"Pension plan valuation and dynamic mortality tables"

Cossette, Hélène ; Delwarde, Antoine ; Denuit, Michel ; Guillot, Frédérick ; Marceau, Etienne

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Pension Plan Valuation and Dynamic Mortality Tables

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\textbf{Abstract:} It is now well documented that the human mortality globally declined during the course of the 20th century. These mortality improvements pose a challenge for the pricing and reserving in life insurance and for the management of public pension regimes. Assuming a further continuation of the stable pace of mortality decline, a Poisson log-bilinear projection model is applied to population mortality data to forecast future death rates. Afterwards, a relational model embedded in a Poisson regression approach is used to merge a dynamic mortality table based on data of a large population (in this case the province of Quebec, Canada) to mortality data of a given pension plan (here the Régie des Rentes du Québec) to create another dynamic mortality table. This last table can be used to make any assessments on the total costs of the pension plan. We provide at the end numerical examples which illustrate the impact of mortality improvements on a pension plan.

\textbf{Keywords:} public and private pension plans; age-sex-specific mortality; projected lifetables.

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1 Introduction and Motivation

Mortality at adult and old ages, as demonstrated in Benjamin & Soliman (1993) and McDonald et al. (1998), reveal decreasing annual death probabilities. Life expectancy is greater then ever before and increasing rapidly. The past 100 years have seen many improvements in life expectancy, but the pattern of the improvement is changing markedly. In the first half of the 20th century, infectious diseases were almost eradicated, and this gave massive improvements in mortality among the young ages. However, cancer and heart disease kept mortality rates stable for older people. Since then, substantial increases in longevity have been achieved at later ages. All of this poses a challenge for the planning of public retirement systems, the long term risk management of supplemental pension plans as well as for the pricing and reserving for life insurance companies. The evaluation of present values of future costs requires an appropriate mortality projection to avoid important misestimation.

Lee & Carter (1992) proposed a simple model for describing the secular change in mortality as a function of a single time index. This model is fit to historical data. The resulting estimate of the time-varying parameter is then modeled and forecast as a stochastic time series using standard Box-Jenkins methods. From this forecast of the general level of mortality, the actual age-specific rates are derived using the estimated age effects. The main statistical tool of Lee & Carter (1992) is least-squares estimation via singular value decomposition of the matrix of the log age-specific observed forces of mortality. This implicitly means that the errors are assumed to be homoskedastic. We refer the reader to Lee (2000) for a review of the theory and applications of this approach. Brouhns et al. (2002a,b) and Renshaw & Haberman (2003a,b) have each implemented similar alternative approaches to mortality forecasting based on heteroskedastic Poisson error structures. A detailed account of the literature devoted to mortality projections can be found in Tuljapurkar & Boe (1998), Pitacco (2004) and Wong-Fupuy & Haberman (2004).

The methodology examined in this paper is an extrapolation of past trends. All purely extrapolative forecasts assume that the future will be in some sense like the past. Some authors (see, e.g., Gutterman & Vanderhoof (1999)) severely criticized this approach because it seems to ignore underlying mechanisms. As pointed out by Wilmoth (2000), such a critique is valid only insofar as such mechanisms are understood with sufficient precision to offer a legitimate alternative method of prediction. The understanding of the complex interactions of social and biological factors that determine mortality levels being still imprecise, the extrapolative approach to prediction is particularly compelling in the case of human mortality. Also, our methodology is flexible enough and can be adapted in order to take into account the opinions of experts in the setting of the assumptions of the forecasting model.

In a later section, we use this methodology to obtain a dynamic life table allowing a better assessment of the total costs of a given private or public pension plan, in our case the Régie des Rentes du Québec (RRQ) pension plan. More precisely, we construct under this approach a dynamic
lifetable with mortality data of the province of Quebec, Canada. To this table based on data of a large population, we combine the mortality data of the RRQ. This results in a dynamic mortality table that can be used by the RRQ to infer on the total cost associated to the RRQ pension plan.

The structure of this paper is as follows. In section 2, we present the notation used throughout the paper. We then introduce various approaches for the forecast of the mortality of the population of Quebec in section 3. We consider a nonparametric method together with two parametric models, namely the Poisson log-bilinear and the binomial models. We pursue with the projection of the time trend and explain the construction of prospective mortality tables from the parametric models. In section 4, we combine with a Poisson regression model the mortality data of the RRQ to a dynamic mortality table obtained with the mortality data of the population of Quebec to create a dynamic mortality table which can be used by the RRQ to make any assessment of its total costs. Numerical examples to illustrate the impact of mortality improvements on a pension plan are provided in section 5.

2 Notation and Assumptions

2.1 Notation

We analyze the changes in mortality as a function of both age $x$ and time $t$. Henceforth,

- $T_x(t)$ is the remaining lifetime of an $x$-aged individual in calendar year $t$; this individual will die at age $x + T_x(t)$ in year $t + T_x(t)$.
- $q_x(t)$ is the probability that an $x$-aged individual dies in calendar year $t$, i.e. $q_x(t) = \Pr[T_x(t) \leq 1]$.
- $p_x(t) = 1 - q_x(t)$ is the probability that an $x$-aged individual in calendar year $t$ reaches age $x + 1$, i.e. $p_x(t) = \Pr[T_x(t) > 1]$.
- $\mu_x(t)$ is the mortality force at age $x$ during calendar year $t$.
- $e_x(t)$ is the expected remaining lifetime of an individual aged $x$ in year $t$.
- $\bar{a}_x(t)$ is the pure premium of a life annuity sold to an $x$-year-old individual in year $t$, assuming that the payments are made at the beginning of the year.
- $\text{ETR}_{xt}$ is the exposure-to-risk at age $x$ during year $t$, i.e. the total time lived by people aged $x$ in year $t$.
- $D_{xt}$ is the number of deaths recorded at age $x$ during year $t$, from an exposure-to-risk $\text{ETR}_{xt}$.
- $L_{xt}$ is the number of individuals aged $x$ on January 1 of year $t$. 

3
2.2 Assumption of Piecewise Constantness for Forces of Mortality

In this paper, we assume that the age-specific forces of mortality are constant within bands of age and time, but allowed to vary from one band to the next. Specifically, given any integer age $x$ and calendar year $t$, it is supposed that

$$
\mu_{x+\xi}(t + \tau) = \mu_x(t) \quad \text{for} \quad 0 \leq \xi, \tau < 1.
$$

This is best illustrated with the aid of a coordinate system that has calendar time as abscissa and age as coordinate (called a Lexis diagram after the German demographer who introduced it). Both time scales are divided into yearly bands, which partition the Lexis plane into squared segments. Model (1) assumes that the force of mortality is constant within each square, but allows it to vary between squares.

3 Mortality Forecast for the Population Data of Quebec

3.1 Description of the Data

In our study, we use the population data of the province of Quebec in Canada, as it is provided by the Régie des Rentes du Québec and Statistics Canada. These data are appropriate for the purpose of our study. We have observations for the integer ages $x = x_1, \ldots, x_n$ and $t = t_1, \ldots, t_m$ where $x_1 = 60$, $x_n = 89$, $t_1 = 1971$, and $t_m = 1999$.

3.2 Time Trends

Under a non-parametric approach, the age specific forces of mortality are estimated with

$$
\hat{\mu}_x(t) = \frac{D_{xt}}{ETR_{xt}},
$$

for $x = x_1, \ldots, x_n$ and $t = t_1, \ldots, t_m$. Figure 1 depicts the evolution of the non-parametric estimates of the forces of mortality over time for some selected ages, separately for men and women. It is clear that the forces of mortality tend to diminish with time. The mortality surfaces for men and women are displayed in Figure 2.

The non-parametric estimates of the age specific forces of mortality help us to illustrate the evolution of the mortality over time. However, we need to model the forces of mortality in order to produce forecasts for the future. We consider two methodologies: the first one is based on a Poisson log-bilinear specification and the second one uses a Binomial-Gumbel bilinear specification.
Fig. 1 – Forces of mortality over time for some selected ages.
Fig. 2 – Mortality surfaces for the population data of Quebec (left: males, right: females).
3.3 Poisson Log-Bilinear Methodology

LEE & CARTER (1992) proposed a simple model for describing the secular change in mortality as a function of a single time index. The method describes the log of a time series of age-specific death rates as the sum of an age-specific component that is independent of time and another component that is the product of a time-varying parameter reflecting the general level of mortality, and an age-specific component that represents how rapidly or slowly mortality at each age varies when the general level of mortality changes.

The approach of BROUHNS ET AL. (2002a) consists in substituting Poisson random variation for the number of deaths for an additive error term on the logarithm of mortality rates. It is worth to mention that the Poisson distribution is well-suited to mortality analyses; see, e.g., MC DONALD (1996a,b,c) for more details. More specifically, we now consider that

\[ D_{xt} \sim \text{Poisson}\left( \text{ETR}_{xt} \mu_x(t) \right) \text{ with } \mu_x(t) = \exp(\alpha_x + \beta_x \kappa_t), \]

for \( x = x_1, \ldots, x_n \) and \( t = t_1, \ldots, t_m \). To ensure identifiability, the parameters are subjected to the constraints

\[ \sum_t \kappa_t = 0 \text{ and } \sum_x \beta_x = 1. \]

Note that the force of mortality has the same log-bilinear form \( \ln \mu_x(t) = \alpha_x + \beta_x \kappa_t \) as in the Lee-Carter model. Interpretation of the parameters is quite simple:

- \( \alpha_x \): \( \exp \alpha_x \) reflects the general shape of the mortality schedule \( (x = x_1, \ldots, x_n) \).
- \( \beta_x \): represents the age-specific patterns of mortality change \( (x = x_1, \ldots, x_n) \). It indicates the sensitivity of the logarithm of the force of mortality at age \( x \) to variations in the time index \( \kappa_t \). In principle, \( \beta_x \) could be negative at some ages \( x \), indicating that mortality at those ages tends to rise when falling at other ages.
- \( \kappa_t \): represents the time trend \( (t = t_1, \ldots, t_m) \). The actual forces of mortality change according to an overall mortality index \( \kappa_t \) modulated by an age response \( \beta_x \). The shape of the \( \beta_x \) profile tells which rates decline rapidly and which decline slowly over time in response of change in \( \kappa_t \). When \( \kappa_t \) is linear in time, mortality at each age changes at its own constant exponential rate; indeed, assume \( \kappa_t = c - \kappa_0 t \) then

\[ \frac{d}{dt} \ln \{\mu_x(t)\} = \beta_x \frac{d}{dt} \kappa_t = -\beta_x \kappa_0. \]

To estimate the parameters \( \alpha = (\alpha_{x_1}, \ldots, \alpha_{x_n}), \beta = (\beta_{x_1}, \ldots, \beta_{x_n}), \) and \( \kappa = (\kappa_{t_1}, \ldots, \kappa_{t_m}) \), the model (2) is fitted to a matrix of age-specific observed forces of mortality using the maximum likelihood principle, i.e. by maximizing the log-likelihood \( L(\alpha, \beta, \kappa) \) based on model (2). Let us denote as

\[ \bar{D}_{xt} = E[D_{xt}] = \text{ETR}_{xt} \exp(\alpha_x + \beta_x \kappa_t) \]
the expected number of deaths at age $x$ during year $t$. Then,

$$L(\alpha, \beta, \kappa) = \ln \left\{ \prod_t \prod_x \left( \frac{\hat{D}_{xt} \exp(-\hat{D}_{xt})}{d_{xt}!} \right) \right\}$$

$$= \sum_t \sum_x \left\{ d_{xt} \ln \hat{D}_{xt} - \hat{D}_{xt} \ln \{d_{xt}!\} \right\}$$

$$= \sum_t \sum_x \left\{ d_{xt}(\alpha_x + \beta_x \kappa_t) - \text{ETR}_{xt} \exp(\alpha_x + \beta_x \kappa) \right\} + \text{constant.}$$

Because of the presence of the bilinear term $\beta_x \kappa_t$, it is not possible to estimate the proposed model with commercial statistical packages that implement Poisson regression. GOODMAN (1979) proposed an iterative procedure to obtain the MLE’s: in iteration step $\nu + 1$, a single set of parameters is updated fixing the other parameters at their current estimates using the following updating scheme:

$$\hat{\theta}^{(\nu+1)} = \hat{\theta}^{(\nu)} - \frac{\partial L^{(\nu)}}{\partial \theta} - \frac{\partial^2 L^{(\nu)}}{\partial \theta^2}$$

where $L^{(\nu)} = L^{(\nu)}(\hat{\theta}^{(\nu)})$.

In our application, there are three sets of parameters; that is, the vectors $\alpha = (\alpha_{x_1}, ..., \alpha_{x_n})$, $\beta = (\beta_{x_1}, ..., \beta_{x_n})$, and $\kappa = (\kappa_1, ..., \kappa_{m})$. The updating scheme is as follows: starting with $\hat{\alpha}_x^{(0)} = 0$, $\hat{\beta}_x^{(0)} = 1$, and $\hat{\kappa}_t^{(0)} = 0$ (random values can also be used), we iterate using the following formulas:

$$\hat{\alpha}_x^{(\nu+1)} = \hat{\alpha}_x^{(\nu)} - \frac{\sum_t (D_{xt} - \hat{D}_{xt}^{(\nu)})}{\sum_t \hat{D}_{xt}^{(\nu)}}, \quad \hat{\beta}_x^{(\nu+1)} = \hat{\beta}_x^{(\nu)}, \quad \hat{\kappa}_t^{(\nu+1)} = \hat{\kappa}_t^{(\nu)},$$

$$\hat{\kappa}_t^{(\nu+2)} = \hat{\kappa}_t^{(\nu+1)} - \frac{\sum_x (D_{xt} - \hat{D}_{xt}^{(\nu+1)}) \hat{\kappa}_t^{(\nu+1)}}{-\sum_x \hat{D}_{xt}^{(\nu)} \left( \hat{\kappa}_t^{(\nu+1)} \right)^2}, \quad \hat{\alpha}_x^{(\nu+2)} = \hat{\alpha}_x^{(\nu+1)}, \quad \hat{\beta}_x^{(\nu+2)} = \hat{\beta}_x^{(\nu+1)},$$

$$\hat{\beta}_x^{(\nu+3)} = \hat{\beta}_x^{(\nu+2)} - \frac{\sum_t (D_{xt} - \hat{D}_{xt}^{(\nu+2)}) \hat{\kappa}_t^{(\nu+2)}}{-\sum_t \hat{D}_{xt}^{(\nu+2)} \left( \hat{\kappa}_t^{(\nu+2)} \right)^2}, \quad \hat{\alpha}_x^{(\nu+3)} = \hat{\alpha}_x^{(\nu+2)}, \quad \hat{\kappa}_t^{(\nu+3)} = \hat{\kappa}_t^{(\nu+2)},$$

where $\hat{D}_{xt}^{(\nu)} = \text{ETR}_{xt} \exp(\hat{\alpha}_x^{(\nu)} + \hat{\beta}_x^{(\nu)} \hat{\kappa}_t^{(\nu)})$, or the estimated number of deaths after iteration step $\nu$. The criterion used to stop the procedure is a very small increase of the log-likelihood function. Let us mention that it is also possible to optimize the Poisson likelihood by monitoring the associated deviance, as described in RENSHAW & HABERMAN (2003b).
The ML estimations of the parameters have to be adapted in order to fulfill the Lee-Carter constraints: specifically, we switch from \((\hat{\alpha}, \hat{\beta}, \hat{\kappa})\) to \((\alpha^*, \beta^*, \kappa^*)\) given by

\[
\kappa_t^* = (\bar{\kappa} - \bar{\alpha}) \sum_x \hat{\beta}_x, \quad \beta_x^* = \frac{\hat{\beta}_x}{\sum_x \hat{\beta}_x} \quad \text{and} \quad \alpha_x^* = \hat{\alpha}_x + \hat{\beta}_x \bar{\kappa}.
\]

The new estimates \(\alpha_x^*, \beta_x^*\) and \(\kappa_t^*\) fulfill the constraints and provide the same \(\hat{D}_{xt}\) since \(\hat{\alpha}_x + \hat{\beta}_x \hat{\kappa}_t = \alpha_x^* + \beta_x^* \kappa_t^*\). Henceforth, we denote as \(\hat{\alpha}_x, \hat{\beta}_x\) and \(\hat{\kappa}_t\) the estimated parameters satisfying (4).

Note that differentiating the loglikelihood with respect to \(x\) gives the equation

\[
\sum_t D_{xt} = \sum_t \hat{D}_{xt} = \sum_t \text{ETR}_{xt} \exp(\hat{\alpha}_x + \hat{\beta}_x \hat{\kappa}_t).
\]

So, the estimated \(\kappa_t\)'s are such that the resulting death rates applied to the actual risk exposure produce the total number of deaths actually observed in the data for each age \(x\). Sizable discrepancies between predicted and actual deaths are thus avoided.

We apply the Poisson modelling to the population data of the province of Quebec. The Poisson parameters \(\alpha_x, \beta_x\) and \(\kappa_t\) involved in (2) are estimated via maximum likelihood. Figure 3 plots the estimated values of \(\alpha = (\alpha_{x_1}, ..., \alpha_{x_n}), \beta = (\beta_{x_1}, ..., \beta_{x_n}),\) and \(\kappa = (\kappa_{t_1}, ..., \kappa_{t_m})\).

Since we work in a regression framework, it is essential to inspect the residuals. With Poisson random components, deviance residuals

\[
\sqrt{2} \times \text{sign}(d_{xt} - \hat{d}_{xt}) \sqrt{d_{xt} \ln \frac{d_{xt}}{d_{xt}} - (d_{xt} - \hat{d}_{xt})}
\]

are appropriate to monitor the quality of the fit. Figure 4 displays the evolution of residuals through time at different ages. The absence of structure supports the model.

### 3.4 Binomial-Gumbel Bilinear Methodology

The Poisson modelling for \(D_{xt}\) can be seen as an approximation of the “true” binomial process generating death counts. The number of deaths \(D_{xt}\) at age \(x\) during year \(t\) has a binomial distribution with parameters \(L_{xt}\) and \(q_x(t)\). The specification for \(\mu_x(t)\) gives

\[
q_x(t) = 1 - \exp\left(-\exp\left(\alpha_x + \beta_x \kappa_t\right)\right),
\]

for \(x = x_1, ..., x_n\) and \(t = t_1, ..., t_m\). It follows that

\[
D_{xt} \sim \text{Binomial} \left(L_{xt}, q_x(t)\right) \quad \text{with} \quad q_x(t) = 1 - \exp\left(-\exp\left(\alpha_x + \beta_x \kappa_t\right)\right).
\]
Fig. 3 – \( \alpha \), \( \beta \) and \( \kappa \) estimates for the Poisson model.
Fig. 4 – Deviance residuals at different ages for the Poisson model.
To ensure identifiability, we adhere to the set of constraints (3) displayed above for the parameters \(\alpha = (\alpha_1, \ldots, \alpha_n)\), \(\beta = (\beta_1, \ldots, \beta_n)\), and \(\kappa = (\kappa_1, \ldots, \kappa_m)\). Assuming independence, the likelihood for the entire data is the corresponding product of binomial probability factors. The log-likelihood is then given by

\[
L(\alpha, \beta, \kappa) = \ln \left\{ \prod_t \prod_x \left( \left( \frac{L_{xt}}{d_{xt}} \right) \left( 1 - \hat{q}_{xt} \right)^{d_{xt} \hat{q}_{xt}^{d_{xt}}} \right) \right\} = \sum_t \sum_x \left\{ d_{xt} \ln \left( 1 - \hat{q}_{xt} \right) + d_{xt} \ln \hat{q}_{xt} \right\} + \text{constant}.
\]

Because of the presence of the bilinear term \(\beta_m \kappa_t\), it is not possible to estimate the proposed model with commercial statistical packages that implement Binomial regression. Therefore, we resort as in the Poisson case to the procedure suggested by Goodman (1979). Denoting as

\[
\mu_{xt}^{(\nu)} = \exp(\tilde{\alpha}_x^{(\nu)} + \tilde{\beta}_x^{(\nu)} q_{xt}^{(\nu)}) \quad \text{and} \quad q_{xt}^{(\nu)} = 1 - \exp \left( - \mu_{xt}^{(\nu)} \right) = 1 - p_{xt}^{(\nu)},
\]

the iterative procedure yielding the MLE’s writes as follows:

\[
\tilde{\alpha}_x^{(\nu+1)} = \tilde{\alpha}_x^{(\nu)} - \frac{\sum_{t=t_{\min}}^{t_{\max}} \mu_{xt}^{(\nu)} \left( D_{xt} \frac{q_{xt}^{(\nu)} \mu_{xt}^{(\nu+1)} - L_{xt}}{q_{xt}^{(\nu)}} \right)}{\sum_{t=t_{\min}}^{t_{\max}} \mu_{xt}^{(\nu)} \left( D_{xt} \frac{q_{xt}^{(\nu)} \mu_{xt}^{(\nu)} + p_{xt}^{(\nu)} - L_{xt}}{q_{xt}^{(\nu)}} \right)},
\]

\[
\tilde{\beta}_x^{(\nu+1)} = \tilde{\beta}_x^{(\nu)}, \quad \tilde{\kappa}_t^{(\nu+1)} = \tilde{\kappa}_t^{(\nu)},
\]

\[
\tilde{\beta}_x^{(\nu+2)} = \tilde{\beta}_x^{(\nu+1)} - \frac{\sum_{t=t_{\min}}^{t_{\max}} \left( \tilde{\kappa}_t^{(\nu+1)} \right)^2 \mu_{xt}^{(\nu+1)} \left( D_{xt} \frac{q_{xt}^{(\nu+1)} \mu_{xt}^{(\nu+1)} + p_{xt}^{(\nu+1)} - L_{xt}}{q_{xt}^{(\nu+1)}} \right)}{\sum_{t=t_{\min}}^{t_{\max}} \left( \tilde{\kappa}_t^{(\nu+1)} \right)^2 \mu_{xt}^{(\nu+1)} \left( D_{xt} \frac{q_{xt}^{(\nu+1)} \mu_{xt}^{(\nu+1)} + p_{xt}^{(\nu+1)} - L_{xt}}{q_{xt}^{(\nu+1)}} \right)},
\]

\[
\tilde{\alpha}_x^{(\nu+2)} = \tilde{\alpha}_x^{(\nu+1)}, \quad \tilde{\kappa}_t^{(\nu+2)} = \tilde{\kappa}_t^{(\nu+1)},
\]

\[
\tilde{\kappa}_t^{(\nu+3)} = \tilde{\kappa}_t^{(\nu+2)} - \frac{\sum_{x=x_{\min}}^{x_{\max}} \left( \tilde{\beta}_x^{(\nu+2)} \right)^2 \mu_{xt}^{(\nu+2)} \left( D_{xt} \frac{q_{xt}^{(\nu+2)} \mu_{xt}^{(\nu+2)} + p_{xt}^{(\nu+2)} - L_{xt}}{q_{xt}^{(\nu+2)}} \right)}{\sum_{x=x_{\min}}^{x_{\max}} \left( \tilde{\beta}_x^{(\nu+2)} \right)^2 \mu_{xt}^{(\nu+2)} \left( D_{xt} \frac{q_{xt}^{(\nu+2)} \mu_{xt}^{(\nu+2)} + p_{xt}^{(\nu+2)} - L_{xt}}{q_{xt}^{(\nu+2)}} \right)},
\]

\[
\tilde{\alpha}_x^{(\nu+3)} = \tilde{\alpha}_x^{(\nu+2)}, \quad \tilde{\beta}_x^{(\nu+3)} = \tilde{\beta}_x^{(\nu+2)}.
\]

We apply the Binomial modelling to the population data of the province of Quebec. The parameters \(\alpha = (\alpha_1, \ldots, \alpha_n)\), \(\beta = (\beta_1, \ldots, \beta_n)\), and \(\kappa = (\kappa_1, \ldots, \kappa_m)\) involved in (6) are estimated via
maximum likelihood. Figure 5 plots the estimated values of $\alpha = (\alpha_1, ..., \alpha_n)$, $\beta = (\beta_1, ..., \beta_n)$, and $\kappa = (\kappa_1, ..., \kappa_m)$.

To monitor the quality of the fit by the Binomial-Gumbel model, the appropriate deviance residuals given by

$$\sqrt{2} \times \text{sign}(d_{xt} - \hat{d}_{xt}) \sqrt{d_{xt} \ln \frac{1 - d_{xt}}{L_{xt}} + d_{xt} \ln \frac{d_{xt}}{L_{xt}\hat{q}_{xt}}}$$

are appropriate to monitor the quality of the fit. The evolution of residuals through time at different ages are illustrated in Figure 6. As one can see, the absence of structure supports the model.

### 3.5 Selection of the Model

To compare the Poisson and the Binomial models, we use the percentages of total deviance explained by each model (i.e. the pseudo $R^2$), which are given in Table 1.

<table>
<thead>
<tr>
<th></th>
<th>Poisson model</th>
<th>Binomial model</th>
</tr>
</thead>
<tbody>
<tr>
<td>Males</td>
<td>86.30%</td>
<td>86.18%</td>
</tr>
<tr>
<td>Females</td>
<td>87.08%</td>
<td>87.00%</td>
</tr>
</tbody>
</table>

**Tab. 1 – Percentages of total deviance by Poisson and Binomial models.**

The Poisson model is slightly superior to its Binomial analogue, and we decide to keep the Poisson model for the death counts stratified by age and time.

### 3.6 Projection of the Time Index

The time factor $\kappa_t$ is intrinsically viewed as a stochastic process and Box-Jenkins techniques are then used to estimate and forecast $\kappa_t$ within an ARIMA($p,d,q$) times series model, which takes the general form

$$\nabla^d \kappa_t = \rho + \frac{\Theta_q(B)\epsilon_t}{\Phi_p(B)},$$

where

- $B$ is the delay operator, $B(\kappa_t) = \kappa_{t-1}$, $B^2(\kappa_t) = \kappa_{t-2}$, ...;
- $\nabla = 1 - B$ is the difference operator, $\nabla \kappa_t = \kappa_t - \kappa_{t-1}$, $\nabla^2 \kappa_t = \kappa_t - 2\kappa_{t-1} + \kappa_{t-2}$, ...;
- $\Theta_q(B)$ is the Moving Average polynomial, with coefficients $\theta = (\theta_1, \theta_2, ..., \theta_q)$;
- $\Phi_p(B)$ is the Autoregressive polynomial, with coefficients $\phi = (\phi_1, \phi_2, ..., \phi_p)$; and
Fig. 5 – $\alpha$, $\beta$ and $\kappa$ estimates for the Binomial model.
Fig. 6 – Deviance residuals at different ages for the Binomial model.
\( \epsilon_t \) is white noise with variance \( \sigma^2 \).

The parameters of the models are \( \mu, \theta, \phi \) and \( \sigma \). The method used to obtain estimates for the ARIMA parameters is conditional least squares. The choice of the optimal model has been based on the AIV values. This yielded to select an ARIMA(1,1,0) for males and an ARIMA(0,1,1) for females. Forecasted values of time parameters \( \kappa_t \) (here, \( t \) is beyond the observation period) can be seen on Figure 7.

As is discussed in the next sections, the parameter estimates of the Poisson model and the \( \kappa_t \) forecasts can be used to obtain projected age-specific mortality rates, life expectancies, and annuities single premiums. Projected deaths rates at some selected ages are displayed in Figure 8.

### 3.7 Completion of the Lifetables

Data at old ages produce suspect results (because of small risk exposures): the pattern at old and very old ages is heavily affected by random fluctuations because of their scarcity. Sometimes, data above some high age are not available at all. Recently, some in-depth demographic studies provided a sound knowledge about the slope of the mortality curve at very old ages. It has been documented that the force of mortality is slowly increasing at very old ages, approaching a rather flat shape. The
deceleration of the rate of mortality increase can be explained by the selective survival of healthier individuals to older ages (see, e.g., Horiuchi & Wilmoth (1998) for more details).

Demographers and actuaries suggested various techniques to complete forces of mortality at old ages. Let us mention the influential works by Lindbergson (2001), Coale & Guo (1989) and Coale & Kisker (1990). We refer the interested reader to Buettner (2002) for an interesting discussion. In this paper, we use a simple and powerful method proposed by Denuit & Goderniaux (2004). This method is briefly presented below.

The starting point is standard: a constrained log-quadratic regression model of the form

$$\ln q_x(t) = a_t + b_t x + c_t x^2 + \epsilon_{xt}$$

with $\epsilon_{xt}$ i.i.d. $\mathcal{N}(0, \sigma^2)$ is fitted separately to each calendar year $t$ ($t = t_1, \ldots, t_m$) and to ages ($x$) 75 and over. The difference lies in the constraints we impose. First, we fix a closure constraint $q_{130}(t) = 1$ for all $t$.

Even if the human life span shows no sign of approaching a fixed limit imposed by biology or other factors (see, e.g., Wilmoth (1997) or Wilmoth et al. (2000)), it seems reasonable to retain as working assumption that the limit age 130 will not be exceeded. Secondly, we assume an inflexion

FIG. 8 – Projected death rates at different ages.
These two constraints yield the following relation between the $a_t$'s, $b_t$'s and $c_t$'s for each calendar time $t$:

$$a_t + b_t x + c_t x^2 = c_t (130 - x)^2,$$

for $x = 75, 76, \ldots, x_n$ and $t = t_1, \ldots, t_m$. The $c_t$'s are then estimated on the basis of the series $\{q_x(t), x = 75, 76, \ldots, x_n\}$ relating to year $t$. It is worth mentioning that the two requirements underlying the modelling of the $q_x(t)$ for high $x$ are in line with the empirical demographic evidence.

The completed data set is then obtained as follows. We keep the original $q_x(t)$ for $x = x_1, \ldots, 85$ and we replace the death probabilities for ages over 85 with the fitted values coming from the constrained quadratic regression. The results for calendar years 1975, 1985 and 1995 can be seen in Figure 9. This furnishes a rectangular array of data displayed in Figure 10.

4 Application to a Pension Plan

We are now interested in the evaluation of the total cost associated to a given pension plan (private or public). More precisely, we want to base this evaluation on a dynamic mortality table obtained from the mortality experience of this pension plan and its characteristics. Often, the pension plan is not large enough to solely base the construction of a dynamic mortality table on its own experience. To obtain such a table, one can combine a dynamic mortality table built from a larger population, and the mortality and the characteristics of the population of the pension plan. In the application that follows, we combine with a Poisson regression model the mortality data of the Régie des Rentes du Québec (RRQ) pension plan to a dynamic mortality table obtained with the mortality data of the whole province of Quebec. The resulting dynamic mortality table can be used by the RRQ to make any assessment of the total cost associated to the plan. This approach could also be used for smaller private pension plans using any appropriate basic dynamic mortality table.

4.1 Description of the RRQ and Data

The Régie des Rentes du Québec (RRQ) is a provincial pension plan which came into effect in January 1966 at the same time as his federal substitute, the Canadian Pension Plan (CPP). The goal of the RRQ is mainly to provide to all members of the paid labour force in the province of Quebec, a minimum retirement income and benefits in the event of disability or death. All workers of the province of Quebec must contribute with his employer to the funding via monthly contributions,
Fig. 9 – Completion of death probabilities for years 1975, 1985 and 1995.
Fig. 10 – Projected mortality surfaces for the population data of Quebec (left: males, right: females).
which are determined by the government of the province of Quebec. Annuities provided by the RRQ depend on the worker’s wage and are indexed each year to compensate for inflation. Disability annuities are made of two parts, which are a uniform rate and an additional benefit for dependent children. Note that a disable individual will see his disability annuity being transformed into a retirement annuity on his 65th birthday. Retirement annuities are strictly proportional to the average wage of the retired annuitant. They are set in order to provide 25% of the annuitant’s income before retirement up to 25% of the average national wage.

Let us briefly describe the data file provided by the RRQ for the analysis that follows. The data file contains the individual data (1,606,141 records) for all members from the inception of the RRQ to 2003. In addition to the year of birth and of death (if applicable), we use as covariates whether the individual retired before 65 (early retirement) or after this age, whether he/she is disabled or not at retirement, the amount of his/her annuity (expressed as a percentage of the maximal amount, corresponding to full service and maximum pensionable earnings) and whether he/she benefited from payments made by the Canada Pension Plan for individuals who spent part of their career in Canada, but outside Quebec.

4.2 Descriptive Analysis

Let us first compare the mortality experienced by the RRQ annuitants to the whole Quebecian population. Figure 11 displays the ratio of the mortality rates for the RRQ and the entire Quebecian population. It can be seen that the situation depends on the gender. For males, the ratio is around 1 and above, showing no significant reduction of mortality for the RRQ annuitants. For females however, Figure 11 suggests a better mortality for RRQ.

4.3 Impact of RRQ Annuitants Characteristics

Let us start with a description of the RRQ database. We see from the histograms displayed in Figures 12 to 14 that

- the vast majority of the RRQ annuitants do not benefit from a pension paid by the Canada pension plan;
- a majority of individuals retired before age 65;
- men benefit from higher pensions than women (modal class for men is 100, that is, a percentage of 100%, whereas modal class for women is 30-50, corresponding to a percentage ≥ 30% and < 50%.

Let us now study the possible marginal effect of the explanatory variables on the mortality experienced by the RRQ annuitants. This is done in Figures 15 to 19 where the mortality rates
Fig. 11 – Ratio of the mortality rates for the RRQ and the entire Quebecian population (ages 65, 75 and 85 above, years 1975, 1985, 1995 below).
by age and gender are represented separately for the subpopulations indexed by the explanatory variables. We can see there that

- early retirement (i.e. before age 65) leads to better mortality rates;
- receiving payments from both RRQ and CPP has only a weak impact on the mortality rates;
- the mortality rates increase for individuals who are disabled at retirement;
- the mortality rates globally seem to decrease with the percentage for men, but the situation is somewhat ambiguous for women.

The next section proposes a Poisson regression model for death counts incorporating exogeneous information.

### 4.4 Poisson Regression for Annual RRQ Death Counts

Let us now explain the annual death counts observed for the RRQ annuitants, in function of the four explanatory variables we have at our disposal. More precisely, we fit a different Poisson regression model for men and women where the annual death counts $D_{zt}^{RRQ}$ for RRQ members aged
Fig. 13 – RRQ annuitants characteristics: early retirement and disability.
Fig. 14 – RRQ annuitants characteristics: amount of annuity and payments made by the Canadian Pension Plan.
Fig. 15 – Marginal effects of the explanatory variables on the RRQ mortality experience: People retired after and before age 65.
Fig. 16 – Marginal effects of the explanatory variables on the RRQ mortality experience: People retired after age 65 and not disabled.
Fig. 17 – Marginal effects of the explanatory variables on the RRQ mortality experience: People retired after age 65 and disabled.
Fig. 18 – Marginal effects of the explanatory variables on the RRQ mortality experience: Canadian financing.
Fig. 19 – Marginal effects of the explanatory variables on the RRQ mortality experience: Amount of annuity.
$x$ in year $t$ as

$$D_{xt}^{RRQ} \sim \text{Poisson} \left( ETR_{xt}^{RRQ} \mu_x(t) \exp \left( \beta_0 + \sum_{j=1}^{s} \beta_j z_{xtj} \right) \right)$$

where $ETR_{xt}^{RRQ}$ is the exposure-to-risk for RRQ at age $x$ in year $t$, $\mu_x(t)$ is the fitted mortality rate for the Quebec population (obtained in the preceding section) and the $z_{xtj}$'s are binary variables coding the four categorical explanatory variables at our disposal. The number $s + 1$ of parameters depends on the sex ($s = 9$ for men and $s = 8$ for women).

For men, we group the percentages of maximal pension amounts in seven classes 0-10, 10-30, 30-50, 50-70, 70-85, 85-100 and 100. For the women, the classes are 0-10, 10-30, 30-50, 70-100 and 100. We also indicate if an individual has retired before ($< 65$) or, given that he/she has retired at/after ($\geq 65$) age 65, whether he/she is disabled or not. Finally, we take into consideration whether the annuity is fully paid by the RRQ or jointly with the CPP. We use the logarithm of the product of the exposure-to-risk with the Quebec mortality rate as offset. As an example, for men who have retired at age 63 and received a pension annuity between 70% and 85% of the maximal pension amount the covariates are coded as follows: $z_{u1} = 0$, $z_{u2} = 0$, $z_{u3} = 0$, $z_{u4} = 1$, $z_{u5} = 0$, $z_{u6} = 0$, $z_{u7} = 0$, $z_{u8} = 0$, and $z_{u9} = 0$.

After having taken into consideration all the elements just mentioned, we have obtained the two models displayed in Tables 2 and 3. For both models, all the regression coefficients significantly differ from 0.

<table>
<thead>
<tr>
<th></th>
<th>Covariates $z_{xtj}$</th>
<th>Estimate of $\beta_j$</th>
<th>Std. Error</th>
<th>$\chi^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>Intercept</td>
<td>0.1463</td>
<td>0.0080</td>
<td>338.00</td>
</tr>
<tr>
<td>1</td>
<td>10≤%Max&lt;30</td>
<td>-0.1086</td>
<td>0.0088</td>
<td>153.09</td>
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<tr>
<td>2</td>
<td>30≤%Max&lt;50</td>
<td>-0.2939</td>
<td>0.0085</td>
<td>572.48</td>
</tr>
<tr>
<td>3</td>
<td>50≤%Max&lt;70</td>
<td>-0.2577</td>
<td>0.0083</td>
<td>967.06</td>
</tr>
<tr>
<td>4</td>
<td>70≤%Max&lt;85</td>
<td>-0.2905</td>
<td>0.0082</td>
<td>1258.48</td>
</tr>
<tr>
<td>5</td>
<td>85≤%Max&lt;100</td>
<td>-0.3655</td>
<td>0.0076</td>
<td>2296.91</td>
</tr>
<tr>
<td>6</td>
<td>%Max=100</td>
<td>-0.2380</td>
<td>0.0130</td>
<td>332.78</td>
</tr>
<tr>
<td>7</td>
<td>Age≥65 &amp; disability = No</td>
<td>0.0370</td>
<td>0.0039</td>
<td>89.06</td>
</tr>
<tr>
<td>8</td>
<td>Age≥65 &amp; disability = Yes</td>
<td>0.6930</td>
<td>0.0057</td>
<td>14814.6</td>
</tr>
<tr>
<td>9</td>
<td>Provincial + Federal (RRQ + CPP)</td>
<td>-0.0452</td>
<td>0.0064</td>
<td>50.04</td>
</tr>
</tbody>
</table>

**Tab. 2 – Parameters of the Poisson regression model for the RRQ annuitants - Males.**

Boxplots of the residuals are displayed in Figure 20. No structure emerges from these figures, so that the data seem to agree with the model.
### Females

<table>
<thead>
<tr>
<th>$j$</th>
<th>Covariates $z_{xtj}$</th>
<th>Estimate of $\beta_j$</th>
<th>Std. Error</th>
<th>$\chi^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>Intercept</td>
<td>-0.0565</td>
<td>0.0070</td>
<td>65.80</td>
</tr>
<tr>
<td>1</td>
<td>10≤%Max&lt;30</td>
<td>-0.1028</td>
<td>0.0078</td>
<td>173.02</td>
</tr>
<tr>
<td>2</td>
<td>30≤%Max&lt;50</td>
<td>-0.1525</td>
<td>0.0084</td>
<td>330.96</td>
</tr>
<tr>
<td>3</td>
<td>50≤%Max&lt;85</td>
<td>-0.1852</td>
<td>0.0077</td>
<td>577.50</td>
</tr>
<tr>
<td>4</td>
<td>85≤%Max&lt;100</td>
<td>-0.2498</td>
<td>0.0089</td>
<td>783.02</td>
</tr>
<tr>
<td>5</td>
<td>%Max=100</td>
<td>-0.2402</td>
<td>0.0325</td>
<td>54.64</td>
</tr>
<tr>
<td>6</td>
<td>Age≥65 &amp; disability = No</td>
<td>0.0720</td>
<td>0.0058</td>
<td>152.01</td>
</tr>
<tr>
<td>7</td>
<td>Age≥65 &amp; disability = Yes</td>
<td>0.7046</td>
<td>0.0122</td>
<td>3323.59</td>
</tr>
<tr>
<td>8</td>
<td>Provincial + Federal (RRQ + CPP)</td>
<td>-0.0440</td>
<td>0.0165</td>
<td>7.08</td>
</tr>
</tbody>
</table>

**Tab. 3 – Parameters of the Poisson regression model for the RRQ annuitants - Females.**

![Boxplots of deviance residuals](image)

**Fig. 20 – Boxplots of deviance residuals.**
As a result, the forces of mortality for the annuitants of the RRQ correspond to the forces of mortality estimated for the population of Quebec multiplied by the factors displayed in Tables 7 and 8, given in Appendix.

5 Numerical Examples

In this section, we will apply prospective mortality tables and the Poisson regression model discussed in the preceding sections in particular to the RRQ data. More precisely, we analyse in three different examples the impact of mortality improvement on the expected residual lifetime, on annuity prices and on the solvency of a general/supplemental pension plan.

5.1 Impact of Mortality Improvement on the Expected Residual Lifetime

We begin our analysis with a look at the temporal evolution of the expected residual lifetime given by

$$
E[K_x(t)] = e_x(t)
= \sum_{k=0}^{\omega-x-1} (kp_x(t))
= \sum_{k=0}^{\omega-x-1} \left( \prod_{i=0}^{k-1} p_{x+i}(t+i) \right),
$$

where $K_x(t)$ is the curtate remaining lifetime corresponding to $T_x(t)$, i.e. the integer part of $T_x(t)$.

We consider three categories of the RRQ pension plan for males and females; namely the category with the worst (category A) and the best (category B) observed mortality and one which mostly represents a Quebec pensioner. This last category (category C) regroups non-disabled pensioners which have always worked in the province of Quebec. Also, these pensioners have all left the workforce after the age of 65 and always gained an income higher to the maximum annual earnings on which contributions can be made to the Quebec Pension Plan (40 500 $CAN. in 2004). The details associated with these categories are given in Tables 4 and 5.

Note that the effect column in Tables 4 and 5 refers to the total multiplicative impact of retirees characteristics on the force of mortality and the proportion column gives the percentage of pensioners observed in each category for the last year of the RRQ data file.

The results of Figure 21 lead to similar conclusions as the ones made with the nonparametric approach performed previously. Mainly, we observe that the mortality of the females is, in each category, better than for the males and that the evolution in time of the mortality at a given
Males

<table>
<thead>
<tr>
<th>Name</th>
<th>Percentage</th>
<th>Financing</th>
<th>AgeRtr</th>
<th>Disability</th>
<th>Effect</th>
<th>Proportion</th>
</tr>
</thead>
<tbody>
<tr>
<td>M-A</td>
<td>0≤%Max&lt;10</td>
<td>RRQ</td>
<td>≥65</td>
<td>Yes</td>
<td>2.3147</td>
<td>0.015%</td>
</tr>
<tr>
<td>M-B</td>
<td>85≤%Max&lt;100</td>
<td>RRQ+CPP</td>
<td>&lt;65</td>
<td>No</td>
<td>0.7677</td>
<td>0.875%</td>
</tr>
<tr>
<td>M-C</td>
<td>%Max=100</td>
<td>RRQ</td>
<td>&lt;65</td>
<td>No</td>
<td>0.9124</td>
<td>12.687%</td>
</tr>
</tbody>
</table>

Tab. 4 – Details associated to categories for males of the RRQ pension plan.

Females

<table>
<thead>
<tr>
<th>Name</th>
<th>Percentage</th>
<th>Financing</th>
<th>AgeRtr</th>
<th>Disability</th>
<th>Effect</th>
<th>Proportion</th>
</tr>
</thead>
<tbody>
<tr>
<td>F-A</td>
<td>0≤%Max&lt;10</td>
<td>RRQ</td>
<td>≥65</td>
<td>Yes</td>
<td>1.9119</td>
<td>0.036%</td>
</tr>
<tr>
<td>F-B</td>
<td>85≤%Max&lt;100</td>
<td>RRQ+CPP</td>
<td>&lt;65</td>
<td>No</td>
<td>0.7045</td>
<td>0.159%</td>
</tr>
<tr>
<td>F-C</td>
<td>10≤%Max&lt;30</td>
<td>RRQ</td>
<td>&lt;65</td>
<td>No</td>
<td>0.8527</td>
<td>8.702%</td>
</tr>
</tbody>
</table>

Tab. 5 – Details associated to categories for females of the RRQ pension plan.

Fig. 21 – Expected residual lifetime for the categories of the RRQ data defined in Tables 4 and 5
age differs. We have noticed that the improvement of the mean residual life is mostly significative between the ages 65 and 85 and becomes less significative at higher ages. This leads, in time, to pensioners living longer lives i.e. reaching an age closer to the limit age without however an increase in the limit age.

We also observe significant differences in the expected residual lifetime of the pensioners in the different categories which confirms the relevance of the Poisson regression model to discriminate the retirees. For example, the expected residual lifetime at age 65 in 2005 for the males of categories A, B and C are respectively -33 %, 13 % and 5 % higher than the ones obtained with the GAM 83 table and for the females -14 %, 25 % and 18 %. In 2015, these percentages become -28 %, 18 % and 11 % for the males and -7 %, 31 % and 24 % for the females. The increase in time of the percentages confirms our previous observation of a better mortality in time.

5.2 Impact of Mortality Improvement on Annuity Prices

In our second example, we look at the impact of the temporal evolution of the mortality on the price of life annuities given by

\[
\tilde{a}_x(t) = \sum_{k=0}^{\omega-x-1} v^k (k p_x(t)) = \sum_{k=0}^{\omega-x-1} v^k \left( \prod_{i=0}^{k-1} p_{x+i}(t+i) \right).
\]

We choose an interest rate of 6 % which is comparable to the observed yield to maturity on long-term bonds. Figure 22 illustrates the evolution in time of annuity prices for the categories of the RRQ data defined in the previous example.

Not surprisingly, the results corroborate the ones obtained on the evolution of the mean residual lifetime given that it can also be seen as a life annuity with a zero interest rate.

Figure 22 allows us to see the effect of mortality improvements on annuity prices i.e. that its improvement is attenuated by an increase in the interest rate. As was done in the first example, we can look at the differences in the price of annuities obtained with a dynamic mortality table and the GAM 83 mortality table (computed with an interest rate of 6 %). For example, an annuity issued at 65 in 2005 for the three male categories are -24 %, 6 % and 2 % higher than the price obtained with the GAM 83. In 2015, these percentages respectively become -20 %, 9 % and 5 %. As for the females in the same categories, the increases in percentage are -9 %, 11% and 8 % for 2005 and -5 %, 14 % and 11 % for 2015.
5.3 Impact of Mortality Improvement on the Solvency of Pension Plans

We now analyze the impact of mortality improvements in time on the solvency of general/supplemental pension plans.

We consider a pension plan with \( n \) members divided in the 78 categories with the same proportions as those observed in the RRQ pension plan between years 1994 and 2003. We assume that at a given evaluation date, the solvency ratio of the pension plan is 100%, which means that the assets are equal to the liabilities which are calculated with a static mortality table (e.g. GAM 83). Such a ratio seems to indicate a good financial situation but one must remember that the mortality improvements are ignored with such a mortality table. In fact, one should expect more important liabilities with the improvement of mortality displayed in dynamic life tables.

To illustrate such a problem, we compute the ruin probability, which corresponds to the probability that the pension assets go below zero in the future. We assume that the future mortality improvements of the pension members will correspond to those predicted with the dynamic mortality table obtained previously for the RRQ pension plan. At valuation date, we assume that the pension assets and liabilities are equal. For sake of simplicity, we also assume that the assets are invested at a deterministic rate of 4 or 6% and that the annuity payments are made at the beginning of each year. In Table 6, we observe that the ruin probabilities increase as the evaluation is made...
at later dates.

\[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
\text{Valuation date} & i = 4\% & i = 6\% & i = 4\% & i = 6\% & i = 4\% & i = 6\% \\
\hline
2000 & 67659522 & 56753274 & 66231698 & 56131312 & 0.9490 & 0.8434 \\
2005 & 68930972 & 57657921 & 66231698 & 56131312 & 0.9990 & 0.9910 \\
2010 & 70153586 & 58521615 & 66231698 & 56131312 & 1 & 1 \\
2015 & 71333727 & 59350133 & 66231698 & 56131312 & 1 & 1 \\
2020 & 72471688 & 60144082 & 66231698 & 56131312 & 1 & 1 \\
\hline
\end{array}
\]

Tab. 6 – Pension liabilities (with projected mortality tables and GAM 83) and ruin probabilities.

These are results are confirmed by Figure 23 which shows the evolution of the expected annuity payments in the future in several different contexts. The computation of these payments is made with the GAM 83 static mortality table and a dynamic mortality table in which the improvement of the mortality begins in years 2000, 2005, 2010, 2015 or 2020. From this figure, we observe that according to the projected tables, the expected annuity payments decrease less rapidly than those computed with the GAM 83 mortality table.

6 Acknowledgements

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Fig. 23 – Projection of the expected pension annuity payments for a representative RRQ portfolio of 1000 annuitants as a function of the year of insurance.
7 References


Tab. 7 – Multiplicative factors to the forces of mortality of the population of Quebec to obtain the forces of mortality of the RRQ annuitants - Males.

<table>
<thead>
<tr>
<th>Class #</th>
<th>Percentage</th>
<th>Financing</th>
<th>AgeRtr</th>
<th>Disability</th>
<th>Multiplicative effect</th>
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<tbody>
<tr>
<td>1</td>
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<td>RRQ</td>
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<td>&gt;65</td>
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<td>1.2012</td>
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<tr>
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<td>0&lt;Max&lt;10</td>
<td>RRQ</td>
<td>&gt;65</td>
<td>Yes</td>
<td>2.3147</td>
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<tr>
<td>4</td>
<td>0&lt;Max&lt;10</td>
<td>RRQ+CPP</td>
<td>&lt;65</td>
<td>No</td>
<td>1.1064</td>
</tr>
<tr>
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<td>0&lt;Max&lt;10</td>
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Tab. 8 – Multiplicative factors to the forces of mortality of the population of Quebec to obtain the forces of mortality of the RRQ annuitants - Females.