose then that there exists $k^1 > 0$ such that $\frac{\partial g(k^1, a_i)}{\partial k} l'_i > 1$. By Assumption 1, clause (c), there exists $k^2 > 0$ such that $\frac{\partial g(k^2, a_i)}{\partial k} l'_i < 1$. Moreover, again by Assumption 1, the first derivative of $g$ with respect to $k$ is decreasing continuously, hence there must exist a unique $k' \in (k^1, k^2]$ such that $\frac{\partial g(k', a_i)}{\partial k} l'_i = 1$.

So far we have demonstrated that $k^*(l; a_i, t_i)$ is a continuous function that assigns to each number in the interval $[0, 1]$ a non-negative, finite real number. Inspecting the construction that generates the function $k^*(l; a_i, t_i)$ reveals that the assignment does not depend on the parameter $t_i \in \mathbb{R}$. Put formally, fixing $a_i$ at any admissible level we have that for each $t^1, t^2 \in \mathbb{R}$ and $l \in [0, 1]$, $k^*(l; a_i, t^1) = k^*(l; a_i, t^2)$. In fact, we may invoke the term lump-sum transfer precisely because its actual value does not matter for the determination of the value of $k$ that maximizes gross consumption. Henceforth, we will just write $k^*(l; a_i)$. We may now proceed to define for each $a_i \geq 0$ and each pair $(l, t) \in [0, 1] \times \mathbb{R}$ the function

$$c(l, t; a_i) = g(k^*(l; a_i), a_i)l - k^*(l; a_i) + t.$$

$^6$Since the technology will be held constant in what follows, let us agree to drop $g$ from the expression.

$^7$The proofs of the lemmata of this section are available upon request.
Having fixed the parameter \( a_i \) at some arbitrary level, in figure 1.3-part (a) the thick curves represent the function \( c(l, t; a_i) \) for two values of \( t \) equal to \( T > 0 \) and \( -T \). We need only know \( c(l, t; a_i) \) for one value of \( t \) equal to \( T \), in order two ascertain its general form. For any \( t^+ \neq T \), \( c(l, t^+; a_i) \) equals \( c(l, T; a_i) + t^+ - T \).

Let us now establish the connection between the function \( c(l, t; a_i) \) and the outcome frontier.

**Lemma 1** For each \( a_i \in \mathbb{R}_+ \), \( t_i \in \mathbb{R} \) and \( l_i \in [0, 1] \), if \( (l_i, c(l_i, t_i; a_i)) \in Z \) then \( (l_i, c(l_i, t_i; a_i)) \in OF(a_i, t_i) \). Conversely, for each \( a_i \in \mathbb{R}_+ \), \( t_i \in \mathbb{R} \) and \( l_i \in [0, 1] \), if \( (l_i, c_i) \in OF(a_i, t_i) \) then \( c_i = c(l_i, t_i; a_i) \).

We have, thus, identified an equivalent definition of the outcome frontier. Applying it we may conclude that for each \( t_i \geq 0 \) the outcome frontier coincides with the function \( c(l_i, t_i; a_i) \) and for each \( t_i < 0 \) the outcome frontier coincides with the segment of \( c(l_i, t_i; a_i) \) that lies above the labour axis. In figure 1.3-part (a) this occurs for values of labour greater or equal to \( l^+ \).

The following two lemmata establish that the outcome frontier is non-decreasing and convex.

**Lemma 2** For each \( a_i \in \mathbb{R}_+ \), \( t_i \in \mathbb{R} \), \( (l^1, c^1), (l^2, c^2) \in OF_i(a_i, t_i) \), if \( l^2 > l^1 \) then \( c^2 \geq c^1 \)

**Lemma 3** For each \( a_i \in \mathbb{R}_+ \), \( t_i \in \mathbb{R} \), \( (l^1, c^1), (l^2, c^2) \in OF(a_i, t_i), \lambda \in (0, 1) \), if \( (l^3, c^3) \in Z \) is such that \( l^3 = \lambda l^1 + (1 - \lambda)l^2 \) and \( c^3 > \lambda c^1 + (1 - \lambda)c^2 \), then \( (l^3, c^3) \notin OF(a_i, t_i) \)

Any feasible bundle can be associated with a budget line and, consequently, an amount of investment in human capital. If, in addition, some bundle \( (l^*_i, c^*_i) \in Z \) lies on the outcome frontier, then the associated budget line is generated by an investment equal to \( k^*(l^*_i; a_i) \). The outcome frontier is the upper envelope of the budget lines generated by \( k^*(l; a_i) \) as \( l \) takes all the values in the \([0, 1]\) interval. Therefore, the outcome frontier can be viewed as a menu of marginal productivities or, equivalently, budget lines. It is non-decreasing because it is the upper envelope of non-decreasing lines. And it is convex because it is supported by intersecting lines of positive slope. These observations are depicted in figure 1.3-part (a). The curve through bundles \( z^0, z^1 \) and \( z^2 \) constitutes a generic outcome frontier. The

![Figure 1.3: A summary of the properties of the outcome frontier.](image)
segment of the outcome frontier that runs parallel to the labour axis, such as the segment between bundles $(0, T)$ and $z^0$ in the figure, is associated with values of labour time that render a positive investment in human capital suboptimal. If such a segment exists, then it constitutes the part of the outcome frontier that is not strictly convex.

Let us, finally, determine how the outcome frontier varies across individuals that differ in innate ability. The following lemma states that innate ability differentials translate bluntly into different outcome frontiers that dominate each other. This is a mere consequence of the fact that for any amount of labour the same investment in human capital produces more gross income the greater innate ability is.

Lemma 4 For each $a_k, a_j \in \mathbb{R}_+$, with $a_k > a_j$ and each $t \in \mathbb{R}$,

1. if $(l_k, c_k) \in OF(a_k, t)$ and $c_k \geq \max\{0, t\}$, then there does not exist $c_j \geq c_k$ such that $(l_k, c_j) \in OF(a_j, t)$, and

2. if $(l_k, c_k) \in OF(a_k, t)$ and $c_k = \max\{0, t\}$, then there does not exist $c_j > c_k$ such that $(l_k, c_j) \in OF(a_j, t)$.

In figure 1.3-part (b) we depict two outcome frontiers corresponding to two individuals that have different innate abilities, yet facing the same transfer. The thick curve depicts the outcome frontier of the more talented individual. Although lemma 4 allows for the possibility of overlaps, this will only occur at a segment of the outcome frontier that runs parallel to the labour axis.

1.4 Pareto Efficiency

The purpose of this section is to investigate the implications of Pareto Efficiency. An allocation rule $\varphi$ is a correspondence that assigns a non-empty subset of $Z(e_N)$ to each economy in the domain $\mathcal{E}$. An allocation rule can be viewed as a social recommendation. It selects in each economy the feasible allocations that are more desirable from the social point of view. We require that all the allocation rules we consider henceforth satisfy the following axiom.

Definition 1 An allocation rule $\varphi$ satisfies Pareto Efficiency (PE) if and only if for each $e_N \in \mathcal{E}$, each $z_N \in \varphi(e_N)$ and each $z'_N \in Z(e_N)$,

$$z'_i R_i z_i, \forall i \in N \Rightarrow z'_i I_i z_i, \forall i \in N$$

Since we will be confining our discussion to allocation rules that always select Pareto efficient allocations, it pays to explore the properties of the set of Pareto efficient allocations. We begin by characterizing it. The following lemma states that an allocation is Pareto efficient if and only if there exists a balanced vector of transfers that decentralizes it.

Lemma 5 For each $e \in \mathcal{E}$, an allocation $z_N \in Z(e_N)$ is Pareto efficient if and only if there exists a vector of transfers $t_N \in \mathbb{R}^{|N|}$ such that

1. for each $i \in N$, $z_i \in OF(a_i, t_i)$,
2. for each $i \in N$, $z_i$ maximizes $R_i$ on $OF(a_i, t_i)$, and
3. $\sum_{i \in N} t_i = 0$.

Proof.

We will only prove that the conditions above are sufficient for Pareto efficiency. Suppose not. Suppose, in particular, that for some economy $e_N \in E$ there exists an allocation $z_N \in Z(e_N)$ and a vector of transfers $t_N \in \mathbb{R}^{|N|}$ such that conditions (1)-(3) are satisfied, and, moreover, let there exist $z'_N \in Z(e_N)$ such that for each $i \in N$, $z'_i R_i z_i$, and for some $j \in N$, $z'_j P_j z_j$. Since $z'_N \in Z(e_N)$ there must exist some $t'_N \in \mathbb{R}^{|N|}$ such that for each $i \in N$, $z'_i \in OS(a_i, t'_i)$. Suppose that $t'_j > t_j$. This means that, since $\sum_{i \in N} t_i = 0$, there must exists $k \in N$ for whom $t_k > t'_k$. In turn this implies that $OS(a, t'_k)$ is strictly included in $OS(a, t_k)$. By assumption, $z_k \in OF(a, t_k)$. Therefore, we obtain $z_k P_k z'_k$, a contradiction. Suppose, then, that $t'_j < t_j$. A similar line of reasoning results in $z_j P_j z'_j$, a contradiction. Hence, we obtain $t'_j = t_j$. Therefore, $OS(a_j, t_j) = OS(a_j, t'_j)$. Thus, $z'_j P_j z_j$ contradicts the assumption that $z_j$ maximizes $R_j$ on $OF(a_j, t_j)$.

The above lemmata and in particular the last one, precisely focus the message we attempted to convey with the example we invoked in the introduction. In any generic economy, ‘equalizing opportunities’ in the straightforward sense conflicts with Pareto efficiency: no sensible policy can force the outcome frontiers which individuals face to coincide. It is impossible to achieve an efficient and fair outcome simply by organizing the investment profile $k_N$ in a way that disregards information on preferences and innate abilities. In figure 1.4 we illustrate an economy comprising two individuals. Individual 2 is more talented than individual 1. The thick curves depict the outcome frontiers in the absence of taxation, namely for the case $t_1 = t_2 = 0$. Dotted curves depict preferences. The allocation $(z_1, z_2)$ has the feature of equalizing marginal productivities. The dashed lines through $z_1$ and $z_2$ are parallel to each other. According to lemma 5 the allocation is not Pareto efficient. Although the sum of transfers equals zero and both $z_1$ and $z_2$ lie on their respective outcome frontiers, preferences are not maximized. In contrast, the allocation $(z'_1, z'_2)$ is Pareto efficient. The marginal productivity of individual 1 is zero, whereas the marginal productivity of individual 2 is very high. Nonetheless, Pareto Efficiency is compatible with lump-sum transfers. Such transfers can be seen as a mechanism for redistributing oppor-
tunities or consumption possibilities among individuals. In fact, as we shall see shortly, underlying any Pareto efficient allocation rule, is a vision on how these transfers should be carried out and the purpose they should strive to fulfill.

1.5 The axioms

Before we move on, let us introduce some further pieces of notation. We denote the maximal elements of the Outcome Set, from the point of view of an individual \( i \in N \), facing a transfer \( t_i \in \mathbb{R} \), with preference relation \( R_i \), by \( m(R_i, OS(a_i, t_i)) \).

Assumption 2 For each \( e_N \in \mathcal{E} \), each \( t_N \in \mathbb{R}^{\mid N\mid} \) and each \( i \in N \), \( z_i \in m(R_i, OS(a_i, t_i)) \) if and only if \( z_i R_i z_i' \), for each \( z_i' \in OS(a_i, t_i) \).

Since individual preferences are monotonic, if \( z_i \in m(R_i, OS(a_i, t_i)) \), then \( z_i \in OF(a_i, t_i) \).

Assumption 3 For each \( e_N \in \mathcal{E} \) and each \( z_N \in Z(e_N) \) define \( t(z_N) = (t_i(z_i))_{i \in N} \) to be such that, for each \( i \in N \):

\[
t_i(z_i) = \min \{ t \in \mathbb{R} : z_i \in OS(a_i, t) \}.
\]

For each \( e_N \in \mathcal{E} \) and each \( z_N \in Z(e_N) \), the set \( \min \{ t \in \mathbb{R} : z_i \in OS(a_i, t) \} \) is non-empty and contains a single element. In fact, an equivalent definition would be to require that \( t_i(z_i) \) is such that \( z_i \in OF(a_i, t_i) \). Figure 1.5 illustrates the construction that determines \( t_i(z_i) \). Consider some arbitrary bundle \( z_i \in Z \) (it does not matter whether it is feasible or not) and some individual \( i \) in the economy. Given his innate ability and the earnings function we may infer his outcome frontier corresponding to a zero transfer. To determine \( t_i(z_i) \) we need to translate the zero-tax outcome frontier vertically, either upwards or downwards, until it reaches the bundle \( z_i \). The transfer that accomplishes that is \( t_i(z_i) \). If an allocation \( z_N \) is feasible we obtain \( \sum_{i \in N} t_i(z_i) \leq 0 \). If an allocation \( z_N \) is Pareto efficient we obtain \( \sum_{i \in N} t_i(z_i) = 0 \).

Axioms 2-4, presented below, proceed from Fleurbaey [17]. We use them to conceptualize Equality of Opportunity, as already noted. We begin with the idea of Compensation for differences in innate ability. We require that no advantage be grounded on parameters that are beyond individual accountability. Individuals with the same preferences should enjoy the same level of satisfaction. Such individuals may potentially differ only with respect to innate ability.
Definition 2 An allocation rule \( \varphi \) satisfies Equal Welfare for Equal Preference (EWEP) if and only if for each \( e_N \in \mathcal{E} \), each \( z_N \in \varphi(e_N) \) and each pair \( i, j \in N \)

\[
R_i = R_j \Rightarrow z_i I_i z_j.
\]

The following two axioms capture the idea of Responsibility. Any two individuals with the same innate ability should have the same consumption opportunities or, more practically, should end up on the same outcome frontier. Differences in preferences alone are not a legitimate reason to elicit unequal treatment. From an ethical point of view, individuals should be neither rewarded nor punished for their preferences. In plain terms, axiom 3 requires that individuals with the same ability should face the same lump-sum transfer.

Definition 3 An allocation rule \( \varphi \) satisfies Equal Frontier for Equal Ability (EFEA) if and only if for each \( e_N \in \mathcal{E} \), each \( z_N \in \varphi(e_N) \) and each pair \( i, j \in N \)

\[
a_i = a_j \Rightarrow t_i(z_N) = t_j(z_N).
\]

A weaker idea (Fleurbaey and Maniquet [19]) would be to posit no envy among individuals who share the same talent. Indeed, it is not necessary for two equally able individuals to face the same transfer in order for the following axiom to be satisfied.

Definition 4 An allocation rule \( \varphi \) satisfies No Envy among the Equally Able (NEEA) if and only if for each \( e_N \in \mathcal{E} \), each \( z_N \in \varphi(e_N) \) and each pair \( i, j \in N \)

\[
a_i = a_j \Rightarrow z_i R_i z_j \text{ and } z_j R_j z_i.
\]

The axioms that follow do not directly relate to Equality of Opportunity. However, Equality of Opportunity as an ethical goal does not justify all possible means. We maintain that an allocation rule should refrain from exploiting individuals. Put simply, we ask that the poorly endowed establish a right over a policy that, while aiming at improving their situation, at the same time does not ‘exploit’ the talented. Axioms 5 and 6 that follow defend the idea of collective ownership of the potential that the parameter \( a_i \) associates with each individual. It relates to Rawls [61] who claims that no one ‘deserves’ his place in the distribution of innate abilities. Thomson [57] and Roemer [62] proposed an axiom that manifests the Rawlsian principle concretely. Ability Solidarity requires that either we all benefit or we all suffer from any arbitrary change in the vector of innate abilities. The consequences of such a change should be borne collectively.

Definition 5 An allocation rule \( \varphi \) satisfies Ability Solidarity (AS) if and only if for each \( e_N = (R_N, a_N, g) \in \mathcal{E} \), each \( e'_N = (R_N, a'_N, g) \in \mathcal{E} \), each \( z_N \in \varphi(e_N) \) and each \( z'_N \in \varphi(e'_N) \)

\[
either z_i R_i z'_i \text{ for each } i \in N \text{ or } z'_i R_i z_i \text{ for each } i \in N.
\]

Ability Solidarity can be interpreted as the impetus that drives an allocation rule towards redistribution. We will see that it takes a clear stand in favor of Compensation objectives. However, Ability Monotonicity, defined below (which is implied by Ability Solidarity, together with Pareto Efficiency), is fully compatible with Equal Frontier for Equal Ability, so much so as to be satisfied by a laissez-faire like allocation rule. It requires that everyone benefit whenever the vector of innate abilities (weakly) increases. It should be interpreted as an axiom that prevents the exploitation of the less talented.\footnote{We write \( a'_N \geq a_N \) to mean \( a'_i \geq a_i \), for each \( i \in N \).}
Definition 6 An allocation rule \( \varphi \) satisfies Ability Monotonicity (AM) if and only if for each \( e_N = (R_N, a_N, g) \in \mathcal{E} \), each \( e'_N = (R_N, a'_N, g) \in \mathcal{E} \), each \( z_N \in \varphi(e_N) \), each \( z'_N \in \varphi(e'_N) \) and each \( i \in N \)

\[
a'_N \geq a_N \Rightarrow z'_i R_i z_i.
\]

While we endorse solidarity, still, we would like to avoid what Dworkin [15], [16] calls the ‘slavery of the talented’. We do not allow for a transfers scheme that causes the talented to ‘resent their gift’. Limited Self-Ownership of Ability requires that a transfer of resources does not go as far as making anyone in the society envious of some least able individual’s consumption possibilities. Let \( N(a; e_N) \equiv \{ i \in N : a_i = a \text{ for some } a \in \mathbb{R}_+ \} \).

Definition 7 An allocation rule \( \varphi \) satisfies Limited Self-Ownership of Ability (LSOA) if and only if for each \( e_N \in \mathcal{E} \), each \( z_N \in \varphi(e_N) \), each \( i \in N \) and each \( j \in N(\min a_N; e_N) \)

\[
z_i R_i m(R_i, OS(a_j, t_j(z_N))).
\]

Minimal Self-Ownership of Ability is a variant of the same idea. No individual should be worse off than he would be if he belonged to the set \( N(\min a_N; e_N) \) and, moreover, there were no transfers.

Definition 8 An allocation rule \( \varphi \) satisfies Minimal Self-Ownership of Ability (MSOA) if and only if for each \( e_N \in \mathcal{E} \), each \( z_N \in \varphi(e_N) \), each \( i \in N \) and each \( j \in N(\min a_N; e_N) \)

\[
z_i R_i m(R_i, OS(a_j, 0)).
\]

Axioms 7 and 8 encompass mild libertarian objectives. They are probably too weak for the taste of philosophers like Nozick [47], who invoke the Kantian principle that one should not serve as a means for another. They can be interpreted as lower bounds. Moulin and Roemer[40] propose axioms that relate to ours. Since we employ a model with constant returns to scale, we do not use bounds à la Moulin [41], [42]. It should be stressed, finally, that Limited and Minimal Self-Ownership of Ability, are logically unrelated. In particular, having all the least able receive a negative transfer is compatible with the former, yet violates the latter. Both of them, however, are a consequence of ‘almost the same’ list of axioms, as Propositions 1 and 2 illustrate.

Proposition 1 If an allocation rule \( \varphi \) satisfies Pareto efficiency, Equal Frontier for Equal Ability and Ability Monotonicity then it satisfies Minimal Self-Ownership of Ability.

Proof,

Let \( \varphi \) satisfy the axioms. Consider \( e_N = (R_N, a_N, g) \in \mathcal{E} \) and let \( e'_N = (R_N, a'_N, g) \in \mathcal{E} \) be such that \( a'_N = (a, a, \ldots, a) \), where \( a \equiv \min a_N \). Suppose \( z_N \in \varphi(e_N) \). By PE and EFSA, for any \( z'_N \in \varphi(e'_N) \) it must be \( z'_i \in m(R_i, OS(\min a_N, 0)) \), for each \( i \in N \). By AM, for each \( i \in N \), \( z_i R_i z'_i \). Hence, we have recovered MSOA.

To recover Limited Self-Ownership of Ability we need to enrich the list of axioms in Proposition 2. In particular, we will appeal to Replication Invariance, which requires that the prescriptions of the allocation rule carry through when the economy is replicated. The formal definition follows Maniquet and Sprumont [34].
Theorem 1 Let \( \rho \) be a positive integer. We will say that an economy \( e_{N^\rho} = (R_{N^\rho}, a_{N^\rho}, g) \in \mathcal{E} \) is a \( \rho \)-replica of \( e_N = (R_N, a_N, g) \in \mathcal{E} \) if and only if the following conditions hold:

1. The function \( g \) is the same in both economies,
2. there exists a mapping \( \xi : N^\rho \to N \) such that for each \( i \in N \), \( |\xi^{-1}(i)| = \rho \),
3. for each \( i \in N \) and each \( j \in \xi^{-1}(i) \), \( R_j = R_i \) and \( a_j = a_i \).

Similarly, we may define the \( \rho \)-replica of an allocation. For any \( e_N, e_{N^\rho} \) and some \( z_N \in \varphi(e_N) \), let \( z_{N^\rho} \) be such that \( z_i = z_{j}^\rho \), for each \( i \in N \) and each \( j \in \xi^{-1}(j) \).

Definition 9 An allocation rule \( \varphi \) satisfies Replication Invariance (RI) if and only if for each \( e_N = (R_N, a_N, g) \in \mathcal{E} \) and each \( e_{N^\rho} = (R_{N^\rho}, a_{N^\rho}, g) \in \mathcal{E} \) such that \( e_{N^\rho} \) is a \( \rho \)-replica of \( e_N \)

\[ z_N \in \varphi(e_N) \Rightarrow z_{N^\rho} \in \varphi(e_{N^\rho}). \]

Proposition 2 If an allocation rule \( \varphi \) satisfies Pareto efficiency, Equal Frontier for Equal Ability, Ability Monotonicity and Replication Invariance then it satisfies Limited Self-Ownership of Ability.

Proof. Let \( \varphi \) satisfy the axioms. Suppose that there exists \( e_N = (R_N, a_N, g) \in \mathcal{E} \), \( z_N \in \varphi(e_N) \), \( k \in N \) and \( j \in N \left( \min a_N ; e_N \right) \) such that

\[ m \left( R_k, OS(a_j, t_j(z_N)) \right) P_k z_k. \]

Define \( t^* \in \mathbb{R} \) to be such that

\[ m \left( R_k, OS(a_j, t^*) \right) I_k z_k. \]

By definition, \( t^* < t_j(z_N) \). Define \( e_{N^\rho} = (R_{N^\rho}, a_{N^\rho}, g) \in \mathcal{E} \) to be a \( \rho \)-replica of \( e_N \). We will derive a contradiction for \( \rho \) large enough. Define \( z_{N^\rho}^\rho \) to be such that \( z_i = z_{j}^\rho \), for each \( i \in N \) and all \( j \in \xi^{-1}(i) \). By RI, \( z_{N^\rho}^\rho \in \varphi(e_{N^\rho}) \). Consider \( e_{N''} = (R_{N''}, a_{N''}, g) \in \mathcal{E} \), where \( a_{N''} \) is such that \( a_k'' = \min a_{N''} = a_j \) and \( a_i'' = a_i'' \) for each \( i \neq k \). By EFEA, we may write \( t(z_{N''}'), \xi(z_{N''}') \) to denote the tax liability of the least talented in \( e_{N''}, e_{N''} \) respectively. Let \( \lambda \equiv |N(\min a_{N''} ; e_{N''})| = \rho |N(\min a_N ; e_N)| \). Letting \( N' = N'' - N(\min a_{N''} ; e_{N''}) - \{k\} \), by PE, for each \( z_{N''}'' \in \varphi(e_{N''}) \) we obtain

\[ t_k(z_{N''}^\rho) + \lambda t(z_{N''}^\rho) + \sum_{i \in \tilde{N}} t_i(z_{N''}^\rho) = (\lambda + 1) t(z_{N''}^\rho) + \sum_{i \in \tilde{N}} t_i(z_{N''}^\rho) = 0. \quad (1.5.1) \]

Moreover, by AM, we have that

\[ t_i(z_{N''}^\rho) \geq t_i(z_{N''}^\rho), \quad \text{for each } i \in \tilde{N}. \quad (1.5.2) \]

From (3) and (4) we deduce that

\[ t_i(z_{N''}^\rho) \leq (1 + \frac{1}{\lambda}) t_i(z_{N''}^\rho) - \frac{1}{\lambda} t_k(z_{N''}^\rho). \]
Moreover, by AM, we get

\[ t(z''_{N\rho}) \leq \frac{1}{\lambda} t(z''_{N\rho}) - \frac{1}{\lambda} t_k(z''_{N\rho}). \]

By letting \( \rho \) tend to infinity we obtain \( t(z''_{N\rho}) \simeq t(z''_{N\rho}). \) By RI, since \( t_j(z_N) > t^* \), \( t(z''_{N\rho}) > t^* \), therefore \( t(z''_{N\rho}) > t^* \). This implies that

\[ z''_{kP} m \left( R_k, OS(a_j, t^*) \right) I_k z''_{k}, \]

which violates AM.

\[ \blacksquare \]

### 1.6 Allocation rules

In this section we propose allocation rules and discuss their implications. In the first part we discuss Responsibility-oriented allocation rules at the expense of Compensation. In the second part, we do the opposite. Before proceeding farther, we need to establish the incompatibilities among the axioms listed above which are pertinent to our discussion.

The proof of Propositions 3-5 can be found in the appendix. For some \( g_\circ \in G \), define \( E|g = g_\circ \) to be the sub-domain of \( E \) for which \( g = g_\circ \).

**Proposition 3** For any \( g_\circ \in G \), there exists no allocation rule, defined on \( E|g = g_\circ \), that satisfies Pareto Efficiency, Equal Welfare for Equal Preference and No-Envy among the Equally Able.

Fleurbaey and Maniquet [19] obtain the same incompatibility in the context of a production economy with exogenous marginal productivities. We corroborate their result, although one would have reasonably hoped otherwise. Interestingly enough, the solution of equalizing budget lines, although it violates Pareto Efficiency, has the property of satisfying both Equal Welfare for Equal Preference and Equal Frontier for Equal Ability. Apparently, by endogenizing productivities, because of Pareto Efficiency, we cannot improve on the conclusion of Fleurbaey and Maniquet [19]. As a consequence, equality of opportunity cannot be given a unique axiomatic definition. It needs to be treated as an elastic concept. Any compromise between the two principles of Compensation and Responsibility, whether it leans more towards the one or the other, will be treated as a legitimate stance on the matter.

**Proposition 4** There exists no allocation rule that satisfies Pareto Efficiency, No-Envy among the Equally Able and Ability Solidarity.

Fleurbaey and Maniquet [20] prove that Pareto Efficiency and Ability Monotonicity are incompatible with Responsibility, in a similar model with exogenous productivities\(^{10}\). Proposition 4 is weaker, thus allowing for a potential compromise between weak collective ownership of ability, namely Ability Monotonicity, and Responsiblility axioms.

Finally, the significance of the following impossibility result will become apparent once we set out to investigate Compensation-oriented allocation rules.

**Proposition 5** There exists no allocation rule that satisfies Pareto Efficiency, Equal Welfare for Equal Preference, Ability Monotonicity and Limited Self Ownership of Ability.

\(^{10}\)Contrary to Fleurbaey and Maniquet[20], we assume a constant returns to scale production technology. In that respect we are less general than they are.
1.6.1 Responsibility

The first allocation rule we will discuss is the *laissez-faire* allocation rule, illustrated in figure 1.6.

**Allocation Rule 1** An allocation rule \( \varphi \) is called *laissez-faire* (\( \varphi^f \)) if and only if for each \( e_N \in \mathcal{E} \), each \( z_N \in \varphi(e_N) \), \( z_N \) is Pareto efficient and, moreover, for each \( i \in N \), we have \( t_i(z_N) = 0 \).

The axiomatization of *laissez-faire* allows us to evaluate its ethical viewpoint. Apart from complying fully with Responsibility, it demonstrates that excluding the possibility of re-distribution across individuals with different innate abilities is compatible with *Ability Monotonicity*. Moreover, it satisfies *Weak Consistency*, a robustness axiom. If, for some economy, the prescribed allocation is such that some individuals receive a zero transfer, the axiom requires that after removing these individuals, the bundles assigned to the rest remain unchanged. Thomson [58] reviews the implication of a series of axioms that relate to *Weak Consistency*.

**Definition 10** An allocation rule \( \varphi \) satisfies *Weak Consistency* (WCON) if and only if for each \( e_N \in \mathcal{E} \) and each \( z_N \in \varphi(e_N) \)

\[
\exists M \subseteq N \text{ s.t. } t_i(z_N) = 0, \forall i \in M \Rightarrow z_{N \setminus M} \in \varphi(R_{N \setminus M}, a_{N \setminus M}, g),
\]

*Weak Consistency* plays an instrumental role in isolating *laissez-faire* within a more general class of rules that are founded on the same ethical premise.

**Proposition 6** An allocation rule \( \varphi \) satisfies Pareto efficiency, *Equal Frontier for Equal Ability*, *Ability Monotonicity* and *Weak Consistency* if and only if \( \varphi = \varphi^f \).

**Proof.**

We omit the proof that \( \varphi^f \) satisfies the axioms. Take any \( e_N \in \mathcal{E} \), and any \( z_N \in \varphi(e_N) \). Let \( M \in N \) be such that \( M \cap N = \emptyset \) and \( |M| = |N| \). Let \( e'_{MUN} = (R_{UNM}, a_{NUM}, g) \in \mathcal{E} \) be such that \( a_{NUM} = (a_N, a_N) \) and \( R_{NUM} = (R_N, R_M) \). Moreover, for each \( i \in M \), \( R_i \in R_M \) is represented by \( u_i(c_i, l_i) = c_i - \theta l_i \), \( \theta > 0 \) being sufficiently large. Hence, by PE, for each \( z'_{MUN} \in \varphi(e'_{MUN}) \), for each \( i \in M \) it must be \( l_i = 0 \). By Proposition 1 we may invoke MSOA: any \( z_{MUN} \in \varphi(e'_{MUN}) \) must be such that, for each \( i \in M \), \( t_i(z_{MUN}) \geq 0 \). Hence, by EFSA, for each \( z'_{MUN} \in \varphi(e'_{MUN}) \), for each \( i \in N \) we obtain \( t_i(z'_{MUN}) \geq 0 \). By PE, these facts imply that for any \( z'_{MUN} \in \varphi(e'_{N}) \) we have \( t_i(z'_{MUN}) \geq 0 \). Hence, by WCON, there exists \( z_N \in \varphi(e_N) \) such that \( t_i(z_N) = 0 \), for each \( i \in N \). By AM, if there exists \( z_N \in \varphi(e_N) \) such that \( z_N \neq z'_{MUN} \), it must be that \( z_i z_j \), for each \( i \in N \). If there exists \( j \in N \) such that either \( t_j(z_N) > t_j(z_{MUN}) = 0 \) or \( t_j(z_N) < t_j(z_{MUN}) = 0 \), by PE, we would obtain respectively \( z_j P_j z_j \), \( z_j P_j z_j \), a contradiction. Therefore, for each \( z_N \in \varphi(e_N) \) and each \( i \in N \) we have \( t_i(z_N) = 0 \).
Figure 1.6: Laissez-faire. The economy comprises 4 persons and, moreover, $R_1 = R_3$, $R_2 = R_4$ and $a_1 = a_2 < a_3 = a_4$. Thick lines represent outcome frontiers while thin lines represent indifference curves.

Figure 1.7: Reference preference. The economy comprises 4 persons and, moreover, $R_1 = R_3$, $R_2 = R_4$ and $a_1 = a_2 < a_3 = a_4$. Thick lines represent outcome frontiers while thin lines represent indifference curves.
Responsibility does not in general preclude redistribution. In fact, the main conclusion of this section will be to acknowledge this fact and, in addition, describe precisely the extent of redistribution that Responsibility will tolerate. Before we can do that, we need first to take some intermediate steps.

For each allocation rule \( \varphi \), each economy \( e_N \in \mathcal{E} \) and each allocation \( z_N \in \varphi(e_N) \) we may identify the upper envelope of all outcome frontiers generated by \( z_N \). Formally,

\[
UE(z_N;e_N) = \left\{ (l',c') \in \mathbb{Z} : \text{there exists } j \in N \text{ s.t. } z'_j = (l',c') \in OF(a_j,t_j(z_N)) \right\}
\]

for each \( i \in N \setminus \{j\} \), \( c(l',t_j(z_N);a_j) \geq c(l',t_i(z_N);a_i) \).

The following axiom, *Access to the Collective Frontier*, defends the idea that every individual should be represented in the collective frontier of consumption possibilities. Put differently, every individual should enjoy the possibility of attaining at least one bundle that lies on the upper envelope of outcome frontiers.

**Definition 11** An allocation rule \( \varphi \) satisfies *Access to the Collective Frontier* (ACF) if and only if for each \( e_N \in \mathcal{E} \), each \( z_N \in \varphi(e_N) \) and each \( i \in N \) there exists \( (l',c') \in OF(a_i,t_i(z_N)) \) such that \( (l',c') \in UE(z_N;e_N) \).

*Access to the Collective Frontier* is stronger than *Equal Frontier for Equal Ability*. Thomson [59] explores ideas similar in spirit. If one is to interpret consumption possibilities as opportunities, the axiom makes a lot of sense in terms of equality of opportunity. It should be stressed, however, that such an interpretation is biased towards Responsibility.

Not surprisingly, the *laissez-faire* allocation rule satisfies *Access to the Collective Frontier*. This is because for each \( e_N \in \mathcal{E} \) and each \( z_N \in \varphi^{lf}(e_N) \),

\[
UE(z_N;e_N) = OF(\max a_N, 0).
\]

Moreover, for each \( i \in N \), \((0,0) \in OF(a_i,0)\). Hence *laissez-faire* satisfies *Access to the Collective Frontier*.

In general, an allocation rule \( \varphi \) satisfies *Access to the Collective Frontier* if and only if for each \( e_N \in \mathcal{E} \) and each \( z_N \in \varphi(e_N) \) there exists \( R \in \mathcal{R} \) such that for each pair \( i,j \in N \)

\[
m\left( \tilde{R}, OS(a_i,t_i(z_N)) \right) \tilde{P} m\left( \tilde{R}, OS(a_j,t_j(z_N)) \right).
\]

Sufficiency is straightforward. One can set \( \tilde{R} = UE(z_N,e_N) \). For each \( e_N \in \mathcal{E} \) and each \( z_N \in Z^{|N|} \) we have \( \tilde{R} = UE(z_N,e_N) \in \mathcal{R} \). To demonstrate necessity let us assume, without loss of generality, that for some \( e_N \in \mathcal{E} \), some \( z_N \in \varphi(e_N) \) and some pair \( i,j \in N \),

\[
\text{for each } \tilde{R} \in \mathcal{R}, \ m\left( \tilde{R}, OS(a_j,t_j(z_N)) \right) \tilde{P} m\left( \tilde{R}, OS(a_i,t_i(z_N)) \right).
\]

This implies that \( OS(a_i,t_i(z_N)) \subseteq OS(a_j,t_j(z_N)) \). Therefore, for each \((l',c') \in OS(a_i,t_i(z_N))\) we obtain \((l',c') \notin UE(z_N;e'_N)\).

We will refer to the family of rules satisfying *Access to the Collective Frontier* as the family of *Reference Preference Egalitarian Equivalent* rules, denoted \( \varphi^R \) (refer to figure 1.7 for an illustration). Notice that for a rule to belong to this family all that is required is that in each economy the selected allocation equalizes welfare from the point of view of some arbitrary reference preference. This reference preference may be either allowed to vary with the economy or set independently of it. For instance, to recover the *laissez-faire*
rule one needs to set the reference parameter equal to the preference that is infinitely averse to labour in each economy. So far we have focused our attention on a family of rules that encompass responsibility in a strong sense. Now we may raise the question: How much redistribution will a rule in this family tolerate?

It turns out that the extent of redistribution that an allocation rule satisfying Access to the Collective Frontier permits will depend on the economy and, in particular, on the preference profile. There is a relation $\succ$, to be read ‘more averse to labour than’, which underlies the set $\mathcal{R}$. It is transitive, yet not complete, as we have not made an assumption on the domain of preferences that will guarantee us a single crossing property in the leisure-consumption space.

**Theorem 2** For each $R, R' \in \mathcal{R}$ we will write $R \succ R'$ if and only if for each $a \in \mathbb{R}_+$, each $T \in \mathbb{R}$, each $(l, c) \in m(R, OS(a, T))$ and each $(l', c') \in m(R', OS(a, T))$ we have $l' \geq l$.

Given an economy $e_N \in \mathcal{E}$ we will generally be able to rank some individuals according to their aversion to labour. Let, for each $e_N \in \mathcal{E}$, $\mathcal{L}(e_N) = \{R \in R_N : \exists R' \in R_N \text{ such that } R' \succ R\}$. This is the set of preferences that exhibit the highest degree of aversion to labour in the profile $R_N$. The set $\mathcal{L}(e_N)$ is non-empty and its cardinality will range from 1 to $|N|$, depending on the economy.

**Proposition 7** An allocation rule $\varphi \in \varphi^\mathcal{R}$ satisfies Pareto Efficiency, Ability Monotonicity and Replication Invariance if and only if the reference parameter $\tilde{\varphi}$ is such that for each $e_N \in \mathcal{E}$, either $\tilde{R} \in \mathcal{L}(e_N)$ or $\tilde{R} \succ R'$ for each $R' \in \mathcal{L}(e_N)$.

**Proof.**

Step 1. If $\varphi$ satisfies the axioms then for each $e_N = (R_N, a_N, g) \in \mathcal{E}$, each $z_N \in \varphi(e_N)$ and each $j, k \in N$ such that $a_j = \min a_N$ and $a_k > \min a_N$ it must be

$$m(R_j, OS(a_k, t_k(z_N))) \succ R_j z_j.$$  

Suppose not. Suppose, in particular, that $\varphi$ satisfies the axioms and let there exist $e_N \in \mathcal{E}$, $z_N \in \varphi(e)$ and $j, k \in N$, with $a_j = \min a_N$ and $a_k > \min a_N$, such that

$$z_j P_j m(R_j, OS(a_k, t_k(z_N))).$$  

Let $e^\rho_{N^\rho} = (R_{N^\rho}, a^\rho_{N^\rho}, g) \in \mathcal{E}^{N^\rho}$ be a $\rho$-replica of $e_N$. Define $z^\rho_j = z^\rho_j$ for each $i \in N$ and each $j \in \mathcal{E}^{-1}(i)$. By RI, $z^\rho_N \in \varphi(e^\rho_{N^\rho})$. We will derive a contradiction for $\rho$ large enough. Let $e^\lambda_{N^\rho} = (R_{N^\rho}, a^\lambda_{N^\rho}, g) \in \mathcal{E}$ be such that $a_j = a_k$ and for each $i \in N^\rho - \{j\}$, $a_j = a_i$. Appealing to EFEA (it is implied by ACF) we will write $t(a; \tilde{z}_N)$ to denote the tax liability of an individual with innate ability equal to $a$ associated with any allocation $\tilde{z}_N$. Let $\lambda \equiv |N(a_k; e^\rho_{N^\rho})| = \rho|N(a_k; e_N)|$. Let $N^\rho = N^\rho - N(a_k; e^\rho_{N^\rho}) - \{j\}$. By PE, for each $z^\rho_{N^\rho} \in \varphi(e^\rho_{N^\rho})$ we obtain

$$t_j(z^\rho_{N^\rho}) + \lambda t(a_k; z^\rho_{N^\rho}) + \sum_{i \in N} t_i(z^\rho_{N^\rho}) = (\lambda + 1)t(a_k; z^\rho_{N^\rho}) + \sum_{i \in N} t_i(z^\rho_{N^\rho}).$$  

Moreover, by AM, it must be

$$\sum_{i \in \hat{N}} t_i(z^\rho_{N^\rho}) \leq \sum_{i \in \hat{N}} t_i(z^\rho_{N^\rho}).$$  

(1.6.3)
We now turn to the other extreme and place full weight on compensation. We consider individuals who are averse to labour, in such an economy we cannot escape a preference for the reference parameter over any other. Finally, to be linear in order to facilitate the exposition. The two individuals have the same abilities. Consider some economy \( e \in E \) and any two reference preferences \( \bar{R}, \bar{R}' \in \mathcal{R} \) such that \( \bar{R} \succ \bar{R}' \succ R \), for each \( R \in \mathcal{L}(e_N) \). For any \( z'_N \in \varphi(\bar{R}')(e_N) \), any \( z''_N \in \varphi(\bar{R}')(e_N) \) it will be \( \sum_{i \in N} \max\{0, t_i(z'_N)\} \geq \sum_{i \in N} \max\{0, t_i(z''_N)\} \). This is illustrated in figure 1.8, where we depict a simple two-person economy. The reference parameters \( \bar{R}', \bar{R}'' \) are chosen to be linear in order to facilitate the exposition. The two individuals have the same preferences which, in addition, exhibit no aversion to labour. Finally, \( a_2 > a_1 \). One can see that with \( \bar{R}' \) more welfare is redistributed to the least able compared to \( \bar{R}'' \).

Another noteworthy point is that the reference parameter can be chosen in a way that makes the rule accommodate some very mild form of Compensation. If everyone in the economy had the same preferences, welfare would ideally be equalized.

**Definition 12** An allocation rule \( \varphi \) satisfies Equal Welfare for Uniform Preference if and only if for each \( e_N \in E \) and each \( z_N \in \varphi(e_N) \)

\[ R_i = R_j, \text{ for each } i, j \in N \Rightarrow z_i I_i z_j, \text{ for each } i, j \in N. \]

In the example of figure 1.8 the economy is such that \( R_1 = R_2 = L(e_N) \). In any such economy, in order to satisfy Equal Welfare for Uniform Preference, we need to have \( R = L(e_N) \).

Consequently, for the reasons offered above, it seems desirable to condition the reference preference on the economy. There are two conclusions to be drawn. First, some \( R \in \mathcal{L}(e_N) \) is a natural candidate for the reference parameter. The least able individuals in the economy prefer that choice for the reference parameter over any \( \bar{R} \notin \mathcal{L}(e_N) \). Second, the economy may not allow for much redistribution at all. If all preferences in \( \mathcal{L}(e_N) \) are very averse to labour, in such an economy we cannot escape laissez-faire.

We have thus concluded with the view of equality of opportunity that leans towards responsibility. We now turn to the other extreme and place full weight on compensation.
1.6.2 Compensation

The first allocation rule we propose is in the tradition of Pazner and Schmeidler [49]. It selects allocations that would be envy-free if everyone in the society had the same innate ability (see figure 1.9 for an illustration).

**Allocation Rule 2** An allocation rule \( \varphi \) is a reference ability egalitarian equivalent rule \( (\varphi^a) \) if and only if for each \( e \in \mathcal{E} \), each \( z \in \varphi(e) \), \( z \) is Pareto efficient and, moreover, for each \( i \in N \) there exists \( \tilde{a} \in \mathbb{R}_+ \) such that

\[
\tilde{z}_i \sim m\left(R_i, OS(\tilde{a}, 0)\right).
\]

Existence follows from Assumption 1, clause (b). The characterization relies on the following axiom. It states that if an allocation rule proposes a certain allocation, it should also propose all other allocations that are welfare equivalent.

**Definition 13** An allocation rule \( \varphi \) satisfies Pareto Indifference (PE) if and only if for each \( e \in \mathcal{E} \), each \( z \in \varphi(e) \) and each \( z' \in Z(e) \), if

\[
\text{for each } i \in N, \; z_i \sim z'_i \Rightarrow z'_N \in \varphi(e).
\]

The proof below relies heavily on Fleurbaey [18]11.

**Proposition 8** An allocation rule \( \varphi \) satisfies Pareto efficiency, Ability Solidarity, Minimal Self-Ownership of Ability and Pareto Indifference if and only if \( \varphi = \varphi^a \).

**Proof.**

Let an allocation rule \( \varphi \) satisfy the axioms. Unless the following statement is true, the reference ability egalitarian equivalent rule is ill defined: For each \( e = (R_N, a_N, g) \in \mathcal{E} \),

11I refer the reader to Proposition 4.6 on page 124. Fleurbaey uses Equal Transfer for Uniform Skills (equal abilities in our setting). This axiom is implied by the combination of PE and MSOA.
each \( z_N \in \varphi(e_N) \), there exists \( \tilde{a} \in [\min a_N, \max a_N] \) such that for some \( \tilde{z}_N \in \varphi^{\tilde{a}}(R_N, (\tilde{a}, \ldots, \tilde{a}), g) \) and some \( z_N^P \) that is Pareto efficient in economy \( e_N \), we have

for each \( i \in N \), \( z_i I_i z_i^P \).

By PE and MSOA, if \( z'_N \in \varphi(R_N, (\tilde{a}, \ldots, \tilde{a}), g) \), then

for each \( i \in N \), \( z'_i \in m(R_i, OS(\tilde{a}, 0)) \).

Therefore, if \( z'_N \in \varphi(R_N, (\tilde{a}, \ldots, \tilde{a}), g) \), then \( z'_N \in \varphi^{\tilde{a}}(R_N, (\tilde{a}, \ldots, \tilde{a}), g) \). We need to show that the converse is also true. If \( z_N^k \in \varphi^{\tilde{a}}(R_N, (\tilde{a}, \ldots, \tilde{a}), g) \) by construction we have

for each \( i \in N \), \( z_N^k I_i m(R_i, OS(\tilde{a}, 0)) \).

Therefore, by PI, if \( z_N^k \in \varphi^{\tilde{a}}(R_N, (\tilde{a}, \ldots, \tilde{a}), g) \), then \( z_N^k \in \varphi(R_N, (\tilde{a}, \ldots, \tilde{a}), g) \). To sum up: \( \varphi^{\tilde{a}}(R_N, (\tilde{a}, \ldots, \tilde{a}), g) = \varphi(R_N, (\tilde{a}, \ldots, \tilde{a}), g) \).

We will now show that if \( z_N \in \varphi(e_N) \), then

for each \( i \in N \), \( z_i I_i z_i^P \).

Suppose not. Without loss of generality let there exist \( j \in N \) for whom \( z_j P_j z_j^p \). By PE, there must also exist \( k \in N \), for whom \( z_k^P P_k z_k \). This contradicts AS. Thus, applying PI as before, we obtain \( \varphi(e_N) = \varphi^{\tilde{a}}(e_N) \).

A factor to take into account is that Ability Solidarity is compatible with Minimal Self-Ownership of Ability even when our priority is Compensation. In view of Proposition 5, if one is to favor Limited Self-Ownership of Ability, one should have to give up on Ability Monotonicity. This possibility is presented below (See figure 1.10 for an illustration).

**Allocation Rule 3** An allocation rule \( \varphi \) is a min\( a_N \)-equivalent rule (\( \varphi^2 \)) if and only if for each \( e_N \in \mathcal{E} \), each \( z_N \in \varphi(e_N) \), \( z_N \) is Pareto efficient and, moreover, for each \( i \in N \) and each \( j \in N(\min a_N; e_N) \)

\[
\text{for each } i \in N, z_i I_i m(R_i, OS(a_j, t_j(z_N))).
\]

In particular, in abandoning solidarity-type axioms while still insisting on full compensation, we may satisfy further libertarian objectives. Policy is often focused on the most disadvantaged individuals. Pareto Efficiency and Limited Self-Ownership of Ability are natural limits to the resources that can be directed to the least able.

**Proposition 9** An allocation rule \( \varphi \), that belongs to the class of allocation rules satisfying Pareto efficiency and Limited Self-Ownership of Ability, has the property of maximizing, for each \( e_N \in \mathcal{E} \), the welfare of the least able, if and only if \( \varphi = \varphi^{\tilde{a}} \).

\[\text{12} \text{By this I mean that } \tilde{z}_N \text{ is proposed by the reference ability egalitarian equivalent rule with reference parameter } \tilde{a}.\]
Proof.
Suppose, first, that $\varphi$ satisfies the axioms and $\varphi \neq \varphi^E$. This implies that there exists $e_N \in \mathcal{E}$, $z \in \varphi(e_N)$, $k \in N$ and $j \in N \setminus \{\min a_N; e_N\}$ such that
\[
zk P_k m\left( R_k, OS(\min a_N, t_j(z_N)) \right).
\]
Moreover, by LSOA, we have that for each $i \in N$, $t_i(z_N) = t_j(z_N)$. Hence, $k \in N - N(\min a_N; e_N)$. There exists a Pareto efficient allocation $z_N'$ compatible with the vector of lump-sum transfers $t(z'_N)$, where $t_k(z'_N) = t_k(z_N) - \epsilon$ and $t_i(z'_N) = t_i(z_N) + \frac{\epsilon}{|N| - 1}$, for each $i \in N$. For $\epsilon$ positive and small enough we have that $z_N'$ respects LSOA. In addition, for each $i \in N(\min a_N; e_N)$, we obtain $z_i^t P_i z_i$. Therefore, $\varphi$ does not maximize the welfare of the least able in the economy $e_N$.

Conversely, suppose that there exists $e_N \in \mathcal{E}$, $z \in \varphi^E(e_N)$, $z'_N \in \varphi(e_N)$ and $k \in N \setminus \{\min a_N; e_N\}$ such that $z'_N P_k z_k$. By PE, this implies that $t_k(z'_N) > t_k(z_N)$. By LSOA, $t_k(z'_N) = t_k(z_N)$ and $t_k(z_N) = t_i(z_N)$ for each $i \in N$. By PE, there exists $j \in N - N(\min a_N; e_N)$ such that $t_j(z'_N) < t_j(z_N)$. By definition
\[
z_j I_j m\left( R_j, OS(\min a_N, t_k(z_N)) \right).
\]
Hence,
\[
m\left( R_j, OS(\min a_N, t_k(z'_N)) \right) P_j z_j',
\]
which constitutes a violation of LSOA. Therefore, the welfare of the least able in $z_N$ is the maximum the axioms will allow for.

No allocation rule we reviewed in this section precludes allocations that transfer resources from the untalented to the talented. We demonstrate this in two examples depicted in figures 1.9 and 1.10. The reference ability egalitarian equivalent rule is depicted on the left. The reference parameter is denoted $\tilde{a}$. The portrayed economy consists of three individuals. One may notice that although $a_2 > a_1$ the allocation rule prescribes $t_1(z_1, z_2, z_3) = 0 < t_2(z_1, z_2, z_3)$. On the right of figure we deal with the min $a_N$-equivalent rule. Once more, although $a'_3 > a'_2 > a'_1$ we obtain $t_3(z'_1, z'_2, z'_3) > 0$ while $t_2(z'_1, z'_2, z'_3) < 0$.

Figure 1.9: Reference innate ability
Another noteworthy feature of these allocation rules is the way in which they treat hard-working individuals. Any generic allocation they propose induces a partition of the set of agents into contributors and receivers. Of two receivers with the same talent, the one who is more prone to labour will end up receiving more. As investment in human capital *ceteris paribus* decreases with the laziness of one’s preference, under Assumption 1, investment in human capital is rewarded. Surprisingly, among the contributors and within the same talent level the effect is reversed. If one is a highly talented individual and thus a contributor, one will envy an individual with the same innate ability if and only if he is more averse to labour. The view of equality of opportunity that is based on Compensation rewards laziness among contributors and punishes it among receivers.

1.7 Concluding Remarks

In our analysis we assumed full information. This served to expose the implications of Pareto efficiency. It leaves a distinct mark on all our results. Investment in human capital needs to be determined by the profile of preferences and innate abilities. This is what Pareto efficiency dictates. A policy, for instance, that would aim at guaranteeing all individuals a minimum positive amount of investment in human capital would clash with efficiency. To see this, consider an individual who is infinitely averse to labour. Any amount spent in building earning capabilities for that particular individual is an amount wasted. The broader conclusion is that differences in innate abilities will translate into differences in marginal productivities. The concept of Equality of Opportunity we put forward acknowledges this premise.

The allocation rules we characterized rationalize policy, in the form of transfers and, in particular, associate it with ethical principles. The Responsibility-minded family of allocation rules $\varphi^R$ we presented sheds light on both the scope and the extent of redistribution. The amount of resources transferred in the direction of the less talented is profile-dependent. Still, there exist $\varphi \in \varphi^R$ that justify public intervention as opposed to *laissez-faire*. On the Compensation side, the rules we propose strive to make a subset of the consumption space equally accessible to all. We provide two solutions that accomplish this and discriminate between them on a moral basis. We summarize our results in the table below.
The question that yet confronts us is how these results carry through to the second best analysis. Let us imagine that innate abilities and investment in human capital are observable, whereas preferences are not. There exists a promising strand of literature that has generalized allocation rules into social welfare orderings. Unlike an allocation rule that selects the best social outcomes in a certain domain, a social welfare ordering ranks all the elements of the domain. Therefore, orderings can be maximized under incentive compatibility constraints (Fleurbaey and Maniquet [21] discuss income taxation from this perspective in a Mirrlees model with preference heterogeneity and exogenous skills). The whole exercise is interesting for the additional reason that investment profiles, if assumed to be observable, become a parameter of important informative value. Needless to say, the second best analysis will provide richer policy recommendations.

1.8 Appendix
Thus, NEEA is contradicted.

Figure 1.12: Pareto efficiency, No-Envy among the Equally Able and Ability Solidarity are incompatible.

Proof of Proposition 3. We present a counter-example. Fix $g_0 \in \mathcal{G}$ and suppose that $\varphi$, defined on $\mathcal{E}|_{g=g_0}$, satisfies the axioms. Consider $e_N = (R_N, a_N, g_0) \in \mathcal{E}|_{g=g_0}$, where $N = \{1, 2, 3, 4\}$, $a_1 = a_3 < a_2 = a_4$ and preferences are homothetic. In addition, for some $\alpha > \gamma \geq 0$, $\beta > \delta \geq 0$ and $d > 0$, the utility representation of preferences is given by

$$u_i(c_i, l_i) = c_i - \begin{cases} \alpha l_i & \text{if } c_i \leq d(1 - l_i) \\ \beta l_i & \text{if } c_i > d(1 - l_i) \end{cases}$$

for $i = 1, 2$, while

$$u_i(c_i, l_i) = c_i - \begin{cases} \gamma l_i & \text{if } c_i \leq d(1 - l_i) \\ \delta l_i & \text{if } c_i > d(1 - l_i) \end{cases}$$

for $i = 3, 4$. Given the fact that the outcome frontier is increasing and convex, for any $g \in \mathcal{G}$, a menu of parameters $(d, \alpha, \beta, \gamma, \delta)$ exists such that, by PE, for each $e_N \in \mathcal{E}|_{g=g_0}$ and each $z_N \in \varphi(e_N)$, $l_i = 0$ for $i = 1, 3$ and $l_i = \frac{\alpha - d}{d}$ for $i = 2, 4$. Suppose that $R_N$ is constructed so as to accomplish this effect. For each $z_N \in \varphi(e_N)$, by EWEP, we need to have $z_1 l_1 z_2$ for $i \in \{1, 2\}$ and $z_3 l_3 z_4$ for $i \in \{3, 4\}$. Figure 1.11 depicts such an allocation. This generates a case of ‘double envy’ among individuals with the same ability. Refer to figure 1.11: $z_2 P_4 z_4$ and $z_3 P_1 z_1$. Hence, by EWEP and PE, for each $z_N \in \varphi(e_N)$ it can be either $t_1(z_N) = t_3(z_N)$ and $t_2(z_N) > t_4(z_N)$ or $t_2(z_N) = t_4(z_N)$ and $t_1(z_N) < t_3(z_N)$. Thus, NEEA is contradicted.

Proof of Proposition 4. We present a counter-example. Suppose that an allocation rule $\varphi$ satisfies the axioms. Let $N = \{1, 2, 3\}$ and consider $e_N = (R_N, (\tilde{a}, \tilde{a}, \tilde{a}), g) \in \mathcal{E}$ such that $R_i$ is quasi-linear and, in addition, for some $0 < \tilde{\alpha} < \tilde{\gamma} < \tilde{\delta} < \tilde{\beta}$, the utility representation of $R_i$ is given by:

$$u_i(c_i, l_i) = c_i - \begin{cases} \hat{\alpha} l_i & \text{if } l_i \leq \tilde{l} \\ \hat{\beta} l_i & \text{if } l_i > \tilde{l} \end{cases}$$
As shown, \( z \) depicted in figure 1.13 belongs to frontiers. We will first demonstrate that if the curvature of the outcome frontier decreases in \( a \). Hence, PE dictates that for each \( z_N \in \varphi(e_N) \), \( t_1 = 0, t_2 = 1 \) and \( t_3 = 1 \). The allocation \( z_N' = (z_1', z_2', z_3') \), depicted in figure 1.12, is Pareto efficient in \( e_N' \), with \( t_4(z_N') = 0 \), for each \( i \in N \). Moreover, by construction, \( ||z_1' - z_1'|| + ||z_2' - z_2'|| = ||\tilde{z} - z_3'|| \). By AS and the fact that PE requires balanced transfers, for any \( z_N \in \varphi(e_N') \) it must be \( t_1(z_N) = ||z_1' - z_1'||, t_2(z_N) = ||z_2' - z_2'|| \) and \( t_3(z_N) = ||\tilde{z} - z_3'|| \). Therefore \((z_1', z_2', \tilde{z}_N') \in \varphi(e_N') \). This violates NEEA: \( z_1''P_2z_2'' \).

\[ u_i(c_i, l_i) = c_i - \begin{cases} \tilde{l}_i & \text{if } l_i \leq \tilde{l} \\ \delta \tilde{l}_i & \text{if } l_i > \tilde{l} \end{cases} \]

for \( i = 2 \). Refer to figure 1.12. Thick lines represent \( R_1 = R_3 \), while thin \( R_2 \). The choice of parameters is such that for each \( z_N \in \varphi(e_N) \), by PE, \( l_i = \tilde{l} \) for each \( i \in N \). Therefore, by NEEA \((\tilde{z}, \tilde{z}, \tilde{z}) = \varphi(e_N)\). Let \( e_N' = (R_N, a_N', g) \in \mathcal{E} \), with \( a_1' = a_2 < \tilde{a} \) and \( a_3' > \tilde{a} \). Figure 1.12 illustrates the outcome frontiers with dotted curves. The function \( g \) is such that the curvature of the outcome frontier decreases in \( a \).

\[ t_1(z_N') > t_1(z_N) \text{ then by EWEP } t_2(z_N') > t_2(z_N). \]

The assumption that \( t_1(z_N') > t_1(z_N) \) implies, by LSOA, that also \( t_4(z_N') > t_4(z_N) \) which finally entails, by EWEP, that \( t_3(z_N') > t_3(z_N) \). This violates PE. Suppose then that \( t_1(z_N') < t_1(z_N) \).

By EWEP, \( t_2(z_N') < t_2(z_N) \). This implies, by LSOA, that \( t_3(z_N') < t_3(z_N) \) which finally entails, by EWEP, that \( t_3(z_N') < t_4(z_N) \). Once more this violates PE. It must be than that \( t_1(z_N') = t_1(z_N) \). By EWEP, \( t_2(z_N') = t_2(z_N) \). Also by EWEP, \( t_3(z_N') = t_4(z_N') \).

**Proof of Proposition 5.** Consider the economy \( e_N = (R_N, a_N, g) \) depicted in figure 1.13. As shown, \( N = \{1,2,3,4\} \). Moreover, \( R_1 = R_2 \) (quasi-linear preferences) and \( R_3 = R_4 \) (linear preferences). Finally, the vector of innate abilities is such that \( a_1 = a_3 < a_2 = a_4 \).

In the figure thin lines represent indifference curves, while thick lines represent outcome frontiers. We will first demonstrate that if \( \varphi \) satisfies the axioms then the allocation \( z_N \) depicted in figure 1.13 belongs to \( \varphi(e_N') \). Suppose not. Let there exist some other allocation \( z_N' \in \varphi(e_N') \). If \( t_1(z_N') > t_1(z_N) \) then by EWEP \( t_2(z_N') > t_2(z_N) \). The assumption that \( t_1(z_N') > t_1(z_N) \) implies, by LSOA, that also \( t_4(z_N') > t_4(z_N) \) which finally entails, by EWEP, that \( t_3(z_N') > t_3(z_N) \). This violates PE. Suppose then that \( t_1(z_N') < t_1(z_N) \).

By EWEP, \( t_2(z_N') < t_2(z_N) \). This implies, by LSOA, that \( t_3(z_N') < t_3(z_N) \) which finally entails, by EWEP, that \( t_3(z_N') < t_4(z_N) \). Once more this violates PE. It must be than that \( t_1(z_N') = t_1(z_N) \). By EWEP, \( t_2(z_N') = t_2(z_N) \). Also by EWEP, \( t_3(z_N') = t_4(z_N') \). By PE,
since \( t_1(z_N) = t_2(z_N) = t_1(z'_N) = t_2(z'_N) \) and \( t_3(z_N) = t_4(z_N) \), \( t_3(z'_N) = t_4(z'_N) \). By construction, for each \( i \in N \) and each \( t \in \mathbb{R} \) the bundle that maximizes \( R_i \) in \( OS(t,a_i) \) is unique. Thus, \( z_N = z'_N \).

Consider now \( c''_N = (R_N, a''_N, g) \), where \( a_1 < a''_1 = a''_3 < a_2 \) and \( a_i = a''_i \) for each \( i = 2, 4 \). In the figure \( O F(a''_1,0), O F(a''_1,t(z_1)) \) are represented by the dotted curves. The construction is such that \( z_4 \) lies on \( O F(a''_1,t(z_1)) \). By AM, \( t(z''_4) \geq t(z_4) \). By AM, given the construction, \( t(z''_4) \geq t(z_3) \). By LSOA, \( t(z''_3) = t(z''_1) \), hence \( t_1(z''_N) \geq t(z_1) \). This latter observation implies, by EWEP, that \( t(z''_2) > t(z_2) \). Combining the inequalities one obtains a contradiction of PE.
Chapter 2

On Sharing the Benefits of Communication

2.1 Introduction

An agent is associated with one of two platforms. Communication between two agents requires that they operate on a common platform. Adoption of a new platform is costly. The cost depends on the agent’s native platform. The benefit of communication depends on a subjective parameter reflecting the value the agent attaches to the collaborative enterprises she may engage in by adopting a new platform. The benefit of communication is increasing in the number of agents one may interact with.

A variety of situations fit this paradigm. Historically, traveling by train between France and Spain required switching trains at the border. In 2010 the completion of an alternative high speed line, operating on the French standard, solved the break of gauge problem. The development of the DVD was made possible only when Philips and Sony abandoned the "Multimedia Compact Disc" format and joined the camp of Toshiba, Time Warner, Pioneer and others developing the "Super Density Disc" format. Finally, to cite a case involving numerous agents, communication among individuals belonging to different language groups in places such as Belgium, Canada and Florida makes necessary the adoption of a foreign language.

The model we propose is the simplest one that adheres to these observations. We build on Selten and Pool [54]. Henceforth, we will use the term language to denote a mode of operation and the term communication to denote the possibilities of interaction that speaking a common language affords. Refer to Figure 2.1. Agents are represented by nodes. The set of agents in partitioned in two language groups. Agent $j$ speaks language $\alpha$ natively. Learning is depicted by an arrow stemming from a node and pointing to a
set of nodes. Individual $j$ learns language $\beta$. This enables her to communicate with three agents speaking $\beta$. Agent $j$’s benefit of communication is the number of ‘foreign’ agents she communicates with, multiplied by a non-negative real number that encompasses the agent’s willingness to communicate. The cost agent $j$ faces is determined by the language group she belongs to. The agent’s net benefit is the difference between the benefit and the cost of communication.

The model depicted in Figure 2.1 will be the backdrop against which we will formulate our discussion. We are interested in mechanisms. Roughly speaking, these objects associate a social outcome to the various values the primitives of the model may take. In particular, our main concern will be to examine the extent to which the following four properties can be attributed to a mechanism:

1. **Assignment Efficiency**: the sum of net benefits should be maximized.

2. **Strategy-Proofness**: all agents, if asked, should have a dominant strategy to reveal their willingness to communicate truthfully.

3. **Individual Rationality**: no agent should enjoy a level of well being that is lower than the level of well being she would enjoy if she was not a constituent of the economy.

4. **Feasibility**: the mechanism should rely exclusively on the resources generated within the economy, i.e. no outside funding should be permitted.

It turns out that no mechanism satisfies all of the above requirements. There are, however, mechanisms that satisfy any three of the properties above. We place particular emphasis on Assignment Efficiency and Strategy-Proofness. Both these properties are shown to be inherently linked with incentives. A mechanism that violates them is prone to deficiencies that undermine the implementation exercise in a fundamental way.

Appealing to a result due to Holmström [30], embracing Assignment Efficiency and Strategy-Proofness entails confining our investigation to the family of Groves mechanisms. The literature discussing such mechanisms does so in three fairly distinct contexts. Pure public goods, excludable public goods and private goods.

In this paper we study Groves mechanisms in a context of private goods that accounts for the effect of an externality. It turns out that much of the conventional wisdom on Groves mechanisms does not carry through to our model. An instance of this phenomenon is the impossibility we cite in the previous paragraph. When all agents value the good positively (as is the case in our framework), the celebrated Pivotal mechanism constitutes an example of a Groves mechanism that satisfies the aforementioned properties whether the context is private or public goods\(^1\). More surprisingly, the Pivotal mechanism here fails feasibility. The nature of the externality we capture in our model causes the Pivotal mechanism to sometimes assign positive transfers, something that is disallowed in the framework of either public or private goods.

Our proposal involves two mechanisms. First we look at individually rational Groves mechanisms. We show that such mechanisms are often in deficit. We characterize the mechanism that minimizes the deficit whenever it occurs. Second, we single out the only feasible Groves mechanism that is not Pareto dominated by another strategy-proof, feasible and anonymous mechanism. In order to do so we restrict our attention to economies comprising two agents. Both our proposals involve rules that do not appear elsewhere in the literature and are thus specific to the model we put forward.

\(^1\)Parkes [48] provides sufficient conditions for a Pivotal mechanism to be individually rational in a pure public good framework.

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Effectively the objective we pursue is to identify the best in class mechanism. When feasibility is out of the picture the criterion that isolates the best mechanism is related to the incidence of the deficit. We do not focus on the worst case scenario (as in Bailey [4], Cavallo [9], Guo and Conizer [26], Moulin [45] ) or on the asymptotic behavior of the deficit (as in Deb Razzolini and Seo [14], Green and Laffont [25], McAfee [36] and Zhou [66]). Rather we propose a mechanism that runs a lower deficit than any other mechanism in each economy where the deficit presents itself.

In order to isolate the best feasible Groves mechanism, we do not base our selection on sums of utilities but rather on their distribution (as in Athanasiou [2]). A mechanism Pareto dominates another one if the former generates, in each economy and for each agent, a higher amount of utility. This criterion turns out to be sharp enough to select a unique feasible Groves mechanism when the discussion is confined to two-agent economies.

The analysis is complemented by a discussion of the discrete nature of the problem. In a separate section, we provide an algorithm that identifies efficient linguistic assignments. Interestingly, at the optimum learning by agents of both linguistic groups (two-sided learning) may ensue. Moreover, efficient linguistic assignments do not necessarily impose full communication among agents.

Aside from the agenda we pursue in this paper, a parallel literature deals with decentralized outcomes that may arise in situations similar to the ones we explore. In their seminal contribution Selten and Pool [54] introduce a general model of language acquisition. They show that an equilibrium of the multi-country multilingual language acquisition model exists. The characterization of an equilibrium is then studied by Church and King [10]. Recently, Ginsburgh et al. [24] and Gabszewicz et al. [23] study qualitative properties of such equilibria in the context of bilingual societies. In our model one may rationalize different Nash Equilibria, exhibiting both one-sided as well as multi-sided learning. However, the efficient outcome does not generically come about as a Nash Equilibrium.

Section 2 introduces the model. Section 3 discusses efficiency. Section 4 introduces the axioms and presents the impossibility. Section 5 discusses individually rational Groves mechanism. Section 6 discusses feasible Groves mechanisms. Section 7 concludes.

2.2 The model

The finite set of agents is denoted \( N \subseteq \mathbb{N} \). There are only two languages, \( \alpha \) and \( \beta \). Each agent speaks originally only one of them, for each \( \lambda \in \{ \alpha, \beta \}, N^\lambda \) is the set of agents speaking language \( \lambda \) so that \( N = N^\alpha \cup N^\beta \). Each agent \( i \in N \), belonging either to \( N^\alpha \) or \( N^\beta \), might take the action of learning the language spoken by the other group or not. This is represented by the dichotomous variable \( l_i \in \{ 0, 1 \} \), namely, \( l_i = 1 \) if she learns, \( l_i = 0 \) otherwise. Let \( l_N = (l_i)_{i \in N} \in \{ 0, 1 \}^N \) denote the vector describing the actions taken by each member of the population. The function

\[
\gamma : N \rightarrow \{ \alpha, \beta \}, \text{ such that for each } \lambda \in \{ \alpha, \beta \} \text{ and each } i \in N, \gamma(i) = \lambda \leftrightarrow i \in N^\lambda
\]

determines the native language of each agent. For each \( i \in N \), let \( N^\gamma(i) \) denote the set of agents whose native language is the same as \( i \). The amount of effort that one needs to exert in order to learn a foreign language depends on her native tongue. The vector \( C = (c_\alpha, c_\beta) \in \mathbb{R}^2_+ \) provides this information. In particular, for each \( i \in N \), \( c_\gamma(i) \) is the learning cost if \( i \)'s native language is \( \gamma(i) \). Our only further assumption is that \( c_\alpha, c_\beta < \infty \).

At a given \( l_N \in \{ 0, 1 \}^N \), the benefit one derives from being able to communicate with agents from the other language group is simply a linear function of the number of agents one can communicate with as a result of the overall linguistic achievement. The marginal
willingness to communicate is measured, for each agent, by the parameter $\theta_i \in \mathbb{R}_+$. Hence, for each $i \in N$ and each $l_N \in \{0, 1\}^N$, the expression

$$\theta_i \sum_{j \in N \setminus N^{(i)}} \min \{1, (1, l_i) \cdot (l_j, 1)\}$$

(2.2.1)

specifies the gross benefit from communication of agent $i$. Let $\theta_N^\alpha \equiv (\theta_i)_{i \in N^\alpha}$, $\theta_N^\beta \equiv (\theta_i)_{i \in N^\beta}$ and $\theta_N \equiv (\theta_i)_{i \in N}$. An economy is denoted by $e = ((\theta_N^\alpha, \theta_N^\beta), C) = (\theta_N, C) \in \mathbb{R}^{N+2}_+ \equiv \mathcal{E}$.

It is important to stress that there is not inherent value in the linearity assumption. Assuming a decreasing marginal benefit from communication would not alter the nature of our results but it would come at the price of tedious calculations. Moreover, any mathematical difficulty the model presents would remain, even if we were to apply a convexification, namely letting $l_i$ take values in an interval, say $[0, 1]$. In fact, the discrete nature of the problem proceeds from the min operator (equation 2.1). Even if two agents share more than one language, the benefit each derives from communicating with the other remains unchanged relative to case of communication through a single language. Therefore, even if the parameter $l_i$ was continuous, the problem of maximizing social welfare would have to be solved under multiple constraints.

The expression above does not include the benefit one derives from communicating with people with whom he shares the same mother tongue. This value is a constant. Whether the gross benefit accounts for it or not, has no bearing on our results. For instance, let $N = \{1, 2, 3, 4\}$ with $N^\alpha = \{1, 2\}$ and $N^\beta = \{3, 4\}$. Agent 1 can already communicate with agent 2 but here the focus is on the benefit deriving from the fact the she might eventually be able to communicate with 3 and 4. Such an utility is generated by the amount of communication alone. Agents do not care neither with whom they communicate, nor in what language they do so. Indeed, as long as $l_3 = l_4$, $(1, l_i)(l_3, 1) = (1, l_i)(l_4, 1)$.

On the other hand, for each $i \in N$, the disutility pertaining to action $l_i$, is $l_i c_{\gamma(i)}$. Hence, the net benefit associated with $l_N = (l_i)_{i \in N}$ is

$$v_i(l_N; \theta_i) = \theta_i \sum_{j \in N \setminus N^{(i)}} \min \{1, (1, l_i) \cdot (l_j, 1)\} - l_i c_{\gamma(i)}$$

For each $e \in \mathcal{E}$ and each $l_N \in \{0, 1\}^N$, let $\pi(l_N; e) = \sum_{i \in N} v_i(l_N; \theta_i)$ be the sum of net benefits generated by the the linguistic assignment $l_N$.

Each agent also consumes a transfer $t_i \in \mathbb{R}$. The final utility of each agent, at $l_N \in \{0, 1\}^N$, is then

$$u_i(l_N, t_i; \theta_i) = v_i(l_N; \theta_i) + t_i,$$

that is, preferences are quasi-linear. Let $t_N = (t_i)_{i \in N} \in \mathbb{R}^N$. An allocation is a list $(l_N, t_N) \equiv (l_i, t_i)_{i \in N}$ where $l_i$ is a linguistic assignment for agent $i$ and $t_i$ is the transfer she receives. Let $Z$ be the set of all allocations. A mechanism is a function $\varphi$ defined over $\mathcal{E}$ that associates with each economy an allocation $(l_N, t_N) \in Z$. Namely

$$\varphi: \mathcal{E} \rightarrow \{0, 1\}^N \times \mathbb{R}^N = Z$$

so that $\varphi(e) = (l_N, t_N)$ and $\varphi_i(e) = (l_i, t_i)$.
2.3 efficient linguistic assignments

In this section we discuss the problem of determining efficient linguistic assignments, that is, assignments that maximize the sum of net benefits. Depending on the particular $e \in \mathcal{E}$ at hand, we need to solve

$$P(e) : \max_{l_N \in \{0,1\}^N} \sum_{i \in N} \left( \theta_i \left( \sum_{j \in N \setminus N_{\gamma(i)}} \min\{1, (1, l_i \cdot (l_j, 1))\} - l_i c_{\gamma(i)} \right) \right).$$

For each $e \in \mathcal{E}$, let $\Sigma(e)$ be the set of linguistic assignments that solve $P(e)$. In what follows we provide an algorithm that produces for each $e \in \mathcal{E}$, one $l_N \in \Sigma(e)$.

Each agent has at most 2 alternatives, she either learns the other language or he does not. There are $N$ agents. Therefore, there are at most $2^N$ candidate solutions. Since the set of candidate solutions is finite, for each $e \in \mathcal{E}, \Sigma(e) \neq \emptyset$.

The construction of the algorithm is founded on the lemmata below. They constitute conditions necessary for optimality. Lemma 6 states that if at the optimum an agent does not learn, then so do all other agents in her language group that have a lower willingness to communicate. Lemmata 7 and 8 relate to a distinctive feature of the problem, namely, the conditions necessary for optimality. Lemma 6 states that if at the optimum an agent does not learn, then so do all other agents in her language group that have a lower willingness to communicate. Lemmata 7 and 8 relate to a distinctive feature of the problem, namely, the effect of the externality that learning entails. Consider some arbitrary linguistic assignment $l_N \in \{0,1\}^N$ such that for some $k \in N^\alpha$, $l_k = 0$. If agent $k$ were to learn, she would both attain a personal utility gain equal to $\theta_k ([N \setminus N_{\gamma(k)}] - \sum_{j \in N \setminus N_{\gamma(k)}} l_j)$, namely, her marginal willingness to communicate times the number of agents from the other language group who have not learned $k$’s language ($j \in N^\beta$ for whom $l_j = 0$), and benefit these very same agents who would be able to communicate with one more agent. Therefore, agent $k$, by learning would generate a marginal contribution equal to $\theta_k ([N \setminus N_{\gamma(k)}] - \sum_{j \in N \setminus N_{\gamma(k)}} l_j) + \sum_{j \in N \setminus N_{\gamma(k)}} \theta_j (1 - l_j) - c_{\gamma(k)}$.

**Lemma 6** For each $e \in \mathcal{E}$, each $\lambda \in \{\alpha, \beta\}$ and each $i_1, i_2 \in N^\lambda$, if $\theta_{i_1} > \theta_{i_2}$, then there exists $l_N \in \Sigma(e)$ such that either

1. $l_{i_1} = l_{i_2} = 0$, or
2. if $l_{i_2} = 1$, then $l_{i_1} = 1$.

**Proof.** Without loss of generality let $\lambda = \alpha$. Suppose, by way of contradiction, that there exists $l_N \in \Sigma(e)$ such that $l_{i_1} = 0$ and $l_{i_2} = 1$. Construct an alternative solution $\tilde{l}_N$ such that

$$\tilde{l}_j = \begin{cases} l_j & \text{if } j \neq i_1, j \neq i_2 \\ 1 & \text{if } j = i_1 \\ 0 & \text{if } j = i_2 \end{cases} \quad (2.3.1)$$

Let also $S \subseteq N^{\beta}$, be such that for each $i \in S$, $l_i = \tilde{l}_i = 0$. By construction, $\pi(\tilde{l}_N; e) - \pi(l_N; e) = (\theta_{i_1} - \theta_{i_2})|S|$. By assumption, $l_N$ is optimal and $l_{i_2} = 1$, with $i_2 \in N^\alpha$. If $S = \emptyset$ then, for each $i \in N^{\beta}$, $l_i = 1$. But if every agent in $N^{\beta}$ learned, it would be suboptimal for agent $i_2$ to learn too. Therefore, $S \neq \emptyset$. This implies that $(\theta_{i_1} - \theta_{i_2})|S| > 0$, the desired contradiction.

**Lemma 7** For each $e \in \mathcal{E}$, each $l_N \in \Sigma(e)$ and each $i \in N$
1. if \( l_i = 1 \), then \( \theta_i([N \setminus N^{\gamma(i)}] - \sum_{j \in N \setminus N^{\gamma(i)}} l_j) + \sum_{j \in N \setminus N^{\gamma(i)}} \theta_j(1 - l_j) - c_{\gamma(i)} \geq 0 \)

2. if \( l_i = 0 \), then \( \theta_i([N \setminus N^{\gamma(i)}] - \sum_{j \in N \setminus N^{\gamma(i)}} l_j) + \sum_{j \in N \setminus N^{\gamma(i)}} \theta_j(1 - l_j) - c_{\gamma(i)} \leq 0 \)

**Proof.** Conditions (1) and (2) are necessary for \( l_N \in \Sigma(e) \). Indeed, if \( l_N \in \Sigma(e) \), then for each \( i \in N \) unilaterally reducing the amount of communication (first condition), or unilaterally increasing the amount of communication (second condition) must decrease the sum of utilities.

**Lemma 8** For each \( e \in \mathcal{E} \), each \( \lambda \in \{\alpha, \beta\} \) and each \( i^* \in N^\lambda \) such that \( \theta_{i^*} \geq \theta_i \) for each \( i \in N^\lambda \), if \( l_N \in \Sigma(e) \) and \( l_{i^*} = 0 \), then

\[
l_i = 0 \quad \text{for all } i \in N^\lambda \quad (2.3.2)
\]

\[
l_i = \begin{cases} 
1 & \text{if } |N^\lambda| \theta_i + \sum_{j \in N^\lambda} \theta_j - c_{\gamma(i)} > 0 \\
0 & \text{otherwise}
\end{cases} \quad \text{for all } i \in N \setminus N^\lambda \quad (2.3.3)
\]

**Proof.** Implication (21) follows from lemma 6. Implication (23) follows from lemma 7.

The algorithm is presented in Table 1. It relies on lemmata 6 and 8. The algorithm performs approximately \( |N^\alpha||N^\beta| \) arithmetic operations. At most \( \frac{N^2}{4} \) operations will be performed.

On the left part of figure 2.2 we elaborate on the algorithm. One may think of the successive iterations of the algorithm as a move by each agent in \( N^\alpha \), starting from the agent with the highest willingness to communicate and proceeding in descending order. Each agent may move in one of two possible directions; she may choose to learn or not to learn. In this vein, any path from the initial node to any of the \( m + 1 \) terminal nodes corresponds to a candidate solution. For each \( k = 1, \ldots, m + 1 \), let \( l_N[k] \) denote any such candidate solution. For instance, at \( k = 3 \) we consider the path \((l_1 = 1, l_2 = 1, l_3 = 0)\) that corresponds to a linguistic assignment \( l_N[3] \) such that \( l_1[3] = l_2[3] = 3 \), for each \( i \in N^\alpha \setminus \{1, 2\} \), \( l_i[3] = 0 \). For each \( i \in N^\beta \), \( l_i[3] = l_i'[3] \), where \( l_i' \) solves the problem

\[P((\theta_{N^\alpha \setminus \{1, 2\}}, \theta_{N^\beta}), C)\] subject to \( l_i = 0 \), for each \( i \in N^\alpha \setminus \{1, 2\} \).

Lemma 8 provides us with the calculation that determines \( l_i' \). Finally, \( \pi\left((l_N[3]; (\theta_{N^\alpha}, \theta_{N^\beta}), C)\right) = M_1 + M_2 + \pi\left((l_N'; (\theta_{N^\alpha \setminus \{1, 2\}}, \theta_{N^\beta}), C)\right) = M_1 + M_2 + V_3 \). The longest path \((l_1 = 1, l_2 = 1, \ldots, l_m = 1)\) corresponds to the linguistic assignment \( l_N[(m + 1)] \), where for each \( i \in N^\alpha \), \( l_i[(m + 1)] = 1 \) and for each \( i \in N^\beta \), \( l_i[(m + 1)] = 0 \) and \( \pi\left((l_N[(m + 1)]; (\theta_{N^\alpha}, \theta_{N^\beta}), C)\right) = M_1 + M_2 + \cdots + M_m \).

Consider the economy \( e = \left(((0.8, 0.1), (2, 0.1, 0)), (1.6, 1.1)\right) \). The algorithm generates three candidate solutions. Refer to the right part of figure 2.2. The corresponding values are \( V_1 = 3.8 \), \( V_2 + M_1 = 3.9 \) and \( M_1 + M_2 = 3.7 \). The optimal linguistic assignment is \((1, 0, 1, 0, 0, 0)\).

The example above, apart from illustrating how the algorithm works, demonstrates two important points regarding an efficient linguistic assignment. Firstly, at the optimum full communication does not necessarily ensue. Secondly, and perhaps more surprisingly, optimal linguistic assignments may entail that agents from both language groups learn.
Table 2.1: The algorithm

1. Let $N^\alpha = \{1, \ldots, m\}$ and $N^\beta = \{m + 1, \ldots, N\}$. Sort agents so that for each $\lambda \in \{\alpha, \beta\}$ and each $i, j \in N^\lambda$, $i > j$ if and only if $\theta_i \geq \theta_j$. Let $M_i = |N \setminus N^\gamma(i)| \theta_i + \sum_{j \in N \setminus N^\gamma(i)} \theta_j - c_\gamma(i)$.

2. Consider the problem $P((\theta_{N^\alpha}, \theta_{N^\beta}), C)$.
   (a) By lemma 6 either agent 1 learns $\beta$ or no one in $N^\alpha$ learns $\beta$. Therefore, create two subproblems.
   (b) In the first subproblem set $l_1 = 0$. By lemma 6, for each $i \in N^\alpha$ we have $l_i = 0$. In this case, the optimal solution is obtained using lemma 8. Call the resulting value $V_1$. Save the solution to this subproblem.
   (c) In the second subproblem set $l_1 = 1$. This completely determines agent 1’s status. Save the value $M_1$, remove agent 1 from the problem.

If $m > 1$, re-run steps (a)-(c) for the reduced problem $P((\theta_{N^\alpha \setminus \{1\}}, \theta_{N^\beta}), C)$.

3. After $t$ iterations, with $t < m$, the algorithm treats the problem $P((\theta_{N^\alpha \setminus \{1, \ldots, t\}}, \theta_{N^\beta}), C)$.
   (a) By lemma 6 either agent $t + 1$ learns $\beta$ or no one in $N^\alpha \setminus \{1, \ldots, t\}$ learns $\beta$. Therefore, create two subproblems.
   (b) In the first subproblem set $l_{t+1} = 0$. By lemma 6, for each $i \in N^\alpha \setminus \{1, \ldots, t\}$ we have $l_i = 0$. In this case, the optimal solution is obtained using lemma 8. Call the resulting value $V_{t+1}$. Save the solution to this subproblem.
   (c) In the second subproblem set $l_1 = 1$. This completely determines agent 1’s status. Save the value $M_{t+1}$, remove agent $t + 1$ from the problem.

If $m > t + 1$, re-run steps (a)-(c) for the reduced problem $P((\theta_{N^\alpha \setminus \{1, \ldots, t+1\}}, \theta_{N^\beta}), C)$.

4. The algorithm generates $m + 1$ solutions. Namely, $V_1, V_2 + M_1, V_3 + M_1 + M_2, \ldots, V_m + M_1 + \cdots + M_{m-1}, M_1 + \cdots + M_m$. Pick the best one among them.
2.4 Axioms

In this section we formally define Strategy-Proofness, Assignment Efficiency and Individual Rationality and discuss their implications. Although, these axioms may be motivated by appealing to normative considerations, we emphasize here their significance in alleviating the incentive problem. In our framework a mechanism can be manipulated in three ways:

1. The agent may misreport relevant information she holds private.
2. The agent may choose not to conform to the prescriptions of the mechanism.
3. The agent may refuse to participate.

The axioms we present in this section each tackles one of the issues list above.

Although the social planner observes the partition of agents in language groups and is aware if the costs learning entails, she does not know the agents’ willingness to communicate. As a consequence, some agents might find it profitable do behave strategically and misreport it. We require that each agent has a weakly dominant strategy to report truthfully her willingness to communicate.

Strategy – Proofness For each $e \in \mathcal{E}$, $i \in N$ and $\theta_i' \in \mathbb{R}_+$,

$$u_i(\varphi_i(\theta_N, C); \theta_i) \geq u_i(\varphi_i(\theta_i', \theta_{N\setminus\{i\}}, C); \theta_i).$$

A mechanism is Assignment Efficient if, for each economy in the admissible domain, it selects an allocation that involves a linguistic assignment that maximizes the sum of net benefits. Assignment Efficiency differs from Pareto Efficiency in that it does not require transfers to sum up to zero.

Assignment – Efficiency For each $e \in \mathcal{E}$, if $(l_N, t_N) = \varphi(e)$ then $l_N \in \Sigma(e)$.

The way Assignment Efficiency wards against manipulation is not immediately apparent. In order to make it explicit we need to emphasize one of the implications of the axiom. Assignment Efficiency implies the following axiom.
**Home-Schooling Proofness:** For each $e \in \mathcal{E}$, if $(l_N, t_N) = \varphi(e)$, then, for each $i \in N$ and $l'_i \in \{0, 1\}$ such that $l'_i > l_i$,\[
theta_i \sum_{j \in N \setminus N^{(i)}} \left[ \min \{1, (1, l_j)(1, (l'_i - l_i))\} \right] - (l'_i - l_i)c_\gamma(i) < 0.
\]

Suppose that an allocation is such that Maggie is instructed not to learn. However, given the pattern of language learning that the allocation envisages, she finds it profitable to unilaterally deviate and learn. Even if there is an institution in place that controls whether the agents comply with the instructions of the social planner, it would be very costly to ensure that Maggie sticks to the plan. For instance, it would be hard, indeed almost implausible, to monitor her every step, in order to make sure that her mother, who happens to speak the language, will not home-school her.

A mechanism satisfies **Individual Rationality** if no agent is coerced into participation. All agents must must enjoy a positive utility as a result of their participation.

**Individual Rationality** For each $e \in \mathcal{E}$ and $i \in N$, $u_i(\varphi_i(e); \theta_i) \geq 0$.

Since the domain of preference profiles is convex (and hence smoothly connected) we know from Holmstrom [30] that a mechanism satisfies Assignment Efficiency and Strategy Proofness if and only if it belongs to the family of Groves mechanisms (see Groves [28]).

Such mechanisms determine a transfer composed of two parts. First, each agent receives the total net benefit obtained by all other agents at the assignment chosen by the mechanism. Second, each agent receives a sum of money that does not depend on her own (announced) willingness to communicate. Let $h_i$ be a real-valued function defined on $\mathbb{R}^{N-1}$ such that for each $i \in N$ and $\theta_N \in \mathbb{R}_+^N$, $h_i$ depends at most on $\theta_{N\setminus\{i\}}$. In other words, the function $h_i$ does not depend on agent $i$’s willingness to communicate.

**The Groves Mechanism** For each $e \in \mathcal{E}$, $(l_N, t_N) = \varphi^g(e)$ if and only if $l_N \in \Sigma(e)$ and, for each $i \in N$$t_i = \sum_{j \neq i} v_{j}(l_N; \theta_i) - h_i(\theta_{N\setminus\{i\}})$.

In general, for Groves mechanisms, the sum of the transfers may be either positive or negative. Generically there is a waste. However, there is a particular difficulty pertaining to a deficit. A Planner will need to finance the Groves scheme using resources that are not generated within the economy. Any mechanism, be it Groves or not, by construction, is silent as to where these outside funds may be found.

**Feasibility** requires that the Planner does not need to resort to outside funding. **Feasible** mechanisms are self-sufficient.

**Feasibility** For each $e \in \mathcal{E}$, if $(l_N, t_N) = \varphi(e)$ then $\sum_{i \in N} t_i \leq 0$.

The exercise we perform in this paper is shaped by the following result. It implies that the Planner may opt for either an individually rational Groves mechanism, at the expense of Feasibility, or a feasible Groves mechanism, at the expense of Individual Rationality.
Proposition 10 There exists no mechanism \( \varphi \) that satisfies Strategy Proofness, Individual Rationality, Feasibility and Home Schooling Proofness.

Proof. We construct a counter-example. Suppose that some mechanism \( \varphi \) satisfies the axioms. Consider an economy \( e \) consisting of two agents, \( N = \{1, 2\} \), speaking distinct languages. We have \( \theta_1 = \theta_2 > c_\alpha = c_\beta > 0 \) and \( c_\alpha + c_\beta > \theta_1, \theta_2 \). By Home Schooling Proofness, for each \( (l_N, t_N) \in \varphi(e) \) either one of the following must be true:

1. \( l_1 = 0 \) and \( l_2 = 0 \), or
2. \( l_1 = 0 \) and \( l_2 = 1 \), or
3. \( l_1 = 1 \) and \( l_2 = 0 \).

Consider case (1), i.e. let there exist \( (l_N, t_N) \in \varphi(e) \) such that \( l_1 = 0 \) and \( l_2 = 0 \). Suppose that \( u_1(l_N, t_1; \theta_1) \geq \theta_1 \) and \( u_2(l_N, t_2; \theta_2) \geq \theta_2 \). Therefore,

\[
\begin{align*}
   u_1(l_N, t_1; \theta_1) + u_2(l_N, t_2; \theta_2) &\geq \theta_2 + \theta_1 \\
   \theta_1 - c_\alpha + \theta_2 + t_1 + t_2 &\geq \theta_1 + \theta_2 \\
   t_1 + t_2 &\geq c_\alpha
\end{align*}
\]

By Feasibility, \( t_1 + t_2 \leq 0 \) and hence inequality (2.4.1) constitutes a contradiction, as by assumption \( c_\alpha > 0 \). The same reasoning applies for cases (2), (3). In conclusion, for each \( (l_N, t_N) \in \varphi(e) \), either \( u_1(l_N, t_1; \theta_1) < \theta_1 \) or \( u_2(l_N, t_2; \theta_2) < \theta_2 \). Without loss of generality, suppose that for each \( (l_N, t_N) \in \varphi(e) \), \( u_1(l_N, t_1; \theta_1) < \theta_1 \).

Consider economy \( e' \), to be one that is identical to economy \( e \), except for the fact that \( \theta_1' = 0 \). By Individual Rationality and Feasibility, since \( c_\alpha + c_\beta > \theta_2 \), there cannot exist \( (l'_N, t'_N) \) such that \( l'_1 = 1 \) and \( l'_2 = 1 \). Therefore, by Home Schooling Proofness, for each \( (l'_N, t'_N) \in \varphi(e') \), either \( l'_1 = 1 \) and \( l'_2 = 0 \), or \( l'_1 = 0 \) and \( l'_2 = 1 \). By Individual Rationality, in the former case \( t'_1 \geq c_\alpha \beta \) and in the latter \( l'_1 \geq 0 \). This implies that from a profile of announcements \( (\theta_1, \theta_2) \) agent 1 can profitably deviate to the profile \((0, \theta_2)\) and obtain a utility level equal to \( \theta_1 \). Thus, the mechanism \( \varphi \) violates Strategy-Proofness, a contradiction.

Proposition 1 is of interest for an additional reason. It marks a stark difference between our framework and other economic environments where it is well known that feasible and individually rational Groves mechanisms exist. Examples of feasible and individually rational Groves mechanisms are also provided by Guo and Conitzer [26] and Moulin [45] (among others) as solutions to the problem of assigning \( p \) identical objects to a group of agents. When a public good is involved Groves mechanism are not in general individually rational. If we consider situations where all agents value the good positively, as for our model, then the Pivotal mechanism also satisfies individual rationality.

2.5 An Individually Rational Groves Mechanism

In this section we focus on Groves mechanisms that satisfy Individual Rationality. As shown before, such mechanisms violate Feasibility. Among them, we isolate a mechanism that minimizes the deficit whenever it occurs. The natural question to raise next concerns the incidence of the deficit. We provide conditions on the economy that, when met, imply that any Individually Rational Groves mechanism will be in deficit in that economy.
Inspection of these sufficient conditions suggest that the deficit is, indeed, a prevalent phenomenon. In fact, running simulations we were not able to find any economy in which an Individually Rational Groves mechanism is not in deficit.

For each \( e \in \mathcal{E} \) and each \( i \in N \) let \( e^i \) denote an economy that is otherwise identical to \( e \), except for the fact that agent \( i \)'s willingness to communicate has been set equal to zero. Formally, if \( e = (\theta_N, C) \), then \( e^i = ((0, \theta_N \setminus \{i\}), C) \). In addition, let \( l_N \in \Sigma(e^i) \).

The Minimal Deficit Mechanism (MDM) For each \( e \in \mathcal{E}, (l_N, t_N) = \varphi^{md}(e) \) if and only if \( l_N \in \Sigma(e) \) and, for each \( i \in N \),

\[
t_i = \sum_{j \neq i} v_j(l_N; \theta_i) - \sum_{j \neq i} v_j(l_N^i; \theta_j) - v_i(l_N; 0).
\]

Roughly speaking, the transfer of the MDM constitutes an assessment of the impact each agent’s willingness to communicate has on the optimal sum of agent net benefits. In order to accomplish that, the mechanism, for each agent \( i \in N \), needs to calculate \( \pi(l_N^i; e^i) \).

In particular, in order to obtain the MDM from within the family of Groves mechanisms one needs to set, for each \( e \in \mathcal{E} \) and each \( i \in N \),

\[
h_i(\theta_N \setminus \{i\}) = \pi(l_N^i; e^i) = \sum_{j \neq i} v_j(l_N^i; \theta_j) + v_i(l_N; 0).
\]

We illustrate the MDM with the help of an example. Consider the economy

\[
e = \left( ((2, 1, 0.1), (0.8, 0.1)), (1.1, 1.6) \right).
\]

Refer to figure 2.3, top left, for a graphical representation of it. Dots represent agents and they are grouped in two columns, each representing a language group. Individuals whose mother-tongue is \( \alpha \) are on the left column, while agents whose mother-tongue is \( \beta \) are on the right column. The number in parentheses are the names of the agents moreover \( \theta_1 = 2, \theta_2 = 1, \theta_3 = 0.1, \theta_4 = 0.8, \theta_5 = 0.1 \). The figure depicts also the five economies we obtain by setting, each time, the willingness to communicate of each agent equal to zero, that is, \( e^1, \ldots, e^5 \). An arrow stemming from a node representing agent \( i \in N \) and pointing to a language group stands for \( l_i = 1 \). The absence of an arrow stands for \( l_i = 0 \). The figure depicts \( l_N \in \Sigma(e) \), as well as \( l_N^1, \ldots, l_N^5 \). For instance, \( l_N = (0, 0, 0, 1, 1) \). Interestingly a slight perturbation in the original problem can drastically change the efficient linguistic assignment as it is made evident by inspecting the figure. Finally, for each \( i \in N \), the figure provides \( \pi(l_N^i; e^i) \) so that it is possible to compute the transfer that the MDM prescribes for \( e \), namely, \( t_N = (-0.2, -0.2, 0, 0.9, 1.6) \). Agents 1 and 2 are taxed, while agents 4 and 5 are subsidized. The MDM produces a deficit equal to 2.1.

The MDM needs to be distinguished from the Clarke mechanism (see Clarke [11] and Moulin [39] for a detailed description of the properties of such mechanism in the pure public good framework). In order to obtain the Clarke mechanism from the family of Groves mechanisms one needs to set, for each \( e \in \mathcal{E} \) and each \( i \in N \),

\[
h_i(\theta_N \setminus \{i\}) = \sum_{j \neq i} v_j(l_N'_{N \setminus \{i\}}; \theta_j),
\]

where \( l_N'_{N \setminus \{i\}} \in \Sigma(\theta_N \setminus \{i\}, C) \). The \( h_i(.) \) component of the Clarke transfer is obtained by removing agent \( i \) from the economy altogether and then calculating the optimal sum of net benefits in her absence. An apparent difference between the Clarke mechanism and
the MDM is that the former, unlike the latter, by removing the agent from the economy, deprives the remaining agents from any benefit they may derive from being able to communicate with her. In the canonical public good provision model whether an agent is removed from the economy or her valuation of the project is set to zero, amounts to the same effect. In our framework, the nature of the externality that each agent produces on the other agents is different. An agent is still a potential source of value for the rest even if her willingness to communicate is equal to zero. Removing her from the economy amounts to more than deducting her net benefit from the total sum. This fact has stark implications as the following lemma demonstrates

**Lemma 9** If for some $e \in \mathcal{E}$ the MDM generates a deficit then, in the same economy, the Clarke mechanism generates a greater or equal deficit.

**Proof.** By construction, for each $e = (\theta_N, C) \in \mathcal{E}$ and each $l_{N \setminus \{i\}} \in \Sigma(\theta_{N \setminus \{i\}}, C)$,

$$\pi((l_{N \setminus \{i\}}, 0); e^i) \geq \pi((l_{N \setminus \{i\}}; (\theta_{N \setminus \{i\}}, C)).$$

If, $l'_N \in \Sigma(e^i)$, then, by definition,

$$\pi(l'_N; e^i) \geq \pi((l_{N \setminus \{i\}}, 0); e^i).$$

Therefore, for each $e = (\theta_N, C) \in \mathcal{E}$ and each $l_{N \setminus \{i\}} \in \Sigma(\theta_{N \setminus \{i\}}, C)$ and each $l'_N \in \Sigma(e^i)$,

$$\pi(l'_N; e^i) \geq \pi((l_{N \setminus \{i\}}; (\theta_{N \setminus \{i\}}, C)). \quad (2.5.1)$$

Let $\varphi^c$ denote the Clarke mechanism. For some $e \in \mathcal{E}$ let $(l^m_N, t^m_N) = \varphi^m(e)$ and $(l^c_N, t^c_N) = \varphi^c(e)$. By definition, using inequality 2.5.1, we obtain, for each $i \in N$, $t^m_i \leq t^c_i$. Therefore,
In the canonical public good provision model the transfers associated with the Clarke mechanism are non-positive. It follows from lemma 9 that this it is no longer the case in our framework. However, both in the canonical public good provision model and our framework the Clarke mechanism satisfies Individual Rationality. The following Proposition generalizes lemma 9.

**Proposition 11** If for some \( e \in \mathcal{E} \) the MDM generates a deficit then, in the same economy, any mechanism satisfying Assignment Efficiency, Strategy-Proofness and Individual Rationality generates a greater or equal deficit.

**Proof.** By Assignment Efficiency and Strategy-Proofness we need to compare our mechanism with other mechanisms belonging to the Groves family of mechanisms. Moreover, by Individual Rationality we need to have, for each \( e \in \mathcal{E} \) and each \( i \in \mathcal{N} \)

\[
u_i(\varphi^0(e); \theta_i) = v_i(l_N; \theta_i) + t_i = \sum_{i \in \mathcal{N}} v_i(l_N; \theta_i) - h_i(\theta_N \setminus \{i\}) \geq 0
\]

with

\[
\sum_{i \in \mathcal{N}} v_i(l_N; \theta_i) = \sum_{i \in \mathcal{N}} \left( \theta_i \sum_{j \in \mathcal{N} \setminus \{i\}} \min \{1, (1, l_i) \cdot (l_j, 1)\} - l_i c_{\gamma(i)} \right)
\]

where \( l_N \in \Sigma(e) \). Hence, for any given profile \( \theta_N \setminus \{i\} \in \mathbb{R}^{N-1} \) the component \( \sum_{i \in \mathcal{N}} v_i(l_N; \theta_i) \), which is the sum of net benefits at an efficient linguistic assignment, reaches its minimum value when agent \( i \)'s willingness to communicate is equal to zero. Hence, in order to satisfy Individual Rationality we need to set, for each \( e \in \mathcal{E} \), \( i \in \mathcal{N} \),

\[
h_i(\theta_N \setminus \{i\}) \leq \sum_{j \neq i} v_j(l_N^i; \theta_j) + v_i(l_N^i; 0)
\] (2.5.2)

where \( l_N \in \Sigma(e) \). Moreover,

\[
\sum_{i \in \mathcal{N}} t_i = \sum_{i \in \mathcal{N}} \sum_{j \neq i} v_j(l_N; \theta_j) - \sum_{i \in \mathcal{N}} h_i(\theta_N \setminus \{i\})
\]

By equation 2.5.2,

\[
\sum_{i \in \mathcal{N}} t_i \geq \sum_{i \in \mathcal{N}} \sum_{j \neq i} v_j(l; \theta_j) - \sum_{i \in \mathcal{N}} \sum_{j \neq N} v_j(l^i; \theta_j) - \sum_{i \in \mathcal{N}} v_i(l_N^i; 0)
\] (2.5.3)

This means that, as soon a mechanism generates a deficit, it generates at least as much deficit as in the the right-end side of equation 2.5.3. Hence in order to minimize the deficit produced by the mechanism we need to set

\[
h_i(\theta_N \setminus \{i\}) = \sum_{j \neq i} v_j(l_N^i; \theta_j) + v_i(l_N^i; 0)
\]

We actually conjecture that, for any conceivable economy, any individually rational Groves mechanism runs a deficit. As a consequence the sum of the transfers pertaining to the MDM would always be non-negative as well. While we are unable to prove the statement in general, first, simulations suggest that it is true and, second, we are able to show it.
in specific environments. In particular, we prove it to be the case in subdomains of \( \mathcal{E} \) that encompass, inter alia, large economies. This is of interest since, as pointed for example by Deb, Razzolini and Seo [14], some members of the Groves family are asymptotically balanced as the number of agents increases in a well-behaved way, for instance by replication. In our framework, if any feasible Groves mechanism is asymptotically balanced, it would need to be the MDM.

The two Propositions that follow state that in a significant sub-domain of the set of admissible economies any Groves mechanism satisfying Individual Rationality runs a deficit. More precisely, Propositions 12 and 13 determine the number of agents that need to have a willingness to communicate greater than some arbitrary non-negative real number \( T \) for an individually rational Groves mechanism to run a deficit. The actual value of the threshold \( T \) depends on all the parameters of the economy, namely its size, the profile of preferences and costs. The proofs can be found in Appendix 1.

**Proposition 12** For each positive real number \( T \), let \( L(T,e) = \{ i \in N \mid \theta_i < T \} \) and \( \zeta(T,e) = |L(T,e)| \). Let

\[
\bar{N}^\beta = \inf_{T > 0} \{ \zeta(T,e) + 2 + \frac{c_\alpha}{T} + \frac{c_\beta}{T} \}. \tag{2.5.4}
\]

If \( N^\beta \geq \bar{N}^\beta \), then any Individually Rational Groves mechanism runs a deficit.

**Proposition 13** For each positive real number \( T \), let \( L(T,e) = \{ i \in N \mid \theta_i < T \} \) and \( \zeta(T,e) = |L(T,e)| \). Let

\[
\bar{N}^\alpha = \inf_{T > 0} \left\{ \frac{c_\beta}{c_\alpha} (\zeta(T,e) + 1) + 1 + \frac{2c_\beta}{T} \right\}. \tag{2.5.5}
\]

If \( N^\alpha \geq \bar{N}^\alpha \), then any Individually Rational Groves mechanism runs a deficit.

The following examples elaborate on Propositions 12 and 13. Consider the economy

\[ e = \left( (3, 2, 1, 0), (3, 3, 2, 2, 1.5, 1.5, 0.5)), (1, 2) \right). \]

The set of agents with \( \theta_i \) less that 1, i.e., \( L(1,e) \) is \{4, 11\}. Hence, \( \zeta(1,e) + 2 + \frac{c_\alpha+c_\beta}{T} = 7 \). Hence, \( \bar{N}^\beta \leq 7 \). Since \( |N^\beta| = 7 \) by Proposition 12, in this economy any Individually Rational Groves mechanism runs a deficit. Consider, next, the economy

\[ e = \left( ((\epsilon, \ldots, \epsilon), (\epsilon, \ldots, \epsilon)), (qe,qe) \right), \]

where \( \epsilon > 0 \) and \( q \in N \). The set of agents with \( \theta_i \) less that \( \epsilon \), i.e., \( L(\epsilon,e) \) is empty. Hence,

\[ \frac{qe}{qe} (\zeta(\epsilon,e) + 1) + 1 + \frac{2qe}{\epsilon} = 2 + 2q. \]

Hence, as soon as \( N^\beta \geq q \), by Proposition 12, any Individually Rational Groves mechanism runs a deficit.

Two comments are in line. First, the domain restrictions ‘almost’ do not rely on preferences. The only consideration is about agents that are entirely averse to the possibility of communication. Secondly, the domain restrictions do not require the economy to be large, unless both costs are enormous. Even in such a case though, one needs only one language group to be adequately large for the result to come through.
2.6 A feasible Groves Mechanism

In this section we drop Individual Rationality in favor of Feasibility. The restriction to feasible Groves mechanisms reflects a physical constraint often imposed on the implementation effort. The social planner is not mandated to resort to outside funding. She may only rely on her power to tax agents which may be, for the purposes of this section, complimented by her ability to coerce participation.

There is a simple class of mechanisms that accomplishes the task. To retrieve it we need to expand on the discussion that ensued in the previous section. In particular, we need to take advantage of a feature of the MDM. For each economy, let the set of linguistic assignments that ensure full communication be denoted $\mathcal{L}(e)$. Consider within this set the subset of linguistic assignments that are the least costly. That is, for each $e \in \mathcal{E}$, let

$$L^{f*}(e) \equiv \operatorname*{argmin}_{l_N \in \mathcal{L}(e)} \sum_{i \in N} l_i c_\gamma(i) \subseteq \mathcal{L}(e)$$

Moreover, for each $l_N \in L^{f*}(e)$, let

$$c = \sum_{i \in N} l_i c_\gamma(i).$$

Notice that in order to compute the value $c$ one does not needs to know the profile of preferences. It turns out that for each economy the total cost pertaining to the efficient linguistic assignment in that economy will be less or equal to the value $c$ for that economy.

The mechanism we present below charges all individuals an amount $c$ on the top of what they were charged by the MDM.

The Translated Minimal Deficit Mechanism (TMDM) For each $e \in \mathcal{E}$, $(l_N, t_N) = \varphi^{md}(e)$ if and only if $l_N \in \Sigma(e)$ and, for each $i \in N$

$$t_i = \sum_{j \neq i} v_j(l_N; \theta_i) - \sum_{j \neq i} v_j(l^*_N; \theta_j) - v_i(l^*_N; 0) - \frac{c}{N}$$

where $l^*_N \in \Sigma(e^i)$.

Relative to the MDM, the TMDM levies an extra amount $c$ that aims at ensuring feasibility. Moreover, the TMDM collects an equal share of this extra amount from each agent. One may imagine alternative ways of distributing this extra burden. However, as long as a Groves mechanism levies an amount $c$ over the amount the MDM levies, it will satisfy Feasibility. This is a direct consequence of the following lemma.

Lemma 10 For each $e \in \mathcal{E}$, if $(l_N, t_N) = \varphi^{md}(e)$, then

$$\sum_{i \in N} t_i \leq \sum_{i \in N} l_i c_\gamma(i)$$

Proof. For each $e \in \mathcal{E}$ and each $i \in N$, if $l_N \in \Sigma(e)$ and $l^*_N \in \Sigma(e^i)$, then

$$\theta_i \sum_{j \in N \setminus \Sigma^*(i)} \min\{1, (1, l_i) \cdot (l_j, 1)\} \geq \sum_{i \in N} v_i(l_N; \theta_i) - \sum_{j \neq i} v_j(l^*_N; \theta_j) - v_i(l^*_N; 0).$$
If that were not true, by rearranging the terms of the inequality one would obtain
\[
\pi(l_N^t; e^t) = \sum_{j \neq i} v_j(l_N^t; \theta_j) + v_i(l_N^t; 0) < \sum_{i \in N} v_i(l_N; \theta_i) - \theta_i \sum_{j \in N \setminus N^t(i)} \min\{1, (1, l_i) \cdot (l_j, 1)\} = \pi(l_N; e^t),
\]
which constitutes a contradiction, as, by assumption, \(l_N^t \in \Sigma(e^t)\). Summing over \(i \in N\) we obtain
\[
\sum_{i \in N} \theta_i \sum_{j \in N \setminus N^t(i)} \min\{1, (1, l_i) \cdot (l_j, 1)\} \geq N\pi(l_N; e) - \sum_{i \in N} \pi(l_N^t; e^t).
\]
The left-hand side of the previous equation represents the total gross benefit deriving from communication at \(l_N\). A simple algebraic manipulation yields
\[
\sum_{i \in N} l_{C(\gamma(i))} \geq (N - 1)\pi(l_N; e) - \sum_{i \in N} \pi(l_N^t; e^t) = \sum_{i \in N} t_i.
\]

Lemma 10 provides a rough idea of the challenge one needs to overcome when designing feasible Groves mechanisms. If the planner knows for each \(e \in \mathcal{E}\), with \((l_N, t_N) = \varphi^{md}(e)\), the value \(\sum_{i \in N} l_{C(\gamma(i))}\), then she has at her disposal a rough rule of thumb that she may apply in order to comply with Feasibility. However, lemma 10 does not do much more than pointing in the right direction. The value \(\sum_{i \in N} l_{C(\gamma(i))}\) varies with the economy and there is no a priori reason to be hopeful that collecting the information required to calculate it complies with Strategy-Proofness. The TMDM circumvents this issue by utilizing the fact that for each \(e \in \mathcal{E}\), each \(l_N \in \Sigma(e)\) and each \(l_N^t \in l^{\star}(e)\), we have \(\sum_{i \in N} l_{C(\gamma(i))} \geq \sum_{i \in N} l_{C(i)}^t\)

And there is one more caveat. To facilitate the analysis we assume that \(N^\alpha = \{1\}\) and \(N^\beta = \{2\}\). Moreover, assume that \(c = c_\alpha < c_\beta\), so that if ever learning is efficient, it is individual 1 that learns ‘beta’. An economy is denoted \(e = (\theta_N, c) \in \mathbb{R}_+^3\). We will adhere to these assumptions for the rest of this section. Figure 2.6 below depicts the TMDM for this constricted set of economies.

![Diagram](image)

Figure 2.4: The vector of transfers \(\{l_N^t\}\) for each profile \((\theta_1, \theta_2) \in \mathbb{R}_+\) according to the TMDM in the two agent case. Agent 1 learns if \(\theta_1 + \theta_2 \geq 0\). She does not otherwise.

At first glance there is nothing outright opposable with the TMDM. It does violate Individual Rationality, but this is a concession we knew we had to make. Nonetheless,
the TMDM depicted in figure 2.4 suffers from a serious flaw. It is Pareto dominated by another mechanism that satisfies among other things Strategy-Proofness and Feasibility. In what follows we isolate, in the two agent case, the only Feasible Groves mechanism that is Pareto undominated.

In building the notion of of Second-Best efficiency we will focus on anonymous mechanisms. Let $\tau : \{1, 2\} \to \{1, 2\}$ be such that $\tau(i) = j$ for each $i, j \in N$ and $\theta_{\tau(N)}$ denote the vector $\theta_N$ permuted according to $\tau$. In the two agent setting Anonymity effectively requires that the distribution of utility that an allocation induces does not depend on the agent specific cost that learning entails. The axiom constitutes a minimal legitimacy requirement that all mechanisms should naturally fulfill. Moreover, it focuses the exercise we are about to perform. A dictatorial mechanism that always assigns all the surplus to a given individual will be Pareto undominated. A criterion that aims at identifying optimal mechanisms needs to exclude such a perverse phenomenon.

**Anonymity** For each $e \in E$, $i \in N$,

$$u_i(\phi_i(\theta_N, c); \theta_i) = u_{\tau(i)}(\phi_{\tau(i)}(\theta_{\tau(N)}, c))$$

We may now define the criterion that isolates the best-in-class mechanism. Let $\Phi$ be the set of Strategy-Proof, Anonymous and Feasible mechanisms. For each pair of mechanisms $\phi, \phi' \in \Phi$, $\phi$ Pareto dominates $\phi'$ if and only if, for each $e = (\theta_N, c) \in \mathbb{R}_+^2$, letting $(l_N, t_N) = \phi(e)$ and $(l'_N, t'_N) = \phi'(e)$,

- for each $i \in N$, $v_i(l_N; \theta_i) + t_i \geq v_i(l'_N; \theta_i) + t'_i$, and
- for some $j \in N$, $v_j(l_N; \theta_j) + t_j > v_j(l'_N; \theta_j) + t'_j$.

A mechanism $\phi \in \Phi$ is **Second Best Efficient** if and only if there does not exist another mechanism $\phi' \in \Phi$ such that $\phi'$ Pareto dominates $\phi$.

![Figure 2.5: The vector of transfers $(t'_1)$ for each profile $(\theta_1, \theta_2) \in \mathbb{R}_+$ according to the SBM in the two agent case. Agent 1 learns if $\theta_1 + \theta_2 \geq 0$. She does not otherwise.](image)

If the domain of economies is reduced to the ones comprising two agents the space from which profiles of preferences are drawn becomes the non-negative real orthant. For each $c \in \mathbb{R}_+$ figure 2.5 defines a feasible mechanism, call it the Second Best Mechanism, or SBM. The mechanism violates Individual Rationality at each $(\theta_N, c) \in E$ such that either $\theta_1 < c$ or $\theta_2 < c$. 

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Proposition 14  *A mechanism satisfying Strategy-Proofness, Anonymity and Assignment Efficiency is Second Best Efficient if and only if it is the SBM.*

The proof can be found in Appendix 2. The subtlety of Proposition 5 becomes apparent by comparing the SBM with the TMDM. The Pareto criterion does not rank the two. The TMDM is preferred to the SBM by agent 1 in economy $e = (\theta_N, c) \in \mathbb{R}_+^3$ such that $\theta_1 \in (0, \frac{c}{2})$, $\theta_2 \in (\frac{c}{2}, c)$ and $\theta_1 + \theta_2 > c$. The mechanism that Pareto dominates the TMDM, hence rendering it not *Second Best Efficient* must be itself not *Second Best Efficient*.

2.7 Concluding remarks

This paper focuses on Groves mechanisms in a model of private goods with externalities. Agents need a common language in order to communicate and the procedure that allows them to acquire such a knowledge is costly. We first provide an algorithm that allows to single out the set of efficient linguistic assignments (i.e., that maximize the sum of net benefits that agents derive from communication) and then we tackle the issue of implementing one of such assignments by mean of a Groves mechanism.

The externality present in the problem radically change the characteristics of well known solutions like the Clarke Mechanism that, in our framework, is no longer feasible. Indeed we show that there is no Groves mechanism that is both *Individually Rational* and *Feasible*. This fact force us to explore to distinct venues.

We first look at Groves mechanisms that are *Individually Rational* and, among them we single out the mechanism that minimizes the amount of money necessary to finance it. We also provide quite general conditions under which an *Individually Rational* Groves mechanism needs to resort on outside funding.

We then look at feasible Groves mechanisms. Even if it is relatively simple to find examples of such mechanisms it proves to be an extremely more demanding task to single out mechanisms that are Pareto Undominated. By focusing on the simpler but still meaningful domain of economies comprising only two agents we are able to single out the only second best efficient and feasible Groves mechanism.

Even if the paper focus solely on Groves mechanisms, it is worth making the point that there are interesting mechanisms not belonging to this class that merit further investigation (see for example Moulin and Shenker [38] or Moulin [44]). To keep things simple let us maintain the simplifying assumptions we made in the previous section. Consider the following simple mechanism defined over two agent economies depicted in figure 2.6.

Effectively, the mechanism asks agents to announce their willingness to communicate. If the announcements are such that either $\theta_1 < \frac{c}{2}$ or $\theta_2 < \frac{c}{2}$ there is no learning and each agent receives a transfer equal to zero. Otherwise, agent 1 learns and $t_1 = \frac{c}{2} = -t_2$. This simple mechanism satisfies *Strategy-Proofness, Individual Rationality* and *Feasibility* (in fact, the sum of transfers equals zero in all two-agent economies). Alas, it violates *Home-Schooling Proofness* and, thus, *Assignment Efficiency*. To see that, suppose that $\theta_2 = 0 < c < \theta_1$. The mechanism prescribes to agent 1 to refrain from learning. Nonetheless, agent 1 can ensure a higher utility if he deviates from the proposed allocation and learn. Aside from this deficiency, one should not be hasty in dismissing the mechanism depicted in figure 2.6. Applying the reasoning employed in the proof of Proposition 5, it can be demonstrated that it is *Second Best Efficient*. This fact alludes to the existence of interesting mechanisms outside the Groves family and it is left for future research.
Figure 2.6: The vector of transfers $(t_1, t_2)$ for each profile $(\theta_1, \theta_2) \in \mathbb{R}^+$. Agent 1 learns if $\theta_1 \geq \frac{c_2}{2}$ or $\theta_2 \geq \frac{c_2}{2}$. She does not otherwise.

2.8 Appendix 1. Proofs of Propositions 3 and 4

By inspecting the proof of Proposition (namely the lower bound set by inequality 2.5.3) the following fact can be easily deduced. Let $\varphi$ be an individually rational Groves mechanism. For each $e = (\theta_N, C) \in \mathcal{E}$, with $(l_N, t_N) = \varphi(e)$, if

$$
\sum_{j \neq i} v_j(l_N; \theta_i) - h_i(\theta_N \setminus \{i\}) = \sum_{i \in N} \sum_{j \neq i} v_j(l_N; \theta_j) - \sum_{i \in N} \sum_{j \in N} v_j(l_N; \theta_j) - \sum_{i \in N} v_i(l_N; 0) \geq 0,
$$

then

$$
\sum_{i \in N} t_i \geq 0.
$$

In what follows we assume without loss of generality that $c_\alpha \leq c_\beta$, that $N^\alpha = \{1, \ldots, m - 1\}$, $N^\beta = \{m, \ldots, |N|\}$ and that, within each language group and for each $i, j \in N^\alpha$, if $i \leq j$, then $\theta_i \geq \theta_j$. Similarly for $N^\beta$.

With $(I, J)$ we denote a structure where a set of $I$ agents learn language $\beta$, and a set of $J$ agents learn language $\alpha$. With $z(I, J)$ we denote the value of such a structure. Similarly, define $(I^i, J^i)$ and $z^i(I^i, J^i)$ where the superscript signifies the fact that $\theta_i$ has been set to zero. Let $\zeta(e) \in \mathbb{N}^+$ be the number of agents in economy $e \in \mathcal{E}$ for whom $\theta_i = 0$.

The following lemmata are needed for the proofs of propositions 12 and 13.

**Lemma 11** For each $i \in N^\alpha$, $\pi(l_N^i; e^i) \leq \pi(l_N^1; e^1) + |N^\beta|(\theta_1 - \theta_i)$.

**Proof.** Let $(I^i, J^i)$ be an optimal solution corresponding to $\pi(l_N^i; e^i)$. There are three cases:

1. $i \in I^i$: Since in this economy $\theta_i$ is set to zero, this implies that $I^i = N^\alpha$ and $J^i = \emptyset$. Therefore

$$
\pi(l_N^i; e^i) - z^1(N^\alpha, \emptyset) = -|N^\beta|\theta_i + |N^\beta|\theta_1
$$

which implies

$$
\pi(l_N^i; e^i) = z^1(N^\alpha, \emptyset) + |N^\beta|(\theta_1 - \theta_i) \leq \pi(l_N^1; e^1) + |N^\beta|(\theta_1 - \theta_i).
$$
2. \(i \notin I^{i}, 1 \in I^{i}\): Then
\[
\pi(l_{N}^{i};e^{i}) = |N^{\beta}| \sum_{p \in I^{i}} \theta_{p} + |N^{\alpha}| \sum_{j \in J^{i}} \theta_{j} + |I^{i}| \sum_{p \in N^{\alpha}\setminus I^{i}} \theta_{p} - |I^{i}|c_{\alpha} - |J^{i}|c_{\beta}.
\]
Now
\[
z^{1}(I^{i} \setminus \{1\} \cup \{i\}, J^{i}) = |N^{\beta}| \sum_{p \in (I^{i}\setminus\{i\})} \theta_{p} + |N^{\alpha}| \sum_{j \notin J^{i}} \theta_{j} + |J^{i}| \sum_{p \in N^{\alpha}\setminus (I^{i}\cup\{i\})} \theta_{p} - |I^{i}|c_{\alpha} - |J^{i}|c_{\beta}.
\]
Therefore,
\[
\pi(l_{N}^{i};e^{i}) - z^{1}(I^{i} \setminus \{1\} \cup \{i\}, J^{i}) = -|N^{\beta}|\theta_{i} + |N^{\beta}|\theta_{1}.
\]
Therefore,
\[
\pi(l_{N}^{i};e^{i}) \leq \pi(l_{N}^{i};e^{i}) + |N^{\beta}|(\theta_{1} - \theta_{i}).
\]
3. \(i \notin I^{i}, 1 \notin I^{i}\): Then in this case \(I^{i} = \emptyset\). In this case,
\[
\pi(l_{N}^{i};e^{i}) = |N^{\alpha}| \sum_{j \in J^{i}} \theta_{j} + |J^{i}| \sum_{p \in N^{\alpha}\setminus \{i\}} \theta_{p} - |J^{i}|c_{\beta}.
\]
Now,
\[
z^{1}(\emptyset, J^{i}) = |N^{\alpha}| \sum_{j \in J^{i}} \theta_{j} + |J^{i}| \sum_{p \in N^{\alpha}\setminus \{1\}} \theta_{p} - |J^{i}|c_{\beta}.
\]
Therefore
\[
\pi(l_{N}^{i};e^{i}) - z^{1}(\emptyset, J^{i}) = |J^{i}|(\theta_{1} - \theta_{i}).
\]
Therefore,
\[
\pi(l_{N}^{i};e^{i}) \leq \pi(l_{N}^{i};e^{i}) + |J^{i}|(\theta_{1} - \theta_{i}) \leq \pi(l_{N}^{i};e^{i}) + |N^{\beta}|(\theta_{1} - \theta_{i}).
\]

**Lemma 12** If \(1 \notin I^{1}\), then \(z(I^{1} \cup \{1\}, J^{1}) - \pi(l_{N}^{1};e^{1}) \geq |N^{\beta}|(\theta_{1} - \theta_{i}) + \sum_{j \notin N^{\alpha}\setminus J^{1}} \theta_{j} - c_{\beta} \forall i \in N^{\alpha}.

*Proof.* Now \(z(I^{1} \cup \{1\}, J^{1}) - \pi(l_{N}^{1};e^{1}) = |N^{\beta}|\theta_{1} + \sum_{j \notin N^{\alpha}\setminus J^{1}} \theta_{j} - c_{\beta}\). Moreover from previous proposition we have that \(\pi(l_{N}^{1};e^{1}) \leq \pi(l_{N}^{1};e^{1}) + |N^{\beta}|(\theta_{1} - |N^{\beta}|\theta_{1})\) and therefore \(z(I^{1} \cup \{1\}, J^{1}) - \pi(l_{N}^{1};e^{1}) \geq |N^{\beta}|\theta_{1} + \sum_{j \notin N^{\alpha}\setminus J^{1}} \theta_{j} - c_{\beta}\).

Let \(J \subseteq N^{\beta}\) be any subset. Note that similarly we can prove that if \(m \notin J^{m}\), then \(\sum_{j \in J}(\pi(l_{N};e) - \pi(l_{N}^{1};e^{1})) \geq \sum_{j \in J}(z(I^{1} \cup \{m\}) - \pi(l_{N}^{1};e^{1})) \geq |N^{\alpha}| \sum_{j \in J} \theta_{j} + |J| \sum_{i \in N^{\alpha}\setminus \{m\}} \theta_{i} - |J|c_{\alpha}\) where \(J\) is any subset of \(N^{\beta}\).

**Lemma 13** If \(I^{1} = N^{\alpha}\) then the conjecture is true.
Proof. First note that it is easily verified that $\pi(l_i^N; e^i) = z^i(N^\alpha, \emptyset) \forall i \in N^\alpha$. This is because using Proposition 11 we have that, $\pi(l_i^N; e^i) \leq \pi(l_i^N; e^1) + |N^\beta|(\theta_i - \theta_i)$. However, $z^i(N^\alpha, \emptyset) = \pi(l_i^N; e^1) + |N^\beta|(\theta_1 - \theta_i)$. Therefore,

$$\sum_{i \in N^\alpha} (\pi(l_N; e) - \pi(l_i^N; e^i)) \geq |N^\beta| \sum_{i \in N^\alpha} \theta_i \quad (2.8.2)$$

Claim: $m \notin J^m$: If $m \in J^m$, we must that $\sum_{i \in N^\alpha \setminus I^m} \theta_i - c_\alpha \geq 0$. This will imply that $J^m = N^\beta$. This is a contradiction since $|N_\alpha|c_\beta \leq |N^\beta|c_\beta$.

Let $\pi(l_i^N; e^m) = z^m(I^m, J^m)$. Therefore, using Proposition 12, we obtain that

$$\sum_{j \in J^m \cup \{m\}} (z(I^m; J^m \cup \{m\}) - \pi(l_i^N; e^i)) \geq |N^\alpha| \sum_{j \in (J^m \cup \{m\})} \theta_j + |J^m \cup \{m\}| \sum_{i \in N^\alpha \setminus J^m} \theta_i - |J^m \cup \{m\}|c_\beta \quad (2.8.3)$$

Adding (2.8.2) and (2.8.3) we obtain

$$\sum_{i \in N^\alpha} (\pi(l_N; e) - \pi(l_i^N; e^i)) + \sum_{j \in J^m \cup \{m\}} (z(I^m; J^m \cup \{m\}) - \pi(l_i^N; e^i)) \geq |N^\beta| \sum_{i \in N^\alpha} \theta_i + |N^\alpha| \sum_{j \in (J^m \cup \{m\})} \theta_j + |J^m \cup \{m\}| \sum_{i \in N^\alpha \setminus J^m} \theta_i - |J^m \cup \{m\}|c_\beta \quad (2.8.4)$$

Claim: $\sum_{j \in N^\alpha \setminus (J^m \cup \{m\})} \theta_j - c_\alpha \leq 0$. Since the optimal solution of corresponding to the $m^{th}$ economy is $(I^m, J^m)$ we obtain that $\theta_i + \sum_{j \in N^\alpha \setminus (J^m \cup \{m\})} \theta_j - c_\alpha \leq 0 \forall i \in N^\alpha \setminus I^m$.

As $\theta_i \geq 0$, $\sum_{j \in N^\alpha \setminus (J^m \cup \{m\})} \theta_j - c_\alpha \leq 0$.

Now we can rewrite (2.8.4) as

$$\sum_{i \in N^\alpha} (\pi(l_N; e) - \pi(l_i^N; e^i)) + \sum_{j \in J^m \cup \{m\}} (z(I^m; J^m \cup \{m\}) - \pi(l_i^N; e)) \geq |N^\beta| \sum_{i \in N^\alpha} \theta_i + |N^\alpha| \sum_{j \in (J^m \cup \{m\})} \theta_j + |J^m \cup \{m\}| \sum_{i \in N^\alpha \setminus J^m} \theta_i - |J^m \cup \{m\}|c_\beta$$

$$= |N^\beta| \sum_{i \in N^\alpha \setminus I^m} \theta_i + z(I^m; J^m \cup \{m\}) \geq z(I^m, J^m \cup \{m\}) \quad (2.8.5)$$

Therefore, we obtain that $\sum_{i \in N^\alpha} (\pi(l_N; e) - \pi(l_i^N; e^i)) + |J^m|z(I^m, J^m \cup \{m\}) - \sum_{j \in J^m \cup \{m\}} \pi(l_j^N; e^j) \geq 0$ implying that $(|N^\alpha| + |N^\beta| - 1)\pi(l_N; e) - \sum_{i \in N^\alpha} \pi(l_i^N; e^i) - \sum_{j \in N^\beta} \pi(l_j^N; e^j) \geq 0$. Similarly for $J^m = N^\beta$. 

\begin{lemma}
Suppose that
\begin{enumerate}
\item $1 \notin I^1$ and $m \notin J^m$,
\item $I^1 \cup \{1\} \supseteq I^m$.
\end{enumerate}

Then the conjecture holds.
\end{lemma}
Since \( I^1 \cup \{1\} \supseteq I^m \), we have that \( N^\alpha \setminus I^m \supseteq N^\alpha \setminus (I^1 \cup \{1\}) \). Thus \( |J^1| \sum_{j \in N^\alpha \setminus I^m} \theta_i \geq |J^1| \sum_{N^\alpha \setminus (I^1 \cup \{1\})} \theta_i \). Therefore, we may rewrite 2.8.6 as,

\[
\sum_{i \in I^1 \cup \{1\}} (z(I^1 \cup \{1\}, J^1) - \pi(l_N^i; e^i)) + \sum_{i \in J^1} (z(I^m, J^m \cup \{m\}) - \pi(l_N^i; e^i)) \\
\geq |N^\beta| \sum_{i \in I^1 \cup \{1\}} \theta_i + |I^1 \cup \{1\}| \sum_{j \in N^\beta \setminus J^1} \theta_j - |I^1 \cup \{1\}| c_\beta \\
+|N^\alpha| \sum_{j \in J^1} \theta_j + |J^1| \sum_{j \in N^\alpha \setminus I^m} \theta_i - |J^1| c_\alpha \\
\geq |N^\beta| \sum_{i \in I^1 \cup \{1\}} \theta_i + |I^1 \cup \{1\}| \sum_{j \in N^\beta \setminus J^1} \theta_j - |I^1 \cup \{1\}| c_\beta \\
+|N^\alpha| \sum_{j \in J^1} \theta_j + |J^1| \sum_{j \in N^\alpha \setminus (I^1 \cup \{1\})} \theta_i - |J^1| c_\alpha \\
= z(I^1 \cup \{1\}, J^1) \tag{2.8.7}
\]

Putting Lemmas 13 and 14 together we obtain that following result.

**Lemma 15** If \( I^1 \cup \{1\} \supseteq I^m \) or \( J^m \cup \{m\} \supseteq J^1 \) then the conjecture is true.

**Proof of Proposition 12:** Let \( \tilde{N}^{\beta,T} = \zeta(T, e) + 2 + \frac{c_\alpha}{T} + \frac{c_\beta}{T} \) for some \( T > 0 \). To prove the result it is sufficient to verify that \( N^\beta \geq \tilde{N}^{\beta,T} \).

Assume that for some \( i \in N \), the optimal structure when \( \theta_i = 0 \) is of the type \((I^1, J^1)\), with \( |J^1| \geq 1 \). By the F.O.C.s it must be \( \sum_{k \in N^\alpha \setminus I^1} \theta_k \leq c_\beta \) and \( \sum_{k \in N^\beta \setminus J^1} \theta_k \leq c_\alpha \), or more succinctly,

\[
\sum_{k \in N^\alpha \setminus I^1} \theta_k + \sum_{k \in N^\beta \setminus J^1} \theta_k \leq c_\alpha + c_\beta. \tag{2.8.8}
\]

We assume that \( i \in N^\alpha \). The same proof is valid when \( i \in N^\beta \). We have that \( \sum_{k \in (N^\alpha \setminus I^1) \setminus (L(T,e) \cup \{i\})} \theta_k \geq T|(N^\alpha \setminus I^1) \setminus (L(T,e) \cup \{i\})| \) and \( \sum_{k \in (N^\beta \setminus J^1) \setminus (L(T,e))} \theta_k \geq T|(N^\beta \setminus J^1) \setminus (L(T,e))| \). Therefore we obtain,

\[
\frac{1}{T} \left( \sum_{k \in (N^\alpha \setminus I^1) \setminus (L(T,e) \cup \{i\})} \theta_k \right) \geq \left| (N^\alpha \setminus I^1) \setminus (L(T,e) \cup \{i\}) \right| + \left| (N^\beta \setminus J^1) \setminus (L(T,e)) \right| 
\]

or by using 2.8.13 we obtain

\[
\zeta(T, e) + 1 + \frac{c_\alpha}{T} + \frac{c_\beta}{T} \geq |N^\alpha \setminus I^1| + |N^\beta \setminus J^1| \tag{2.8.10}
\]

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or

\[ |I| + |J| \geq |N^\alpha| + |N^\beta| - \zeta(T, e) - \frac{c_\alpha}{T} - \frac{c_\beta}{T} - 1. \]  

(2.8.11)

Now if \( N^\beta \geq \tilde{N}^{\alpha,T} \), then

\[ |I| + |J| > |N^\alpha|, \]

(2.8.12)

which is a contradiction to the fact that \(|J| \geq 1\). By applying lemma 15 the statement follows.

**Proof of Proposition 13:** Let \( \tilde{N}^{\alpha,T} = \frac{c_\beta}{c_\alpha} \left( \zeta(T, e) + 1 \right) + 1 + \frac{2c_\beta}{T} \) for some \( T > 0 \). It is enough to verify the result if \( N^\alpha \geq \tilde{N}^{\alpha,T} \).

Assume that for some \( i \in \mathbb{N} \), the optimal structure when \( \theta_i = 0 \) is of the type \((I^i, J^i)\), with \(|I^i| \geq 1\). By the F.O.C.s it must be \( \sum_{k \in N^\alpha \setminus I^i} \theta_k \leq c_\beta \) and \( \sum_{k \in N^\beta \setminus J^i} \theta_k \leq c_\alpha \), or more succinctly

\[ c_\alpha \sum_{k \in N^\alpha \setminus I^i} \theta_k + c_\beta \sum_{k \in N^\beta \setminus J^i} \theta_k \leq 2c_\alpha c_\beta. \]  

(2.8.13)

We assume that \( i \in N^\alpha \). The same proof is valid when \( i \in N^\beta \). We have \( \sum_{k \in (N^\alpha \setminus I^i) \setminus (L(T, e) \cup \{i\})} \theta_k \geq T|(N^\alpha \setminus I^i) \setminus (L(T, e))| \) and \( \sum_{k \in (N^\beta \setminus J^i) \setminus (L(T, e))} \theta_k \geq T|(N^\beta \setminus J^i) \setminus (L(T, e))| \). Therefore we obtain,

\[ \frac{1}{T}(c_\alpha \sum_{k \in (N^\alpha \setminus I^i) \setminus (L(T, e) \cup \{i\})} \theta_k + c_\beta \sum_{k \in (N^\beta \setminus J^i) \setminus (L(T, e))} \theta_k) \geq c_\alpha |(N^\alpha \setminus I^i) \setminus (L(T, e) \cup \{i\})| + c_\beta |(N^\beta \setminus J^i) \setminus (L(T, e))| \]  

(2.8.14)

or using (2.8.13)

\[ c_\beta \zeta(T, e) + c_\beta + \frac{2c_\beta c_\alpha}{T} \geq c_\alpha |N^\alpha \setminus I^i| + c_\beta |N^\beta \setminus J^i| \]

(2.8.15)

or

\[ c_\alpha |I^i| + c_\beta |J^i| \geq c_\alpha |N^\alpha| + c_\beta |N^\beta| - c_\beta \zeta(T, e) - \frac{2c_\alpha c_\beta}{T} - c_\beta. \]  

(2.8.16)

Now if \( N^\alpha \geq \tilde{N}^{\alpha,T} \), then

\[ c_\alpha |I^i| + c_\beta |J^i| > c_\beta |N^\beta| \]

(2.8.17)

which is a contradiction to the fact that \(|I^i| \geq 1\). By applying lemma 15 the statement follows.

### 2.9 Appendix 2. Proof of Proposition 5

A mechanism \( \varphi \in \Phi \) satisfies **Condition A** if and only if for each \( e = (\theta_N, c) \in \mathbb{R}_+^3 \) and each \( i \in N \), there does not exist \( \epsilon > 0 \) such that

\[ (l_N(x), t_N(x)) = \varphi((x, \theta_N \setminus \{i\}), c), \]

and for each \( x \geq 0 \), \( \sum_{j \in N} t_j(x) + \epsilon > 0. \)
**Proposition 15** A mechanism \( \varphi \in \Phi \) is Second Best Efficient only if it satisfies Condition A.

**Proof.**
Suppose that \( \varphi \in \Phi \) is **Second Best Efficient**. By way of contradiction let there exist \( c' = (\theta'_N, c') \in \mathbb{R}_+^3 \) and \( \epsilon > 0 \) such that

\[
(t'_N(x), t'_N(x)) = \varphi((x, \theta'_{N \setminus \{j\}}), t'_N, C'),
\]

and similarly, for each \( i \) each \( \hat{e} \), by construction, so does \( \hat{e} \). Therefore, \( \varphi(e) = \varphi(e) \). Otherwise, \( \varphi((x, \theta'_{N \setminus \{i\}}), c') = (t'_N(x), t'_N(x)) \), where, for each \( x \geq 0 \),

1. \( t'_N(x) = t'_N(x) \),
2. for each \( i \in N \setminus \{j\} \), \( t'_l(x) = t'_l(x) \),
3. \( t'_j(x) = t'_j(x) + \epsilon \),

and similarly, for each \( x \geq 0 \), we construct \( \varphi((\theta'_{N \setminus \{i\}}, x), c') \) from \( \varphi((\theta'_{N \setminus \{i\}}, x), c') \). By assumption, \( \varphi \) satisfies Strategy-Proofness and Anonymity. Hence, by construction, so does \( \varphi \). By assumption, \( \varphi \) satisfies Feasibility. Hence, since the negation of **Condition A** is true, by construction, so does \( \varphi \). Therefore, \( \varphi \in \Phi \). By construction, for each \( e \in E \) and each \( i \in N \),

\[
u_i(\varphi(e) ; \theta_i) \geq u_i(\varphi(e) ; \theta_i).
\]

Moreover, by construction, for each \( x \geq 0 \),

\[
u_j(\varphi((x, \theta'_{N \setminus \{i\}}), t'_N, C') ; \theta_j) > u_j(\varphi((x, \theta'_{N \setminus \{i\}}), t'_N, C') ; \theta_j).
\]

Therefore, \( \varphi \) cannot be **Second Best Efficient**, a contradiction.

**Proposition 16** If a mechanism \( \varphi \) satisfies Strategy-Proofness, Anonymity and Assignment Efficiency, then there exists some function \( f : \mathbb{R}_+ \rightarrow \mathbb{R} \), such that for each \( e = (\theta_N, c) \in E \), with \( (l_N, t_N) = \varphi(e) \),

\[
t_1 = \begin{cases} 
  v_2(l_N; \theta_2) + f(\theta_2) & \text{if } \theta_2 \leq c, \\
  f(\theta_2) & \text{if } \theta_2 > c.
\end{cases}
\]

and

\[
t_2 = \begin{cases} 
  v_1(l_N; \theta_1) + f(\theta_1) & \text{if } \theta_1 \leq c, \\
  f(\theta_1) - c & \text{if } \theta_1 > c.
\end{cases}
\]

**Proof.**
The set of economies is convex. Once more we appeal to Holmstrom’s characterization [30]. A mechanism \( \varphi \) satisfies Strategy Proofness and Assignment Efficiency if and only if there
exists some function $h_i : \mathbb{R}_+ \to \mathbb{R}$ such that for each $e = (\theta_N, c) \in \mathcal{E}$, with $(l_N, t_N) = \varphi(e)$, we have $l_N \in \Sigma(e)$, and, for each $i \in N$,

$$t_i = v_{N \setminus \{i\}}(l_i; \theta_{N \setminus \{i\}}) - h_i(\theta_{N \setminus \{i\}})$$

Setting

$$h_i(\theta_{N \setminus \{i\}}) = \begin{cases} -f_i(\theta_{N \setminus \{i\}}) & \text{if } \theta_{N \setminus \{i\}} \leq c, \\ v_{N \setminus \{i\}}(l_i; \theta_{N \setminus \{i\}}) - f_i(\theta_{N \setminus \{i\}}) & \text{if } \theta_{N \setminus \{i\}} > c, \end{cases}$$

we obtain, for each $i \in N$,

$$t_i = \begin{cases} v_{N \setminus \{i\}}(l_i; \theta_{N \setminus \{i\}}) + f_i(\theta_{N \setminus \{i\}}) & \text{if } \theta_{N \setminus \{i\}} \leq c, \\ f_i(\theta_{N \setminus \{i\}}) & \text{if } \theta_{N \setminus \{i\}} > c, \end{cases} \quad (2.9.1)$$

Let $\theta \in [0, \frac{c}{2})$ and $(l_N, t_N) = \varphi((\theta, \theta), c)$. By Assignment Efficiency, $l_N = (0, 0)$. Hence, using (2.9.1), we obtain $u_1(\varphi_i(l_N, t_N)) = f_1(\theta)$ and $u_2(\varphi_i(l_N, t_N)) = f_2(\theta)$. By Anonymity, $u_1(\varphi_i(l_N, t_N)) = u_2(\varphi_i(l_N, t_N))$. Hence,

for each $\theta \in [0, \frac{c}{2}]$, $f_1(\theta) = f_2(\theta). \quad (2.9.2)$

Let $\theta \in [\frac{c}{2}, c]$ and $(l_N, t_N) = \varphi((\theta, \theta), c)$. By Assignment Efficiency, $l_N = (1, 0)$. Hence, using (2.9.1), we obtain $u_1(\varphi_i(l_N, t_N)) = \theta - c + [f_1(\theta)]$ and $u_2(\varphi_i(l_N, t_N)) = \theta + [\theta - c + f_2(\theta)]$. By Anonymity, $u_1(\varphi_i(l_N, t_N)) = u_2(\varphi_i(l_N, t_N))$. Hence,

for each $\theta \in [\frac{c}{2}, c]$, $f_1(\theta) = f_2(\theta). \quad (2.9.3)$

Let $\theta \in (c, +\infty)$ and $(l_N, t_N) = \varphi((\theta, \theta), c)$. By Assignment Efficiency, $l_N = (1, 0)$. Hence, using (2.9.1), we obtain $u_1(\varphi_i(l_N, t_N)) = \theta - c + f_1(\theta)$ and $u_2(\varphi_i(l_N, t_N)) = \theta + f_2(\theta)$. By Anonymity, $u_1(\varphi_i(l_N, t_N)) = u_2(\varphi_i(l_N, t_N))$. Hence, for each $\theta \in (c, +\infty)$, $f_1(\theta) = f_2(\theta) + c.$

for each $\theta \in (c, +\infty)$, $f_1(\theta) = f_2(\theta) + c. \quad (2.9.4)$

Setting for each $\theta \in \mathbb{R}_+$, $f(\theta) = f_1(\theta)$ and combining (2.9.1), (2.9.2), (2.9.3), (2.9.4), we obtain the desired result.

\textbf{Proposition 17} If a mechanism $\varphi$ satisfying Strategy-Proofness and Assignment Efficiency is Second Best Efficient, then for each $c > 0$ the function $f$ corresponding to $\varphi$ is such that for each $\theta_1, \theta_2 \in [0, c]$, with $\theta_1 < \theta_2$,

$$f(\theta_2) + \theta_2 \geq f(\theta_1) + \theta_1$$

\textbf{Proof.}

Suppose not. Let there exist $\theta', \theta'' \in [0, c]$, with $\theta' < \theta''$, and $\epsilon > 0$ such that

$$f(\theta'') + \theta'' + \epsilon = f(\theta') + \theta'.$$

Consequently, $f(\theta') - [f(\theta'') + \epsilon] = \theta'' - \theta'$. Since, by assumption, $\theta' - \theta' > 0$, we obtain

$$f(\theta') > f(\theta'') + \epsilon \quad (2.9.6)$$

Define $g : \mathbb{R}_+ \to \mathbb{R}$ to be such that $g(\theta) = f(\theta)$, for each $\theta \in \mathbb{R}_+ \setminus \{\theta''\}$, and $g(\theta'') = f(\theta'') + \epsilon$. 

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Using (2.9.5) and (2.9.6), we obtain for each $\theta$ that

$$
S^f(\theta_1, \theta_2) = \begin{cases} 
    f(\theta_2) + f(\theta_1) & \text{if } \theta_1 + \theta_2 \leq c, \\
    f(\theta_2) + \theta_2 + f(\theta_1) + \theta_1 - c & \text{if } \theta_1 + \theta_2 > c \text{ and } \theta_1, \theta_2 \leq c, \\
    f(\theta_2) + \theta_2 + f(\theta_1) & \text{if } \theta_1 > c \text{ and } \theta_2 \leq c, \\
    f(\theta_2) + f(\theta_1) + \theta_1 - c & \text{if } \theta_1 \leq c \text{ and } \theta_2 > c, \\
    f(\theta_2) + f(\theta_1) - c & \text{if } \theta_1 > c \text{ and } \theta_2 > c.
\end{cases}
$$

By construction, for each $(\theta_1, \theta_2) \in \mathbb{R}_+^2$, $S^g(\theta_1, \theta_2) \leq S^f(\theta_1, \theta_2)$.

Similarly, for each $(\theta_2) \in \mathbb{R}_+$, $S^g(\theta_0, \theta_2) \leq S^f(\theta_0, \theta_2)$.

By construction, for each $(\theta_1, \theta_2) \in \mathbb{R}_+^2$, such that $\theta_1 \neq \theta_0$ and $\theta_2 \neq \theta_0$, $S^g(\theta_1, \theta_2) = S^f(\theta_1, \theta_2)$. Therefore, for each $(\theta_1, \theta_2) \in \mathbb{R}_+^2$, $S^g(\theta_1, \theta_2) \leq 0$. The groves mechanism induced by $g$ satisfies Feasibility.

By construction,

$$(\theta_1, \theta_2) \in \{(\theta_1, \theta_2) \in \mathbb{R}_+^2 : \text{either } \theta_1 = \theta_0 \text{ or } \theta_2 = \theta_0 \text{ or both}\}$$

**Lemma 16** For each $\theta > c$, $f(\theta) = \frac{c}{2}$.

**Proof.**

Suppose first, by way of contradiction, that for some $\tilde{\theta} > c$, $f(\tilde{\theta}) > \frac{c}{2}$. By Lemma 16, letting $(t_N, t_N) = \varphi((\tilde{\theta}, \tilde{\theta}), c)$, $t_1 + t_2 = 2f(\tilde{\theta}) - c$. By assumption, $f(\tilde{\theta}) > \frac{c}{2}$, therefore $t_1 + t_2 > 0$, which contradicts Feasibility.

Suppose then that for some $\tilde{\theta} > c$, $f(\tilde{\theta}) < \frac{c}{2}$. In particular, let $f(\tilde{\theta}) = \frac{c}{2} - \epsilon$, for some $\epsilon > 0$.

Consider first profile $(\tilde{\theta}, \tilde{\theta})$, for some $\theta \in [0, c]$. By Lemma 16, letting $(t_N, t_N) = \varphi((\tilde{\theta}, \tilde{\theta}), c)$, $t_1 + t_2 = \frac{c}{2} - \epsilon + \theta - c + f(\theta)$. By Lemma 17,

$$
\frac{c}{2} - \epsilon + \theta - c + f(\theta) \leq \frac{c}{2} - \epsilon + f(c) + c = \frac{c}{2} - \epsilon + f(c).
$$

Note that, by Feasibility, $f(c) \leq -\frac{c}{2}$. Hence,

$$
\frac{c}{2} - \epsilon + f(c) \leq \frac{c}{2} - \epsilon - \frac{c}{2} = -\epsilon < 0.
$$

Consider, finally, profile $(\tilde{\theta}, \tilde{\theta})$, for some $\theta \in [c, +\infty)$. Letting again $(t_N, t_N) = \varphi((\tilde{\theta}, \tilde{\theta}), c)$, and appealing to Lemma 16, we obtain $t_1 + t_2 = \frac{c}{2} - \epsilon + f(\theta) - c$. We have already established earlier in the proof that $f(\theta) \leq \frac{c}{2}$. Hence, $t_1 + t_2 \leq \frac{c}{2} - \epsilon + \frac{c}{2} - c = -\epsilon < 0$.

Hence, for each $\theta \in \mathbb{R}_+$, $t_1 + t_2 < 0$, which contradicts Condition A.

**Lemma 17** For each $\theta \in [\frac{c}{2}, c]$, $f(\theta) \leq \frac{c}{2} - \theta$.

**Proof.**

By Lemma 16, $S(\theta, \theta) = 2f(\theta) + 2\theta - c$. By Feasibility, $S(\theta, \theta) \leq 0$. Hence, $f(\theta) \leq \frac{c}{2} - \theta$.

$$
S(\tilde{\theta}, \tilde{\theta}) \leq f(\tilde{\theta}) + f(\tilde{\theta}) + \tilde{\theta} - c.
$$

**Lemma 18** For each $\theta \in [0, \frac{c}{2}]$, $f(\theta) = 0$.
Proof.
Suppose not. Let there exist some \( \tilde{\theta} \in [0, \frac{c}{2}] \) such that \( f(\tilde{\theta}) \neq 0 \). If \( f(\tilde{\theta}) > 0 \) we would obtain \( S(\tilde{\theta}, \theta) = 2f(\tilde{\theta}) > 0 \), a violation of Feasibility. Therefore, it must be \( f(\tilde{\theta}) < 0 \).

**Step 1.** Let \( \theta \in (c, +\infty) \). By Lemma 16, using the fact that \( \tilde{\theta} \in [0, \frac{c}{2}] \) and \( \theta \in (c, +\infty) \), we obtain
\[
S(\tilde{\theta}, \theta) = f(\theta) + \tilde{\theta} - c + f(\tilde{\theta}).
\]
By Lemma 16, using the fact that \( \theta \in (c, +\infty) \),
\[
S(\tilde{\theta}, \theta) = \frac{c}{2} + \tilde{\theta} - c + f(\tilde{\theta}) = -\frac{c}{2} + \tilde{\theta} + f(\tilde{\theta}).
\]
By assumption, \( f(\tilde{\theta}) < 0 \) and \( -\frac{c}{2} + \tilde{\theta} \leq 0 \), hence
for each \( \theta \in (c, +\infty) \), \( S(\tilde{\theta}, \theta) < 0 \).

**Step 2.** Let \( \theta \in [0, \frac{c}{2}] \). By Lemma 16, using the fact that \( \tilde{\theta}, \theta \in [0, \frac{c}{2}] \), we obtain
\[
S(\tilde{\theta}, \theta) = f(\theta) + f(\tilde{\theta}).
\]
By Lemma 16, using the fact that \( \theta \in [0, \frac{c}{2}] \), \( \theta, \theta \in [0, \frac{c}{2}] \), we obtain
\[
S(\tilde{\theta}, \theta) = \frac{c}{2} + \tilde{\theta} - c + f(\tilde{\theta}) = -\frac{c}{2} + \tilde{\theta} + f(\tilde{\theta}).
\]
By assumption, \( f(\tilde{\theta}) < 0 \) and \( -\frac{c}{2} + \tilde{\theta} \leq 0 \), hence
for each \( \theta \in [0, \frac{c}{2}] \), \( S(\tilde{\theta}, \theta) < 0 \).

**Step 3.** Let \( \theta \in (\frac{c}{2}, c] \). By Lemma 16, using the fact that \( \tilde{\theta} \in [0, \frac{c}{2}] \) and \( \theta \in (\frac{c}{2}, c] \), we obtain
\[
S(\tilde{\theta}, \theta) = f(\theta) + \theta - c + f(\tilde{\theta}) + \tilde{\theta}.
\]
By Lemma 17, substituting for \( f(\theta) \), we obtain
\[
S(\tilde{\theta}, \theta) \leq f(\tilde{\theta}) + \tilde{\theta} - \frac{c}{2}.
\]
By assumption, \( f(\tilde{\theta}) < 0 \) and \( \tilde{\theta} \in [0, \frac{c}{2}] \), hence
for each \( \theta \in (\frac{c}{2}, c] \), \( S(\tilde{\theta}, \theta) < 0 \).

**Step 3.** From Steps 1-3 we obtain that for each \( \theta \in \mathbb{R}_+ \), \( S(\tilde{\theta}, \theta) < 0 \). This constitutes a violation of Condition A.

\[\blacksquare\]

**Lemma 19** For each \( \theta \in (\frac{c}{2}, c] \), \( f(\theta) = \frac{c}{2} - \theta \).

*Proof.* Suppose not. Let there exist some \( \tilde{\theta} \in (\frac{c}{2}, c] \) such that \( f(\tilde{\theta}) \neq \frac{c}{2} - \theta \). By Lemma 17, it can’t be \( f(\theta) > \frac{c}{2} - \theta \). Therefore, it must be \( f(\tilde{\theta}) < \frac{c}{2} - \theta \).

**Step 1.** Let \( \theta \in (c, +\infty) \). By Lemma 16, using the fact that \( \tilde{\theta} \in (\frac{c}{2}, c] \) and \( \theta \in (c, +\infty) \), we obtain
\[
S(\tilde{\theta}, \theta) = f(\theta) + f(\tilde{\theta}) + \tilde{\theta} - c.
\]
By Lemma 16, using the fact that \( \theta \in (c, +\infty) \) and substituting for \( f(\theta) \), this becomes

\[ S(\tilde{\theta}, \theta) = f(\tilde{\theta}) + \tilde{\theta} - \frac{c}{2}. \]

By assumption, \( f(\tilde{\theta}) < \frac{c}{2} - \theta \), hence

for each \( \theta \in (c, +\infty) \), \( S(\tilde{\theta}, \theta) < 0 \).

**Step 2.** Let \( \theta \in [0, c] \) and \( \hat{\theta} + \theta \leq c \). By Lemma 16,

\[ S(\hat{\theta}, \theta) = f(\theta) + f(\hat{\theta}). \]

Since \( \hat{\theta} + \theta \leq 0 \) and \( \hat{\theta} \in (\frac{c}{2}, c] \) it must be \( \theta < \frac{c}{2} \). Therefore, by Lemma 18, \( f(\theta) = 0 \). Consequently, \( S(\hat{\theta}, \theta) = f(\hat{\theta}) \). By assumption, \( f(\hat{\theta}) < \frac{c}{2} - \theta \) and \( \hat{\theta} > \frac{c}{2} \). Hence,

if \( \hat{\theta} + \theta \leq c \), then for each \( \theta \in [0, c] \), \( S(\tilde{\theta}, \theta) < 0 \).

**Step 3.** Let \( \theta \in [0, c] \) and \( \hat{\theta} + \theta > c \). By Lemma 16,

\[ S(\hat{\theta}, \theta) = f(\theta) + \theta + f(\hat{\theta}) + \hat{\theta} - c. \]

By assumption, \( f(\hat{\theta}) < \frac{c}{2} - \theta \). Hence, substituting for \( f(\hat{\theta}) \) we obtain

\[ S(\hat{\theta}, \theta) < \theta + f(\theta) - \frac{c}{2}. \]

Since \( \theta < c \), by Lemma 17, \( f(c) + c \geq f(\theta) + \theta \). Substituting for \( f(\theta) + \theta \) we obtain

\[ S(\hat{\theta}, \theta) < \frac{c}{2} + f(c). \]

By Lemma 17, \( f(c) \leq -\frac{c}{2} \), so that

if \( \hat{\theta} + \theta > c \), then for each \( \theta \in [0, c] \), \( S(\tilde{\theta}, \theta) < 0 \).

**Step 4.** From Steps 1-3 we obtain that for each \( \theta \in \mathbb{R}_+ \), \( S(\tilde{\theta}, \theta) < 0 \). This constitutes a violation of Condition A.

The function \( f \) corresponding to the mechanism depicted in Figure 2.5 is

\[ f(\theta) = \begin{cases} 
0 & \text{if } \theta \leq \frac{c}{2} \\
\frac{c}{2} - \theta & \text{if } \frac{c}{2} < \theta \leq c \\
\frac{c}{2} & \text{if } \theta > c
\end{cases} \]

The preceding results prove the only if part. Let us then prove that the SBM is indeed Second Best Efficient.

**Lemma 20** If a mechanism \( \varphi \) is Strategy-Proof, then for each profile \( (\theta_1, \theta_2) \in \mathbb{R}^+_2 \) such that \( (l_N, t_N) = \varphi((\theta_1, \theta_2), c) \) and \( l_N = ((1, 1), (0, 1)) \), there exists \( (p_1, p_2) \in \mathbb{R}_+ \) such that for each \( i = 1, 2 \),

- for each \( x < p_i \), letting \( (l_N(x), t_N(x)) = \varphi((x, \theta_N \setminus \{i\}), c) \), \( l_N(x) = ((1, 0), (0, 1)) \) and \( t_i(x) = t_i(0) \),

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Pareto dominates the mechanism depicted in Figure 2.5. Therefore, for each

\[ \phi \in \Phi \text{, } \phi \text{ Pareto dominates the SBM, then } \phi' \in \Phi \text{ coincides with the SBM at profiles } (\theta_1, \theta_2) \in \mathbb{R}_+^2 \text{ such that either } \theta_1, \theta_2 \geq \frac{5}{2} \text{ or } \theta_1, \theta_2 \leq \frac{5}{2}. \]

**Proof.**

This follows straightforwardly from the fact that the mechanism depicted in Figure 2.5 is first best, i.e. assignment efficient and exactly feasible, at those profiles.

**Proposition 18** There does not exist a mechanism \( \phi' \in \Phi \) that Pareto dominates the SBM.

By Lemma 6 and Anonymity, if \( \phi' \) Pareto dominates the mechanism depicted in Figure 2.5 it must be so for some profile \((\theta_1, \theta_2) \in [0, \frac{5}{2}] \times \mathbb{R}_+\). For each \( c \geq 0 \), each \( \theta_1 \in \mathbb{R}_+ \) and each \( x \geq 0 \), let \((t_1^0(x), t_2^0(x)) = \phi'((\theta_1, x), c)\). Suppose that for some \( \theta_1 \in [0, \frac{5}{2}] \) and each \( x \geq 0 \),

\[ (t_1^0(x), t_2^0(x)) = \phi'((\theta_1, x), c) \text{ and } t_2^0(x) = ((1, 0)(0, 1)). \]

By Strategy-Proofness, for each \( \theta_1 \in [0, \frac{5}{2}] \) and each \( x > 0 \), \( t_2^0(x) = \hat{t}_2^0(0) \). By Lemma 6, \( \hat{t}_2^0(0) = 0 \). At each profile \((\theta_1, x)\), with \( x > c \), the utility of individual 2, under the mechanism depicted in Figure 2.5 is positive. By contrast the utility of individual 2, under \( \phi' \), is equal to zero. This contradicts the fact that \( \phi' \) Pareto dominates the mechanism depicted in Figure 2.5. Therefore, for each \( \theta_1 \in [0, \frac{5}{2}] \) and some \( y \geq 0 \),

\[ (t_1^0(y), t_2^0(y)) = \phi'((\theta_1, y), c) \text{ and } t_2^0(y) = ((1, 0)(0, 1)). \]

By Lemma 5 there exists \( p \geq 0 \) such that

- if \( x < p \), then \( \hat{t}_1^0(x) = ((1, 0)(0, 1)) \) and \( \hat{t}_2^0(x) = t_2(0) \), and
- if \( x \geq p \), then \( \hat{t}_1^0(x) = ((1, 0)(0, 1)) \) and \( \hat{t}_2^0(x) = -p + t_2(0) \).

By Lemma 6, \( t_2(0) = 0 \). If \( p > c - \theta_1 \), then the utility of individual 2, under \( \phi' \), at profile \((\theta_1, \theta_2)\), with \( \theta_2 > p \), equals \( \theta_2' - p < \theta_2' + \theta_1 - c \). The utility of individual 2, under the mechanism depicted in Figure 2.5, at profile \((\theta_1, \theta_2)\), with \( \theta_2 > p \), equals \( \theta_2' + \theta_1 - c \). This contradicts the fact that \( \phi' \) Pareto dominates the mechanism depicted in Figure 2.5. If \( p < c - \theta_1 \), then the utility of individual 2, under \( \phi' \), at profile \((\theta_1, \theta_2)\), with \( c - \theta_1 < \theta_2' < p \), equals \( \theta_2' + \theta_1 - c > 0 \). This contradicts the fact that \( \phi' \) Pareto dominates the mechanism depicted in Figure 2.5. Therefore, for each \( \theta_1 \in [0, \frac{5}{2}] \),

- if \( x < \theta_1 - c \), then \( \hat{t}_1^0(x) = ((1, 0)(0, 1)) \) and \( \hat{t}_2^0(x) = 0 \), and
- if \( x \geq \theta_1 - c \), then \( \hat{t}_1^0(x) = ((1, 0)(0, 1)) \) and \( \hat{t}_2^0(x) = \theta_1 - c \).

Moreover, by Strategy-Proofness, for each \( \theta_1 \in [0, \frac{5}{2}] \) and each \( x \geq 0 \), such that \( \theta_1 + x > c \), \( \hat{t}_1^0(x) = \hat{t}_2^0(x) \). By lemma 5, for each \( \theta_1 \in [0, \frac{5}{2}] \) and each \( x > \frac{5}{2} \), such that \( \theta_1 + x \leq c \), \( \hat{t}_1^0(x) = \theta_1 - c + \hat{t}_2^0(x) \). By Lemma 6, \( \hat{t}_2^0(x) = \frac{5}{2} \). Finally, by Lemma 6 and the assumption that \( \phi' \) Pareto dominates the mechanism depicted in Figure 2.5, for each \( \theta_1 \in [0, \frac{5}{2}] \) and each \( x \leq \frac{5}{2} \), \( \hat{t}_1^0(x) = 0 \). Therefore, for each profile \((\theta_1, \theta_2) \in [0, \frac{5}{2}] \times \mathbb{R}_+ \), \( \phi' \) is Pareto indifferent to the mechanism depicted in Figure 2.5, a contradiction.

\[ \square \]
Chapter 3

A Solomonic Solution to the Problem of Assigning a Private Indivisible Object

3.1 Introduction

The Friday morning flight from Pittsburgh to Philadelphia has been overbooked. Two individuals lay claim to the last available seat. One will be later boarding an overseas flight operated by a different airline. The other needs to attend to a pressing business engagement later in the day. They have both made their case to the representative of the airline and they are awaiting her decision. To whom should the last seat be assigned?

This example captures the core features of the problem we study in this paper. A benevolent Planner wishes to assign an indivisible private good to \( n \) claimants. Individuals attach different values to the good. Nonetheless all individuals attach the same value to money. Pecuniary transfers are, therefore, possible. If such transfers are to be carried out, they will need to be financed by the surplus the assignment generates.

Our two morning travelers to Philadelphia have incentives to misreport their valuation of the good. The representative of the the airline may legitimately wonder whether, for instance, either individual is telling the truth about the predicament they will be in if they miss their flight. A mechanism that solves the problem needs to overcome this challenge. All the mechanisms we study in this paper will be Strategy-Proof, that is, they will induce, for each individual, a dominant strategy to tell the truth.

Strategy-Proofness constitutes a robust form of implementation. It is more likely that any given individual will act on a dominant strategy compared to a strategy that becomes optimal only under a certain presumption on how the other claimants will act. Moreover, Strategy-Proof implementation is prior free. It does not require knowledge of the distribution of individual valuations.

When king Solomon was presented with the problem of assigning custody of a newborn child, he opted for a seemingly paradoxical solution: splitting the baby. His mechanism proved successful both in eliciting information (the actual mother revealed herself, albeit indirectly) and in assigning the child to the person who valued it the most. The biblical story emphasizes the importance of Assignment Efficiency. This property requires of mechanisms to always assign the good and, moreover, to the individual that values it the most. King Solomon’s commitment to Assignment Efficiency is far from unwavering. He seemed keen on uncovering the true mother, yet in doing so he employed a method that
could have resulted in the child’s death. Thus, the solomonic solution does not fully comply with both the requisites of Assignment Efficiency.

In economic domains the family of Vickrey-Clarke-Groves mechanism (see Vickrey [65], Clarke [11], Groves [28]) is characterized by Strategy-Proofness and Assignment Efficiency (see Holmstrom [30]). Generically, Vickrey-Clarke-Groves mechanisms fail to balance the budget (see Green and Laffont [25]). As a consequence, adhering to Strategy-Proofness and Assignment Efficiency produces a welfare loss. In particular, in the framework we are exploring, the waste takes the form of a budget deficit.

Bailey [4] in the framework of public goods and Cavallo [9] in a context similar to ours discuss Vickrey-Clarke-Groves mechanisms that involve both payments from the individuals to the center along with a redistribution scheme that alleviates the problem of the deficit. Moulin [45] and Guo and Conitzer [27] follow up on this idea to provide us with the optimal way of carrying out this exercise. The significance of these two papers lies in the fact that they paint a precise picture of the minimal welfare loss Vickrey-Clarke-Groves mechanisms entail.

A loss of welfare in the presence of informational constraints manifests in two ways: either in the form of a deficit, as in the case of Vickrey-Clarke-Groves, or in the form of an inefficient assignment. Therefore, a natural question to raise is whether mechanisms that do not commit to Assignment Efficiency are welfare superior to Vickrey-Clarke-Groves.

DeClippel et al. [12] take up this issue. They show that the welfare superior mechanism among those satisfying Strategy-Proofness and other desirable properties involves destroying the good at times. The upshot is that by abandoning Assignment Efficiency a welfare improvement ensues.

A common feature of the papers above is that they base their conclusions on a metric that captures the welfare loss. Mechanisms are compared with each other on the basis of the sum of utilities the allocations they prescribe produce. The approach we employ diverges from this path. Rather than carrying out comparisons on the basis of welfare dominance, we resort instead to Pareto dominance.

Roughly speaking, a mechanism Pareto dominates another when all individuals agree that the outcome under the first is better than the outcome under the second. A mechanism is Second-Best Efficient if it is Strategy-proof and, moreover, there does not exist another Strategy-proof mechanism that Pareto dominates it.

The question we raise then is:

What is the set of Second-Best Efficient mechanisms?

Similar to DeClippel et al. [12], we pose the question in a way that creates no commitment to Assignment Efficiency. A priori, there is no way of knowing whether a mechanism is Second-Best because it is in deficit or because it violates Assignment Efficiency or both. Our project involves investigating the way a violation of Assignment Efficiency trades off with the deficit.

In models of provision of a public good the possibility of exclusion addresses the incentive problem. A concrete example is the Serial Cost Sharing mechanism [43]. A series of papers (Moulin [44], Olszewski [52] and Maniquet and Sprumont [35] among others) discuss the merits of such a mechanism. For our purposes it suffices to note that these papers forcefully demonstrate that exclusion is not detrimental and, under certain circumstances, necessary for identifying the class of Second-Best Efficient mechanisms.

In figure 3.1 we juxtapose a mechanism that sometimes destroys the good, a Solomonic mechanism, with a Vickrey-Clarke-Groves mechanism, in a simple economy comprising two individuals. The non-negative real orthant is the set of admissible economies, each
point specifying a valuation for the object by each individual 1 and 2. At each profile of valuations the information in parentheses specifies the allocation the mechanism prescribes. The first number is 1 if the individual obtains the good and 0 if he does not. The second number is his transfer.

The mechanism on right hand side is the standard Vickrey (often called pivotal) mechanism. The individual with the highest valuation obtains the good and pays the second highest valuation (the other valuation in our simple setting). The individual with the lowest valuations neither pays nor receives any money.

The mechanism on the left hand side sets some price \( p \) and destroys the good if both individuals report a valuation less than \( p \). If at least one individual announces a valuation higher than \( p \) the good is assigned and, moreover, to the individual that values it the most. Whenever the good is assigned, transfers between individuals are carried out. These transfers depend on \( p \). This \( p \)-dependent mechanism accomplishes two things. First, in most profiles it wastes less money that the mechanism on the right. In some profiles it is balanced. Second, and perhaps more importantly, it compensates the "loser", the individual that ends up empty handed.

Figure 2 elaborates the relative advantages of each mechanism. On the left hand side, shaded, we isolate the profiles for which the individual with the highest valuation prefers Vickrey over Solomon. If \( \theta_2 > \theta_1 \) and \( \theta_1 < p \), individual 2 foregoes a utility equal to \( \theta_2 - \theta_1 \) if \( \theta_2 < p \), or equal to \( \theta_2 - p \) if \( \theta_2 \geq p \). Contrary to that, if \( \theta_2 > \theta_1 \), individual 1, the 'loser', always prefers Solomon over Vickrey. Along the bold line the two mechanisms are welfare equivalent. This is so even though on part of the bold line (the one that coincides with the 45° line) the two mechanisms disagree on the issue of the assignment. On the right hand side we identify the set of profiles for which Vickrey generates a smaller waste compared to Solomon. This occurs if \( \theta_1 + \theta_2 < 2p \). Otherwise, Solomon is less wasteful.

![Figure 3.1: The Solomonic solution (left) versus the standard pivotal mechanism (right), in a two person economy.](image)

Which, if any, between the two solutions depicted in figure 3.1 is Second-Best efficient? In order to answer this question we need to take an intermediate step.

Moulin [45], Guo and Conitzer [27] and deClippel [12], all study a more general model than the one we investigate. Their model comprises \( m \) homogeneous goods and \( n \) individuals, with \( m < n \), the so called rationing problem. The benefit of restricting our attention
to a single object scenario is that in this framework Anonymity, the requirement that the assignment does rely on the names of individuals, plays a crucial role. Strategy-Proofness and Anonymity imply Weak Assignment Efficiency, a property that requires the good to be assigned to the individual that values it the most at any profile at which the good is not destroyed. Effectively, for Strategy-Proof and Anonymous mechanisms, a departure from First-Best occurs either by means of a deficit or by destroying the good, but not by assigning the good inefficiently. Therefore, a Strategy-Proof and Anonymous Second-Best mechanism may violate Assignment Efficiency only in so far as it destroys the good.

Our main result can be summarized as follows: we identify three conditions that are necessary for a mechanism to be Second-Best efficient. Moreover, if a Strategy-Proof and Anonymous mechanism satisfies these three conditions together with Voluntary Participation, the requirement that no one is coerced into participating, then there does not exist another Strategy-Proof and Anonymous mechanism that Pareto dominates it. This result isolates a single Vickrey-Clarke-Groves mechanism, the one depicted in figure 3.1, as well as a class of mechanisms that rely on some price \( p \) and that destroy the good at certain profiles, such as the one depicted in figure 3.1. The crux of the matter is that destroying the good is in line with Second-Best efficiency.

A corollary result that comes out of the analysis is that, under the proviso that the good needs to be assigned always, Strategy-Proofness and Anonymity imply Assignment Efficiency and, therefore, by Holmstrom [30], Vickrey-Clarke-Groves. This result does depend on the assumption of transferable utilities. In section 6 we extend this result to a model comprising 2 heterogeneous goods and two individuals.

The solution we depict on the left hand side of figure 3.1 needs to be distinguished from the second-price auction with a reservation price, which features prominently in the auctions literature (see Krishna [31]). Myerson [46] shows that optimal selling mechanisms take the form of auctions that incorporate a reservation price. In figure 3.3 we demonstrate this possibility. We reproduce the Vickrey-Clarke-Groves mechanism of figure 3.1 with the qualification that if both individuals announce a valuation lower than the reservation price \( p \) the good is not assigned. In the auctions setting the reservation price is intended to defend the interests of the seller. In our setting the seller is a benevolent planner whose interests fully align with the interests of the individuals. That is a prima facie difference. The solution we propose as an alternative to Vickrey-Clarke-Groves is profoundly different from the solution depicted in figure 3.3. This latter alternative is Pareto dominated by any other Vickrey-Clarke-Groves mechanism featuring a lower reservation price. In contrast, our proposal uses the threat of not assigning the good in order to alleviate the incentive problem, rather than in order to oppose the interests of the individuals (or buyers in the context of an auction). Although there are combinations of announcements for which the good is destroyed, an event that is undesirable from a welfare point of view, this very fact creates the opportunity to improve welfare for other combinations of announcements. This is a crude intuition that explains the reason why mechanisms that do not always assign the good are nonetheless Second-Best Efficient.

The model we explore also appears in the literature of fair allocation (Tadenuma and Thomson [56]). Altamaz and Yengin [1] discuss fair Groves mechanisms. A challenge pertaining to this exercise is what to make of the deficit fair Groves mechanisms entail. Our approach can shed some light on this discussion. Once the set of Second-Best Efficient mechanisms has been determined, issues of fairness can be explored more fruitfully, as one may confine his search for fair mechanisms to this set.

Section 2 introduces the model. Section 3 states the axioms and discusses the implication of Anonymity. Section 4 discusses the three conditions. Section 5 contains the main
Figure 3.2: A comparison between Solomon and Vickrey.

Figure 3.3: The second-price auction with reservation price p.
Anonymity names of the individuals. An economy is a profile if and only if is captured by a parameter. Section 6, as noted above, briefly discusses the implications Anonymity. Section 7 concludes.

3.2 Notation

Let \( n \) be a finite positive integer greater than 2. The set of individuals is \( N = \{1, 2, \ldots, n\} \). Each individual \( i \in N \) has a private valuation over the indivisible good that is captured by a parameter \( \theta_i \) belonging to \( \mathbb{R}_+ \), the set of non-negative real numbers. An economy is a profile \( \theta_N = (\theta_i)_{i \in N} \in \mathbb{R}_+^n \). For each \( i \in N \), define a function \( f_i : \mathbb{R}_+^n \to \{0, 1\} \). An assignment is a profile \( f_N(\theta_N) = (f_i(\theta_N))_{i \in N} \). An assignment is feasible at \( \theta_N \in \mathbb{R}_+^n \) if and only if \( \sum_{i \in N} f_i(\theta_N) \leq 1 \). For each \( i \in N \), define a function \( t_i : \mathbb{R}_+^n \to \mathbb{R} \). A vector of transfers is denoted \( t(\theta_N) = (t_i(\theta_N))_{i \in N} \). It is feasible if and only if \( \sum_{i \in N} t_i(\theta_N) \leq 0 \). An allocation is a feasible assignment coupled with a feasible vector of transfers. A mechanism \( \varphi \) is a function that assigns an allocation to each economy in the domain. Individual \( i \)'s bundle in \( \theta_N \in \mathbb{R}_+^n \) according to \( \varphi \) is denoted \( \varphi_i(\theta_N) = (f_i(\theta_N), t_i(\theta_N)) \). Individuals have quasi-linear preferences. For each \( i \in N \) and each bundle \( \varphi_i(\theta_N) \in \{0, 1\} \times \mathbb{R} \), \( i \)'s utility is given by the expression \( u_i(\varphi_i(\theta_N); \theta_i) = \theta_i \times f_i(\theta_N) + t_i(\theta_N) \). Often we will write \( u_i(\varphi_i(\theta_i', \theta_N \setminus \{i\}); \theta_i) \) to denote the value of bundle \( \varphi_i(\theta_N \setminus \{i\}'; \theta_i) \) from the point of view of preferences \( \theta_i \).

The following notation comes in handy when defining Anonymity. Let \( \Pi_N \) be the family of functions \( \pi : N \to N \). Let \( \theta_{\pi(N)} \) denote the permuted profile \( \theta_N \). The lower contour set for each bundle in the consumption space is the set of bundles that the individual deems no worse given his preferences. Formally, for each \( \theta_N \in \mathbb{R}_+^n \) and each \( i \in N \), \( LC(\varphi_i(\theta_N); \theta_i) \equiv \{ w \in \{0, 1\} \times \mathbb{R} : u_i(\varphi_i(\theta_N)) \geq u_i(w) \} \). The notion is used in Lemma 22. Finally, wherever sets are concerned, \( \subset \) and \( \subseteq \) denote strict and weak inclusion respectively. For two sets \( A, B \), with \( A \subset B \), \( A \setminus B \equiv \{x \in A \text{ and } x \notin B\} \). Along the same lines, for any set \( A \subset N \), \( \theta_N \setminus A \equiv (\theta_i)_{i \in N \setminus A} \).

3.3 The axioms

In this section we state our axioms and explore the logical relations among them. Strategy-Proofness requires that all individuals have a dominant strategy to reveal their type truthfully. The principal advantage of strategy-proof mechanisms lies in the fact that they are prior-free, that is they do not rely on information regarding the joint distribution of types.

**Strategy-Proofness:** For each \( \theta_N \in \mathbb{R}_+^n \) and each \( i \in N \)

\[
u_i(\phi_i(\theta_i, \theta_N \setminus \{i\}); \theta_i) \geq u_i(\phi_i(\theta_i', \theta_N \setminus \{i\}); \theta_i)
\]

for each \( \theta_i' \in \mathbb{R}_+ \) and each \( \theta_N \setminus \{i\} \in \mathbb{R}_+^{n-1} \).

**Anonymity** is a minimal fairness requirement. The allocation does not depend on the names of the individuals.

**Anonymity:** For each \( \theta_N \in \mathbb{R}_+^n \), each \( \pi \in \Pi_N \), and each \( i \in N \)

\[
u_i(\phi_i(\theta_N); \theta_i) = u_{\pi(i)}(\phi_{\pi(i)}(\theta_{\pi(N)}); \theta_i).
\]
The two preceding axioms embed the notion of Second-Best efficiency. A partial order underlies the definition. A mechanism $\varphi'$ Pareto dominates $\varphi$, if and only if,

1. for each $\theta_N \in \mathbb{R}_+$ and each $i \in N$, $u_i(\varphi'(\theta_N); \theta_i) \geq u_i(\varphi(\theta_N); \theta_i)$,

2. for some $\tilde{\theta}_N \in \mathbb{R}_+$ and some $j \in N$, $u_j(\varphi'(\tilde{\theta}_N); \theta_j) > u_j(\varphi(\tilde{\theta}_N); \theta_j)$.

A mechanism is **Second-Best Efficient** if and only if it strategy-proof and, moreover, there does not exist another strategy-proof mechanism that Pareto dominates it.

**Voluntary Participation** requires that all individuals derive a non-negative benefit from being present in the economy. In other words, all individuals have an incentive to participate.

**Voluntary Participation**: For each $\theta_N \in \mathbb{R}_n^+$ and each $i \in N$

$$u_i(\phi_i(\theta_N); \theta_i) \geq 0.$$

As noted in the introduction, we provide three conditions that are necessary for a mechanism to be Second-Best. However, we do not prove that these conditions are sufficient. Rather, we show that a strategy-proof and anonymous mechanism satisfying these conditions, together with Voluntary Participation, is not Pareto dominated by another strategy-proof and anonymous mechanism. The exercise carried out in this paper would be of little interest were we to discard Anonymity from the definition above. Mechanisms of the sort “individual 1 always obtains the good” are politically impossible to begin with. Consequently, it would also be improper to remove a mechanism from the set of Second-Best mechanisms because it is Pareto dominated by a strategy-proof mechanism that violates nonetheless Anonymity.

A natural question one may raise is whether Second-Best Efficiency and Assignment Efficiency are related and, if so, how. Assignment Efficiency entails two things. First, the good is never destroyed. Second, the good is assigned to the individual that values it the most.

**Assignment Efficiency**: For each $\theta_N \in \mathbb{R}_n^+$ there exists $j \in N$ for whom $f_j(\theta_N) = 1$ and $\theta_j \geq \theta_i$, for each $i \in N \setminus \{j\}$.

It has to be conceded that in certain situations there exist compelling ethical reasons that preclude the possibility of destroying the good. However, a case against the ‘Solomonic solution’ cannot be based on the grounds of Pareto inferiority. Assignment Efficiency is only cited as a point of reference. Our analysis henceforth will rely on a weakening of Assignment Efficiency, namely Weak Assignment Efficiency. If the good is assigned, it must be assigned efficiently. Contrary to Assignment Efficiency, Weak Assignment Efficiency permits the destruction of the good in some or even all profiles.

**Weak Assignment Efficiency**: For each $\theta_N \in \mathbb{R}_n^+$ and each $i \in N$ such that $f_i(\theta_N) = 0$, if for some $j \in N$, $f_j(\theta_N) = 1$, then $\theta_j \geq \theta_i$.

\[^1\text{Although this paper concerns itself with incentives, we would amiss not to mention that Voluntary Participation has also a considerable normative appeal since it precludes instances of exploitation.}\]
Moreover, Weak Assignment Efficiency is implied by Strategy-Proofness and Anonymity. In order to formally demonstrate that we need first to establish Lemma 22, a well-known result in the literature on Strategy-Proofness (Tadenuma and Thomson [56], Schummer [53]).

**Lemma 22** If a mechanism $\varphi$ satisfies Strategy Proofness then for each $\theta_N \in \mathbb{R}_+^n$, each $i \in N$ and each $\theta'_i \in \mathbb{R}_+$, if

$$LC(\varphi_i(\theta_N); \theta_i) \setminus \{\varphi_i(\theta_N)\} \subset LC(\varphi_i(\theta_N); \theta'_i),$$

then $\varphi_i(\theta_N) = \varphi_i(\theta'_i, \theta_{N \setminus \{i\}})$.

**Proof.** Let $\theta_N \in \mathbb{R}_+^n$ and suppose that for some $i \in N$ and some $\theta'_i \in \mathbb{R}_+$ we obtain $LC(\varphi_i(\theta_N); \theta_i) \subset LC(\varphi_i(\theta_N); \theta'_i) \setminus \{\varphi_i(\theta_N)\}$. This case is depicted in figure 3.4, part (a).

By Strategy-Proofness,

$$\varphi_i(\theta'_i, \theta_{N \setminus \{i\}}) \in LC(\varphi(\theta_N); \theta_i)$$

and

$$\varphi(\theta_N) \in LC(\varphi_i(\theta'_i, \theta_{N \setminus \{i\}}); \theta'_i).$$

Define the set of bundles that individual $i$, with preference $\theta'_i$, deems at least as good as $\varphi_i(\theta_N)$,

$$UC(\varphi_i(\theta_N); \theta'_i) = \{w \in \{0, 1\} \times \mathbb{R}: u_i(\varphi_i(\theta_N); \theta'_i) \leq u_i(w; \theta'_i)\}$$

An equivalent statement of 3.3.1 is

$$\varphi_i(\theta'_i, \theta_{N \setminus \{i\}}) \in LC(\varphi_i(\theta_N); \theta_i)$$

and

$$\varphi_i(\theta'_i, \theta_{N \setminus \{i\}}) \in UC(\varphi_i(\theta_N); \theta'_i).$$

Refer to figure 3.4, part (a). By construction,

$$LC(\varphi_i(\theta_N); \theta_i) \cap UC(\varphi_i(\theta_N); \theta'_i) = \{\varphi_i(\theta_N)\}.$$  

Hence, by 3.3.2, $\varphi_i(\theta_N) = \varphi_i(\theta'_i, \theta_{N \setminus \{i\}})$.

![Figure 3.4: A lower contour set expansion (left) and the proof of Proposition 19 (right).](image)

**Proposition 19** If a mechanism satisfies Strategy Proofness and Anonymity, then it satisfies Weak Assignment Efficiency.
Proof. Let there exist some \( \theta_N \in \mathbb{R}^n_+ \) and some pair \( j, k \in N \) such that \( f_k(\theta_N) > f_j(\theta_N) \) and \( \theta_k < \theta_j \). We will obtain a contradiction. Refer to figure 3.4, part (b). By assumption, \( \mathcal{LC}(\varphi_k(\theta_N); \theta_k) \setminus \{\varphi_k(\theta_N)\} \subset \mathcal{LC}(\varphi_k(\theta_N); \theta_j) \). Hence, appealing to lemma 22,
\[
\varphi_k(\theta_N) = \varphi_k(\theta_j, \theta_N \setminus \{k\}).
\] (3.3.3)
Moreover, in \( (\theta_j, \theta_N \setminus \{k\}) \in \mathbb{R}^n_+ \) individuals \( j, k \in N \) have the same preferences. By Anonymity (which implies Equal Treatment of Equals),
\[
u_k(\varphi_k(\theta_j, \theta_N \setminus \{k\}); \theta_k) = u_j(\varphi_j(\theta_j, \theta_N \setminus \{k\}); \theta_k).
\] (3.3.4)
Bundles \( \varphi_j(\theta_j, \theta_N \setminus \{k\}) \) and \( \varphi_k(\theta_j, \theta_N \setminus \{k\}) = \varphi_k(\theta_N) \) are depicted in figure 3.4, part (b). Suppose then that, starting from profile \( (\theta_j, \theta_N \setminus \{k\}) \), individual \( j \) decreases his valuation from \( \theta_j \) to \( \theta_k \). The resulting economy is \( \theta_{\pi(N)} \), where \( \pi' \in \Pi \) is such that \( \pi'(k) = j, \pi'(j) = k \) and \( \pi'(i) = i \), for each \( i \in N \setminus \{j, k\} \). We obtain
\[
\mathcal{LC}(\varphi_j(\theta_j, \theta_N \setminus \{k\}); \theta_j) \setminus \{\varphi_j(\theta_j, \theta_N \setminus \{k\})\} \subset \mathcal{LC}(\varphi_j(\theta_j, \theta_N \setminus \{k\}); \theta_{\pi'(k)}).
\]
By lemma 22,
\[
\varphi_j(\theta_j, \theta_N \setminus \{k\}) = \varphi_{\pi'(k)}(\theta_{\pi'(N)}).
\] (3.3.5)
Equivalently, \( \varphi_j(\theta_j, \theta_N \setminus \{k\}) = \varphi_j(\theta_{\pi'(N)}) \). Therefore, combining 3.3.3, 3.3.4 and 3.3.5 we obtain \( u_{\pi'(k)}(\varphi_{\pi'(k)}(\theta_{\pi'(N)}); \theta_{\pi'(k)}) \neq u_k(\varphi(\theta_N); \theta_k) \), which contradicts Anonymity.

\[\ \]

3.4 The Three Conditions

In this section we state and discuss the three conditions that characterise the set of Second-Best efficient mechanisms.

**Condition A:** For each \( \bar{\theta}_N \in \mathbb{R}^n_+ \) such that \( \sum_{i \in N} f_i(\bar{\theta}_N) = 0 \) and each \( j \in N \), there exist \( \bar{\theta}_j < x' < +\infty \) such that \( f_j(x', \bar{\theta}_N \setminus \{j\}) = 1 \).

**Condition B:** For each \( \theta_N \in \mathbb{R}^n_+ \) and each \( i \in N \), there does not exist \( \epsilon > 0 \) such that
\[
\sum_{j \in N} t_j(x, \theta_N \setminus \{i\}) + \epsilon \leq 0, \text{ for each } x \geq 0.
\]

**Condition C:** For each \( \bar{\theta}_N \in \mathbb{R}^n_+ \) such that \( \sum_{i \in N} f_i(\bar{\theta}_N) = 0 \) and each \( j \in N \), if \( y = \inf \{ x \geq \bar{\theta}_j : f_j(x, \bar{\theta}_N \setminus \{j\}) = 1 \} \) \( < +\infty \), then there does not exist \( \epsilon > 0 \) such that
\[
\sum_{i \in N} t_i(x', \bar{\theta}_N \setminus \{j\}) + \epsilon \leq 0, \text{ for each } x' \geq y.
\]

Put differently, **Condition C** requires that
\[
\sup_{x \geq y} \sum_{i \in N} t_i(x, \bar{\theta}_N \setminus \{j\}) = 0.
\]
The three conditions are logically unrelated. Let us move one step farther and demonstrate that any two conditions do not imply the third by discussing three distinct mechanisms that all satisfy Strategy-Proofness, Anonymity and Voluntary Participation.

\[
\begin{array}{ccc}
A & B & C \\
\times & \checkmark & \checkmark & \text{Case 1} \\
\checkmark & \times & \checkmark & \text{Case 2} \\
\checkmark & \checkmark & \times & \text{Case 3}
\end{array}
\]

Consider, first, a mechanism that always destroys the good and always assigns a transfer of zero to each individual. Condition A is violated, whereas Conditions B and C hold, albeit trivially so. That covers case 1.

Consider, next, a two-individual economy and the mechanism depicted in figure 3.4. It satisfies condition A and, moreover, as it is balanced in the regions for which \( \theta_2 \geq 2p, \theta_1 \leq p \) and \( \theta_2 \leq p, \theta_1 \geq 2p \), it also satisfies Condition C. However, by fixing \( \theta_2 \) at some value in the range \((p, \frac{3p}{2})\) so that we can express the sum of transfers as a function of \( \theta_1 \) alone, we observe that \( \sum_{i \in \{1, 2\}} t_i(\theta_1) > 0 \), for each \( \theta_1 \geq 0 \). Therefore, Condition B is violated and, thus, case 2 is covered.

Finally, consider a three individual economy. Let us denote by \( \alpha_N \in \mathbb{R}_+ \) the profile that proceeds from reorganizing the elements of some \( \theta_N \in \mathbb{R}_+ \) in descending order. Since we are only interested in Anonymous mechanisms we need not worry about the fact that as many as three distinct profiles \( \theta_N \) are associated with the same profile \( \alpha_N \); they are all permutations of each other. For some arbitrary non-negative real number \( p \), let us define the following mechanism:

- If \( \alpha_1 < p \), then the vector of transfers is \((0, 0, 0)\) and the good is destroyed\(^2\).
- If \( \alpha_1 \geq p > \alpha_2 \), then the vector of transfers is \((-p, \frac{p}{3}, \frac{p}{3})\) and the good is assigned to \( i \in N \), for whom \( \theta_i = \alpha_1 \).
- If \( \alpha_2 \geq p \), then the vector of transfers is \((\frac{p}{3} - \alpha_2, \frac{p}{3}, \frac{p}{3})\) and the good is assigned to \( i \), for whom \( \theta_i = \alpha_1 \).

This mechanism violates Condition C. If \( \alpha_1 \geq p > \alpha_2 \), then the sum of the transfers is strictly smaller than zero, independently of the actual value of \( \alpha_1 \). On the contrary, Conditions A and B are satisfied.

### 3.5 The Theorem

This section contains the bulk of our results. The proof of the theorem stated below is broken up in four propositions. Proposition 2 deals with ‘the only if’ part, whereas Propositions 3-5 establish the ‘if’ part.

**Theorem** A mechanism is Second-Best Efficient only if it satisfies Conditions A, B and C. Moreover, if a Strategy-Proof and Anonymous mechanism satisfies Conditions A, B, C and Voluntary Participation, then there does not exist another Strategy-Proof and Anonymous mechanism that Pareto dominates it.

\(^2\)In the vector of transfers we are listing, from left to right, the transfer of the individual with the highest valuation, then that of the individual with the second highest valuation, etc.
Figure 3.5: A counter-example: Conditions A and C do not imply Condition B. The transfer of individual 1 is cited above that of individual 2. The underlined number denotes the individual that obtains the good. The good is destroyed if both individuals announce a valuation lower than $p$. 

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Lemma 23 explores an implication of Strategy-Proofness that we appeal to often in the proofs that follow. Consider any profile \( \hat{\theta}_N \in \mathbb{R}^n_+ \) and choose an individual \( k \in N \) who does not obtain the good at this profile. Consider then the continuum of profiles \((x, \hat{\theta}_{N \setminus \{k\}})\), where \( x > \hat{\theta}_k \). By Strategy-Proofness there exists at most a unique cut-off value \( x \) may take, call it \( y \), such that for all profiles \((x, \hat{\theta}_{N \setminus \{k\}})\), with \( x > y \), individual \( k \) obtains the good, while for all profiles \((x, \hat{\theta}_{N \setminus \{k\}})\), with \( x < y \), he does not. Moreover, if his transfer for those profiles generated by values of \( x \) below the cut-off is \( t \), then his transfer for those profiles generated by values of \( x \) above the cut-off is \(-y + t\). The value of \( y \) can be interpreted as the price he pays for the good.

**Lemma 23** Suppose that \( \varphi \) satisfies Strategy-Proofness. For each \( \hat{\theta}_N \in \mathbb{R}^n_+ \) and each \( k \in N \) such that \( f_k(\hat{\theta}_N) = 0 \), if \( y = \inf \{x \geq \hat{\theta}_k : f_k(x, \hat{\theta}_{N \setminus \{k\}}) = 1\} < +\infty \), then for each \( x', x'' \prec y \)

\[ \varphi(x', \hat{\theta}_{N \setminus \{k\}}) = \varphi(x'', \hat{\theta}_{N \setminus \{k\}}) \text{ and } f_k(x', \hat{\theta}_{N \setminus \{k\}}) = 0, \]

and for each \( x^1, x^2 \geq y \)

\[ \varphi(x^1, \hat{\theta}_{N \setminus \{k\}}) = \varphi(x^2, \hat{\theta}_{N \setminus \{k\}}) \text{ and } f_k(x^1, \hat{\theta}_{N \setminus \{k\}}) = 1. \]

Moreover, \( t_k(x^1, \hat{\theta}_{N \setminus \{k\}}) = -y + t_k(x', \hat{\theta}_{N \setminus \{k\}}). \)

**Proof.** The first part of the statement is a straightforward application of lemma 22. Letting \( x' < x'' \) and noting that \( f_k(x', \hat{\theta}_{N \setminus \{k\}}) = 0 \) we conclude that the move from profile \((x^2, \hat{\theta}_{N \setminus \{k\}})\) to profile \((x^1, \hat{\theta}_{N \setminus \{k\}})\) constitutes a lower set expansion. Similarly for the other case.

Let us then prove the latter part of the proposition. By way of contradiction, suppose that for some \( x^1, x' \) such that \( x^1 \geq y \succ x' \) and some \( \mu > 0 \)

\[ t_k(x^1, \hat{\theta}_{N \setminus \{k\}}) = -y + \mu + t_k(x', \hat{\theta}_{N \setminus \{k\}}). \]

Consider profile \((y - \epsilon, \hat{\theta}_{N \setminus \{k\}})\), where \( \epsilon < \mu \). Invoking the first part of the statement, truth telling yields a utility equal to \( t_k(x', \hat{\theta}_{N \setminus \{k\}}) \). Individual \( k \) by lying may produce profile \((y + \epsilon, \hat{\theta}_{N \setminus \{k\}})\). Again invoking the first part of the statement we obtain

\[ u_k(\varphi_k(y + \epsilon, \hat{\theta}_{N \setminus \{k\}}); y - \epsilon) = y - \epsilon + t_k(x^1, \hat{\theta}_{N \setminus \{k\}}) = \mu - \epsilon + t_k(x', \hat{\theta}_{N \setminus \{k\}}). \]

The difference in utility between lying and telling the truth is \( \mu - \epsilon > 0 \). This contradicts Strategy-Proofness.

Assume then that

\[ t_k(x^1, \hat{\theta}_{N \setminus \{k\}}) = -y - \mu + t_k(x', \hat{\theta}_{N \setminus \{k\}}). \]

Consider profile \((y + \epsilon, \hat{\theta}_{N \setminus \{k\}})\), where \( \epsilon < \mu \). Applying the same reasoning as before, truth yields a utility equal to \( \epsilon - \mu + t_k(x', \hat{\theta}_{N \setminus \{k\}}) \). Lying yields

\[ u_k(\varphi_k(y - \epsilon, \hat{\theta}_{N \setminus \{k\}}); y + \epsilon) = t_k(x', \hat{\theta}_{N \setminus \{k\}}). \]

Again, we obtain a contradiction with Strategy-Proofness.

\[ \blacksquare \]
Proposition 20 A mechanism is Second-Best Efficient only if it satisfies Conditions A, B and C.

Proof. Suppose that a mechanism \( \varphi \) fails Condition A. This implies that there exists a profile \( \hat{\theta}_N \in \mathbb{R}_+ \), with \( \sum_{i \in N} f_i(\hat{\theta}_N) = 0 \), and \( j \in N \) such that

\[
\sum_{i \in N} f_i(x, \hat{\theta}_{N \setminus \{j\}}) = 0, \text{ for each } x \geq 0.
\]

Construct \( \varphi' \) in the following way. Choose some \( p > 0 \). If \( \theta_{N \setminus \{j\}} \neq \hat{\theta}_{N \setminus \{j\}} \), or if \( \theta_{N \setminus \{j\}} = \hat{\theta}_{N \setminus \{j\}} \) and \( \theta_j < p \), set \( \varphi'(\theta_N) = \varphi(\theta_N) \). If \( \theta_{N \setminus \{j\}} = \hat{\theta}_{N \setminus \{j\}} \) and \( \theta_j \geq p \), set \( \varphi'(\hat{\theta}_{N \setminus \{j\}}) = \varphi(\theta_{N \setminus \{j\}}) \), for each \( i \in N \setminus \{j\} \), and \( \varphi'(\hat{\theta}_{N \setminus \{j\}}) = (1, -p + t_j(\theta_N)) \). Since \( \varphi \) satisfies Strategy-Proofness, by construction, \( \varphi' \) also satisfies Strategy-Proofness. Moreover, again by construction, \( \sum_i f_i(\theta_N) \geq \sum_i f_i(\theta_N) \), for each \( \theta_N \in \mathbb{R}_+ \). Hence, \( \varphi' \) is well defined, i.e. satisfies the requirement of feasibility. Finally, for each profile \( (x', \theta_{N \setminus \{j\}}) \in \mathbb{R}_+ \), with \( x' \geq p \), we obtain \( u_j(\varphi'(x', \hat{\theta}_{N \setminus \{j\}})) > u_j(\varphi(x', \theta_{N \setminus \{j\}})) \). For any other profile and any other individual the two mechanisms choose the same allocation. Therefore, \( \varphi' \) Pareto dominates \( \varphi \), and, consequently, \( \varphi \) is not Second-Best Efficient.

Suppose that a mechanism \( \varphi \) fails Condition B, i.e. there exists \( \hat{\theta}_N \in \mathbb{R}_+ \), \( k \in N \) and \( \epsilon > 0 \) such that \( \sum_{j \in N} t_j(x, \hat{\theta}_{N \setminus \{k\}}) + \epsilon \leq 0 \), for each \( x \geq 0 \). Construct \( \varphi' \) in the following way. For each \( \theta_N \in \mathbb{R}_+^N \) and each \( i \in N \), \( f'_i(\theta_N) = f_i(\theta_N) \). \( \theta_{N \setminus \{k\}} \neq \hat{\theta}_{N \setminus \{k\}} \), then for each \( i \in N \), \( t'_i(\theta_N) = t_i(\theta_N) \). \( \theta_{N \setminus \{k\}} = \hat{\theta}_{N \setminus \{k\}} \), then for each \( i \in N \setminus \{k\} \), \( t'_i(\theta_N) = t_i(\theta_N) \) and \( t'_k(\theta_N) = t_k(\theta_N) + \epsilon \). By construction, \( \varphi' \) is Strategy-Proof, well-defined in the sense that it selects feasible allocations, and, moreover, no-one is worse-off relative to \( \varphi \), while individual \( k \) is strictly better-off if \( \theta_{N \setminus \{k\}} = \hat{\theta}_{N \setminus \{k\}} \). Consequently, \( \varphi \) is not Second-Best Efficient.

Suppose that a mechanism fails Condition C. There exists \( \hat{\theta}_N \in \mathbb{R}_+ \), \( j \in N \) such that \( y = \inf \{ x \geq \hat{\theta}_j : f_j(x, \hat{\theta}_{N \setminus \{j\}}) = 1 \} < +\infty \) and, moreover, for each \( x \geq y \) and some \( \epsilon > 0 \), \( \sum_{i \in N} t_i(x, \hat{\theta}_{N \setminus \{j\}}) + \epsilon < 0 \). Construct \( \varphi' \) as follows. If \( \theta_{N \setminus \{j\}} \neq \hat{\theta}_{N \setminus \{j\}} \), then for each \( i \in N \), \( \varphi'_i(\theta_N) = \varphi_i(\theta_N) \). \( \theta_{N \setminus \{j\}} = \hat{\theta}_{N \setminus \{j\}} \), then for each \( i \in N \setminus \{j\} \), \( \varphi'_i(\theta_N) = \varphi_i(\theta_N) \). Moreover, if \( \theta_j < y - \epsilon \), then \( f'_j(\theta_N) = 0 \) and \( t'_j(\theta_N) = t_j(\theta_N) \), and if \( \theta_j \geq y - \epsilon \), then \( f'_j(\theta_N, \hat{\theta}_{N \setminus \{j\}}) = 1 \) and \( t'_j(\theta_N) = t_j(\theta_N) + \epsilon \). Effectively, individual \( j \) is facing a lower price for the indivisible good. Equivalently, we may say that \( \varphi' \) destroys the good less often. By construction, \( \varphi' \) is feasible. Since \( \varphi \) is Strategy-Proof then, by construction, \( \varphi' \) is Strategy-Proof. Therefore, we conclude that \( \varphi' \) Pareto dominates \( \varphi \).

The proof of sufficiency is broken up in three Propositions. We examine the ways in which two mechanisms \( \varphi \) and \( \varphi' \) both satisfying Strategy-Proofness and Anonymity (and therefore Weak Assignment Efficiency) may disagree on the issue of the assignment of the good. A key observation is that if \( \varphi \) and \( \varphi' \) disagree on some point, yet still agree in its neighborhood, then the two mechanism are welfare equivalent at the point of disagreement. In general, if for each profile \( \hat{\theta}_N \in \mathbb{R}_+^N \) such that \( \sum_{i \in N} f_i(\hat{\theta}_N) = 0 \), there exists another profile arbitrarily close to \( \theta_N \) such that the assignment coincides between \( \varphi \) and \( \varphi' \), then the mechanisms are not distinct. This is demonstrated in Proposition 5.

Proposition 3 deals with the case of a mechanism \( \varphi' \) which assigns the good at profile \( \hat{\theta}_N \), as opposed to mechanism \( \varphi \) which does not. Moreover, mechanism \( \varphi \) does not assign
the good in a profile in which an individual with the highest valuation in \( \tilde{\theta}_N \), we call him \( k \), increases his valuation by an infinitesimal amount, while the rest value the object as in \( \tilde{\theta}_N \). Proposition 3 demonstrates that if \( \varphi \) satisfies the conditions we presented above, then \( \varphi' \) does not Pareto dominate it.

**Proposition 21** For each pair of mechanisms \( \varphi, \varphi' \) both satisfying Strategy-Proofness and Anonymity, if \( \varphi \) satisfies Condition A, Condition C and there exists a profile \( \tilde{\theta}_N \in \mathbb{R}_+^n \), an individual \( k \in N \), for whom \( \tilde{\theta}_k \geq \tilde{\theta}_i \) for each \( i \in N \), and an arbitrarily small positive real number \( \delta \), such that for each \( \epsilon \in [0, \delta] \) we obtain \( \sum_{i \in N} f_i(\tilde{\theta}_k + \epsilon, \tilde{\theta}_N \setminus \{k\}) < \sum_{i \in N} f_i(\tilde{\theta}_N) \), then \( \varphi' \) does not Pareto dominate \( \varphi \).

**Proof.** Let \( \varphi \) be a mechanism that satisfies Strategy-Proofness, Anonymity (and therefore Weak Assignment Efficiency courtesy of Proposition 19), Condition A and Condition C. Suppose, moreover, that there exists some other mechanism \( \varphi' \), satisfying Strategy-Proofness and Anonymity, that Pareto dominates \( \varphi \). Suppose, finally, that there exists some profile \( \tilde{\theta}_N \in \mathbb{R}_+^n \), some individual \( k \in N \), for whom \( \tilde{\theta}_k \geq \tilde{\theta}_i \) for each \( i \in N \), and some \( \delta > 0 \) such that

\[
\text{for each } \epsilon \in [0, \delta], \quad \sum_{i \in N} f_i(\tilde{\theta}_k + \epsilon, \tilde{\theta}_N \setminus \{k\}) < \sum_{i \in N} f_i(\tilde{\theta}_N). \quad (3.5.1)
\]

By Condition A, we may define \( y \equiv \inf \{ x \geq \tilde{\theta}_k : f_k(x, \tilde{\theta}_N \setminus \{k\}) = 1 \} \), a finite positive real number. That is, if individual \( k \) increases his valuation above a certain threshold, mechanism \( \varphi \) will assign the good as a consequence of Condition A, and since \( \varphi \) satisfies Weak Assignment Efficiency, individual \( k \) will obtain it. On the left hand side of figure 3.5 we depict the number \( y \). By assumption, \( y > \tilde{\theta}_k \).

Similarly, we may define \( y' \equiv \inf \{ x \geq 0 : f'_k(x, \tilde{\theta}_N \setminus \{k\}) = 1 \} \). The number \( y' \) is a threshold below which individual \( k \) loses the object. We need to show that \( y > y' \). First, we will argue that \( \tilde{\theta}_k \geq y' \). If \( f'_k(\tilde{\theta}_N) = 1 \) it follows straightforwardly that it is so. Suppose then that the profile \( \tilde{\theta}_N \) is such that there exists \( j \in N \), with \( \tilde{\theta}_k = \tilde{\theta}_j \) and \( f'_j(\tilde{\theta}_N) = 1 \). By Strategy-Proofness, for each \( x^+ > \tilde{\theta}_j \) we obtain \( f'_j(x^+, \tilde{\theta}_N \setminus \{j\}) = 1 \). By Anonymity, for each \( x^+ > \tilde{\theta}_k \) we obtain \( f'_k(x^+, \tilde{\theta}_N \setminus \{k\}) = 1 \). It must be then the case that \( y' \) is smaller or at most equal to \( \tilde{\theta}_k \). Therefore, given that \( y > \tilde{\theta}_k \), we obtain \( y > y' \).
The numbers \(y, y'\) can be interpreted as the price individuals pay when obtaining the good. By lemma 2, for each \(x \geq y\),
\[
t_k(x, \tilde{\theta}_{N \setminus \{k\}}) = -y + t_k(0, \tilde{\theta}_{N \setminus \{k\}}). \tag{3.5.2}
\]
Similarly, by lemma 2, for each \(x \geq y'\),
\[
t'_k(x, \tilde{\theta}_{N \setminus \{k\}}) = -y' + t'_k(0, \tilde{\theta}_{N \setminus \{k\}}). \tag{3.5.3}
\]

By Condition C, there exists \(x^1 \geq y\), such that the sum of transfers associated to mechanism \(\phi\) at profile \((x^1, \tilde{\theta}_{N \setminus \{k\}})\) if not equal to zero, it is arbitrarily close to zero from below. Suppose then that \(t'_k(x^1, \tilde{\theta}_{N \setminus \{k\}}) > t_k(x^1, \tilde{\theta}_{N \setminus \{k\}})\). We will arrive at a contradiction. Since both \(\phi\) and \(\phi'\) satisfy Weak Assignment Efficiency we obtain \(f_i(x^1, \tilde{\theta}_{N \setminus \{k\}}) = f'_i(x^1, \tilde{\theta}_{N \setminus \{k\}})\), for each \(i \in N\). That is, individual \(k\) obtains the good at profile \((x^1, \tilde{\theta}_{N \setminus \{k\}})\) under both mechanisms. Consequently, given that \(\phi'\) Pareto dominates \(\phi\), we infer that for some \(m > 0\)
\[
\sum_{i \in N} t'_i(x^1, \tilde{\theta}_{N \setminus \{k\}}) \geq \sum_{i \in N} t_i(x^1, \tilde{\theta}_{N \setminus \{k\}}) + m. \tag{3.5.4}
\]
By Condition C, \(\sum_{i \in N} t_i(x^1, \tilde{\theta}_{N \setminus \{k\}}) + m > 0\), since, as noted above, the sum of transfers is arbitrarily close to zero. Moreover, by feasibility, \(0 \geq \sum_{i \in N} t'_i(x^1, \tilde{\theta}_{N \setminus \{k\}})\). These two latter observations, together with 3.5.4 produce the contradiction.

Thus, \(t'_k(x^1, \tilde{\theta}_{N \setminus \{k\}}) = t_k(x^1, \tilde{\theta}_{N \setminus \{k\}})\), so that 3.5.2, 3.5.3 yield \(-y' + t'_k(0, \tilde{\theta}_{N \setminus \{k\}}) = -y + t_k(0, \tilde{\theta}_{N \setminus \{k\}})\), or \(t'_k(0, \tilde{\theta}_{N \setminus \{k\}}) = y' - y + t_k(0, \tilde{\theta}_{N \setminus \{k\}})\). Since \(y > y'\), we obtain \(t_k(0, \tilde{\theta}_{N \setminus \{k\}}) + t'_k(0, \tilde{\theta}_{N \setminus \{k\}})\), which contradicts the fact that \(\phi'\) Pareto dominates \(\phi\).

Proposition 3 dealt with one type of disagreement between to mechanisms that has welfare implications. Proposition 4 treats the complementary case. A mechanism \(\phi\) assigns the good at profile \(\tilde{\theta}_N\), whereas mechanism \(\phi'\) does not. Moreover, mechanism \(\phi\) assigns the good in a profile in which an individual with the highest valuation in \(\tilde{\theta}_N\), we call him \(k\), decreases his valuation by an infinitesimal amount, while the rest value the object as in \(\tilde{\theta}_N\).

**Proposition 22** For each pair of mechanisms \(\phi, \phi'\) both satisfying Strategy-Proofness and Anonymity, if \(\phi\) satisfies Voluntary Participation, and there exists a profile \(\tilde{\theta}_N \in \mathbb{R}^n_+\), an individual \(k \in N\), for whom \(\tilde{\theta}_k \geq \tilde{\theta}_i\) for each \(i \in N\), and an arbitrarily small positive real number \(\delta\), such that for each \(\epsilon \in [0, \delta]\) we obtain \(\sum_{i \in N} f_i(\tilde{\theta}_k - \epsilon, \tilde{\theta}_{N \setminus \{k\}}) > \sum_{i \in N} f'_i(\tilde{\theta}_N)\), then \(\phi'\) does not Pareto dominate \(\phi\).

**Proof.** Let \(\phi\) be a mechanism that satisfies Strategy-Proofness, Anonymity (and therefore Weak Assignment Efficiency courtesy of Proposition 19) and Voluntary Participation. Suppose, moreover, that there exists some other mechanism \(\phi'\), satisfying Strategy-Proofness and Anonymity, that Pareto dominates \(\phi\). Suppose, finally, that there exists some profile \(\tilde{\theta}_N \in \mathbb{R}^n_+\), some individual \(k \in N\), for whom \(\tilde{\theta}_k \geq \tilde{\theta}_i\) for each \(i \in N\), and some \(\delta > 0\) such that
\[
\text{for each } \epsilon \in [0, \delta], \text{ } \sum_{i \in N} f_i(\tilde{\theta}_k - \epsilon, \tilde{\theta}_{N \setminus \{k\}}) > \sum_{i \in N} f'_i(\tilde{\theta}_N). \tag{3.5.5}
\]
Let \(y \equiv \inf \{x \geq 0 : f_k(x, \tilde{\theta}_{N \setminus \{k\}}) = 1\}\). By assumption, \(y < \infty\). Mechanism \(\phi'\) does not necessarily satisfy Condition A and consequently the existence of a finite threshold
Let \( y' \equiv \inf \{ x \geq \bar{\theta}_k : f'_k(x, \bar{\theta}_{N\setminus\{k\}}) = 1 \} \) is not guaranteed. If \( y' < \infty \) then \( y' > y \) by assumption 3.5.5. Otherwise, we obtain \( y' > y \) for free. In any case, the interval \((y, y')\) is non-empty.

Choose any \( x' \in (y, y') \) and consider the profile \((x', \bar{\theta}_{N\setminus\{k\}}) \in \mathbb{R}^n_+ \). By assumption, \( f'_k(x', \bar{\theta}_{N\setminus\{k\}}) = 0 \), so that \( u_k(\varphi'(x', \bar{\theta}_{N\setminus\{k\}})) = t'_k(x', \bar{\theta}_{N\setminus\{k\}}) \). Appealing to lemma 23 we obtain

\[
u_k(\varphi(x', \bar{\theta}_{N\setminus\{k\}})) = x' - y + t_k(0, \bar{\theta}_{N\setminus\{k\}}) .\]

By Voluntary Participation, \( t_k(0, \bar{\theta}_{N\setminus\{k\}}) \geq 0 \), hence, given that \( x' > y \), we obtain

\[
u_k(\varphi(x', \bar{\theta}_{N\setminus\{k\}})) > 0 .\]

Therefore, by the fact that \( \varphi' \) Pareto dominates \( \varphi \), we infer that \( t'_k(x', \bar{\theta}_{N\setminus\{k\}}) > 0 \). By feasibility, this latter fact implies that there exists some individual \( j \in N \) such that \( t'_j(x', \bar{\theta}_{N\setminus\{k\}}) < 0 \). By Weak Assignment Efficiency, \( f'_k(x', \theta_{N\setminus\{k\}}) = 0 \) implies that \( f'_j(x', \bar{\theta}_{N\setminus\{k\}}) = 0 \). Therefore,

\[
u_k(\varphi'(x', \bar{\theta}_{N\setminus\{k\}})) = t'_j(x', \bar{\theta}_{N\setminus\{k\}}) < 0 .\]

Since \( \varphi' \) Pareto dominates \( \varphi \), individual \( j \) must receive a negative utility under \( \varphi \) as well, which is in violation of Voluntary Participation.

\[\square\]

Two mechanisms are not distinct in welfare terms unless their disagreement over the assignment at some profile persists in the neighborhood of that profile. If for any \( \bar{\theta}_N \in \mathbb{R}^n_+ \) at which the two mechanisms assign the good differently there always exists another profile arbitrarily close to \( \bar{\theta}_N \), on which the assignment coincides, the two mechanism are welfare equivalent.

**Proposition 23** For each pair of mechanisms \( \varphi, \varphi' \) both satisfying Strategy-Proofness and Anonymity, if \( \varphi \) satisfies Condition A, Condition B, and for each \( \theta_N \in \mathbb{R}^n_+ \), each \( k \in N \), for whom \( \theta_k \geq \theta_i \) for each \( i \in N \), and some arbitrarily small positive real number \( \delta \), such that for each \( \epsilon \in (0, \delta) \) we obtain either \( \sum_{i \in N} f_i(\theta_k + \epsilon, \theta_{N\setminus\{k\}}) = \sum_{i \in N} f'_i(\theta_N) \) or \( \sum_{i \in N} f_i(\theta_k - \epsilon, \theta_{N\setminus\{k\}}) = \sum_{i \in N} f'_i(\theta_N) \), then \( \varphi' \) does not Pareto dominate \( \varphi \).

**Proof.** Let \( \varphi \) be a mechanism that satisfies Strategy-Proofness, Anonymity (and therefore Weak Assignment Efficiency courtesy of Proposition 19), Condition A and Condition B. Suppose, moreover, that there exists some other mechanism \( \varphi' \), satisfying Strategy-Proofness and Anonymity, that Pareto dominates \( \varphi \). Suppose, finally, that for each profile \( \theta_N \in \mathbb{R}^n_+ \), each individual \( k \in N \), for whom \( \theta_k \geq \theta_i \) for each \( i \in N \), and some \( \delta > 0 \) such that

\[
\text{for each } \epsilon \in (0, \delta), \text{ either } \sum_{i \in N} f_i(\theta_k + \epsilon, \theta_{N\setminus\{k\}}) = \sum_{i \in N} f'_i(\theta_N) \text{ or } \sum_{i \in N} f_i(\theta_k - \epsilon, \theta_{N\setminus\{k\}}) = \sum_{i \in N} f'_i(\theta_N) .\]

We divide the proof in three steps.

**Step 1.** We begin by proving that for each \( \theta_N \in \mathbb{R}_+ \) and each \( i \in N \), we have \( t'_i(\theta_N) \geq t_i(\theta_N) \).
Suppose not. For some \( \hat{\theta}_N \in \mathbb{R}_+ \) and some \( j \in N, t_j(\hat{\theta}_N) > t'_j(\hat{\theta}_N) \). Let \( f_j(\hat{\theta}_N) = 1 \) and \( y \equiv \inf \{ x \leq \hat{\theta}_j : f_j(x, \hat{\theta}_{N\setminus\{j\}}) = 1 \} \). By assumption, if \( x^+ \) is a real number greater than \( \hat{\theta}_k \) by an arbitrarily small amount, it will be \( f'_k(x^+, \hat{\theta}_{N\setminus\{k\}}) \). Hence, by assumption, \( y = y' \equiv \inf \{ x \geq 0 : f'_j(x, \hat{\theta}_{N\setminus\{j\}}) = 1 \} \). By lemma 2, this entails \( t_j(0, \hat{\theta}_{N\setminus\{j\}}) > t'_j(0, \hat{\theta}_{N\setminus\{j\}}) \), which contradicts the fact that \( \varphi' \) Pareto dominates \( \varphi \). The same reasoning applies for the case \( f_j(\hat{\theta}_N) = 0 < f'_j(\hat{\theta}_N) \). We need finally to consider the case \( f_j(\hat{\theta}_N) = f'_j(\hat{\theta}_N) = 0 \). This case is trivial as \( j \)'s utility under both rules at profile \( \hat{\theta}_N \in \mathbb{R}_+ \) is precisely his transfer.

**Step 2.** For each \( \theta_N \in \mathbb{R}_+ \) and each \( i \in N, f_i(\theta_N) = 1 \), then \( u_i(\varphi_i(\theta_N)) = u_i(\varphi'_i(\theta_N)) \). Suppose that for some \( \hat{\theta}_N \in \mathbb{R}_+ \) and \( k \in N, f_k(\hat{\theta}_N) = 1 \) and \( u_k(\varphi'_k(\hat{\theta}_N)) > u_k(\varphi_k(\hat{\theta}_N)) \). Let \( y \equiv \inf \{ x \leq \hat{\theta}_k : f_j(x, \hat{\theta}_{N\setminus\{j\}}) = 1 \} \). By assumption, \( y = y' \equiv \inf \{ x \geq 0 : f'_j(x, \hat{\theta}_{N\setminus\{j\}}) = 1 \} \). If \( u_k(\varphi_k(\theta_N)) < u_k(\varphi'_k(\theta_N)) \), then, by Lemma 2, for each \( x \geq 0, t_k(x, \hat{\theta}_{N\setminus\{k\}}) > t'_k(x, \hat{\theta}_{N\setminus\{k\}}) \). By Step 1, for each \( x \geq 0, \sum_{i \in N} t'_i(x, \hat{\theta}_{N\setminus\{k\}}) < \sum_{i \in N} t_i(x, \hat{\theta}_{N\setminus\{k\}}) \). Moreover, feasibility requires that \( \sum_{i \in N} t'_i(x, \hat{\theta}_{N\setminus\{k\}}) \leq 0 \). Hence, we obtain \( 0 > \sum_{i \in N} t_i(x, \hat{\theta}_{N\setminus\{k\}}), \) for each \( x \geq 0 \). This contradicts the fact that \( \varphi \) satisfies Condition B.

**Step 3.** From Step 2, and since \( \varphi' \) Pareto dominates \( \varphi \), there must exist some \( \theta'_N \in \mathbb{R}_+ \) and some \( j \in N \) such that \( f'_j(\theta'_N) = f_j(\theta'_N) = 0 \) and \( t'_j(\theta'_N) > t_j(\theta'_N) \). By Condition A, there exists a finite \( y = \inf \{ x \geq 0 : f_j(x, \theta'_N\setminus\{j\}) = 1 \} \). By assumption, \( y' = \inf \{ x \geq 0 : f'_j(x, \theta'_N\setminus\{j\}) = 1 \} \). By lemma 2, for any \( x \geq y \),

\[
\begin{align*}
\varphi(x, \theta'_N\setminus\{j\}) &= -y + t_j(\theta'_N), \\
\varphi'(x, \theta'_N\setminus\{j\}) &= -y + t'_j(\theta'_N).
\end{align*}
\]

Hence, \( u_j(\varphi(x, \theta'_N\setminus\{j\})) < u_j(\varphi'(x, \theta'_N\setminus\{j\})) \), which contradicts the claim we proved in step 2.

\[\blacksquare\]

### 3.6 Groves

![Figure 3.7](image)
One may feel that the scope of the exercise we perform in this paper is somewhat limited because more often than not there exist compelling reasons to refrain from destroying the good. It turns out that our findings have consequences even if that is the case.

Suppose that the planner must always assign the good. Proposition 1 then entails that if a mechanism satisfies Strategy-Proofness and Anonymity it must also satisfy Assignment Efficiency. Consequently, as noted in the introduction, appealing to Holmstrom [30] we may conclude that the mechanism belongs to the Vickrey-Clark-Groves family.

This result is anticipated by Serizawa [55]. He considers a more general model comprising $m$ identical indivisible goods and $n$ individuals, with $m < n$, the so-called rationing problem. He shows that Anonymity, Strategy-Proofness and Voluntary Participation imply Assignment Efficiency. He works under the assumption that all transfers are non-positive and that all goods need to be assigned.

The reasoning underlying Proposition 1 carries through to the case of two individuals laying claim over two heterogeneous goods, an instance of matching with quasi-linear preferences. This result may be compared with findings in the literature on divisible goods. Barbera and Jackson [5] characterize their fixed-price trading rule in a two person-two good exchange economy with Strategy-Proofness and Anonymity. Unlike us they consider a broad domain of preferences. Moreover, individuals are endowed with quantities of the two goods to begin with. However, as we also do in this section, they assume that the final assigned quantities must equal the sum of initial endowments. Before we proceed, we need to introduce some pieces of notation.

The two goods are denoted $\alpha$ and $\beta$. An economy is a profile $\theta_N = (\theta_1^\alpha, \theta_1^\beta, \theta_2^\alpha, \theta_2^\beta) \in \mathbb{R}^{2n}$. The assignment function is defined as before, only now it needs to carry a superscript $\alpha$ or $\beta$. An assignment is a profile $(f_i^\alpha, f_i^\beta)_{i \in N}$. Finally, in this current context, feasibility requires that for each $\theta_N \in \mathbb{R}^{2n}_+$ and each $i \in N$, $f_i^\alpha(\theta_N) + f_i^\beta(\theta_N) \leq 1$.

**Proposition 24** If a mechanism $\varphi$ satisfies Strategy-Proofness and Anonymity and moreover it never destroys the goods then it satisfies Assignment Efficiency.

**Proof.**

By way of contradiction let us assume that there exists some profile $\theta_N \in \mathbb{R}^{2n}_+$ such that the assignment is not efficient. Necessarily, there exists some $j \in \{1, 2\}$ and some $\kappa \in \{\alpha, \beta\}$ such that $f_j^\kappa(\theta_N) = 1$ and $\theta_\kappa^\beta < \theta_\kappa^\beta_N \setminus \{k\}$. Without loss of generality, let $k = 1$ and $\kappa = \alpha$.

We obtain $\theta_1^\alpha < \theta_2^\alpha$. Moreover, by assumption, $\theta_1^\alpha + \theta_2^\beta < \theta_1^\beta + \theta_2^\alpha$. Therefore,

$$\theta_1^\alpha < \theta_2^\alpha \text{ and } \theta_2^\beta < \theta_1^\beta + (\theta_2^\alpha - \theta_1^\alpha).\tag{3.6.1}$$

Refer to figure 3.6-part (a). The bundle $\varphi_1(\theta_N)$ is depicted by point $Z$. The indifference curve corresponding to $(\theta_1^\alpha, \theta_1^\beta)$ through bundle $\varphi_1(\theta_N)$ are represented by the continuous lines. In the figure, the distance $(\Gamma \Delta)$ corresponds to $(\theta_2^\alpha - \theta_1^\alpha)$, $(\Omega \Delta)$ corresponds to $\theta_2^\beta$ and $(\Omega \Delta)$ corresponds to $\theta_1^\beta$. Individual 2’s indifference curve through $Z$ is depicted by the dashed lines. By 3.6.1, if individual 2 changes his preferences so that they are equal to 1’s, the resulting economy $((\theta_2^\alpha, \theta_2^\beta), (\theta_1^\alpha, \theta_1^\beta))$ is such that individual 1’s lower contour set through point $Z$ has expanded. By Strategy-Proofness, $\varphi_1((\theta_2^\alpha, \theta_2^\beta), (\theta_1^\alpha, \theta_1^\beta))$ is at point $Z$. By Anonymity, the $\varphi_2((\theta_2^\alpha, \theta_2^\beta), (\theta_1^\alpha, \theta_1^\beta))$ is at point $H$. Consider then preferences $(\theta_2^\alpha, \theta_2^\beta + \theta_1^\alpha - \theta_1^\alpha - \epsilon)$, where $\epsilon$ is some arbitrarily small positive real number. These preferences are depicted in figure 3.6-part (b) by the combination of the dashed and dotted lines. By construction, going from $((\theta_2^\alpha, \theta_2^\beta), (\theta_1^\alpha, \theta_1^\beta))$ to $((\theta_2^\alpha, \theta_2^\beta), (\theta_1^\alpha, \theta_1^\beta + \theta_2^\alpha - \theta_1^\alpha - \epsilon))$ implies that individual 2’s lower contour set through point $H$ expands. By
Strategy-Proofness, \( \varphi_2((\theta_2^0, \theta_2^0), (\theta_2^0, \theta_1^0 + \theta_2^0 - \theta_1^0 - \epsilon)) \) is at point \( H \). However, by Strategy-Proofness, applying the same reasoning, \( \varphi_1((\theta_2^0, \theta_1^0 + \theta_2^0 - \theta_1^0 - \epsilon), (\theta_2^0, \theta_2^0)) \) is at point \( Z \). This contradicts Anonymity.

\[ \]

3.7 Concluding Remarks

The most natural extension of our model is the one comprising \( m \) identical goods and \( n \) individuals, with \( m < n \), the so called rationing problem \([?]\). Although this project is deferred to future research, in this current work we have drawn insights that may prove to be pertinent.

The conditions we put forward, namely A, B and C, are relevant for the exercise of characterizing the set of Second-Best mechanisms in the rationing problem. One can easily demonstrate, for instance, that they are necessary also in this context. However, it does not follow, even if we conjecture that indeed the characterization we offered in this paper carries through to the rationing problem, that there will exist mechanisms that destroy the good which satisfy them.

What is then crucially different between the model we discussed and the rationing problem, is the facility with which we could relate abstract conditions with a particular class of mechanisms. This facility comes in question in the rationing problem. Thus, it is not so much a matter of whether a characterization can be obtained, as it is a matter of whether this characterization can be readily applied in order to offer practical proposals.

What complicates matters even more is that Proposition 1, or rather the line of reasoning that supports the proof, does not suffice to demonstrate a similar result in the rationing problem. One may appeal to Serizawa [55], although his treatment involves the assumption that all transfers are non-positive. Under this assumption it is questionable whether the set of Second-Best mechanisms will include anything beyond Groves.
Bibliography


