"Exponential growth of lie algebras of finite global dimension"

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ABSTRACT

Let \( L \) be a connected finite type graded Lie algebra. If \( \dim L = \infty \) and \( \text{gldim} L < \infty \), then \( \log \text{index} L = \alpha > 0 \). If, moreover, \( \alpha < \infty \), then for some \( \sum_{i=1}^{d-1} \dim L_{k+i} = e(k \alpha k) \), where \( \alpha(k) \to \log \text{index} L \) as \( k \to \infty \).

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EXPONENTIAL GROWTH OF LIE ALGEBRAS OF FINITE GLOBAL DIMENSION

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Abstract. Let $L$ be a connected finite type graded Lie algebra. If $\dim L = \infty$ and $\text{gldim } L < \infty$, then $\log \text{index } L = \alpha > 0$. If, moreover, $\alpha < \infty$, then for some $d$, $\sum_{i=1}^{d-1} \dim L_{k+i} = e^{k \alpha_k}$, where $\alpha_k \to \log \text{index } L$ as $k \to \infty$.

We work with graded vector spaces $V$ over a field $k$ of characteristic $\neq 2$ and denote by $V(p,q)$ the subspace $\{V_i \mid p < i < q\}$. The logarithmic index of any graded vector space $V$ is defined by

$$\log \text{index } V = \limsup_{k} \frac{\log \dim V_k}{k},$$

and an infinite sequence $(q_i)$ is a quasi-geometric growth sequence for $V$ if for some fixed $n$, $q_i < q_{i+1} \leq nq_i$, for all $i$, and if

$$\frac{\log \dim V_{q_i}}{q_i} \to \log \text{index } V.$$

Now consider graded Lie algebras as defined in [4]; in particular we suppose $[x,[x,x]] = 0$, $x \in L_{\text{odd}}$, if char $k = 3$. $L$ is connected finite type (a cft graded Lie algebra) if $L = \{L_i\}_{i \geq 1}$ and each $L_i$ is finite dimensional. The global dimension ($\text{gldim } L$) and depth of a cft graded Lie algebra, $L$, are defined, respectively, by

$$\text{gldim } L = \max \{ k \mid \text{Ext}^k_{UL}(k,k) \neq 0 \}$$

and

$$\text{depth } L = \min \{ k \mid \text{Ext}^k_{UL}(k,UL) \neq 0 \}.$$  

It is easy to see that depth $L \leq \text{gl dim } L$.

Our main result reads:

**Theorem.** Suppose $L$ is a cft graded Lie algebra and $\dim L = \infty$. If $\text{gldim } L < \infty$, then $\log \text{index } L > 0$. If, moreover, $\log \text{index } L < \infty$, then for some $d$,

$$\sum_{i=1}^{d-1} \dim L_{k+i} = e^{k \alpha_k},$$

where $\alpha_k \to \log \text{index } L$, as $k \to \infty$.
Remark. Theorem 3 of [5] establishes the same conclusion when the hypothesis gldim \(L < \infty\) is weakened to depth \(L < \infty\), but certain additional growth conditions on \(L\) are assumed.

Now suppose \(X\) is a simply connected topological space with each \(H_\ast(X; \mathbb{Q})\) finite dimensional. Then the loop space homology, \(H_\ast(\Omega X; \mathbb{Q})\), is the universal enveloping algebra of an \(\infty\)-dimensional \(\mathrm{cft}\) graded Lie algebra \(L_X\), isomorphic to \(\pi_\ast(\Omega X) \otimes \mathbb{Q}\).

**Corollary.** If \(\dim L_X = \infty\), gldim \(L_X < \infty\) and log index \(L_X < \infty\), then for some \(d\),
\[
\sum_{i=1}^{d-1} \dim \pi_{k+i}(X) \otimes \mathbb{Q} = e^{\log N},
\]
where \(\alpha_k \rightarrow \log \dim L_X\) as \(k \rightarrow \infty\). In particular \(\sum_{i=1}^{d-1} \dim \pi_{k+i}(X) \otimes \mathbb{Q}\) grows exponentially in \(k\).

**Proof of the Theorem.** First we establish

**Lemma 1.** An infinite-dimensional \(\mathrm{cft}\) graded Lie algebra \(L\) of finite global dimension has a quasi-geometric growth sequence.

**Proof.** We use the same argument as in the proof of Theorem 2 in [5]: Put \(m = \mathrm{gldim} L\), \(a = \left(\frac{1}{2(m+1)}\right)^{m+1}\) and \(\alpha = \log \dim L\).

The Cartan-Eilenberg-Serre cochain complex \(C^\ast(L)\) is in fact a Sullivan algebra \(\mathfrak{g}^\ast\mathfrak{h}\) of the form \(\bigwedge (sL)^\#\), \((sL)^\#\) denoting the dual of the suspension of \(L\) and \(\bigwedge V\) denoting the free graded commutative algebra on \(V\). The differential in \(\bigwedge (sL)^\#\) increases the length of word gradation by 1 and so gives a second gradation \(H^{(p)}\) in \(H(\bigwedge (sL)^\#)\). As shown in [1], \(\mathrm{Ext}_{UL}^p(k, k) \cong H^{(p)}\), and so our hypothesis implies \(H^{(p)} = 0\) for \(p > m\).

Note that for each \(k\), \(C^\ast(L_{>k})\) is obtained from \(C^\ast(L)\) by dividing by the ideal generated by elements in \((sL)^\#\) of degree \(\leq k\). Since gldim \(L_{>k} \leq m\) it follows that these quotient cochain complexes also satisfy \(H^{(p)} = 0\) for \(p > m\). The argument of [2], section 4, can therefore be applied verbatim to \(\bigwedge (sL)^\#\) (with \(\mathrm{cat} \bigwedge X \leq m\) replaced by gldim \(L \leq m\)) to conclude that \(\alpha > 0\).

The same argument in the proof of Theorem 2 in [5] now shows that each \(L\) has a quasi-geometric growth sequence. Let \(n_i\) be an increasing sequence such that \((\dim L_{n_i})^{\frac{1}{n_i}}\) converges to \(e^\alpha\). By starting the sequence at some \(n_j\) we may assume \(\dim (L_{n_i}) > \frac{1}{a}\), for all \(i\). Thus the formula in ([2], top of page 189) gives a sequence \(n_i = q_0 < q_1 < \cdots < q_k = n_{i+1}\) such that \(q_{j+1} \leq (m+1)q_j\) and such that
\[
\left(\dim L_{q_j}\right)^{\frac{1}{q_j}} \geq (a \dim L_{n_i})^{\frac{1}{n_i}}, \quad j < k.
\]
Hence interpolating the sequences \(n_i\) with the sequences \(q_j\) gives a quasi-geometric growth sequence \(\{r_j\}\).

We now revert to the proof of the Theorem. Since gldim \(L < \infty\) we may choose a non-zero element \(x \in L\) of even degree \(d\). Let \(N\) be the sub-Lie algebra of elements of degree \(> d\) that commute with \(x\). Then
\[
\mathrm{gldim} L \geq \mathrm{gldim} (kx + N) = 1 + \mathrm{gldim} N.
\]
If \(\log N = \log L = \alpha\), then certainly \(\dim N < \infty\), and so \(N\) satisfies the hypotheses of the Theorem. By induction on global dimension it satisfies the
conclusion. In particular, if $\alpha < \infty$, then for some $d$,

$$\sum_{j=1}^{d-1} \dim N_{k+j} = e^{k\beta_k}$$

with $\beta_k \to \alpha$. Write

$$\sum_{j=1}^{d-1} \dim L_{k+j} = e^{k\alpha_k}.$$

Then $\alpha_k \geq \beta_k$ and $\limsup \alpha_k = \alpha$ because $\alpha = \log \text{index} L$. Thus $\alpha_k \to \alpha$, and the Theorem holds in this case.

**Lemma 2.** There is a sequence of finitely generated sub-Lie algebras $E(i) \subset L$ such that $\log \text{index} E(i) \to \alpha$.

**Proof.** Otherwise for some $\varepsilon > 0$ we have $\log \text{index} E \leq \alpha - \varepsilon$ for every finitely generated sub-Lie algebra $E \subset L$. Construct an increasing sequence of finitely generated sub-Lie algebras, $F(i)$, and increasing sequences $(k_i)$ and $(\ell_i)$, as follows. Set $F(0) = 0$, and if $F(i)$ is constructed choose $k_i$ and $\ell_i$ so that

(i) $\dim F(i) \leq e^{\ell_i(\alpha - \varepsilon / 2)}$, $k \geq k_i$,

(ii) $\dim L_{\ell_i} \geq e^{\ell_i(\alpha - 1/i)}$,

(iii) $\ell_i > (m + 1)k_i$.

Then let $F(i + 1)$ be the sub-Lie algebra generated by $F(i)$ and $L_{\ell_i}$.

Now let $F = \bigcup_i F(i)$. Since $\dim F \leq e^{\ell_i(\alpha - \varepsilon / 2)}$ it follows that $\log \text{index} F = \alpha$. Moreover, because $F \subset L$, $\text{gldim} F \leq m$. Thus by Lemma 1 there is an infinite sequence $q_j$ such that for all $j$, $q_j < q_{j+1} \leq (m + 1)q_j$ and $\dim F_{q_j} \geq e^{q_j(\alpha - \varepsilon / 2)}$.

In particular we may choose $i$ and $j$ so that $q_j < k_i < q_{j+1}$. But then $q_{j+1} \leq (m + 1)q_j \leq (m + 1)k_i < \ell_i$, and it follows that $F_{q_{j+1}} = F(i)_{q_{j+1}}$. This implies that $\dim F_{q_{j+1}} < e^{q_{j+1}(\alpha - \varepsilon / 2)}$, a contradiction. \hfill \Box

Finally, we complete the proof of the theorem. It remains to consider the case $\log \text{index} N < \log \text{index} L$. Let $E(i) \subset L$ be finitely generated sub-Lie algebras such that $\log \text{index} E(i) \to \log \text{index} L$. Moreover, $\text{gldim} E(i) \leq m$ and, according to Lemma 1, each $E(i)$ has a quasi-geometric growth sequence. Since $E(i)$ is finitely generated, Theorem 3 of [5] applies and states that for some $d$, $\frac{\log \text{dim} E(i)}{k}$ converges to $\log \text{index} E(i)$.

Fix $\varepsilon > 0$ and choose $i$ so that $\log \text{index} E(i) \geq \alpha - \varepsilon / 4$. Then choose $k_0$ so that

$$\frac{\log \text{dim} E(i(k,k+d_i))}{k} \geq \alpha - \varepsilon / 3, \quad k \geq k_0.$$ 

This implies that $k_0$ extends to an infinite sequence $(k_\ell)$ such that $k_\ell < k_{\ell+1} < k_\ell + d_i$ and such that

$$\frac{\log \text{dim} L_{k_\ell}}{k_\ell} \geq \alpha - \varepsilon / 2, \quad \ell \geq 0.$$ 

On the other hand, since $\log \text{index} N < \log \text{index} L$ we may assume (for $k_0$ sufficiently large and $\varepsilon$ sufficiently small) that

$$\sum_{j \leq d_i / d} \dim N_{k_\ell + jd} \leq \frac{1}{2} \dim L_{k_\ell}, \quad \text{for all } \ell.$$
Since $N = (\ker \text{ad}x) > d$ we have
\[
\dim L_{k_\ell} + pd \geq \dim L_{k_\ell} - \sum_{j=0}^{p-1} \dim N_{k_\ell + j} \geq \frac{1}{2} \dim L_{k_\ell}, \quad p \leq \frac{d_i}{d}.
\]
It follows that for $p \leq \frac{d_i}{d}$ and $k_\ell$ sufficiently large
\[
\frac{\log \dim L_{k_\ell} + pd}{k_\ell + pd} \geq \frac{1}{2} \log \frac{1}{2} \frac{\log \dim L_{k_\ell}}{k_\ell} \frac{k_\ell}{k_\ell + pd} \geq \alpha - \varepsilon.
\]
This establishes the Theorem. □

References


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