"Time and space (4D) homogenization for viscoelastic-viscoplastic solids under large numbers of cycles"

Haouala, Sarra

Abstract
The principal objective of this thesis is to predict the response of inelastic materials under a large number of cycles, while simulating a much smaller number. For coupled viscoelastic-viscoplastic (VE-VP) homogeneous solids subjected to large numbers of cycles, a two-scale time homogenization formulation and the corresponding algorithms are proposed. The main aim is to predict the long time response while reducing the computational cost considerably. The method is based on the definition of macro and micro-chronological time scales, and on asymptotic expansions of the unknown variables. First, the VE-VP constitutive model is formulated based on a thermodynamical framework. Next, the original VE-VP initial-boundary value problem is decomposed into coupled micro-chronological (fast time scale) and macro-chronological (slow time-scale) problems. The former is purely VE, and solved once for each macro time step, whereas the latter problem is nonlinear and solved iteratively using fully...

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Time and space (4D) homogenization for viscoelastic-viscoplastic solids under large numbers of cycles

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Thesis submitted in fulfillment of the requirements for the Ph.D. degree in Engineering Sciences

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Sarra.
Abstract

The principal objective of this thesis is to predict the response of inelastic materials under a large number of cycles, while simulating a much smaller number.

For coupled viscoelastic-viscoplastic (VE-VP) homogeneous solids subjected to large numbers of cycles, a two-scale time homogenization formulation and the corresponding algorithms are proposed. The main aim is to predict the long time response while reducing the computational cost considerably. The method is based on the definition of macro and micro-chronological time scales, and on asymptotic expansions of the unknown variables. First, the VE-VP constitutive model is formulated based on a thermodynamical framework. Next, the original VE-VP initial-boundary value problem is decomposed into coupled micro-chronological (fast time scale) and macro-chronological (slow time-scale) problems. The former is purely VE, and solved once for each macro time step, whereas the latter problem is nonlinear and solved iteratively using fully implicit time integration. For micro-scale time averaging, one-point and multi-point integration algorithms are developed. Several numerical simulations on uniaxial and multiaxial cyclic loadings illustrate the computational efficiency and the accuracy of the proposed methods.

For composite materials, a multiscale computational strategy is proposed for the analysis of structures, which are described at a refined level both in space and in time. The proposal is applied to two-phase VE-VP composite materials subjected to large numbers of cycles. The main aim is to predict the effective long time response while reducing the computational cost considerably. The proposed computational framework is a combination of the mean-field space homogenization based on the generalized incrementally affine formulation for VE-VP composites, and the asymptotic time homogenization approach for coupled VE-VP homogeneous solids under large numbers of cycles. The time homogenization method is based on the definition of micro- and macro-chronological time scales, and on asymptotic expansions of the unknown variables. Firstly, the original anisotropic VE-VP initial-boundary value problem of the composite material is decomposed into coupled micro-chronological (fast time scale) and macro-chronological (slow time-scale) problems. The former corresponds to a VE composite, and is solved once for each macro time step, whereas the latter problem is a nonlinear composite and solved
iteratively using fully implicit time integration. Secondly, mean-field space homogenization is used for both micro- and macro-chronological problems to determine the micro- and macro-chronological effective behavior of the composite material. The response of the matrix material is VE-VP with $J_2$ flow theory assuming small strains. The formulation exploits the return-mapping algorithm for the $J_2$ model, with its two steps: viscoelastic predictor and plastic corrections. The proposal is implemented for an extended Mori-Tanaka scheme for a number of polymer composite materials subjected to large numbers of cycles.

An extension of the two-scale time homogenization approach to VE-VP homogeneous materials coupled with ductile damage (VE-VP-D) under large numbers of cycles is proposed. An asymptotic approach allows to decouple the boundary problem into macro-chronological and micro-chronological problems. A different multiscale decomposition is introduced to account for irreversible inelastic deformation. The method is applied to the fatigue of thermoplastic polymers.
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<tr>
<td>EP</td>
<td>Elasto-Plastic(ity)</td>
</tr>
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<td>EVP</td>
<td>Elasto-Viscoplastic(ity)</td>
</tr>
<tr>
<td>FE</td>
<td>Finite Element</td>
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<tr>
<td>LATIN</td>
<td>Large Time Increment</td>
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<td>MFD</td>
<td>Main Flow Direction</td>
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<td>MFH</td>
<td>Mean-Field Homogenization</td>
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<td>MT</td>
<td>Mori-Tanaka</td>
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<td>RH</td>
<td>Relative Humidity</td>
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<td>RVE</td>
<td>Representative Volume Element</td>
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<tr>
<td>SEM</td>
<td>Scanning Electron Microscopy</td>
</tr>
<tr>
<td>SGFR</td>
<td>Short Glass Fiber Reinforced</td>
</tr>
<tr>
<td>PMC</td>
<td>Polymer Matrix Composite</td>
</tr>
<tr>
<td>VE</td>
<td>Viscoelastic(ity)</td>
</tr>
<tr>
<td>VE-VP</td>
<td>Viscoelastic(ity)-Viscoplastic(ity)</td>
</tr>
<tr>
<td>VP</td>
<td>Viscoplastic(ity)</td>
</tr>
<tr>
<td>VE-VP-D</td>
<td>Viscoelastic(ity)-Viscoplastic(ity)-Damage</td>
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<tr>
<td>ODF</td>
<td>Orientation Distribution Function</td>
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Notations

Symbols

\( \dot{x} \)  
Time derivative.

\((\cdot)_M\)  
Average in time.

\(\tilde{(\cdot)}\)  
Fluctuation in time.

\(\langle \cdot \rangle\)  
Average value over all the RVE.

\(\langle \cdot \rangle_{\Omega}\)  
Average value over all the volume \(\Omega\).

\(\langle \cdot \rangle_{\Omega_r}\)  
Average value over phase \(r\).

\(\langle \cdot \rangle_{\Omega_{p.g.}}\)  
Average value over the pseudo grain domain \(\Omega_{p.g.}\).

\(\langle \cdot \rangle_{\Omega_{r.p.g.}}\)  
Average value over the phase \(r\) in the pseudo grain domain \(\Omega_{p.g.}\).

Scalars and scalar-valued functions

\(t_M\)  
Macro-chronological time.

\(\tau\)  
Micro-chronological time.

\(E\)  
Young’s modulus.

\(\nu\)  
Poisson’s ratio.

\(\sigma_{eq}\)  
von Mises equivalent stress.

\(\sigma_{eq}^{pred}\)  
Trial equivalent stress after a viscoelastic time step.

\(\sigma_y\)  
Yield stress.

\(f\)  
Yield surface.

\(p, \dot{p}\)  
Accumulated plastic strain and its corresponding rate.

\(\psi\)  
Free energy.

\(\psi_{ve}\)  
Viscoelastic free energy.

\(\psi_{vp}\)  
Viscoplastic free energy.

\(\Phi\)  
Dissipation function.

\(\Phi_{ve}\)  
Viscoelastic dissipation function.
Φ_{vp}  Viscoplastic dissipation function.
R(p)    Hardening function.
g_v     Viscoplastic function.
G_i     Shear modulus (Weight).
g_i     Shear relaxation time.
K_j     Bulk modulus (Weight).
k_j     Bulk relaxation time.
D       Damage scalar variable.

**Second-order tensors**

1       Second-order identity tensor.
σ       Cauchy stress tensor.
ε       Infinitesimal strain tensor.
ε^{ve}  Viscoelastic strain tensor.
ε_{vp}, \dot{ε}_{vp} Viscoplastic strain tensor and its corresponding rate.

**Fourth-order tensors**

C^{ve}  Relaxation tensor.
I       Symmetric fourth-order identity tensor.
I^{vol} Volumetric operator: I^{vol} \equiv \frac{1}{3} 1 \otimes 1.
I^{dev} Deviatoric operator: I^{dev} \equiv I - I^{vol}.

Throughout the report, the following notations and conventions are used. Boldface symbols designate second- or fourth-rank tensors as indicated by the context.

The different products are expressed as:

\[ \mathbf{a} : \mathbf{b} = a_{ij}b_{ji}, \quad (\mathbf{C} : \mathbf{D})_{ijkl} = C_{ijmn}D_{nmkl}, \quad (\mathbf{a} \otimes \mathbf{b})_{ijkl} = a_{ij}b_{kl}, \]

where summation over a repeated index is supposed.
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1.0.1 Industrial context and motivations

Nowadays, polymer matrix composites (PMCs) contribute to the improvement of properties and reliability of many components. They find widespread use in various industries such as aerospace, automotive and consumer electronics. A polymer matrix composite (PMC) designates a polymer matrix reinforced with short, long or continuous fibers. The matrix phase of a PMC can be classified as either thermoset (e.g. epoxies, phenolics) or thermoplastic (e.g. High Density Polyethylene (HDPE), polypropylene, polycarbonate, polyamide). In this work, we are interested in short fiber reinforced thermoplastic polymers.

In the context of CO$_2$ emission reduction, the automotive industry tries to substitute heavy metallic parts by lightweight composite structures. Short glass fiber reinforced (SGFR) polymer materials, among them SGFR polyamide-66, are a cost-efficient solution which combines both the reduction of weights and the increase of production rates. Furthermore, the use of this kind of materials is particularly appreciated for the case of obtaining complex shaped components.

During service, many of these polymer-based parts are exposed to cyclic loading conditions resulting in the degradation of material properties. The design of these complex components or structures still requires expensive testing and experiments which are generally limited to small structural components. One of the main issues for engineers, therefore, lies in the prediction of the fatigue life duration under complex loadings. Two steps are involved to solve the problem: first, the investigation of the cyclic behavior of PMCs. Second, the design of a model to predict the fatigue life duration.

1.0.2 Modeling of thermoplastic polymers

Thermoplastic materials is a class of polymer materials that can be melted or made to flow with heat and allows to be re-shaped. Therefore they are characterized by their glass and melting temperature $T_g$ and $T_m$, respectively.
Above the melting temperature the thermoplastics are in amorphous state and the polymer chains acquire a bundled structure. In some polymers, the chains rearrange upon cooling and form partly ordered regions. This phenomenon is called crystallization. If some fraction of the polymer remains un-crystallized, or, amorphous when the polymer is cooled to room temperature, the material is called semi-crystalline. The degree of crystallinity influences the properties of polymers.

Polymeric materials such as semi-crystalline thermoplastics exhibit a complex inelastic behavior and they are time and rate-dependent at all stages. The typical stress-strain curve of a sample of High Density Polyethylene (HDPE) under drawing tension is summarized in figure 1.1. According to [Balieu et al., 2013a] the behavior of semi-crystalline polymers is viscoelastic-viscoplastic. The stress increases at low deformation, the behavior of the material is linear viscoelastic, and it becomes progressively nonlinear viscoelastic until reaching the yield-point. The stress passes over a maximum and a neck forms somewhere in the sample. The behavior of the material becomes viscoplastic and a strain softening appears. For higher deformation, the neck will extend over the sample while the tensile stress keeps essentially unchanged. Finally, after the whole cold-drawing, the stress increases again until the break point.

Defining the yield stress as the point just located before the strain softening, is not the unique interpretation of the beginning of the viscoplastic state. [Raghava et al., 1973] determined the yield strengths from a true stress-true strain plot, using a strain displacement that is equivalent to a 0.3% offset in the usual terminology (figure 1.2). With this definition, the viscoplastic state occurs earlier without to be in a nonlinear viscoelastic state.

[Zhang and Moore, 1997] show the behavior of High Density Polyethylene (HDPE) under uniaxial tension followed by unloading. The behavior of the material is inelastic for both loading and unloading and it depends on both strain and strain rate (figure 1.3). The mechanism responsible for this behavior in polymeric materials is the sliding and relative motion of molecular chains in the material. As a result of this behavior, the polymer specimen can exhibit the phenomenon known as relaxation: a gradual decrease in stress with time under a constant strain (figure 1.4).

Semi-crystalline polymers exhibit also a strong dependency to the temperature. Figure 1.5 shows the behavior of High Density Polyethylene
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Figure 1.1: Typical stress-strain curve of HDPE. The changes in the shape of the sample are schematically indicated. 1) viscoelastic region, 2) yield transient, 3) strain softening, 4) hardening. [Men, 2001].

Figure 1.2: Tensile true stress-true strain plot for polycarbonate (PC) illustrating the selection of "yield strength" as based upon an offset of 0.3% (i.e. a strain of 0.003). [Raghava et al., 1973].
Figure 1.3: Stress-Strain experimental curves for HighDensity Polyethylene under uniaxial tension followed by unloading of [Zhang and Moore, 1997].

Figure 1.4: Typical relaxation behavior.
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Figure 1.5: True stress-strain curves, measured at a constant Hencky strain rate of $5 \times 10^{-3} \text{s}^{-1}$ at the temperatures indicated in the plot for High Density Polyethylene (HDPE). [Men, 2001].

(HDPE) at different temperatures. Increasing the temperature leads to a reduction of the material stiffness, and the stretching necessary for the fracture becomes very large.

The behavior of polymers is also pressure dependent ([G’Sell et al., 2002, G’Sell et al., 2004]). By using a novel video-controlled testing system and under uniaxial tension test at constant true strain rate, the evolution of volume strain is determined in polyethylene terephthalate (PET) by measuring in real time the three principal strain components in a small volume element ([G’Sell et al., 2002]). An increase of the volume variation at increasing axial strain is observed.

The mechanical behavior of thermoplastic polymers presents an important number of dependent parameters (time, strain rate, temperature pressure). Thereby proper engineering design with these materials is subject to the development of an adequate modeling using physical or phenomenological approaches, able to reproduce their mechanical behavior and also the possibility to simulate it numerically. The study of such materials began long before the macromolecular nature of polymers was understood. The mechanical behavior of polymeric materials is generally characterized in terms of their time-dependent properties in shear or in simple tension. In the literature, some authors describe the behavior of polymer materials by linear viscoelastic (VE) models, to capture the time and rate dependent material behavior which leads to relaxation and creep phenomena for constant applied strain or
stress states. Starting from simple uniaxial rheological models of viscoelasticity, e.g. the Maxwell model (series combination of a spring and a dashpot, figure 1.6a) and the Kelvin-Voigt model (a parallel combination of a spring and a dashpot, figure 1.6b).

Generally, a real polymer does not relax with a single relaxation time as predicted by previous models. Thus, to better depict the behavior of most VE materials, generalized series-parallel models are needed. Two forms are considered: the generalized Maxwell models which are useful when the excitation is a strain (figure 1.7a), and the generalized Voigt models which are to be used when the excitation is a stress (figure 1.7b). Christensen, 1971 [Kaliske and Rother, 1997] show that linear viscoelasticity constitutes a reasonable approximation to the time-dependent behavior of a large number of materials.

One of the earliest models, able to predict the behavior of amorphous polymers was proposed by [Haward and Thackray, 1968] in the uniaxial case. It is a physically-based model for amorphous polymer, where the constitutive equations are based on the molecular structure of the polymer material. The model consists of a Hookean spring in series with an Eyring dashpot (to model the plastic flow) and rubber elasticity spring.
Figure 1.7: Generalized Series-Parallel models: (a) N Maxwell elements in parallel, (b) N Kelvin-Voigt elements in series. $E_i$ and $\zeta_i$ are the stiffness and the viscosity of the $i^{th}$ element, (Tschoegl, 1989).
in parallel. This 1D model was extended later to the three-dimensional case by [Boyce et al., 1988] [Arruda and Boyce, 1993] [Wu and van der Giessen, 1995] and [Anand and Gurtin, 2003]. More recently, and based on the work of [Boyce et al., 1988], [Anand and Ames, 2006] develop a new continuum model for the viscoelasticplastic deformation of amorphous polymeric solids. The behavior is modeled by one resistance due to the molecular network interaction in parallel with a generalized Kelvin-Voigt model to describe the inter-molecular interaction. Multi-scale constitutive models have been proposed to capture the VE, VP behavior of semi-crystalline polymer materials. [Nikolov and Doghri, 2000], developed a micromechanically-based constitutive model for HDPE in small deformations. In this model, the microstructure of HDPE consists of closely packed crystalline lamellae separated by layers of amorphous polymer. The macroscopic behavior of the material is then described while taking into account the micro-structure evolution using homogenization techniques. Physical approaches have been also used to model the behavior of semi-crystalline polymers. A physically-based inelastic model under finite strain formulation has been proposed by [Ayoub et al., 2010] [Ayoub et al., 2011] to describe the mechanical behaviour of HDPE. The semi-crystalline polymer is considered as a heterogeneous medium, and the model is based on a two-phase representation of the microstructure where the crystalline and amorphous phases are considered as two separate resistances. A physically-based hyperelastic-viscoplastic model was developed by [Zairi et al., 2011] for large deformation stress-strain response and anisotropic damage in rubber-toughened glassy polymers. Physics-based modeling is very attractive since it takes into consideration the micro-structure of the material, but in the other hand it implies the need for a greater number of experimental data points to generate models with good predictive capability. Thus phenomenological models were investigated. Some authors tried to model the behavior of polymers within continuum mechanics and a thermodynamic framework from which state and evolution laws are derived. Classical models, initially developed for metallic materials, can then be extended for polymers. When the material is subjected to very slow or very small deformation, the behavior can be described by a linear viscoelasticity model. In this case, the constitutive equations are derived from Boltzmann’s superposition principle. A complete description of VE behavior is available in ([Christensen, 1971]),
Introduction

( [Salençon, 1983] ) and ( [Tschoegl, 1989] ). Nonlinear viscoelastic models were also developed to describe the behavior of polymeric materials ( [Han, 1985] ). Although the strain rate dependence can be considered to be the predominant characteristic of the polymer material behavior, VE models are not sufficient to quantitatively describe the mechanical behavior of polymers. In fact, several experimental tests, including monotonic tensile tests and relaxation tests, of polymer materials such as polypropylene ( [Kästner et al., 2012] ), indicate a viscoplastic material behavior with a rate dependency. Several elasto-viscoplastic (EVP) constitutive models have been also developed to describe the strain rate dependence of thermoplastics ( [Regrain et al., 2009] , [Drozdov, 2009] , [Drozdov et al., 2013] , [Balieu et al., 2013b] and [Vecchio et al., 2014] ).

An inelastic strain accumulation induced by cyclic loading, i.e., ratcheting, which occurs in materials subjected to cyclic loading, especially under stress-controlled mode with nonzero mean stress ( [Tao and Xia, 2007] , [Kang et al., 2009] , [Launay et al., 2011] , [da Costa Mattos and de Abreu Martins, 2013] , [Drozdov et al., 2013] , [Vecchio et al., 2014] and [Lu et al., 2014] ), is important in designing structural components. Ratcheting tests with and without intermediate holding times, were performed to characterize the mechanical behavior of polymer materials such as polycarbonate ( [Jiang et al., 2013] ). A test procedure is proposed to discriminate the contribution between viscous recovery and accumulated unrecoverable deformation during the cyclic loading of polymeric materials. It has been shown that, for cyclic loading, a permanent deformation still exists even if the peak stress of cyclic loading is below the yield strength of polycarbonate. Then to better reproduce the rate dependence in both elastic and inelastic deformation, coupled viscoelastic-viscoplastic (VE-VP) models have been proposed.

A coupled VE-VP rheological model has been developed by [Khan and Zhang, 2001] in order to describe the uniaxial response of polytetrafluoroethylene. Another phenomenological model based on rheological equations was proposed by [Khan et al., 2006] to describe the mechanical response of Adiprene-L100, by using a combination of linear and nonlinear springs with dashpots.

Later [Kim and Muliana, 2010] and [Miled et al., 2011] proposed a coupled VE-VP model in the case of $J_2$ viscoplasticity theory. The model [Miled et al., 2011] can reproduce in an acceptable way the response of polymers under small deformation, it supposes a decomposition of the total strain into a sum of VE and VP parts.
1.0.3 Modeling of composite materials subjected to high numbers of cycles

The prediction of the long-term behavior of PMC structures involves theoretical and numerical modeling, and it requires significant computational resources, in particular for cyclic loadings when taking into account the nonlinear behavior of the polymer composite materials. Actually, the problem of structures subjected to rapidly oscillatory loading exhibits multiple temporal and spatial scales. It is a multiscale phenomenon in space (due to the presence of heterogeneities in the microstructure of the material) and time (because the load period could be in the order of seconds whereas the component life may span years). The numerical solution process for complex, time-dependent non-linear problems requires, if one uses classical finite element (FE) codes, a computation time which turns out to be prohibitive. Thus new reliable and efficient strategies taking into account the multi-scale aspects in space and time are needed.

Spatial multi-scale mechanics is rooted in the analysis of the homogenized response of heterogeneous materials. To model the behavior of composite materials, two approaches are in general considered. The first one, is to consider the composite material as homogeneous and to develop a phenomenological constitutive law to predict the macroscopic response of the material. The second one, is to consider the composite material as a structure made of distinct phases (matrix and inclusions) and to focus on the equivalent or effective response of a finite volume of the material. Characteristic volumes were identified as unit cells for periodic materials and representative volume elements (RVE) for statistically heterogeneous media. The mechanical response of such volume is assumed to be equivalent to the macroscopic response of the composite material. Spatial homogenization techniques were first developed within the framework of elasticity and then extended to nonlinear composites. Among the developed homogenization approaches, there is the asymptotic homogenization which constitutes an elegant technique for predicting the effective properties of heterogeneous media with periodic microstructure. It was developed by Bensoussan et al., 1978 and Sanchez-Palencia, 1980 and has been applied successfully on linear elastic or weakly nonlinear composites Jansson, 1992. The generalized method of cells, it was introduced by Aboudi, 1989. Paley and
Aboudi, 1992] and [Dvorak, 1992]. It consists on dividing a repeating unit cell into an arbitrary number of generic cells and it has been shown to be more computationally efficient than finite element analysis based approaches for a range of composites. [Wilt, 1995, Aboudi, 1996]. This approach enables to compute the effective properties of heterogeneous inelastic materials [Llorca et al., 1991, Kwon and Berner, 1995]. A detailed literature review about cell and subcells methods is available in [Bednarcyk et al., 2004]. The mean-field (MF) based approach, was developed several decades ago and its aim is to reduce the computational cost. These so called semi-analytical mean-field homogenization (MFH) methods, enable to give a macroscopic response as well as mean-field information within the phases, based on assumptions of the interaction laws between the different phases. Most of the MFH schemes are based on the [Eshelby, 1957a] result. They are very efficient for a computational point of view and give a good accuracy in the linear elastic regime. MFH schemes are still under investigation for inelastic rate independent and dependent materials and damage.

Relatively few works have been devoted to multi-time-scale phenomena, it is one of the areas in which there has been relatively little research and documentation. [Smolinski et al., 1996] and [Combescure and Gravouil, 2002], introduced the so-called multi-time-step methods which allow to take into account different temporal discretizations in separate regions of the structure. This approach is used when a limited area of space requires a fine temporal solution or in the case of multiphysics problems for which different physical equations do not involve the same time scales.

However, treating problems at a very local scale with the technique presented above remains very costly. A method called variational multiscale method in time, was used by [Hughes and Stewart, 1996] and [Bottasso, 2002]. This approach is based on variational formulations in time and follows the same principles as those for the spatial aspects, where the basis functions form a partition of unity of the studied interval.

None of these strategies involves a true time-homogenization technique. Such techniques seem to have been introduced only for the case of cyclic loadings. There are two techniques that provide a temporal description suited to this type of problems. The first one is a strategy based on the large time increment method (acronym: LATIN) developed by [Ladevèze, 1985a, Ladevèze, 1985b, Cognard and Ladevèze, 1993], which proposes a particular representation of the variables on two time-
scales: it is a temporal FE method. This method is known to be non incremental, i.e. at each iteration it generates an approximation of the solution on the entire time interval. The second technique is called two-scale time homogenization, which is a direct extension of the asymptotic spatial homogenization. It was developed by [Guennouni, 1988] for elasto-viscoplastic homogeneous materials. It is based on asymptotic expansions in time of all the unknown fields and leads to a homogenized time behavior.

1.0.4 Objectives

This work is supported by the Région Wallonne through the "Fatigue of Fiber-Reinforced Polyamides and Industrial Applications on Structural Parts" (DURAFIP) project. This project seeks to expedite greater replacement of metal parts in the automotive industry by using reinforced polyamide and falls within the framework of the development of a package of solutions for the material modeling of durability of short glass fiber reinforced (SGFR) polyamide.

SGFR polyamides have a complicated behavior, their response to a given condition depends on their microstructure: fiber orientation and distribution ([Arif et al., 2014a, Arif et al., 2014b]). Figure 1.8 represents the fiber orientation distribution through the thickness of an injected specimen. The material is a SGFR polyamide-66 with 30% of fiber mass fraction (PA66/GF3). One can clearly see the particular microstructure of this material. It is characterized by a skin-shell-core structure. It can be seen that the fiber orientation throughout the sample is as follows: the fibers are randomly oriented in the skin layers, parallel in the shell layers and normal in the core layers to the main flow direction (MFD).

The main objective is then, the high cycle fatigue life prediction of SGFR polyamide-66 by developing numerical tools able to predict how SGFR polyamide will behave over an extended time period, taking into account the microstructure of the material.

The phenomenon of fatigue of materials under cyclic loadings has been widely examined especially for metals. Predicting fatigue behavior of PMC is still under investigation. Little information is available in the literature and the present knowledge is far from complete.

At the macroscopic scale the behavior of polymer materials under cyclic loadings has in common with that of metals a decrease in stiffness during the excitation. In the case of metals this phenomenon derives from the gradual development of plasticity and damage. Whereas for polymer
Figure 1.8: Skin-shell-transition-core microstructure formation of PA66/GF30 observed by $\mu$CT technique [Arif et al., 2014b].
materials, the stiffness drop may also be caused by the VE response of the material. Indeed, unlike metals, PMC exhibit mechanical behaviors significantly dependent on time and temperature under operation because of the viscoelastic behavior of the polymer matrix. Hence, the evolution of the material properties is time and temperature dependent.

There are a number of important differences between the fatigue behavior of metals and of fiber-reinforced polymer composites. In metals, the stage of invisible deterioration of the material takes a large part of the total life of the material. Microscopic cracks will begin to form. Then these small cracks will produce a large crack which will reach a critical size, and will propagate suddenly to cause the failure of the structure (Head, 1953).

The macroscopic and microscopic fatigue damage behavior of PA66/GF30 have been studied by Arif et al., 2014a and Arif et al., 2014b using scanning electron microscopy (SEM) and X-ray micro-computed tomography ($\mu$CT) techniques. Figure 1.9 describes completely the whole chronology of damage mechanisms in a PA66/GF30 specimen. This figure illustrates clearly the anisotropic nature of the damage at local scale. The deterioration by fatigue starts by the formation of ”damage zones” at fiber ends. It consists on fibre/matrix interface debonding and the appearance of voids at fiber ends. Afterwards, the debonding propagates along the fiber-matrix interface. This is followed by fiber breakage leading to the failure of the material.
The S-N curve is one of the most popular tools used to predict the fatigue life of metals, and it is based on the assumption that fatigue life depends on cycles, but not on time. Moreover, the evolution of the mechanical properties of metals under high cycle fatigue is often small. Thus, to expedite the fatigue test, the cyclic loads can be applied at much higher frequencies than the actual loading. Contrarily to metals, polymers will present a greater sensitivity to the test frequency, therefore simply applying the S-N curve to PMC will not provide accurate prediction of the fatigue life.

To design a part with such materials, two methods can be followed, one can use a large safety coefficient and neglect material variability. This leads to over-designed parts, with a far from optimal weight. In this context, an original approach which takes into account the multi-scale phenomena in both spatial and temporal levels is proposed in this thesis. First a generalization of the time homogenization method from elasto-viscoplasticity (EVP) to viscoelasticity-viscoplasticity (VE-VP) and then to viscoelasticity-viscoplasticity coupled with ductile damage (VE-VP-D) was developed. Second a new approach which combines asymptotic temporal homogenization with mean-field homogenization for coupled VE-VP composites was proposed.

1.0.5 Thesis structure

The manuscript has the following outline. In Chapter 2, we give a general, non-exhaustive summary of the multiscale computational strategies in time. In Chapter 3, a two-scale time homogenization approach for coupled viscoelastic-viscoplastic (VE-VP) homogeneous solids and structures subjected to large numbers of cycles is proposed. The main aim is to predict the long time response while reducing the computational cost considerably. The method is based on the definition of macro and micro-chronological time scales, and on asymptotic expansions of the unknown variables. In Chapter 2.3 the most important features of multiscale modeling in space are presented. In Chapter 4, we propose a multiscale computational strategy for the analysis of composite structures, which are described at a refined level both in space and in time. And finally, in Chapter 5 a new approach for multiscale modeling of fatigue is proposed and an extension of the time homogenization method to VE-VP materials with ductile damage is developed.
CHAPTER 2

State of the art: Multi-scale computational strategies in time and in space

2.1 Introduction

This chapter is a general, non-exhaustive summary of the multiscale computational strategies developed recently. Here, we try to describe the most important features of such strategies. We will focus first on the temporal aspect. Three categories can be distinguished: multi-time-step methods, variational multiscale method in time and methods dedicated to cyclic loadings. For the spatial aspect, the selected strategies are grouped into three categories: Direct finite Element (FE) computation, asymptotic homogenization, method of cells and subcells and mean-field homogenization (MFH).

2.2 Multi-scale computational strategies in time

2.2.1 Multi-time-step methods

This approach was introduced by [Belytschko et al., 1979, Belytschko et al., 1985], and it offers a great flexibility in the choice of temporal descriptions of different regions of space (see figure 2.1). This method is based upon dividing the nodes of the mesh into groups with different time steps. It is used when a limited area of space requires a fine temporal solution or in the case of multiphysics problems for which different physical equations do not involve the same time scales ([Smolinski et al., 1996, Combescure and Gravouil, 2002]). In addition to multiple time steps, these methods allow different integration rules to be used in different elements. To illustrate the method of multi-time steps, [Combescure and Gravouil, 2002] propose to determine the motion, in the time interval \([0, T]\), of a deformable solid occupying a domain \(\Omega\). The idea is to decompose the whole domain \(\Omega\) into sub-domains and to associate to each sub-domain a temporal discretization and an integration scheme.
In figure 2.1 we consider a domain $\Omega$ decomposed into two parts $\Omega_1$ and $\Omega_2$, separated by the interface $\Gamma$, and to which two temporal discretization $\tau_1 = \{ I_i = (t_i, t_{i+1}) \}_{i=0}^{N-1}$ and $\tau_2 = \{ I_j' = (t_j', t_{j'+1}) \}_{j=0}^{N'-1}$ are associated, respectively. The continuity of the unknowns is ensured by the Lagrange multiplier $\Lambda$. The equation expressing the equilibrium of each domain is written as follows:

$$M^k \dddot{U}_k + K^k U_k = F_{\text{ext}}^k + F_{\text{link}}^k, \quad \text{with} \quad F_{\text{link}}^k = C^k \Lambda, \quad k = 1, 2,$$

$$\sum_{k=1}^2 C^k W^k = 0,$$

(2.1)

where $U^k$ and $\dddot{U}^k$ represent the displacement and acceleration vectors, $M^k$, $K^k$ and $C^k$ represent the mass, stiffness and constraints matrices, respectively. $F_{\text{ext}}^k$ designates the externally applied forces and $F_{\text{link}}^k$ are the link loads. $W^k$ is the displacement, velocity or acceleration vector, depending on the choice of the kinematic interface constraints, and the upper script "′" denotes a transpose.

It can be seen from equation (2.1-a) that the displacement vector $U^k$ is the sum of two terms: a displacement obtained from external $U^k_{\text{free}}$ and internal loads $U^k_{\text{link}}$. The equilibrium equation is then decomposed into
two problems, a ”free problem” and a ”link problem”:

\[
\begin{align*}
M^k \ddot{U}^k_{\text{free}} + K^k U^k_{\text{free}} &= F^{\text{ext}}_k, & k = 1, 2, \\
M^k \ddot{U}^k_{\text{link}} + K^k U^k_{\text{link}} &= F^{\text{link}}_k, & k = 1, 2, \\
U^k &= U^k_{\text{free}} + U^k_{\text{link}}, & k = 1, 2, \\
\sum_{k=1}^{2} C^k W^k &= 0.
\end{align*}
\]

The problem can then be solved for each time-step using the dual Schur algorithm as follows:

(a) Solve the free problem (2.2-a) on each sub-domain

(b) Calculate the Lagrange multiplier \( \Lambda \) using the condensed global problem (2.2-d)

(c) Solve the link problem (2.2-b) on each sub-domain

(d) Calculate the displacement (2.2-c) for the global problem

The multi-time-step methods offer an effective way to couple resolutions at different time scales. It is particularly appropriate for parallel computers (i.e. Equations (2.2-a), (2.2-b) and (2.2-c) can be solved in a parallel way in each sub-domain). However, it may not be regarded as a real time multiscale strategy, as a given spatial area is only mono-scale in time.

2.2.2 Variational multiscale method (VMS) in time

Numerous problems from physics can be modeled by partial differential equations (i.e. elliptic problems), called also strong formulation. The variational multiscale method (VMS) begins with the definition of an appropriate variational form of the problem. Using the variational formulation, the boundary value problem (e.g. elliptic problem) is transformed into an entirely different kind of problem which is significantly easier to treat.

Then this variational formulation is followed by a multiscale decomposition and the solution space of the problem is decomposed into large (considered as visible) and small (considered as invisible or unresolvable) scales. Once the large scales space is fixed, the fine scales space is chosen
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ensuring the splitting to be univocal.

The variational multiscale method (VMS) in time, was proposed by [Hughes and Stewart, 1996] and [Bottasso, 2002]. This approach follows the same principles as for the spatial aspect ([Hughes, 1995]). The method of partitioning of unity of the studied interval can be applied to the temporal level. Then the basis functions form a partition of unity of the studied interval and it is therefore possible to construct approximation fields, in the same manner as in the spatial case.

Consider a time interval $I = [0, T]$, its boundary $\gamma = \{0, T\}$, and a family of grids $\{I_h\}_{h>0}$ of $I$. A generic element is denoted $I_e$, and its boundary $\gamma_e$. The grid is obtained defining a partition of $N$ intervals $T_i = \{(t_i, t_{i+1})\}_{i=0}^{N-1}$ of size $h$. $\gamma_h = \{t_i\}_{i=0}^{N}$ is the set of nodes.

Following the classical discontinuous Galerkin scheme the continuous problem is transformed into a discrete one, ([Bottasso, 2002]). First the global domain is decomposed, then a finite element discretization is applied. Stepping in time from the first to the last time element, the problem can be solved in an element by element fashion.

Certain fine scale modes will not be accurately captured by the coarse mesh $\{I_h\}_{h>0}$ of $I$. The main idea of the VMS is to decompose the solution $\phi$ within each element $I_e$ into two parts:

$$\phi = \phi_M + \tilde{\phi},$$  \hspace{1cm} (2.3)

where $\phi_M$ and $\tilde{\phi}$ represent the macro scale viewed as resolvable and micro or fine scale viewed as unresolvable, respectively.

The new information $\tilde{\phi}$ added to the problem, represents a true enrichment because it is not already contained in the visible scales.

The basic idea is to decompose the original problem into two coupled sub-problems. The first one allows to compute the fine scale (considered as invisible). The second computes the coarse scales while taking into account the fine ones. This method is conventionally used to enhance the quality of the classical FE approximation by superimposing the solution of localized functions by element. Microscopic solution is then taken into account explicitly. The method thus depends on the choices that are made to approximate the micro scale.

In ([Bottasso, 2002]), the author uses the variational multiscale method for a hyperbolic problem. The micro solution is sought as a linear combination of polynomial functions orthogonal to those of the macro space. This condition of orthogonality ensures that the new information $\tilde{\phi}$ added to the problem is not already contained in the visible scales,
2.2. Multi-scale computational strategies in time

Figure 2.2: Finite element discretization of space-time ([Hughes and Stewart, 1996]).

and represents a true enrichment.

In ([Hughes and Stewart, 1996]), the authors generalized the VMS method from the steady to the time-dependent case. The method was applied to a problem of parabolic evolution. The macro level is described by finite elements in space and in time. The proposed enrichment functions for the micro scale, are located by space-time element (bubble functions). Figure 2.2 shows the finite element discretization in the space-time domain of a generic slab \( Q \). We denote by \((\Omega_0, \Gamma_0)\) and \((\Omega_T, \Gamma_T)\) the boundaries at \( t = 0 \) and \( t = T \), respectively. \( Q_n \), with lateral boundary \( P_n \), is the space-time domain bounded by spatial hyper-surfaces at times \( t_n \) and \( t_{n+1} \).

The formulations obtained by VMS in times are very close to the reg-


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ularization methods. They can be considered more as a p-version of the FE method in time, in which the mesh is kept fixed and the degree $p$ of the polynomial approximation used, is progressively increased until some desired level of precision is reached, than a real multiscale calculation strategy in time.

2.2.3 Time-homogenization techniques for cyclic loadings

There are three techniques that provide a temporal description suited to the problem of structure subjected to cyclic loadings. The first one is a strategy based on the LATIN method, the second one is the cycle jump method and the third one is the two scale time homogenization method.

2.2.3.1 Problem statement

Consider the quasi-static and isothermal evolution in the time interval $[0, T_F]$ of a spatial domain $\Omega$ subjected to body forces $\vec{f}$. $\Gamma_u$ and $\Gamma_f$ correspond to the parts of the boundary where displacements $\vec{u}_b$ and tractions $\vec{g}$ are prescribed, respectively, such as $\Gamma_u \cap \Gamma_f = \emptyset$ and $\Gamma_u \cup \Gamma_f = \partial \Omega$. We denote by $\vec{n}$ the outer unit vector normal to the boundary and by $\vec{u}^I$ and $\sigma^I$ the initial displacement and stress respectively.

Assuming small deformations, the initial-boundary value problem is expressed as follows:

Boundary conditions:

$$\vec{u} (\vec{x}, t) = \vec{u}_b (\vec{x}, t), \quad \text{on } \Gamma_u \times [0, T_F] \quad (2.4)$$

$$\vec{t} \sigma \cdot \vec{n} = \vec{g} (\vec{x}, t), \quad \text{on } \Gamma_f \times [0, T_F] \quad (2.5)$$

Initial conditions:

$$\vec{u} (\vec{x}, t = 0) = \vec{u}^I (\vec{x}), \quad \text{in } \Omega \quad (2.6)$$

$$\sigma (\vec{x}, t = 0) = \sigma^I (\vec{x}), \quad \text{in } \Omega \quad (2.7)$$

Equilibrium equations:

$$\nabla \cdot \sigma (\vec{x}, t) + \vec{f} (\vec{x}, t) = \vec{0}, \quad \text{in } \Omega \times [0, T_F] \quad (2.8)$$
2.2. Multi-scale computational strategies in time

Kinematic compatibility:

\[ \varepsilon (\bar{u}) = \frac{1}{2} (\nabla \bar{u} + \nabla \bar{u}^T), \quad \text{in } \Omega \times [0, T_F] \]  \hspace{1cm} (2.9)

Constitutive equations:

\[ \varepsilon (\bar{u}) = \varepsilon^{ve} (\vec{x}, t) + \varepsilon^{an} (\vec{x}, t), \quad \text{in } \Omega \times [0, T_F] \]  \hspace{1cm} (2.10)

\[ \sigma (\vec{x}, t) = \wp (\varepsilon (\vec{x}, \tau \leq t)), \quad \text{in } \Omega \times [0, T_F] \]  \hspace{1cm} (2.11)

\[ \dot{\varepsilon}^{an} (\vec{x}, t) = \mathbf{B} (\vec{x}, t, \sigma) \quad \text{in } \Omega \times [0, T_F] \]  \hspace{1cm} (2.12)

Where, \( \varepsilon^{an} \) designates the inelastic part of the total strain \( \varepsilon \) and \( \varepsilon^{ve} \) its viscoelastic (or elastic) part. \( \sigma \) represents the Cauchy stress. \( \wp \) and \( \mathbf{B} \) are a functional and an operator representing the constitutive law. \( \nabla \cdot \) and \( \nabla \) denote the divergence and gradient operators, respectively, and upper script "\( t \)" denotes a transpose.

The state of the structure is fully defined by determining the fields \( (\dot{\varepsilon}^{an}, \sigma) \) and \( \bar{u} (\vec{x}, t) \).

The initial-boundary problem (2.4)-(2.12) is continuous in space and time. Generally, to obtain the numerical solution, the problem is discretized in space and in time. In this Chapter we will focus only on the temporal evolution.

2.2.3.2 LArge Time INcrements method (LATIN)

The LATIN method was developed by Ladev`eze, 1985a, Ladev`eze, 1985b, Ladev`eze and Zienkiewicz, 1992, Cognard and Ladev`eze, 1993, Ladev`eze, 1999. This method is known to be non incremental; the study interval \([0, T_F]\) does not have to be partitioned into small pieces, i.e. at each iteration the method generates an approximation of the solution on the entire time interval, which significantly reduces the computational cost. It is an iterative method that often starts with a relative gross approximation (generally coming from an elastic analysis).

Principle of the method

Consider the problem (2.4)-(2.12) The resolution of the initial boundary problem consists on the determination of the fields \( (\dot{\varepsilon}^{an}, \sigma) \) and \( \bar{u} (\vec{x}, t) \) at each instant of the study interval \([0, T_F]\).
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We denote by \( s = (\ddot{\varepsilon}^a_n, \dot{\sigma}(\ddot{x}, t)) \) the variable representing the set of the solution elements of the mechanical problem, defined in \( \Omega \times [0, T_F] \).

This variable is called a field-process.

The field process must verify the boundary conditions, equilibrium equations and the constitutive law on each point \( x \) of \( \Omega \) and on the whole studied time interval.

The first principle of the LATIN approach is to separate the equations of the problem into two groups to which two solution sub-spaces are associated:

- The linear subspace \( A_d \) of admissible field-processes \( s_n \) verifying the set of linear equations (boundary conditions and equilibrium equations), possibly global.

- The nonlinear subspace \( A_\ell \) of field-processes \( \hat{s}_n \) verifying the set of local equations (constitutive law), possibly local.

So the problem is to find the intersection of \( A_d \) and \( A_\ell \). The final solution is obtained by using an iterative scheme in two stages, called "local stage" and "linear stage". Figure (2.3) shows three iterations of the LATIN algorithm.

To solve the reference problem, two search directions \( E^+ \) and \( E^- \) are defined. The initialization of the solution \( s_0 \) can be constructed starting from an elastic calculation on \([0, T_F]\).

Knowing an element solution \( s_n \) of \( A_d \), we determine an element \( \hat{s}_{n+\frac{1}{2}} \) of \( A_\ell \) along a given search direction \( E^+ \). \( \hat{s}_{n+\frac{1}{2}} \) satisfies the constitutive relation and belongs to \( A_\ell \), i.e., \( (\hat{s}_{n+\frac{1}{2}} - s_n) \in E^+ \).

Next, an element solution \( s_{n+1} \) of \( A_d \) is computed from the local solution following a global search direction \( E^- \), i.e., \( (s_{n+1} - \hat{s}_{n+\frac{1}{2}}) \in E^- \).

Ultimately, the algorithm converges to the exact solution \( s_{ex} \).

Figures (2.4) to (2.8) from (Ladevèze, 1985b) show the different steps of the method and the results obtained at each step for a beam in traction. The material of the beam is described by the viscoplastic model of Chaboche. The method of large time increments is applied to the interval \([0, T_F]\) that corresponds to two cycles of loading (figure 2.4).

For the iteration 0 an elastic initialization is done. Figures (2.5) and (2.6) show the evolution of the set of solution elements \( s_1 = (\sigma_1, \dot{\varepsilon}_1, p_1) \) on \([0, T_F]\) which corresponds to the first iteration in the global step.

Figures (2.7) and (2.8) show the evolution of the set of the final solution elements \( s_{ex} = (\sigma_{ex}, \dot{\varepsilon}_{ex}, p_{ex}) \) on \([0, T_F]\) which corresponds to 11
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Figure 2.3: Principle of the LATIN algorithm, after (Ladevèze, 1985b)

Figure 2.4: Applied loading (Ladevèze, 1999)

iterations in the global step. A good convergence of the method can be noted. As the calculations continue, the error continues to decrease (figure 2.9). Only 11 iterations are necessary to obtain an error less than 3%.

Extension to cyclic loadings

The LATIN method was extended for the resolution of structures subjected to cyclic loadings with large numbers of cycles (Ladevèze, 1999, Ladevèze and Zienkiewicz, 1992). It requires only the calculation of the solution on a limited number of cycles, while taking into account the influence of the intermediate cycles.
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Figure 2.5: Evolution of the stress $\sigma_1$. First iteration- global step ($s_1$), ([Ladevèze, 1999]).

Figure 2.6: Evolution of the inelastic strain $\varepsilon_1^p$ and the accumulated plastic strain $p_1$. First iteration- global step ($s_1$), ([Ladevèze, 1999]).
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Figure 2.7: Evolution of the stress $\sigma_{ex}$. Iteration 11- global step ($s_{ex}$), (Ladevèze, 1999).

Figure 2.8: Evolution of the inelastic strain $\varepsilon_{ex}^p$ and the accumulated plastic strain $p_{ex}$. Iteration 11- global step ($s_{ex}$), (Ladevèze, 1999).
In other words, the method consists in splitting the time interval \([0, T_F]\) into sub-intervals \(I_i=1..N\) (figure 2.10). Each interval \(I_i\) can contain tens of cycles or perhaps, possibly, hundreds of cycles of period \(T\). Then the LATIN approach is applied in each time interval \(I_i\). We present in the following a summary of the approach.

Two time scales are defined. The first corresponds to the macro time scale \(t_M\) associated with a coarse partition of the study interval. The second is a fast time-scale \(\tau\) describing the rapid evolution on a cycle whose period \(T\) is given. The rapid evolution on a cycle is supposed to be described by a \(T\)-periodic function.

A FE representation is proposed to describe cyclic phenomena. Let us define, for each macro-interval \(I_i\), the set \(\{f_i(t_M)\}_{i=0}^N\) of finite element linear basis functions associated with nodes \(\{t_{Mi}\}_{i=0}^N\). Each mechanical field \(\Phi(t)\) of the problem is interpolated as follows:

\[
\Phi(t) \equiv \Phi(t_M, \tau) = \sum_{i=0}^{N} \phi_i(\tau) f_i(t_M),
\]

where \(\phi_i(\tau)\) is a \(T\)-periodic nodal function associated with node \(t_{Mi}\).

For a linear slow time interpolation between \(t_{Mi}\) and \(t_{Mi+1}\), (within the interval \(I_i\)), any time function is defined as follows:

\[
\Phi(t_M, \tau) = \phi_i(\tau)(1 - \frac{t_M - t_{Mi}}{t_{Mi+1} - t_{Mi}}) + \phi_{i+1}(\tau)(1 - \frac{t_M - t_{Mi+1}}{t_{Mi} - t_{Mi+1}}).
\]

The functions \(\Phi(t)\) in equation (2.13) are then sought and calculated.
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Figure 2.10: Principle of the time representation: (a) Decomposition of the study interval $[0, T_F]$ into sub-intervals $I_i = 1..N$. (b) Finite element representation of a function $\Phi(t)$.

Using a finite element approach in time as described above. The integration points introduced in the conventional finite element method are replaced by "integration cycles".

Next all the response fields involved in the problem ($\dot{\varepsilon}^{an}(\vec{x}, t), \sigma(\vec{x}, t)$) defined on the space-time $\Omega \times [0, T_F]$ are approximated in each time interval $I_i$ by:

$$\sum_{j=1}^{m} g_j(t) \alpha_j(\vec{x}), \text{ in } \Omega \times [0, T_F]$$  \hspace{1cm} (2.15)

where $g_j(t), (j = 1..m)$ are functions of time defined by equation (2.13), and $\alpha_j(\vec{x})$ a spatial operator. Equation (2.15) represents the “radial loading” approximation of order $m$, (Ladevèze, 1999).

The problem can then be solved using the LATIN method (figure 2.3) in each time interval $I_i$.

It is then clear that Ladevèze, 1999 described the cyclic phenomena in an economic way by using a combination of the LATIN approach and FEA in time. The calculations are carried out only for few chosen cycles. The FE approach in time presented in this section is not limited to the LATIN method only and can be adapted to incremental calculation methods.

Even though the theory behind the LATIN method is interesting, the
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implementation into commercial FEA software tends to be too cumbersome in its current form to be of practical interest.

2.2.3.3 Cycle Jump

To simulate high numbers of cycles [Lesne and Savalle, 1989] propose an original approach called "Cycle Jump". This technique was applied in [Kruch, 1992] for the simulation of a turbine disk subjected to large numbers of thermomechanical cycles. This technique assumes that the evolution of the mechanical fields, from one cycle to another, is slow. The idea is to make the computation for a certain set of loading cycles at chosen intervals. The evolution between these loading cycles is deduced by extrapolation over the corresponding intervals. A simple cycle-jump scheme was proposed by [Lemaitre and Doghri, 1994], [Paepegem et al., 2001] based on extrapolation of the damage parameter by using the explicit Euler integration formula. The cycle jump method proposed by [Cojocaru and Karlsson, 2006] consists of simulating several reference cycles using FEA to derive a function called "global evolution function" and extrapolate stresses, strains and displacements according to this function over several cycles. Then the extrapolated state is used as an initial condition for the following FEA simulation after the cycle jump.

Consider a vector \( \mathbf{Y} \) having as components the internal variables expressed as a function of number of cycles \( n \). \( \mathbf{y} \) is a vector having as components the same internal variables but expressed as a function of time \( t \). The evolution of one of the vector component is represented in figure 2.11.

The relation between \( \mathbf{Y} \) and \( \mathbf{y} \) can be expressed as follows:

\[
\mathbf{Y}(n) = \mathbf{y}((n-1)T + \tau),
\]

where \( T \) is the load period and \( \tau \) is an instant in the cycle \( (0 < \tau < T) \). As the evolution of the mechanical variables is slow from one cycle to another, it is possible to find an approximation of the evolution of \( \mathbf{Y} \) through a second-order Taylor series expansion over an interval \( \Delta n \):

\[
\mathbf{Y}(n + \Delta n) \approx \mathbf{Y}(n) + \mathbf{Y}'(n) \Delta n + \mathbf{Y}''(n) \frac{(\Delta n)^2}{2},
\]

Consider three cycles \( M, N \) and \( K (K < N < M) \), for which the components of \( \mathbf{Y}_M, \mathbf{Y}_N \) and \( \mathbf{Y}_K \) are known. We can estimate \( \mathbf{Y}' \) and \( \mathbf{Y}'' \) by interpolation as follows:

\[
\mathbf{Y}'(N) = \frac{\mathbf{Y}(N) - \mathbf{Y}(K)}{N - K}.
\]
2.2. Multi-scale computational strategies in time

The number of cycles $\Delta n$ is then chosen by assuming that the second order term is negligible compared to the first order term. By introducing a precision factor $\alpha$ we have:

$$\Delta n = 2\alpha Y'Y'' - 1. \quad (2.20)$$

The technique of cycle jumps has been introduced to treat reasonable numbers of cycles. Generally, for the first cycles the mechanical variables are changing rapidly. The cycle jump technique becomes inaccurate. It is then necessary to make a full calculation for the first cycles during which the evolution of the mechanical fields is important. Although the cycle jump approach is effective to simulate the evolution of state variables, nevertheless this approach does not guarantee the satisfaction of the governing equations.

2.2.3.4 Two scale time homogenization method

The two-scale time homogenization, is a direct extension of the asymptotic spatial homogenization. It was developed by [Guennouni, 1988] for elasto-viscoplastic homogeneous materials. It is based on a representation of the external loads on two time scales: macro time ($t_M$)
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which describes the variations on the entire time interval \( T \), and a micro time \( (\tau) \) which describes the variations in a load cycle. This technique is valid only for relatively small load periods \( (T) \), in other words, when \( T \ll T_F \). The unknown fields are being sought in the form of asymptotic expansions in \( T \). Then the original elasto-viscoplastic initial-boundary problem is decomposed into coupled micro-chronological (fast time scale) and macro-chronological (slow time-scale) problems. The micro-chronological problem is elastic, whereas the macro-chronological one is viscoplastic.

This technique was applied in the case of cyclic loadings by Yu and Fish for thermo-viscoelastic composites (Yu and Fish, 2001), and for homogeneous materials following the Maxwell viscoelastic model and the power-law viscoplastic model (Yu and Fish, 2002). It was used by Aubry and Puel to predict the long-term behavior of elasto-viscoplastic materials subjected to two-frequency periodic loads (Aubry and Puel, 2010). It was also used by Devulder et al., 2010 to study the fatigue damage evolution in cortical bone and Manchiraju et al., 2007, Manchiraju et al., 2008, and by Chakraborty et al., 2011 to study fatigue response of Ti alloys. Given that the asymptotic time homogenization approach is considered as the most elegant and mathematically rigorous, it will be adopted in this work and will be explained in detail in Chapter 3.

2.3 Multi-scale modeling in space of undamaged composites

Multi-scale modeling in space is connected with calculation of material properties or system behavior on one level using information or models taken from different levels. In this section two scale analyses are presented. At the macro scale, the material can be considered as homogeneous and presents some effective mechanical response. Whereas at the micro scale heterogeneities are due to the presence of distinct phases. The objective is to establish the relation between macroscopic properties of heterogeneous materials and their micro-constituents properties. In order to predict the macroscopic behavior of composites.

This section is a general, non-exhaustive summary of the multiscale spatial computational strategies. Here, we try to describe the most important features of such strategies. For the spatial aspect, the selected
strategies are grouped into four categories: Direct finite Element (FE) computation, asymptotic homogenization, method of cells and subcells and mean-field homogenization (MFH). The choice of the method depends on several criteria: arrangement of the microstructure, accuracy of the predictions, computational cost, desired information on the local field...

2.3.1 Representative Volume Element (RVE)

The prediction of the macroscopic stress-strain response of composite materials is related to the description of their complex micro-structural behavior illustrated by the interaction between the constituents. The microstructure of heterogeneous materials is, at any given length scale, complex. In practice, there are only certain averaged effects of the microstructure which are of interest. In this context, the microstructure of the material under consideration is basically taken into account by representative volume elements (RVE) as shown on figure 2.12. The overall properties of each RVE represent the overall properties of the composite material. At each material point \( \bar{x} \) at the macroscopic scale is associated an RVE (\( \Omega \)) which at smaller -micro- scale contains a finite number of constituents. The concept of RVE was introduced by [Hill, 1963] to relate the macro-properties of a composite material point to its micro-properties. It must be "sufficiently large" to represent the microstructure of the heterogeneous composite material, and "sufficiently small" as compared to the characteristic length of the sample, \( (\ell \ll L) \) so that the structure can be considered as continuous medium.

The strain and the stress fields are not necessary uniform within the RVE. Consequently in order to describe those fields at the micro level, average quantities of the strain and stress tensors have to be considered:

\[
\begin{align*}
\langle \varepsilon (\bar{x}, x) \rangle_\Omega &= \frac{1}{V} \int_\Omega \varepsilon (\bar{x}, x) \, dV, \\
\langle \sigma (\bar{x}, x) \rangle_\Omega &= \frac{1}{V} \int_\Omega \sigma (\bar{x}, x) \, dV,
\end{align*}
\]  

(2.21)

where, \( x \) is the vector position attached to the RVE and \( \langle \cdot \rangle_\Omega \) the volume average over the total volume \( \Omega \) of the RVE. In the following, dependence on macro coordinates \( \bar{x} \) will be omitted for simplicity.

The chosen RVE must be subjected to the correct boundary conditions in order to represent the macromechanical problem. Three kinds of
boundary conditions are then usually considered: Linear displacement, uniform traction, and periodic conditions.

### 2.3.1.1 Linear displacement boundary conditions

Consider that each point, on the boundary $\partial \Omega$ of the RVE, is subjected to linear boundary displacements $u_i(x)$ as shown in figure 2.13:

$$u_i(x) = \bar{G}_{ij} x_j; \quad x \in \partial \Omega,$$

(2.22)

where $\bar{G}$ is the macro displacement gradient corresponding to a macroscopic strain $\bar{\varepsilon}$:

$$\bar{\varepsilon} = \frac{1}{2} (\bar{G} + \bar{G}^T).$$

(2.23)
Figure 2.13: A Representative Volume Element (RVE) subjected to linear displacement $u_i$.

as follows:

$$< \varepsilon_{ij} >_{\Omega} = \frac{1}{2V} \int_{\Omega} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) dV$$

$$= \frac{1}{2V} \int_{\partial \Omega} (u_i n_j + u_j n_i) dS$$

$$= \frac{1}{2V} \int_{\partial \Omega} (\overline{G}_{ik} n_k n_j + \overline{G}_{jk} n_k n_i) dS$$

$$= \text{Gauss} \left( \frac{1}{2V} \left( \overline{G}_{ik} \int_{\Omega} \frac{\partial m_k}{\partial n_j} dV + \overline{G}_{jk} \int_{\Omega} \frac{\partial m_k}{\partial n_i} dV \right) \right)$$

$$= \overline{\varepsilon}_{ij},$$

where $n_j$ is the outward normal to the RVE boundary, $\delta$ is the Kronecker symbol and $S$ is the outer surface.

It can be concluded that the volume average of the microscopic strain $\varepsilon(x)$ in the associated RVE is equal to the macroscopic strain $\overline{\varepsilon}$. 
2.3.1.2 Uniform traction boundary conditions

The RVE is subjected to a uniform macro-stress \( \bar{\sigma} \) corresponding to a uniform traction \( t_i(x) \) (see figure 2.14):

\[
t_i(x) = \sigma_{ij} n_j, \quad x \in \partial \Omega,
\]

(2.25)

where \( n_j \) is the outward normal to the RVE boundary.

Using equilibrium equation without body forces \( (\nabla \cdot \sigma = 0) \) and using Gauss theorem, the average stress over the RVE in equation (2.21) writes:

\[
< \sigma_{ij} >_\Omega = \frac{1}{V} \sigma_{ik} \int_{\partial \Omega} n_k x_j dS = \frac{1}{V} \sigma_{ik} \int_{\Omega} \frac{\partial x_j}{\partial x_k} dV = \bar{\sigma}_{ij},
\]

(2.26)

Thereby, it can be concluded also that the macroscopic stress \( \bar{\sigma} \) tensors is defined as the volume average of the microscopic stress \( \sigma(x) \) in the associated RVE.

2.3.1.3 Periodic boundary conditions

For RVEs with periodic microstructure and due to the repetition of the cell in all directions before and after application of the load, periodic
boundary deformation and anti-periodic tractions are applied at each corresponding pair of nodes lying on opposite faces of the RVE boundary:

\begin{align}
    u^+ - u^- &= \bar{G}(x^+ - x^-), \\
    t^+ &= -t^-,
\end{align}

(2.27)

where notations \((\cdot)^+\) and \((\cdot)^-\) refer to quantities which correspond to each other on opposite sides of the RVE.

Periodic boundary conditions, give more accurate response than prescribed displacements or tractions. This has been observed in numerous studies [Kanit et al., 2003, Jiang et al., 2001, Jiang et al., 2002, Ostoja-Starzewski, 2006] for different constitutive behaviors.

### 2.3.1.4 Hill’s Lemma

Consider an RVE \(\Omega\), with volume \(V\) and boundary \(\partial\Omega\), subjected to a prescribed boundary condition (linear displacements or uniform tractions or periodic condition), one has the following result:

\[
    \langle \sigma(x) : \varepsilon(x) \rangle_{\Omega} = \bar{\sigma} : \bar{\varepsilon}
\]

(2.28)

This result is commonly called Hill-Mandell condition or macrohomoogeneity condition.

### 2.3.2 Multi-scale modeling approaches

#### 2.3.2.1 Finite Element (FE) computation of RVE

Homogenization techniques, as stated before, are often based on direct finite element analysis of RVE at micro scale using macroscopic values as the boundary conditions. Then computed results are returned to macro scale by averaging techniques. This approach is very accurate and gives detailed micro fields. If periodic microstructure is assumed, FE simulations can be performed on so-called unit cells and periodic boundary conditions can be applied.

However, especially for nonlinear problems, it is computationally very expensive. The creation of a discrete model of the RVE is also necessary and generation of good meshes can be difficult. Indeed, the preparation of the discrete representation of composite’s microstructure can lead to additional difficulties.

Figure (2.15) shows the accumulated plastic strain in the matrix phase
2.3.2.2 Asymptotic homogenization

Asymptotic homogenization is the most rigorous and elegant method. This method was introduced by [Bensoussan et al., 1978] and it is sometimes referred to as ”periodic homogenization” or ”mathematical homogenization”. This approach is applied to periodic composite materials (laminates, fabrics, masonry, lattice materials) (figure 2.16) and suppose that the mechanical response of the periodic microstructure is governed by linear elasticity. For historical review of the method, see [Chung et al., 2001].

2.3.2.3 Methods of cells and subcells

This method was introduced by ([Aboudi, 1989], [Paley and Aboudi, 1992]) and it consists on discretizing the microstructure into simple cells.
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Figure 2.16: Periodic microstructure, x: macro coordinates, y: micro unit cell coordinates

Figure 2.17: Typical discretization of a repeating unit cell, generic cell and subcell of the generalized method of cells. [Pierard, 2006]

Each cell can be subdivided into subcells which may contain a distinct homogeneous material (figure 2.17). Imposing continuity of displacements and tractions between adjacent subcells and repeating unit cells, the global response is computed by a classical volume average. To better represent the real geometry of the microstructure, a large number of cells is needed, which increases the CPU time and it becomes comparable to FE calculations and this is the most serious limitation to the method’s applicability.
2.3.2.4 Mean-field homogenization (MFH)

Mean-field approximations are based on assumed relations between the average responses of different phases in inclusion-reinforced composites. The effective stress and strain are related to the average stress and strain of each phase through their respective stress and strain concentration tensors. However, contrary to other numerical approaches, MFH schemes are unable to predict detailed micro strain or stress fields. The mean-field based methods have obtained good results in modeling the behavior of various materials in the linear elastic regime, mainly polymer composites and they are the less expensive in terms of user’s time and CPU time. Extension of MFH schemes to inelastic behavior has been the subject of extensive research and is still under investigation.

In the following section some of mean-field homogenization models are presented.

2.3.3 Effective behavior of linear elastic two phase composites

The expressions of the effective elastic properties are generally obtained from the relation between average stress and average strain in a chosen RVE. Consider a RVE (Ω) of a linearly elastic two phase composite. One phase is the matrix denoted by the subscript m and the other phase is inclusions denoted by the subscript I. The inclusions are aligned randomly distributed and identically shaped (the inclusions may be short or long fibers, spherical particles or platelets). The matrix occupies a region Ω_m and the inclusions a domain Ω_I and have uniform stiffness given by C^el_m and C^el_I, respectively.

It can easily be shown that:

\[<\varepsilon>_\Omega = \nu_m <\varepsilon>_\Omega_m + \nu_I <\varepsilon>_\Omega_I, \quad (2.29)\]

\[<\sigma>_\Omega = \nu_m <\sigma>_\Omega_m + \nu_I <\sigma>_\Omega_I, \quad (2.30)\]

where \(\nu_I\) and \(\nu_m\) are the volume fractions of the inclusion and matrix phases respectively, \(<\bullet>_\Omega\) denotes a volume average over the whole RVE. Similarly, \(<\bullet>_\Omega_r\) is a volume average over phase \(r, r = I, m\).

The aim is to find a relationship between the macroscopic and microscopic quantities. Two fourth-order strain concentration tensors are
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defined, $A^\varepsilon$ and $B^\varepsilon$. The first relates the average strain in the inclusion phase to the macro strain as follows:

$$< \varepsilon >_{\Omega I} = A^\varepsilon :< \varepsilon >_{\Omega} .$$

(2.31)

the second, relates inclusion and matrix average strains:

$$< \varepsilon >_{\Omega I} = B^\varepsilon :< \varepsilon >_{\Omega m} .$$

(2.32)

Tensors $A^\varepsilon$ and $B^\varepsilon$ are linked to each-other:

$$A^\varepsilon = B^\varepsilon : [(1 - \upsilon_I)B^\varepsilon + \upsilon_m \mathbf{I}]^{-1} ; \quad B^\varepsilon = (1 - \upsilon_I)(\mathbf{I} - \upsilon_I A^\varepsilon)^{-1} : A^\varepsilon .$$

(2.33)

For a two phase linear elastic composite subjected to linear displacements, an interesting result that can be obtained by using the concentration tensor $A^\varepsilon$ is the effective or homogenized elastic tensor operator $\overline{C}_{el}$ which relates the average stress and strain tensors as follows:

$$\overline{C}_{el} = C_{el m} + \upsilon_I \left( C_{el I} - C_{el m} \right) : A^\varepsilon .$$

(2.34)

2.3.3.1 Various mean-field homogenization models

Several models are proposed to determine the concentration tensors $A^\varepsilon$ and $B^\varepsilon$. We recall, hereafter, some homogenization schemes that are used in this thesis. These MFH models are classified into two groups: models which rely on the simple law of mixture (e.g. Voight, Reuss, etc.) and models based on the Eshelby result [Eshelby, 1957a]. This result was developed in the framework of linear elastic materials, it allows to solve the problem of a single ellipsoidal inclusion of uniform elastic modulus which is embedded in infinite matrix of uniform elastic modulus.

Voigt and Reuss models

Voigt and Reuss models are the simplest homogenization schemes. These two models do not take into account neither the shape nor the orientation of inclusions. They only involve a single topography information about the microstructure, the volume fraction.

In the case of Voigt model, the strain field in each phase $r$ is uniform and equal to the average strain $\bar{\varepsilon}$. The following results are immediately found:

$$B^\varepsilon = \mathbf{I} , \quad \overline{C}_{Voigt} = (1 - \upsilon_I)C_{el m} + \upsilon_I C_{el I} .$$

(2.35)
Reuss model assumes uniform stress in the RVE. The following expressions are found:

\[ \mathbf{B}^\mathcal{E} = (\mathbf{C}_I^{-1})^{-1} : \mathbf{C}_m, \quad \mathbf{C}_{\text{Reuss}} = \left[ (1 - \nu_I) \left( \mathbf{C}_m^{-1} \right) + \nu_I \left( \mathbf{C}_I^{-1} \right) \right]^{-1}. \]  

(2.36)

**Eshelby’s result**

Consider an infinite domain \( \Omega \) formed by a matrix \( m \) occupying a domain \( \Omega_m \) in which is embedded an ellipsoidal inclusion \( I \) occupying a domain \( \Omega_I \) undergoing a stress-free strain \( \mathcal{E}^* \). Both are made from the same isotropic and homogeneous material. Later [Withers, 1989] extended this result to transversely isotropic medium. Eshelby [Eshelby, 1957a] has shown that the resulting strain field in the inclusion is uniform and determined as follows:

\[ \forall x \in (I): \quad \mathcal{E}(x) = \mathbf{S} : \mathcal{E}^*, \]  

(2.37)

where \( \mathbf{S} \) is a fourth-order Eshelby’s tensor depending only on the geometry of the inclusion and the Poisson’s ratio \( \nu \) of the medium. It has minor symmetries. In the particular case of spherical inclusion the tensor has major symmetries. In this case Eshelby’s tensor is completely analytically defined ([Eshelby, 1957a, Eshelby, 1957b]). Whereas, for non-isotropic materials other Eshelby tensors were developed based on extensions of the original Eshelby’s result, ([Faivre, 1971, Sevostianov et al., 2005]).

**Isolated inclusion model**

Consider an infinite matrix with stiffness \( \mathbf{C}_m \) containing an ellipsoidal inclusion \( (I) \) with stiffness \( \mathbf{C}_I \). Far from the inclusion, a uniform strain is imposed \( \mathcal{E}^\infty \) (figure 2.18). Using Eshelby’s result and the superposition principle of linear elasticity, the strain in the inclusion is found to be uniform and related to the imposed stress via a fourth order strain concentration tensor \( \mathbf{H}^\mathcal{E} \) as follows:

\[ \forall x \in (I), \mathcal{E}(x) = \mathbf{H}^\mathcal{E} : \mathcal{E}^\infty, \]  

(2.38)

where,

\[ \mathbf{H}^\mathcal{E} = (\mathbf{I} + \mathbf{S} : (\mathbf{C}_m^{-1} : \mathbf{C}_I^{-1} - \mathbf{I}))^{-1}. \]  

(2.39)
Dilute inclusion model

The dilute inclusion model assumes very low concentrations of inclusions. It provides good estimates if the inclusion’s concentration does not exceed 10%. The basic assumption is to neglect interactions between inclusions, so that each inclusion is considered as isolated in an infinite medium having the same properties of the matrix and subjected to the same far-field strain as the one acting on the real RVE. The strain concentration tensor $A^{(DM)}$ for the dilute inclusion model (DM) is given by the following expression:

$$A^e = A^{(DM)} = \left( I + S : \left( C_m^{el} \right)^{-1} : C_I^{el} - I \right)^{-1} \quad (2.40)$$

Where $S$ is Eshelby’s tensor [Eshelby, 1957a].

Mori-Tanaka model

The non-interaction assumption considered in the dilute inclusion model is not representative of real composite. In fact when the volume fraction of inclusions increases, nearby inclusions start to interact, affecting the overall behavior. The Mori-Tanaka (MT) scheme was proposed by Mori and Tanaka (1973), and takes into account these interactions in an
average sense.
Consider a RVE (Ω) with \( N \) identical and aligned inclusions uniformly distributed in it. The RVE is subjected to uniform macro strain \( \varepsilon^\infty \) corresponding to uniform boundary conditions applied at infinity (see figure 2.19). A fictitious material called equivalent homogeneous medium is defined, which is subjected as in the original problem to linear displacements imposed at infinity and in which the inclusions are made of the same material as the one of the matrix and undergo an eigenstrain \( \varepsilon^* \). Applying the Eshelby’s result and superposition principle, the stress and strain tensors in each inclusion are written as:

\[
\varepsilon_I = \varepsilon^\infty + S : \varepsilon^* + \varepsilon'(x), \quad \sigma_I = \mathbf{C}_{em} : (\varepsilon^\infty + S : \varepsilon^* + \varepsilon'(x) - \varepsilon^*), \quad (2.41)
\]

where \( \varepsilon'(x) \) is the perturbation strain in a given inclusion.

To ensure equivalence between original and equivalent homogeneous problems, same strain in the inclusions of both problems is imposed and we have:

\[
\mathbf{C}_{em} : (\varepsilon^\infty + S : \varepsilon^* + \varepsilon'(x) - \varepsilon^*) = \mathbf{C}_{el} : (\varepsilon^\infty + S : \varepsilon^* + \varepsilon'(x)). \quad (2.42)
\]

Given that \( \varepsilon' \) is not homogeneous, the previous equivalence condition can not be fulfilled at each point of the material. Thus an average over all the inclusions is considered (i.e \( < \varepsilon'(x) >_{\Omega_I} \) instead of \( \varepsilon' \)). Rearranging, the following expression of the eigenstrain is obtained:

\[
\varepsilon^* = - \left[ \left( \mathbf{C}_{em}^{-1} : \mathbf{C}_{el}^{-1} \right) -1 + S \right]^{-1} : (\varepsilon^\infty + < \varepsilon'(x) >_{\Omega_I}). \quad (2.43)
\]

Combining this expression with equation (2.41), the average strain in the inclusion is found as:

\[
< \varepsilon >_{\Omega_I} = \left[ I - S : \left( \left( \mathbf{C}_{em}^{-1} : \mathbf{C}_{el}^{-1} \right) -1 + S \right) -1 \right] : (\varepsilon^\infty + < \varepsilon'(x) >_{\Omega_I}). \quad (2.44)
\]

Due to the presence of a number of inclusions randomly dispersed in the RVE, we can assume that the average of the perturbation field \( \varepsilon' \) is the same in the inclusion and in the matrix. So we can write that:

\[
< \varepsilon >_{\Omega_m} = \varepsilon^\infty + < \varepsilon'(x) >_{\Omega_m} = \varepsilon^\infty + < \varepsilon'(x) >_{\Omega_I}. \quad (2.45)
\]

Combining equation (2.44) and (2.45), we obtain the expression of the strain concentration tensor \( \mathbf{B}^\varepsilon \):

\[
\mathbf{B}^\varepsilon = \mathbf{B}^{(MT)} = \left( I + S : \left[ \mathbf{C}_{em}^{-1} : \mathbf{C}_{el}^{-1} \right] -1 \right)^{-1} \quad (2.46)
\]
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Figure 2.19: Mori-Tanaka model hypothesis: (a) Original problem, (b) Equivalent Homogeneous Medium

The strain concentration $\mathbf{A}^\varepsilon$ tensor is then given by the following expression:

$$
\mathbf{A}^\varepsilon = \mathbf{A}^{(MT)} = \left( \mathbf{I} + (1 - \nu_f) \mathbf{S} : \left[ \left( \mathbf{C}_f^{el} \right)^{-1} : \mathbf{C}_m^{el} - \mathbf{I} \right] \right)^{-1}.
$$

(2.47)

In practice, [Benveniste, 1987] provided a simplified interpretation of the MT method based on the Equivalent Inclusion Problem (EIP): "each inclusion behaves like an isolated inclusion in the matrix seeing $<\varepsilon>_{\Omega_m}$ as a far-field strain", (figure 2.20). According to this interpretation, the relation in equation (2.38) takes this form:

$$
<\varepsilon>_{\Omega_f} = \mathbf{H}^\varepsilon : <\varepsilon>_{\Omega_m}.
$$

(2.48)

Consequently, the strain concentration tensor $\mathbf{B}^\varepsilon$ is defined as follows:

$$
\mathbf{B}^\varepsilon = \mathbf{H}^\varepsilon = (\mathbf{I} + \mathbf{S} : (\mathbf{C}_m^{el-1} : \mathbf{C}_f^{el} - \mathbf{I}))^{-1}
$$

(2.49)

The MT scheme provides good estimates of the stiffness tensor for two phase composites with low to moderate volume fraction of aligned inclusions- namely up to 25% – 30%.

For high inclusion concentrations, the material behaves as if inclusions of the matrix material are embedded into a matrix of inclusion material. The properties of both phases are switched, this gives the inverse Mori-Tanaka scheme (IMT).

Self-consistent scheme
The Self-consistent model (SC) was introduced first by Hershey, 1954 for crystalline aggregates. The self-consistent method is an approach that incorporates the interaction between inclusions by assuming that each inclusion is isolated and embedded in a fictitious infinite matrix having a stiffness operator $\overline{C}_{el}$ corresponding to the homogenized RVE. The strain concentration tensor is given by:

$$A^e = A^{(SC)} = \left( I + S : \left( \left( \overline{C}^{el} \right)^{-1} : C^{el}_I - I \right) \right)^{-1} \tag{2.50}$$

### 2.3.4 Effective behavior of linear thermo-elastic two phase composites

Consider a heterogeneous linear thermo-elastic material, the matrix has elastic stiffness $C^{el}_m$ and thermal expansion $\alpha_m$, and the inclusions are identical and have the same aspect ratio, orientation and properties $C^{el}_I$ and thermal expansion $\alpha_I$. The objective of this section is to determine the macroscopic effective properties $\overline{C}$ and $\overline{\alpha}$, using homogenization model (e.g Lielens, 1999a) which takes into account thermo-elastic behavior.
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At each point \( x \) of the material, for a given change of temperature \( \Delta \theta \) and a total strain \( \varepsilon_r \), the stress for each phase \( r \) \((r = m, I)\) is given by:

\[
\begin{align*}
\sigma_r(x) &= C_{cl}^r : (\varepsilon_r(x) - \varepsilon_r^{th}(x)), \\
\varepsilon_r^{th}(x) &= \alpha_r \Delta \theta(x), \\
\beta_r(x) &= -C_{cl}^r : \alpha_r \Delta \theta(x),
\end{align*}
\]

(2.51)

the strain tensor can then be expressed as follows:

\[
\varepsilon_r(x) = (C_{cl}^r)^{-1} : \sigma_r(x) + \varepsilon_r^{th}(x).
\]

(2.52)

where \( \varepsilon_r^{th} \) is the thermal strain tensor.

For isotropic materials stiffness and thermal expansion tensors depend on only two and one scalars, respectively. Equations (2.51) and (2.52) are rewritten as follows:

\[
\begin{align*}
\sigma_r(x) &= \lambda_r \left[ \text{tr}(\varepsilon_r(x) - \varepsilon_r^{th}(x)) \right] \mathbf{1} + 2\mu_r(\varepsilon_r(x) - \varepsilon_r^{th}(x)), \\
\varepsilon_r(x) &= 1 + \nu_r \left( E_r \sigma_r(x) - \frac{\nu_r}{E_r} \text{tr}(\sigma_r(x)) \right)\mathbf{1} + \varepsilon_r^{th}(x),
\end{align*}
\]

(2.53)

where \( \mu_r \) and \( \lambda_r \) are the Lamé coefficients, \( E_r \) and \( \nu_r \) represent the Young’s modulus and the Poisson’s ratio, respectively. \( \mathbf{1} \) is the second-order unit tensor and "\( \text{tr} \)" is the trace of a tensor.

The macroscopic thermo-elastic properties can be obtained for any homogenization model defined in the isothermal case by the strain concentration tensor \( B^\varepsilon \) or \( A^\varepsilon \), (Lielens, 1999a).

The following expression for the average strain in the inclusions phase, assuming linear boundary conditions corresponding to a macroscopic total strain \( \bar{\varepsilon} \) and uniform temperature change \( \Delta \theta \), is found:

\[
\begin{align*}
< \varepsilon >_{\Omega_I} &= A^\varepsilon : \bar{\varepsilon} + a^\varepsilon, \\
a^\varepsilon &= (A^\varepsilon - \mathbf{1}) : (C_{cl}^I - C_{cl}^m)^{-1} : (\beta_I - \beta_m).
\end{align*}
\]

(2.54)

where \( A^\varepsilon \) is the strain concentration tensor defined in equation (2.33).

Finally, the macroscopic thermo-elastic response can be written as follows:

\[
\begin{align*}
\bar{\sigma} &= \bar{C} : (\bar{\varepsilon} - \bar{\alpha} \Delta \theta) \\
\bar{\varepsilon} &= \bar{C} : \bar{\sigma} + \bar{\beta}.
\end{align*}
\]

(2.55)

\[
\bar{\beta} = (1 - \nu_I) \beta_m + \nu_I \beta_I + \nu_I (C_{cl}^I - C_{cl}^m) : a^\varepsilon.
\]

(2.56)

Here \( \bar{C} \) is given by the isothermal expression in equation (2.34), and the macroscopic thermal expansion \( \bar{\alpha} \) is defined as follows:

\[
\bar{\alpha} = -\frac{1}{\Delta \theta} \bar{C}^{-1} : \bar{\beta}
\]

(2.57)
2.3.5 Extending Mean-field approaches to nonlinear materials: Direct linearization of the constitutive laws

When the constitutive behavior of either matrix or the inclusion is nonlinear the Eshelby reasoning does not apply as such. Several formulations were proposed to solve the problem of inelastic composites. The direct linearization of the constitutive laws consists in the linearization of the constitutive laws at both macro and micro scales. It is based on the definition of Linear Comparison Composite (LCC), with uniform instantaneous stiffness per phase and whose overall mechanical response should be representative of the actual composite.

The most popular linearization methods are presented hereafter:

2.3.5.1 Secant formulation

The secant methods (figure 2.21) relate directly for each phase $r$ of the nonlinear composite material, the total strain and the total stress via a secant operator $C^{\sec(r)}$:

$$\sigma_r = C^{\sec(r)} : \varepsilon_r.$$  \hspace{1cm} (2.58)

The overall stress-strain relation writes:

$$\bar{\sigma} = \bar{C} : \bar{\varepsilon}.$$  \hspace{1cm} (2.59)

Estimates of overall response can be obtained from classical linear schemes [Berveiller and Zaoui, 1979, Tandon and Weng, 1988], supposing uniform
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secant operators in each phase. For isotropic local materials, a reference secant operator is computed for the first-order moment of the equivalent strain \(\bar{\varepsilon}_{req}\):

\[
\bar{\varepsilon}_{req} = \left[ \frac{2}{3} < \xi >_{\Omega_r} < \xi >_{\Omega_r} \right]^{1/2},
\]

where \(\xi\) is the deviatoric part of the strain tensor.

Later, the classical secant method was modified by [Suquet, 1995], and the secant operator is computed from the second order moment of the effective strain in the phase:

\[
\bar{\varepsilon}_{req} = \left[ \frac{2}{3} I^{dev} :: < \varepsilon \otimes \varepsilon >_{\Omega_r} \right]^{1/2},
\]

When applied to two-phase elasto-plastic composites, [Pierard et al., 2007] showed that the modified secant method was more accurate than the original one. The major drawback of this method is that it is limited to monotonic and proportional loadings.

2.3.5.2 Incremental formulation

This approach is based on a rate formulation of the local problem (figure 2.22). It was proposed by [Hill, 1965] and it allows to predict both macroscopic and per-phase responses of nonlinear materials. Various results obtained with this formulation are presented in [Doghri and

Consider an elasto-plastic composite, the per-phase constitutive laws are linearized as follows:

\[ \sigma_r = C^{ep}_r : \dot{\varepsilon}_r, \]

(2.62)

where \(C^{ep}_r\) is the continuum tangent operator. A discretization over a time interval gives:

\[ \Delta \sigma_r \approx C^{alg}_r : \Delta \varepsilon_r, \]

(2.63)

where \(\Delta \sigma_r\) and \(\Delta \varepsilon_r\) are stress and strain increments of phase \(r\) and \(C^{alg}_r\) is the algorithmic tangent operator: \(C^{alg}_r = \frac{\partial \sigma_r}{\partial \varepsilon_r}\).

The homogenization is performed step by step: once the constitutive equations are linearized for each phase over the time step, homogenization models valid in linear elasticity can apply over this time interval, and the effective relation reads:

\[ \Delta \bar{\sigma} = \bar{C} : \Delta \bar{\varepsilon}. \]

(2.64)

### 2.3.5.3 Incrementally affine linearization

The affine approach consists in applying the MFH on the total strain field as proposed by ([Masson et al., 2000]) for the prediction of the effective properties of elasto-plastic composites and polycrystals. It was enhanced for two-phase elasto-viscoplastic composites by [Pierard and Doghri, 2006].

Then [Doghri et al., 2010b] proposed a general incrementally affine linearization which is valid for multi-axial, non monotonic and non-proportional loading histories. More recently, [Miled et al., 2013] proposed a generalization of this linearization method to the coupled VE-VP model.

The main purpose of this method is to relate the increments of stress and strain via a tangent operator:

\[ \Delta \sigma = C^{alg} : (\Delta \varepsilon - \Delta \varepsilon^{af}), \]

(2.65)

where \(\Delta \varepsilon^{af}\) is the affine strain increment.

Actually, this equation is form-similar to linear thermoelasticity, so that available homogenization models for linear thermo-elastic composites can be applied at each time step.

For VE-VP materials the method is presented in details in Chapter 4.
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2.3.5.4 Incremental-secant method

The incremental-secant MFH formulation (figure 2.24) was developed by [Wu et al., 2013a] for EP composite materials without considering damage and was extended recently to material exhibiting damage by [Wu et al., 2013b]. At a given strain-stress state of the composite material, [Wu et al., 2013a] proposed to apply an unloading step to evaluate the residual stresses $\sigma^r_{n}^{res}$ in each phase, before applying the MFH process. Which leads to an incremental-secant formulation with per-phase residual strains. Considering a time interval $[t_n, t_{n+1}]$, the stress tensor in the different composite material phases is computed as follows:

For the residual-incremental-secant method (figure 2.24(a))

$$\sigma_{n+1} = \sigma_{n}^{res} + C^{sr} : \Delta \varepsilon^r_{n+1}, \quad (2.66)$$

and for the zero-incremental-secant method (figure 2.24(b))

$$\sigma_{n+1} = C^{sr} : \Delta \varepsilon^r_{n+1}, \quad (2.67)$$

where $C^{sr}$ is the residual-incremental-secant operator.

The method predicts accurate results, for a broad range of composites and it is valid for general non-monotonic and non-proportional loading.
Figure 2.24: Definition of the incremental-secant formulation, without considering damage. (a) Definition of the residual strain and stress and of the residual-secant operator. (b) Definition of the zero-secant operator. [Wu et al., 2013b]

2.3.5.5 Variational approach

The aim of the approach is to propose estimates for the overall potential $W^M$, from which the stress-strain relation may be derived. This method is mathematically elegant and allows more complex definitions of LCC [Castañeda, 1991]. It gives rigorous bounds for the overall behavior. It was recently proposed by [Lahellec and Suquet, 2013] for elasto-viscoplastic composites with isotropic and kinematic hardening laws, and by [Brassart et al., 2011] for elasto-(visco-)plastic composites.

2.3.6 Modeling of composites containing misaligned fibers

2.3.6.1 Fiber orientation

For short fiber reinforced composites produced by the injection molding, inclusions are randomly oriented and supposed to be ellipsoidal and straight. The inclusion can then be presented by a unit vector $\mathbf{p}$ directed along its axis (figure 2.25).

The properties of composite materials are influenced by the orientation distribution of inclusions. A probability density function $\psi(p) \equiv \psi(\theta, \phi)$, usually called Orientation Distribution Function (ODF) informs about the density of probability for a particular orientation and must verify
Figure 2.25: Coordinate system for specifying a general state orientation of a single fiber.

The following equation:

\[ \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \psi(\theta,\phi) \sin \theta d\phi d\theta = 1. \]  

(2.68)

The ODF can be either derived from experimental measurements (e.g. tomography technique \cite{Bernasconi2008}) or reconstructed using the so-called orientation tensor in its second or fourth-order form \( \mathbf{a} \) and \( \mathbf{A} \). Those tensors are defined as the following ODF-weighted averages (\cite{Advani1987}):

\[ \mathbf{a} \equiv \langle \mathbf{p} \otimes \mathbf{p} \rangle_\psi, \quad \mathbf{A} \equiv \langle \mathbf{p} \otimes \mathbf{p} \otimes \mathbf{p} \otimes \mathbf{p} \rangle_\psi. \]  

(2.69)

From the definition of \( \mathbf{a} \) and \( \| \mathbf{p} \| = 1 \), we have the following properties:

\[ a_{ij} = a_{ji}; \quad a_{11}, a_{22}, a_{33} \geq 0; \quad a_{ii} = 1; \]  

(2.70)

The fourth-order orientation tensor \( \mathbf{A} \) can be constructed from \( \mathbf{a} \). Exact formula exist in the case of aligned or randomly oriented fibers (2D or 3D), using the so-called linear and quadrature closure approximations (\cite{Advani1987,Advani1990}). For other cases, only so-called closure formula exist. For more details, reader can be referred to (\cite{Doghri2006}).

Reconstruction of the ODF
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When unavailable, the ODF must be recovered from orientation tensors \( a \) and \( A \). In the case of 2D orientations and if one supposes the following expression of the second order orientation tensor \( a \):

\[
a = \begin{pmatrix} a & 0 \\ 0 & 1 - a \end{pmatrix}, \quad a \in [0, 1],
\]

the following formula has been proposed by Verleye and Dupret, 1993 using the Natural Closure Approximation:

\[
\psi(\phi) = \frac{1}{2\pi} \left( \frac{1-a}{a} \cos^2 \phi + \frac{a}{1-a} \sin^2 \phi \right)^{-1}.
\]

**Numerical implementation**

In practice, we proceed as follows. Orientations tensors \( a \) and \( A \), are known data. \( A \) is usually derived from \( a \). In the case of 2D orientations, the ODF formula in equation (2.72) is computed at discrete angle values \( \phi_i \in [0, \pi] \), since \( \psi(p) = \psi(-p) \).

Any mechanical variable \( \mu \) can be computed in function of \( p \) as follows:

\[
< \mu(p) >_\psi = \int_0^{2\pi} \mu(\phi)\psi(\phi)d\phi
\]

\[
< \mu(p) >_\psi \simeq 2\Delta \phi \sum_{i=0}^{N} \mu(\phi_i)\psi(\phi_i).
\]

Where \( \Delta \phi \) is a constant angle increment with a total number of increments \( N = \pi/\Delta \phi \).

**2.3.6.2 Pseudo-grain concept and two step homogenization procedure**

A general two-step homogenization procedure was originally proposed by Camacho et al., 1990, Lielens, 1999b and Pierard et al., 2004. Composites made of two materials may be regarded as multi-phased when the identically shaped inclusions do not have the same orientation.

Let the RVE (\( \Omega \)) be a matrix with volume fraction \( \upsilon_m \) reinforced with misaligned fibers (figure 2.26). The numerical RVE can be regarded as a set of pseudograins. Each pseudograin occupies a domain \( \Omega_{p.g} \) with the same matrix concentration \( \upsilon_m \) as the RVE and with aligned inclusions.
Figure 2.26: Two step homogenization of a two-phase composite containing misaligned fibers. RVE decomposition into pseudograins. First homogenization step (homogenization of each pseudograin). Second homogenization step (homogenization over all pseudograins), [Kammoun, 2011]
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The numerical RVE is homogenized in two steps: each pseudo-grain is first homogenized individually with a suitable scheme. Afterwards the set of homogeneous pseudograins is itself homogenized.

For the first homogenization step the MT model is used in this work, but in the second homogenization step one can choose between two alternatives the Voigt model or the self-consistent model.

The first approach MT/Voigt supposes that the strain is uniformly partitioned among pseudograins. It was proposed by Doghri and Tinel, 2005 for materials reinforced with distributed-orientation fibers.

Although the MT/Voigt strategy shows satisfying results for a large range of composite materials, the second approach MT/self-consistent is more realistic. It allows to partition the strain among pseudograins according to the mechanical properties of each pseudograin.

At the end, a volume average over the entire RVE is obtained as an average over orientations of the volume averages over the pseudo-grains.

2.4 Conclusion

In this chapter, we presented some computational strategies for the analysis of structures that take into account the multi-scale aspects in time or in space.

For the temporal level, the multi-time-step approach and the variational multiscale method (VMS) in time, are multiscale strategies in time and they allow to reduce the computation time, but they are not adequate to resolve the problem of structures subjected to cyclic loadings with large numbers of cycles.

There are three methods presented in this chapter which can resolve this kind of problem: the LATIN method, the cycle jump and the time homogenization approach. The first one, coupled with finite element approach, allows the resolution of the problem of structures subjected to large numbers of cycles in an economic way. Nevertheless its FE implementation is still too complicated. The second one is very efficient, but it does not guarantee the satisfaction of the governing equations of the problem. A real time homogenization approach can be provided by the asymptotic time-homogenization in the case of cyclic phenomena with an elegant and rigorous way. This approach will be adopted in this work and will be explained in details in Chapter 3 since we are limited to this kind of application.

For the spatial level, the strategies most commonly used are those from
the homogenization theory. In this work, we propose a new multiscale computational strategy in both space and time for coupled viscoelastic-viscoplastic composites.
CHAPTER 3

Modeling and algorithms for two-scale time homogenization of viscoelastic-viscoplastic solids under large numbers of cycles

3.1 Introduction

A two-scale time homogenization formulation and the corresponding algorithms are proposed for coupled viscoelastic-viscoplastic (VE-VP) homogeneous solids subjected to large numbers of cycles. The main aim is to predict the long time response while reducing the computational cost considerably. The method is based on the definition of macro and micro-chronological time scales, and on asymptotic expansions of the unknown variables. First, the VE-VP constitutive model is formulated based on a thermodynamical framework. Next, the original VE-VP initial-boundary value problem is decomposed into coupled micro-chronological (fast time scale) and macro-chronological (slow time-scale) problems. The former is purely VE, and solved once for each macro time step, whereas the latter problem is nonlinear and solved iteratively using fully implicit time integration. For micro-scale time averaging, one-point and multi-point integration algorithms are developed. Several numerical simulations on uniaxial and multiaxial cyclic loadings illustrate the computational efficiency and the accuracy of the proposed methods.

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In this chapter, we extend the two-scale time homogenization theory proposed by [Guennouni, 1988] from EVP materials to a constitutive model more suitable for thermoplastic polymers. The constitutive VE-VP model couples VE (with arbitrary Prony series for time-dependent shear and bulk moduli) and VP (with arbitrary isotropic hardening and general Perzyna-type VP functions).

The present work presents the following important novelties. First, the VE-VP model is developed within a thermodynamic framework from which state and evolution laws are derived (Section 3.2 and Appendix B). Second, computational fully implicit time integration algorithms are proposed (Section 3.4). Third, new numerical simulations are presented in section 3.5 including multiaxial ones and a comparison with experimental data.

The chapter has the following outline. In Section 3.2, a coupled VE-VP constitutive model which satisfies the Clausius-Duhem inequality is formulated within the framework of small strain theory and isothermal process. In Section 3.3, a time homogenization scheme for coupled VE-VP solids is presented. In Section 3.4, the computational algorithm is detailed and studied. In Section 3.5, the time homogenization approach is verified against reference full-time solution for several loading cases and compared to experimental results in one case.

### 3.2 Thermodynamic formulation of a coupled viscoelastic-viscoplastic (VE-VP) constitutive model

The observed behavior of thermoplastic polymers is generally time and strain dependent at all levels of deformation. A coupled VE-VP model was proposed by [Miled et al., 2011] which can reproduce those dependencies, but the authors did not write the model within the formalism of thermodynamics of irreversible processes. In the present work, the theory of Generalized Standard Materials is adopted (Lemaitre and Chaboche, 1990, Halphen and Nguyen, 1975), which implies that constitutive equations are derived from two potentials: a free energy and a dissipation function. The presentation is restricted to isothermal and small strain conditions.
3.2. Thermodynamic formulation of a coupled viscoelastic-viscoplastic (VE-VP) constitutive model

3.2.1 Clausius-Duhem inequality

The second law of thermodynamics combined with the first law can be expressed by the Clausius-Duhem inequality which states that the change in entropy, in a closed system, is always non-negative (e.g. [Christensen, 1971]) and given in the isothermal case by:

\[-\rho \dot{\psi} + \sigma : \dot{\varepsilon} \geq 0, \quad (3.1)\]

where, \(\rho \left[ \text{kg/m}^3 \right]\) is the mass density, \(\psi\) the Helmholtz free energy \([\text{J/kg}]\), \(\sigma\) and \(\varepsilon\) denote the Cauchy stress and strain tensors, respectively, and the superposed dot means a time derivative.

3.2.2 State laws

In the isothermal case, the observable state variable is the total strain. Internal state variables take into account irreversible phenomena in order to reproduce the history of the material.

The total strain \(\varepsilon\) is assumed to be subdivided into VE (\(\varepsilon^{ve}\)) and VP (\(\varepsilon^{vp}\)) portions ([Miled et al., 2011], [Nikolov et al., 2002]):

\[\varepsilon = \varepsilon^{ve} + \varepsilon^{vp}. \quad (3.2)\]

Furthermore, the splitting between VE and VP behaviour in equation (3.2) allows us to assume that the Helmholtz free energy \(\psi\) may be decomposed into the sum of VE part \(\psi^{ve}\) that represents the VE stored energy and VP part \(\psi^{vp}\) which represents the energy stored due to material hardening,

\[\psi = \psi^{ve} + \psi^{vp}. \quad (3.3)\]

This decomposition of the free energy was already used in ([Lemaitre and Chaboche, 1990]) in the case of EP and EVP.

As demonstrated in Appendix B, the expression of \(\psi^{ve}\) is:

\[\rho \psi^{ve}(t) = \frac{1}{2} \int_{-\infty}^{t} \int_{-\infty}^{t} \frac{\partial \varepsilon^{ve}(\xi)}{\partial \xi} : C^{ve}(2t - \xi - \tau) \frac{\partial \varepsilon^{ve}(\tau)}{\partial \tau} d\xi d\tau, \quad (3.4)\]

where \(C^{ve}\) is a fourth-order relaxation tensor, whose properties are laid down in Appendix B and \(\xi\) and \(\tau\) are the integral arguments. This follows the work of [Christensen and Naghdi, 1967] in linear VE except...
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The VE part of the Helmholtz free energy $\psi_{ve}$ which is a function of the hardening variable $r$ (e.g., [Lemaitre and Chaboche, 1990]) is defined as:

$$\rho \psi_{vp}(r(t)) = \int_0^{r(t)} R(\xi) d\xi. \quad (3.5)$$

The time derivative of $\psi(t)$ is given as follows (see Appendix B):

$$\rho \dot{\psi} = \frac{1}{2} \int_{-\infty}^{t} \int_{-\infty}^{t} \frac{\partial \varepsilon_{ve}(\xi)}{\partial \xi} : \frac{\partial C_{ve}}{\partial t} (2t - \xi - \tau) : \frac{\partial \varepsilon_{ve}(\tau)}{\partial \tau} d\xi d\tau + \left( \int_{-\infty}^{t} C_{ve}(t - \tau) : \frac{\partial \varepsilon_{ve}(\tau)}{\partial \tau} d\tau \right) : \dot{\varepsilon}_{ve} + R\dot{r}. \quad (3.6)$$

Substituting equation (3.6) in the Clausius-Duhem inequality (3.1), and under isothermal conditions we get the state laws:

$$\sigma(t) = \int_{-\infty}^{t} C_{ve}(t - \xi) : \frac{\partial \varepsilon_{ve}(\xi)}{\partial \xi} d\xi, \quad (3.7)$$

$$R = R(r). \quad (3.8)$$

Consequently, by rewriting the Clausius-Duhem inequality (3.1), we get the following relation, which expresses that the total dissipation $\Phi$ is necessarily non-negative:

$$\Phi = \int_{-\infty}^{t} \int_{-\infty}^{t} \frac{\partial \varepsilon_{ve}(\xi)}{\partial \xi} : \frac{\partial C_{ve}}{\partial t} (2t - \xi - \tau) : \frac{\partial \varepsilon_{ve}(\tau)}{\partial \tau} d\xi d\tau \geq 0. \quad (3.9)$$

Given that isothermal conditions are assumed in this paper, thermal dissipation due to the conduction of heat is neglected and only the mechanical dissipation power is considered. The dissipation can be decomposed into VE ($\Phi_{ve}$) and VP ($\Phi_{vp}$) components which are defined in equation (3.9). Unlike the work of [Zhu and Sun, 2013], in which the VE dissipation was not taken into account, it is noticed that the VE transformation is characterized by an intrinsic dissipation $\Phi_{ve}$ which has
3.2. Thermodynamic formulation of a coupled viscoelastic-viscoplastic (VE-VP) constitutive model

A quadratic expression in $\dot{\varepsilon}^{ve}$.

A more restrictive version of the Clausius-Duhem inequality requires that both VE and VP contributions are non-negative.

$$\Phi_{ve} \geq 0 \quad \text{and} \quad \Phi_{vp} \geq 0.$$  \hspace{1cm} (3.10)

3.2.3 Viscoplastic constitutive relations

In order to describe the evolution of internal variables and the VP dissipation process, a pseudo-dissipation potential $\Theta^*_{vp}(\sigma, R; r)$ is defined. This potential is a non-negative, continuous and convex scalar-valued function that depends on thermodynamic forces $\sigma$ and $R$ and also on the state variable $r$ which appears as a parameter. The potential is equal to zero at the origin, $\Theta^*_{vp}(0, 0; 0) = 0$. The flow rules are derived from the potential as follows:

$$\dot{\varepsilon}^{vp} = \frac{\partial \Theta^*_{vp}}{\partial \sigma},$$  \hspace{1cm} (3.11)

$$\dot{r} = -\frac{\partial \Theta^*_{vp}}{\partial R}.$$  \hspace{1cm} (3.12)

Assuming that the VP deformation is an isochoric phenomenon, the stress appears only through the $J_2$ invariant of its deviator and the variable $R$ represents the size of the current VE domain. The potential $\Theta^*_{vp}$ is then expressed as a function of a yield function $f(\sigma, R)$ (e.g., [Lemaitre and Chaboche, 1990]), which defines the VE domain: when the transformation is VE, then $f(\sigma, R) \leq 0$, but if VP strains evolve, then $f(\sigma, R)$ may be positive. Here the chosen function corresponds to the well-known von Mises yield surface $f$:

$$f(\sigma, R) = \sigma_{eq} - \sigma_y(\dot{\varepsilon}) - R(r),$$  \hspace{1cm} (3.13)

where $\sigma_y > 0$ is the initial yield stress. In a uniaxial tension test, one unloads to zero stress and waits for "long" time. If the total strain returns to zero, $\sigma_y$ was not reached yet. A permanent (thus VP) strain after a zero stress is held for a long time defines $\sigma_y$. The latter may depend on the strain rate (e.g., [Richeton et al., 2005], [Howard, 1999], [Rault, 1998]). $\sigma_{eq}$ is the von Mises equivalent stress defined by:

$$\sigma_{eq} \equiv \left( \frac{3}{2} s : s \right)^{\frac{1}{2}},$$  \hspace{1cm} (3.14)
where $s$ is the deviatoric part of the Cauchy stress:

$$s \equiv \sigma - \frac{1}{3} (\text{tr} \sigma) \mathbf{1}. \quad (3.15)$$

Defining the potential $\Theta^*_{vp}$ as a function of $f$, the fluxes are directly derived as follows:

$$\dot{\varepsilon}_{vp} = \frac{\partial \Theta^*_{vp}}{\partial f} \frac{\partial f}{\partial \sigma} = \lambda \frac{\partial f}{\partial \sigma}, \quad (3.16)$$

$$\dot{r} = -\frac{\partial \Theta^*_{vp}}{\partial R} = -\lambda \frac{\partial f}{\partial R}, \quad (3.17)$$

where $\lambda$ is a non-negative VP multiplier.

The generalized normality laws (3.16) and (3.17) then yield the following equations:

$$\dot{\varepsilon}_{vp} = \frac{3}{2} \frac{s}{\sigma_{eq}}, \quad (3.18)$$

$$\dot{r} = \dot{\lambda}. \quad (3.19)$$

Using the definition of the accumulated plastic strain rate $\dot{p}$:

$$\dot{p} = \left(\frac{2}{3} \dot{\varepsilon}_{vp} : \dot{\varepsilon}_{vp}\right)^{\frac{1}{2}}, \quad (3.20)$$

equations (3.18) and (3.19) become:

$$\dot{r} = \dot{\lambda} = \dot{p}, \quad (3.21)$$

$$\dot{\varepsilon}_{vp} = \frac{3}{2} \frac{s}{\sigma_{eq}}. \quad (3.22)$$

The precise sign of the VP multiplier $\dot{p}$ is determined by the following conditions:

$$\dot{p} = g_v(\sigma_{eq}, p, \dot{\varepsilon}) > 0, \text{ if } f > 0; \quad \dot{p} = 0, \text{ if } f \leq 0, \quad (3.23)$$

where $g_v(\sigma_{eq}, p)$ is a VP function.

There are many viscoplastic laws that can describe the VP evolution of polymer materials. Generally semi-crystalline polymers, as well as amorphous polymers, have the yield behavior described by the Eyring’s viscosity theory (Richeton et al., 2005, Cortet et al., 2007, Ghorbel).
3.2. Thermodynamic formulation of a coupled viscoelastic-viscoplastic (VE-VP) constitutive model

This theory considers the yield behavior as a thermally activated process. It accounts for temperature and strain rate effects. In this model, macroscopic deformation is assumed to be a result of basic processes that are either intermolecular (e.g. chain-sliding) or intramolecular (e.g. a change in the conformation of the chain). The molecular motions can be carried on only if an energy barrier is overcome. Eyring’s theory ([Eyring, 1936]) was modified by several authors such as ([Govaert et al., 1999, Richeton et al., 2005, Klompen et al., 2005]).

Other authors describe the yield behavior by Norton’s power law ([Miled et al., 2011, Miled, 2011]).

For example using Norton’s power law:

\[
g_v = g_v(f) = \frac{\sigma_y}{\zeta} \left(\frac{f}{\sigma_y}\right)^m, \quad \text{if } f > 0; \quad g_v = 0, \quad \text{if } f \leq 0, \quad (3.24)
\]

where the two parameters \( \zeta \) and \( m \) represent the VP modulus and exponent, respectively, the expression of the pseudo-potential \( \Theta_{vp}^* \) is given as follows:

\[
\Theta_{vp}^*(f) = \frac{\sigma_y^2}{\zeta(m+1)} \left(\frac{f}{\sigma_y}\right)^{(m+1)}, \quad \text{if } f > 0; \quad \Theta_{vp}^*(f) = 0, \quad \text{if } f \leq 0. \quad (3.25)
\]

3.2.4 Viscoelastic-Viscoplastic constitutive model

In summary and based on the derivations before, the model is described by the following set of equations:

\[
\left\{
\begin{array}{l}
\varepsilon = \varepsilon^{ve} + \varepsilon^{vp}, \\
\sigma(t) = \int_{-\infty}^{t} C^{ve}(t - \xi) : \frac{\partial \varepsilon^{ve}(\xi)}{\partial \xi} d\xi, \\
f(\sigma, R) = \sigma_{eq} - \sigma_y(\dot{\varepsilon}) - R(p), \\
\dot{\varepsilon}^{vp} = \dot{p}N, \\
\dot{p} = g_v(f) > 0, \quad \text{if } f > 0; \quad \dot{p} = 0, \quad \text{if } f \leq 0,
\end{array}
\right. \quad (3.26)
\]

where the following notation is introduced:

\[
N = \frac{\partial f}{\partial \sigma} = \frac{3}{2} \frac{s}{\sigma_{eq}}. \quad (3.27)
\]

For an isotropic material, the fourth rank relaxation tensor takes the following form:

\[
C^{ve}(t) = 2G(t)I^{dev} + 3K(t)I^{vol}, \quad (3.28)
\]
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$G(t)$ and $K(t)$ are the shear and bulk relaxation moduli, respectively, which can be expressed in the form of Prony series:

$$G(t) = G_\infty + \sum_{i=1}^{I} G_i \exp \left( -\frac{t}{g_i} \right), \quad (3.29)$$

$$K(t) = K_\infty + \sum_{j=1}^{J} K_j \exp \left( -\frac{t}{k_j} \right). \quad (3.30)$$

Here, $G_\infty$ and $K_\infty$ are the elastic shear and bulk long-term moduli, respectively; $g_i$ $(i = 1 \ldots I)$ and $k_j$ $(j = 1 \ldots J)$ are shear and bulk relaxation times respectively; and $G_i$ $(i = 1 \ldots I)$ and $K_j$ $(j = 1 \ldots J)$ are shear and bulk weights respectively.

By substituting equations (3.28), (3.29) and (3.30) into equation (3.26-b), a decomposition of the stress tensor into deviatoric ($s(t)$) and dilatational ($\sigma_H(t)$) parts, and the strain tensor into deviatoric ($\xi(t)$) and dilatational ($\epsilon_H(t)$) parts is obtained:

$$\begin{cases}
    s(t) = 2G_\infty \xi^{ve}(t) + \sum_{i=1}^{I} s_i(t), \\
    \sigma_H(t) = 3K_\infty \epsilon^{ve}_H(t) + \sum_{j=1}^{J} \sigma_H_j(t),
\end{cases} \quad (3.31)$$

where the viscous stresses are defined by:

$$\begin{cases}
    s_i(t) = 2G_i \exp \left( -\frac{t}{g_i} \right) \int_{-\infty}^{t} \exp \left( \frac{\eta}{g_i} \right) \frac{\partial \xi^{ve}(\eta)}{\partial \eta} d\eta, \\
    \sigma_H_j(t) = 3K_j \exp \left( -\frac{t}{k_j} \right) \int_{-\infty}^{t} \exp \left( \frac{\eta}{k_j} \right) \frac{\partial \epsilon^{ve}_H(\eta)}{\partial \eta} d\eta.
\end{cases} \quad (3.32)$$

Then, using equation (3.31), the stress tensor can be written as:

$$\sigma(t) = C_\infty : \epsilon^{ve}(t) + \sum_{i=1}^{I} s_i(t) + \sum_{j=1}^{J} \sigma_H_j(t) 1, \quad (3.33)$$

where, $C_\infty = 2G_\infty 1^{dev} + 3K_\infty 1^{vol}$, is the long-term elastic Hooke’s operator.
The only hardening function \( R(p) \) considered in the subsequent simulations is a power-law which is defined as:

\[
R(p) = \begin{cases} 
kp^n, & \text{if } p > 0; \\
0, & \text{if } p = 0,
\end{cases}
\]  
(3.34)

where the two parameters \( k \) and \( n \) represent the hardening modulus and exponent, respectively.

### 3.3 Two scale time homogenization for coupled VE-VP solids under cyclic loadings

Predicting the response of nonlinear materials under high frequency loadings requires significant computational resources, due to the very large number of required time integration increments. The time homogenization method, developed by [Guenouni, 1988] for EVP homogeneous solids, enables to predict their response under a large number of cycles while simulating a much smaller number. In this work, we extend Guennoui’s formulation to solids whose material obeys the coupled VE-VP model of Section 3.2.

#### 3.3.1 Definition of two time scales

In figure 3.1 (a), a cyclic loading is plotted as a function of the physical time, \( t^* \). If we look at the loading on a “macroscopic” time scale, \( t_M^* \), we will see a smooth curve (figure 3.1 (b)). Finally, if we ”zoom” in we will see periodic fluctuations of period \( T^* \) (figure 3.1 (c)). For instance, the macro-time \( t_M^* \) can be seen as a multiplier of \( T^* \), and might correspond to the peaks of the imposed loads or displacements.

We thus define two time scales. The first one corresponds to a natural time scale \( t_M^* \) that is characteristic of the long-term behaviour of the solution. The second one is a fine time-scale \( \tau \) associated with the rapidly varying behaviour of the evolving variables resulting from the oscillating load.

The relation between the physical time \( t^* \) and the two time scales \( t_M^* \) and \( \tau \) is defined as:

\[
t^* = t_M^* + T^* \tau, \quad t_M^* \in [0, T_F^*] \text{ and } \tau \in [0, 1].
\]  
(3.35)

Equation (3.35) is illustrated in figure 3.2, which is inspired from figure 1, Chapter 3 in [Debordes, 2001]), for spatial scales.
Figure 3.1: Macro and micro time coordinates ($t^*$: physical time, $t^*_M$: macro-time, $T^*$: load period, $\tau$: micro-time $\tau \in [0, 1]$).

Figure 3.2: Relation between different time coordinates ($t^*$: physical time, $t^*_M$: macro-time, $T^*$: load period, $\tau$: micro-time $\tau \in [0, 1]$).
3.3. Two scale time homogenization for coupled VE-VP solids under cyclic loadings

Each mechanical variable $\Psi_{T^*} (\vec{x}, t^*)$, at a given spatial location $\vec{x}$ is then supposed to be $T^*$-periodic (with respect to $t^*$) and to depend on macro and micro scales $t_{M}^*$ and $\tau$:

$$\Psi_{T^*} (\vec{x}, t^*) = \Psi (\vec{x}, t_{M}^*, \tau).$$  \hfill (3.36)

A local 1-periodicity assumption with respect to $\tau$ is then made for field variables $\Psi_{T^*} (\vec{x}, t^*)$.

Consider the observation time interval $[0, T_F^*]$, such as $T^* << T_F^*$. Let us define a dimensionless time variable $t$:

$$t = \frac{t^*}{T_F^*}, \quad t \in [0, 1],$$  \hfill (3.37)

a dimensionless macro time variable $t_M$:

$$t_{M} = \frac{t_{M}^*}{T_F^*}, \quad t_M \in [0, 1],$$  \hfill (3.38)

and a small scaling parameter $T$ as follows:

$$T = \frac{T^*}{T_F^*}.$$  \hfill (3.39)

Then the following function substitution is made:

$$\phi_T (\vec{x}, t) = \Psi_{T^*} (\vec{x}, t^*),$$  \hfill (3.40)

then $\phi_T (\vec{x}, t)$ is $T$-periodic with respect to the dimensionless time $t$.

| Subscripts $T^*$ or $T$ associated with a variable denote its association with the two scales. A new dimensionless relation between both time scales is obtained:

$$t = t_{M} + T\tau, \quad t_{M} = [0, 1]; \quad t \in [0, 1]; \quad \tau \in [0, 1].$$  \hfill (3.41)

A local periodicity assumption, with respect to $\tau$ is made for all field variables $\phi_T (\vec{x}, t)$. Each periodic variable $\phi_T (\vec{x}, t)$ is assumed to exhibit

$$\phi_T (\vec{x}, t + T) = \phi_T \left( \vec{x}, \frac{t^* + T^*}{T_F^*} \right) = \Psi_{T^*} \left( \vec{x}, \frac{t_{M}^* + T^*}{T_F^*} \right),$$

$$= \Psi_{T^*} (\vec{x}, t^* + T^*) = \Psi_{T^*} (\vec{x}, t^*),$$

$$= \phi_T (\vec{x}, t).$$
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dependence on both dimensionless time scales. It is then considered as a function of the two time variables $t^*_M$ and $\tau$, and is expressed as:

$$\phi_T (\vec{x},t) = \phi (\vec{x},t^*_M,\tau).$$

(3.42)

Using equation (3.41) and the chain, the time derivation in the two time scales is given as:

$$\dot{\phi}_T (\vec{x},t) = \frac{\partial \phi (\vec{x},t^*_M,\tau)}{\partial t^*_M} + \frac{1}{T} \frac{\partial \phi (\vec{x},t^*_M,\tau)}{\partial \tau},$$

(3.43)

where the superposed dot denotes the total derivative with respect to the dimensionless physical time $t$.

The average value $\phi_M (\vec{x},t^*_M)$ of the function $\phi (\vec{x},t^*_M,\tau)$ with respect to the fast time scale $\tau$ and at time $t^*_M$ is defined as:

$$\phi_M (\vec{x},t^*_M) = \int_0^1 \phi (\vec{x},t^*_M,\tau) d\tau \equiv < \phi (\vec{x},t^*_M,\tau) >.$$  

(3.44)

The symbol $< . >$ corresponds to the micro time averaging operator.

A general decomposition of all variables $\phi$ into macroscopic $\phi_M$ and oscillatory $\widetilde{\phi}$ portions is proposed in the form:

$$\phi (\vec{x},t^*_M,\tau) = \phi_M (\vec{x},t^*_M) + \widetilde{\phi} (\vec{x},t^*_M,\tau),$$

(3.45)

3.3.2 Initial-boundary VE-VP problem

In this work, we consider a typical initial-boundary value problem of a coupled VE-VP solid occupying spatial domain $\Omega$ and subjected to cyclic boundary displacements $\vec{u}_b^T$ and tractions $\vec{g}_T$ of the following forms:

$$\vec{g}_T (\vec{x},t) = \vec{g}^* (\vec{x}) \lambda (t) \quad \text{and} \quad \vec{u}_b^T (\vec{x},t) = \vec{u}_b^* (\vec{x}) \beta (t),$$

(3.46)

where $\lambda(t)$ and $\beta(t)$ are the sum of a rapid harmonic oscillation and a slow loading with a very small load period $T$ compared to the observation time $T_F^*$, and they are of the following general form:

$$a(t) + b(t) \left( \sin \left( \frac{2\pi t}{T^*} \right) + c \right).$$

(3.47)

It is implicit that all variables are functions of position $\vec{x}$, so from now on and without loss of generality, the dependence of all variables on $\vec{x}$ is omitted for simplicity.
A VE-VP solid body occupying a domain $\Omega$ with boundary $\Gamma = \Gamma_u \cup \Gamma_f$ is subjected to forces per unit volume $\vec{f}$ in $\Omega$, boundary displacements $\vec{u}_b$ on $\Gamma_u$ and tractions $\vec{g}$ on $\Gamma_f$. The initial-boundary VE-VP problem is expressed using the two time scales as follows:

Initial conditions:
\[
\vec{u}(\vec{x}, t_M = 0, \tau = 0) = \vec{u}^I(\vec{x}), \quad \text{in } \Omega
\]
\[
\sigma(\vec{x}, t_M = 0, \tau = 0) = \sigma^I(\vec{x}), \quad \text{in } \Omega
\]

Boundary conditions:
\[
\vec{u}(t_M, \tau) = \vec{u}_b(t_M, \tau), \quad \text{on } \Gamma_u \times [0, 1] \times [0, 1]
\]
\[
\sigma \cdot \vec{n} = \vec{g}(t_M, \tau), \quad \text{on } \Gamma_f \times [0, 1] \times [0, 1]
\]

Equilibrium equations:
\[
\nabla \cdot \sigma(t_M, \tau) + \vec{f}(t_M, \tau) = \vec{0}, \quad \text{in } \Omega \times [0, 1] \times [0, 1]
\]

Kinematic compatibility:
\[
\varepsilon(\vec{u}) = \frac{1}{2} \left( \nabla \vec{u} + \tau \nabla \vec{u} \right), \quad \text{in } \Omega \times [0, 1] \times [0, 1]
\]

Constitutive equations:
\[
\varepsilon(\vec{u}) = \varepsilon^{ve}(t_M, \tau) + \varepsilon^{vp}(t_M, \tau), \quad \text{in } \Omega \times [0, 1] \times [0, 1]
\]
\[
\sigma(t_M, \tau) = C_{\infty} : \varepsilon^{ve}(t_M, \tau) + \sum_{i=1}^{I} s_i(t_M, \tau) + \sum_{j=1}^{J} \sigma_{Hj}(t_M, \tau) \mathbf{1}, \quad \text{in } \Omega \times [0, 1] \times [0, 1]
\]
\[
\dot{\varepsilon}^{vp} = B(\sigma, p), \quad \text{in } \Omega \times [0, 1] \times [0, 1]
\]
\[
\dot{p} = C(\sigma, p), \quad \text{in } \Omega \times [0, 1] \times [0, 1]
\]

Where $\nabla \cdot$ and $\nabla$ denote the divergence and gradient operators, respectively, and upper script $'t'$ denotes a transpose. $\vec{n}$ designates the outer unit normal vector to $\Gamma_f$. $B$ and $C$ are viscoplastic operators corresponding to equations (3.22) and (3.23), respectively.
3.3.3 Asymptotic expansion of the initial-boundary VE-VP problem

It is supposed that each mechanical variable \( \phi(t_M, \tau) \) is regular enough and that \( T = \frac{T^*}{T} << 1 \). Therefore, \( \phi(t_M, \tau) \) can be expanded into an asymptotic series of powers of \( T \) (Bensoussan et al., 1978):

\[
\phi(t_M, \tau) = \sum_{i=0}^{\infty} T^i \phi_i(t_M, \tau),
\]

(3.58)

where \( \phi_i(t_M, \tau) \) are \( \tau \)-periodic functions.

\( \phi(t_M, \tau) \) can be regarded as consisting of a leading term \( \phi_0(t_M, \tau) \), plus a series of terms of rapidly decreasing amplitude.

If a nonlinear operator \( C(\sigma, p) \) admits a gradient at a point, its first order asymptotic expansion can be written as follows:

\[
C(\sigma, p) = C(\sigma_0, p_0) + T DC(\sigma_0, p_0) \cdot (\sigma_1, p_1) + O(T^2),
\]

(3.59)

where \( O \) is the Landau notation and \( DC(\sigma_0, p_0) \) is the gradient of \( C \) expressed in \( (\sigma_0, p_0) \).

As demonstrated in Appendix A and if one supposes that \( s_i(t) \) is regular enough and that

\[
\int_{-\infty}^{t} \exp \left( \frac{\eta}{g_i} \right) \xi^{ve}(\eta) d\eta \text{ exists},
\]

then the asymptotic expansion to the first order of equation (3.32-a), in the neighborhood of \( T = 0 \), is given by the following expression:

\[
s_i(t_M + T\tau) = s_{i0}(t_M, \tau) + s_{i1}(t_M, \tau)T + O(T^2),
\]

(3.60)

where \( s_{i0}(t_M, \tau) \) and \( s_{i1}(t_M, \tau) \) are given by the following expressions:

\[
s_{i0}(t_M, \tau) = s_i(t_M) + 2G_i (\xi_0^{ve}(t_M, \tau) - \xi_0^{ve}(t_M, \tau = 0)) ,
\]

(3.61)

\[
s_{i1}(t_M, \tau) = 2G_i \xi_1^{ve}(t_M, \tau) + \frac{2G_i}{g_i} \left[ -\tau \int_{-\infty}^{t_M} \exp \left( \frac{\eta}{g_i} \right) \xi^{ve}(\eta) d\eta + (1 - \tau) (\xi_0^{ve}(t_M, \tau) - \xi_0^{ve}(t_M, \tau = 0)) \right].
\]

(3.62)

\(^3\)If different components of forces and displacements do not have the same period, the asymptotic expansion is done with respect to the smallest period.
3.3. Two scale time homogenization for coupled VE-VP solids under cyclic loadings

By replacing each variable by its asymptotic expansion into equations (3.48) to (3.57) and gathering terms of equal order (i.e. equal powers of $T$), the initial-boundary value problem can be rewritten at the orders $(-1)$ and $(0)$ as follows:

- Order $-1$ problem:

$$
\frac{\partial}{\partial \tau} \varepsilon_{\text{vp}}^0(t_M, \tau) = 0, \quad \text{in } \Omega \times [0, 1] \times [0, 1] (3.63)
$$

$$
\frac{\partial}{\partial \tau} p_0(t_M, \tau) = 0, \quad \text{in } \Omega \times [0, 1] \times [0, 1] (3.64)
$$

- Order $0$ problem:

$$
\vec{u}_0(x, t_M = 0, \tau = 0) = \vec{u}_I(x), \quad \text{in } \Omega (3.65)
$$

$$
\sigma_0(x, t_M = 0, \tau = 0) = \sigma_I(x), \quad \text{in } \Omega (3.66)
$$

$$
\vec{u}_0(t_M, \tau) = \vec{u}_0(t_M, \tau), \quad \text{on } \Gamma_u \times [0, 1] \times [0, 1] (3.67)
$$

$$
\sigma_0(t_M, \tau). \vec{n} = \vec{g}(t_M, \tau), \quad \text{on } \Gamma_f \times [0, 1] \times [0, 1] (3.68)
$$

$$
\nabla \cdot \sigma_0(t_M, \tau) + \vec{f}(t_M, \tau) = \vec{0}, \quad \text{in } \Omega \times [0, 1] \times [0, 1] (3.69)
$$

$$
\varepsilon(\vec{u}_0) = \frac{1}{2} \left( \nabla \vec{u}_0 + \frac{1}{3} \nabla \vec{u}_0^T \right), \quad \text{in } \Omega \times [0, 1] \times [0, 1] (3.70)
$$

$$
\varepsilon(\vec{u}_0) = \varepsilon_{\text{ve}}^0(t_M, \tau) + \varepsilon_{\text{vp}}^0(t_M, \tau), \quad \text{in } \Omega \times [0, 1] \times [0, 1] (3.71)
$$

$$
\sigma_0(t_M, \tau) = C_\infty : \varepsilon_{\text{ve}}^0(t_M, \tau) + \sum_{i=1}^{I} s_{i0}(t_M, \tau) + \sum_{j=1}^{J} \sigma_{H,j0}(t_M, \tau) \mathbf{1}, \quad \text{in } \Omega \times [0, 1] \times [0, 1] (3.72)
$$
Modeling and algorithms for two-scale time homogenization of VE-VP solids

\[
\frac{\partial}{\partial t_M} \varepsilon_{vp}^0(t_M, \tau) + \frac{\partial}{\partial \tau} \varepsilon_{vp}^1(t_M, \tau) = B(\sigma_0, p_0), \quad \text{in } \Omega \times [0,1] \times [0,1]
\] (3.73)

\[
\frac{\partial}{\partial t_M} p_0(t_M, \tau) + \frac{\partial}{\partial \tau} p_1(t_M, \tau) = C(\sigma_0, p_0), \quad \text{in } \Omega \times [0,1] \times [0,1]
\] (3.74)

Equations (3.63) and (3.64) show that at order 0, \( \varepsilon_{vp}^0 \) and \( p_0 \) are function of the slow time variable \((t_M)\) only, which means that at the order 0 the rapid evolution of VP deformation is blocked:

\[
\varepsilon_{vp}^0(t_M, \tau) = \varepsilon_{vp}^0(t_M),
\] (3.75)

\[
p_0(t_M, \tau) = p_0(t_M).
\] (3.76)

Using equations (3.63) and (3.71), we have:

\[
\frac{\partial}{\partial \tau} \varepsilon_0(t_M, \tau) = \frac{\partial}{\partial \tau} \varepsilon_{ve}^0(t_M, \tau), \quad \text{in } \Omega \times [0,1] \times [0,1]
\] (3.77)

Equation (3.77) shows that at the zero-order the rapid evolution of total deformation is equal to that of its VE part.

### 3.3.4 Macro and micro chronological problems

In order to solve the zero-order problem of equations (3.65) to (3.74), we follow the additive decomposition into mean and fluctuation parts defined in equation (3.45). Taking the micro-time average as defined by equation (3.44) of equations (3.65) to (3.74) we obtain a decomposition of the original problem into macro- and micro-chronological problems:

- **Macro-chronological problem:**

  \[
  \tilde{\mathbf{u}}_{0M}(t_M) = \tilde{\mathbf{u}}_{bM}(t_M), \quad \text{on } \Gamma_u \times [0,1]
  \] (3.78)

  \[
  ^t \sigma_{0M}(t_M) \cdot \tilde{\mathbf{n}} = \tilde{\mathbf{g}}_M(t_M), \quad \text{on } \Gamma_f \times [0,1]
  \] (3.79)

  \[
  \nabla \cdot \sigma_{0M}(t_M) + \tilde{f}_M(t_M) = 0 \quad \text{in } \Omega \times [0,1]
  \] (3.80)
3.3. Two scale time homogenization for coupled VE-VP solids under cyclic loadings

\[ \varepsilon_M(\vec{u}_0) = \frac{1}{2} \left( \nabla \vec{u}_0M + ' \nabla \vec{u}_0M \right), \text{ in } \Omega \times [0, 1] \]  
\[ (3.81) \]

\[ \varepsilon_M(\vec{u}_0) = e_{0M}^{ve}(t_M) + e_{0M}^{vp}(t_M), \text{ in } \Omega \times [0, 1] \]  
\[ (3.82) \]

\[ \sigma_{M_0}(t_M) = C_\infty : e_{0M}^{ve}(t_M) + \sum_{i=1}^{I} s_{i0M}(t_M) + \sum_{j=1}^{J} \sigma_{H_{j0M}}(t_M)1, \text{ in } \Omega \times [0, 1] \]  
\[ (3.83) \]

\[ \frac{d}{dt_M} e_{0}^{vp}(t_M) = B_M((\sigma_{0M} + \bar{\sigma}_0), p_0), \text{ in } \Omega \times [0, 1] \]  
\[ (3.84) \]

\[ \frac{d}{dt_M} p_0(t_M) = C_M((\sigma_{0M} + \bar{\sigma}_0), p_0), \text{ in } \Omega \times [0, 1] \]  
\[ (3.85) \]

Given that \( \bar{\varepsilon}_0^{vp}(t_M, \tau) = 0 \) from equation \( (3.63) \), the micro-chronological problem is only VE and it is defined as follows:

- **Micro-chronological problem:**
  \[ \tilde{u}_0(t_M, \tau) = \tilde{u}_b(t_M, \tau), \text{ on } \Gamma_u \times [0, 1] \times [0, 1] \]  
\[ (3.86) \]

\[ l\tilde{\sigma}_0(t_M, \tau) \cdot \vec{n} = \tilde{g}(t_M, \tau), \text{ in } \Gamma_f \times [0, 1] \times [0, 1] \]  
\[ (3.87) \]

\[ \nabla \cdot \tilde{\sigma}_0(t_M, \tau) + \tilde{f}(t_M, \tau) = 0, \text{ in } \Omega \times [0, 1] \times [0, 1] \]  
\[ (3.88) \]

\[ \tilde{\varepsilon}(\tilde{u}_0) = \frac{1}{2} \left( \nabla \tilde{u}_0 + ' \nabla \tilde{u}_0 \right), \text{ in } \Omega \times [0, 1] \times [0, 1] \]  
\[ (3.89) \]

\[ \tilde{\varepsilon}(\tilde{u}_0) = \tilde{e}_{0}^{ve}(t_M, \tau), \text{ in } \Omega \times [0, 1] \times [0, 1] \]  
\[ (3.90) \]

\[ \tilde{\varepsilon}_0(t_M, \tau) = C_\infty : \tilde{e}_0(t_M, \tau) + \sum_{i=1}^{I} \tilde{s}_{i0}(t_M, \tau) + \sum_{j=1}^{J} \tilde{\sigma}_{H_{j0}}(t_M, \tau)1, \text{ in } \Omega \times [0, 1] \times [0, 1] \]  
\[ (3.91) \]

\[ ^4 \text{Given the fast time periodicity assumption, } \frac{\partial}{\partial \tau} p_1(t_M, \tau) = \int_0^1 \frac{\partial}{\partial \tau} p_1(t_M, \tau) d\tau = p_1(t_M, 1) - p_1(t_M, 0) = 0. \]  
Similarly, the average of \( \frac{\partial}{\partial \tau} \bar{\varepsilon}_1^{vp}(t_M, \tau) \) vanishes.
Equations (3.86) to (3.91) correspond to the resolution of a VE problem only, since the VP flow rule does not depend on the fast time variable explicitly.

The structure of the two problems shows that in practice, for each \( t_M \), one solves the micro-time problem (3.86) to (3.91). Next the macro-chronological problem (3.78) to (3.85) completed by the following initial conditions can be solved:

\[
\tilde{u}_0M(\tilde{x}, t_M = 0) + \tilde{u}_0(\tilde{x}, t_M = 0, \tau = 0) = \tilde{u}_I(\tilde{x}), \quad \text{in } \Omega \quad (3.92)
\]

\[
\sigma_0M(\tilde{x}, t_M = 0) + \tilde{\sigma}_0(\tilde{x}, t_M = 0, \tau = 0) = \sigma_I(\tilde{x}), \quad \text{in } \Omega \quad (3.93)
\]

The resolution of the micro-chronological problem, before the macro-chronological one is necessary, given that the latter depends on the zero-order fluctuation of the stress \( \tilde{\sigma}_0 \), (equations (3.84)-(3.85)).

In what follows, the following concepts are used:

- Zero-order homogenized solution \( \phi_0(t_M, \tau) \) (corresponds to the zero-order term in the asymptotic expansion of \( \phi(t) \));

- Zero-order macro-chronological solution \( \phi_{0,M}(t_M) \): average with respect to \( \tau \) of \( \phi_0(t_M, \tau) \), \( \phi_{0,M}(t_M) = \langle \phi_0(t_M, \tau) \rangle \);

- Zero-order micro-chronological solution (fluctuation): \( \tilde{\phi}_0(t_M, \tau) \);

- We have:

\[
\phi_0(t_M, \tau) = \phi_{0,M}(t_M) + \tilde{\phi}_0(t_M, \tau).
\]

### 3.4 Computational algorithm

A numerical time discretization algorithm is proposed in this work, for the time homogenization of a coupled VE-VP material subjected to large numbers of cycles. The algorithm is a combination of techniques used separately for VE and EVP models (e.g., Simo and Hughes, 1998) and for coupled VE-VP (Miled et al., 2011), and the method of time homogenization as explained before.
3.4. Computational algorithm

3.4.1 Time discrete form of the stress update

First, the constitutive equations (3.31)-(3.33) are written in a time discrete form which is more suitable for numerical treatment. Using a mid-point time integration rule as in ([Miled et al., 2011]), the discrete form of (3.32) over a time interval \([t_n, t_{n+1}]\) is:

\[
\begin{align*}
    s_i(t_{n+1}) &= \exp\left(-\frac{\Delta t}{g_i}\right) s_i(t_n) + 2\hat{G}_i(\Delta t)\Delta \xi^{ve}, \\
    \sigma_{H_j}(t_{n+1}) &= \exp\left(-\frac{\Delta t}{k_j}\right) \sigma_{H_j}(t_n) + 3\hat{K}_j(\Delta t)\Delta \varepsilon^{ve}_H,
\end{align*}
\]  

(3.94)

where increments are designated by, \(\Delta t = t_{n+1} - t_n\), \(\Delta \xi^{ve} = \xi^{ve}(t_{n+1}) - \xi^{ve}(t_n)\) and \(\Delta \varepsilon^{ve}_H = \varepsilon^{ve}_H(t_{n+1}) - \varepsilon^{ve}_H(t_n)\). The expressions of \(\hat{G}_i(\Delta t)\) and \(\hat{K}_j(\Delta t)\) are given by:

\[
\begin{align*}
    \hat{G}_i(\Delta t) &= G_i \exp\left(-\frac{\Delta t}{2g_i}\right), \\
    \hat{K}_j(\Delta t) &= K_j \exp\left(-\frac{\Delta t}{k_j}\right).
\end{align*}
\]  

(3.95)

For convenience, the following incremental relaxation moduli are defined:

\[
\hat{G}(\Delta t) = G_\infty + \sum_{i=1}^{I} \hat{G}_i(\Delta t); \quad \hat{K}(\Delta t) = K_\infty + \sum_{j=1}^{J} \hat{K}_j(\Delta t); \quad \hat{E} = 2\hat{G} I^{dev} + 3\hat{K} I^{vol}.
\]  

(3.96)

The time discrete form of equation (3.33) is then:

\[
\sigma(t_{n+1}) = C_\infty : \varepsilon^{ve}(t_n) + \hat{E}(\Delta t) : \Delta \varepsilon^{ve} + \sum_{i=1}^{I} \exp\left(-\frac{\Delta t}{g_i}\right) s_i(t_n) \\
+ \sum_{j=1}^{J} \exp\left(-\frac{\Delta t}{k_j}\right) \sigma_{H_j}(t_n) I.
\]  

(3.97)

Variables \(s_i\), \(\sigma_{H_j}\), \(\xi^{ve}\) and \(\varepsilon^{ve}_H\) depend on both time scales \(t_M\) and \(\tau\). Their asymptotic expansions are injected in equation (3.94). Calculating all the terms at the same micro-time \(\tau\) we have, \(\Delta t = \Delta t_M\). Consequently, the zeroth order stress update is now given by the following:
discrete form of equation (3.72).

\[
\sigma_0(t_{M_{n+1}}, \tau) = C_\infty : \varepsilon_0^v(t_{M_n}, \tau) + \sum_{i=1}^I \exp \left( \frac{-\Delta t_M}{g_i} \right) s_{i0}(t_{M_n}, \tau) + \hat{E}(\Delta t_M) : \Delta \varepsilon_0 + \sum_{j=1}^J \exp \left( \frac{-\Delta t_M}{k_j} \right) \sigma_{Hj0}(t_{M_n}, \tau) \mathbf{1},
\]

(3.98)

with the zero-order terms of these viscous stresses over \([t_{M_n}, t_{M_{n+1}}]\):

\[
\begin{align*}
    s_{i0}(t_{M_{n+1}}, \tau) &= \exp \left( \frac{-\Delta t_M}{g_i} \right) s_{i0}(t_{M_n}, \tau) + 2 \hat{G}_i(\Delta t_M) \Delta \varepsilon_0^v, \\
    \sigma_{Hj0}(t_{M_{n+1}}, \tau) &= \exp \left( \frac{-\Delta t_M}{k_j} \right) \sigma_{Hj0}(t_{M_n}, \tau) + 3 \hat{K}_j(\Delta t_M) \Delta \varepsilon_{H0}.
\end{align*}
\]

(3.99)

The macro zeroth-order VP strain and accumulated plastic strain increments are computed according to the backward Euler scheme applied to equations (3.84) and (3.85):

\[
\Delta \varepsilon_0^p \approx \frac{d\varepsilon_0^p}{dt_M} |_{t_{M_{n+1}}} \Delta t_M, \quad \text{and} \quad \Delta p_0 \approx \frac{dp_0}{dt_M} |_{t_{M_{n+1}}} \Delta t_M.
\]

(3.100)

Following the decomposition into mean and fluctuation terms defined in equation (3.45), and taking the time average as defined by equation (3.44) of equations (3.98), (3.99) and (3.100), the macro and micro-chronological decomposition of discretized problem are obtained and presented in the following subsections.

### 3.4.2 Micro-chronological problem

The micro-chronological problem of equations (3.86) to (3.91) is purely VE. For each macro-time interval \([t_{M_n}, t_{M_{n+1}}]\), the micro-time problem is solved only once, and the stresses are updated explicitly (no iterations required) according to the following discrete form of equation (3.91):

\[
\tilde{\sigma}_0(t_{M_{n+1}}, \tau) = C_\infty : \tilde{\varepsilon}_0(t_{M_n}, \tau) + \hat{E}(\Delta t_M) : \Delta \tilde{\varepsilon}_0 + \sum_{i=1}^I \exp \left( \frac{-\Delta t_M}{g_i} \right) \tilde{s}_{i0}(t_{M_n}, \tau) + \sum_{j=1}^J \exp \left( \frac{-\Delta t_M}{k_j} \right) \tilde{\sigma}_{Hj0}(t_{M_n}, \tau) \mathbf{1},
\]

(3.101)
where the viscous stresses are updated according to:

\[
\tilde{s}_{i0}(t_{n+1}, \tau) = \exp\left(\frac{-\Delta t_M}{g_i}\right) \tilde{s}_{i0}(t_n, \tau) + 2\tilde{G}_i(\Delta t_M) \Delta \tilde{\xi}_0, \tag{3.102}
\]

\[
\tilde{\sigma}_{Hj0}(t_{n+1}, \tau) = \exp\left(\frac{-\Delta t_M}{k_j}\right) \tilde{\sigma}_{Hj0}(t_n, \tau) + 3\tilde{K}_j(\Delta t_M) \Delta \tilde{\varepsilon}_{H} \tag{3.103}
\]

3.4.3 Macro-chronological problem

The macro-chronological problem of equations (3.78) to (3.85) has to be solved for each time interval \([t_n, t_{n+1})\]. The discrete form of equations (3.82) to (3.85) reads:

\[
\Delta \varepsilon_{0M} = \Delta \varepsilon_{0M}^{ve} + \Delta \varepsilon_{0M}^{vp}, \tag{3.104}
\]

\[
\sigma_{0M}(t_{n+1}) = C_\infty : \varepsilon_{0M}^{ve}(t_n) + \tilde{E}(\Delta t_M) : \Delta \varepsilon_{0M}^{vp} + \sum_{i=1}^{l} \exp\left(-\frac{-\Delta t_M}{g_i}\right) s_{i0M}(t_n) + \sum_{j=1}^{J} \exp\left(-\frac{-\Delta t_M}{k_j}\right) \sigma_{Hj0M}(t_n) 1, \tag{3.105}
\]

\[
s_{i0M}(t_{n+1}) = \exp\left(-\frac{-\Delta t_M}{g_i}\right) s_{i0M}(t_n) + 2\tilde{G}_i(\Delta t_M) \Delta \varepsilon_{0M}^{ve}, \tag{3.106}
\]

\[
\sigma_{Hj0M}(t_{n+1}) = \exp\left(-\frac{-\Delta t_M}{k_j}\right) \sigma_{Hj0M}(t_n) + 3\tilde{K}_j(\Delta t_M) \Delta \varepsilon_{H0M}^{ve}, \tag{3.107}
\]

\[
\Delta \varepsilon_{0M}^{vp} = B_M((\sigma_{0M} + \tilde{\sigma}_0)_{|t_{n+1}}) \Delta t_M, \tag{3.108}
\]

\[
\Delta p_0 = C_M((\sigma_{0M} + \tilde{\sigma}_0)_{|t_{n+1}}) \Delta t_M. \tag{3.109}
\]

The macro-chronological problem (3.104)-(3.109) is nonlinear and solved iteratively at each macro time step using the Newton Raphson method.

A return-mapping algorithm is developed in order to solve the macro-chronological problem. Knowing all history variables at \(t_n\) and the
total zero-order macro-strain increment \( \Delta \varepsilon_{0M} \), the values of the zeroth-order variables at \( t_{M_{n+1}} \) are found. The first step is called VE predictor; we assume that the increment is entirely VE (\( \Delta \varepsilon_{0M} = \Delta \varepsilon_{0M}^{ve} \) and \( \Delta \varepsilon_{0M}^{vp} = 0 \)). The zeroth-order VE predictor stress \( \sigma_{0}^{pred} \) is defined from equation (3.105) as:

\[
\sigma_{0}^{pred}(t_{M_{n+1}}, \tau) = \tilde{\sigma}_{0}(t_{M_{n+1}}, \tau) + C_{\infty} : \varepsilon_{0M}^{ve}(t_{M_{n}}) + \tilde{E}(\Delta t_{M}) : \Delta \varepsilon_{0M} + \sum_{i=1}^{J} \exp \left( \frac{-\Delta t_{M}}{g_{i}} \right) s_{0M}(t_{M_{n}}) + \sum_{j=1}^{J} \exp \left( \frac{-\Delta t_{M}}{k_{j}} \right) \sigma_{H_{0M}}(t_{M_{n}}) \mathbf{1},
\]

(3.110)

where the stress fluctuation \( \tilde{\sigma}_{0}(t_{M_{n+1}}, \tau) \) was computed in the micro-time problem, equation (3.101).

If this trial stress satisfies the yield condition:

\[
f_{0}^{pred}(t_{M_{n+1}}, \tau) = \sigma_{0}^{eq}(t_{M_{n+1}}, \tau) - \sigma_{y} - R(p_{0}(t_{M_{n}})) \leq 0,
\]

(3.111)

then the basic assumption is valid and the zeroth-order VE predictor is the solution at \( t_{M_{n+1}} \):

\[
\sigma_{0}(t_{M_{n+1}}, \tau) = \sigma_{0}^{pred}(t_{M_{n+1}}, \tau),
\]

(3.112)

\[
\varepsilon_{0}^{vp}(t_{M_{n+1}}) = \varepsilon_{0}^{vp}(t_{M_{n}}), \quad p_{0}(t_{M_{n+1}}) = p_{0}(t_{M_{n}}).
\]

(3.113)

If \( f_{0}^{pred}(t_{M_{n+1}}, \tau) > 0 \) then the assumption is incorrect and a VP corrector is needed. In this latter case, the solution at \( t_{M_{n+1}} \) has to satisfy the following equation:

\[
\sigma_{0}(t_{M_{n+1}}, \tau) = \sigma_{0}^{pred}(t_{M_{n+1}}, \tau) - \tilde{E}(\Delta t_{M}) : \Delta \varepsilon_{0M}^{vp},
\]

(3.114)

together with equations (3.108) and (3.109).

3.4.4 Application to \( J_{2} \) viscoplasticity

If \( f(\sigma_{0}^{pred}(t_{M_{n+1}}, \tau), p_{0}(t_{M_{n}})) > 0 \), then we must enter the VP corrector phase. For the rate-dependent \( J_{2} \) VP model defined in equations (3.22) and (3.23), equations (3.108) and (3.109) become:

\[
\Delta \varepsilon_{0}^{vp} \approx \Delta p_{0} < N_{0}(t_{M_{n+1}}, \tau),
\]

(3.115)
\[ \Delta p_0 \approx g_v (\sigma_{eq}(t_{M_{n+1}}, \tau), p_0(t_{M_{n+1}})) > \Delta t_M. \] (3.116)

Given that \( N_0 \) is deviatoric and \( \hat{E}(\Delta t_M) \) is isotropic, equation (3.114) can be rewritten as follows:

\[ \sigma_0(t_{M_{n+1}}, \tau) = \sigma_0^{pred}(t_{M_{n+1}}, \tau) - 2\hat{G}(\Delta t_M) < N_0(t_{M_{n+1}}, \tau) > \Delta p_0. \] (3.117)

Note that the trace of \( \sigma_0(t_{M_{n+1}}, \tau) \) is known:

\[ \text{tr} \sigma_0(t_{M_{n+1}}, \tau) = \text{tr} \sigma_0^{pred}(t_{M_{n+1}}, \tau). \] (3.118)

Only the deviatoric part \( s_0(t_{M_{n+1}}, \tau) \) needs to be computed:

\[ s_0(t_{M_{n+1}}, \tau) = s_0^{pred}(t_{M_{n+1}}, \tau) - 2\hat{G}(\Delta t_M) < N_0(t_{M_{n+1}}, \tau) > \Delta p_0. \] (3.119)

Equations (3.116) and (3.119) must be solved for the unknowns: \( p_0(t_{M_{n+1}}) \) and \( s_0(t_{M_{n+1}}, \tau) \). The main difficulty arises from the numerical evaluation of the averages \( < \bullet > \). In the following subsections we discuss two numerical integration techniques. The first one is a multi-point integration and it is applied for simplicity to a uniaxial stress state. The second method is one point integration, applied to multiaxial stresses.

### 3.4.4.1 Multi-point integration

The average of a variable \( \Phi \) can be approximated using a quadrature rule as follows:

\[ < \Phi(t_M, \tau) > = \int_0^1 \Phi(t_M, \tau) d\tau \approx \sum_{j=1}^N \omega_j \Phi(t_M, \tau_j), \] (3.120)

where \( \tau_j \) and \( \omega_j \) are the chosen integration points and their weights, respectively, and \( N \) is the number of integration points.

Considering a \( J_2 \) VE-VP model written in its uniaxial expression, the problem is reduced to solving the following nonlinear equations:

\[
\left\{ \begin{array}{l}
\Delta p_0 = < g_v (|\sigma_0(t_{M_{n+1}}, \tau)|, p_0(t_{M_{n+1}})) > \Delta t_M, \\
\sigma_0(t_{M_{n+1}}, \tau) = \sigma_0^{pred}(t_{M_{n+1}}, \tau) - \hat{E}(\Delta t_M) \Delta p_0 < \text{sign}(\sigma_0(t_{M_{n+1}}, \tau)) > .
\end{array} \right. \] (3.121)

Here, "sign" is a sign function, defined as \( \text{sign} \Phi = \frac{\Phi}{|\Phi|} \) and \( \hat{E}(\Delta t_M) \) is the incremental relaxation modulus, expressed as a function of the
long-term elastic Young modulus $E_\infty$, the stiffness constant $E_i$ and the relaxation time $\lambda_i$.

$$E(\Delta t_M) = E_\infty + \sum_i E_i \exp\left(\frac{-\Delta t_M}{2\lambda_i}\right).$$  \hspace{1cm} (3.122)

Evaluating $< \bullet >$ according to equation (3.120), problem (3.121) is transformed into the resolution of $(N + 1)$ nonlinear equations with $(N + 1)$ unknowns $p_0(t_{M+1})$ and $\sigma_{0j}(t_{M+1}, \tau_j); j = 1 \cdots N$:

$$\begin{align*}
k_{p_0} &\equiv \Delta p_0 - \sum_{j'=1}^N \omega_j' \alpha_j' g_{v'} \left( \left| \sigma_{0j'}(t_{M+1}, \tau_{j'}) \right|, p_0(t_{M+1}) \right) \Delta t_M = 0, \\
k_{\sigma_{0j}} &\equiv \sigma_{0j}(t_{M+1}, \tau_j) - \sigma_{0j}^{\text{pred}}(t_{M+1}, \tau_j) \\
&+ \hat{E}(\Delta t_M) \Delta p_0 \sum_{j'=1}^N \omega_j' \alpha_j' \text{sign}(\sigma_{0j'}(t_{M+1}, \tau_{j'})) = 0, \\
\end{align*}$$

(3.123)

where $\alpha_j'$ is a VP factor: $\alpha_j' = 1$ if $f_{0j}^{\text{pred}} \left( \left| \sigma_{0j'}^{\text{pred}}(t_{M+1}, \tau_{j'}) \right|, p_0(t_{M+1}) \right) > 0$ and $\alpha_j' = 0$ otherwise.

These equations are nonlinear and can be solved iteratively using Newton-Raphson method. At first we suppose that the $\text{sign}(\sigma_{0j'}(t_{M+1}, \tau_{j'}))$ is known and equals $\text{sign}(\sigma_{0j}^{\text{pred}}(t_{M+1}, \tau_{j'}))$. After convergence we check the assumption and conduct a second run if some signs have changed.

The iterative corrections $C_{p_0}$ and $C_{j}$ on $p_0$ and $\sigma_{0j}$, respectively, are given by the expressions below:

$$C_{p_0} = \frac{-k_{p_0} - \Delta t_M \sum_{j'=1}^N \omega_j' \alpha_j' k_{\sigma_{0j}} g_{\sigma_{0j}}}{\hat{E}(\Delta t_M) \Delta t_M \sum_{j'=1}^N \omega_j' \alpha_j' \text{sign}(\sigma_{0j'}(t_{M+1}, \tau_{j'})) \sum_{j'=1}^N \omega_j' \alpha_j' g_{\sigma_{0j'}}} + 1 - \Delta t_M \sum_{j'=1}^N \omega_j' \alpha_j' g_{\sigma_{0j'}}$$

\hspace{1cm} (3.124)

$$C_{j} = -k_{\sigma_{0j}} - \hat{E}(\Delta t_M) \left( \sum_{j'=1}^N \omega_j' \alpha_j' \text{sign}(\sigma_{0j'}(t_{M+1}, \tau_{j'})) \right) C_{p_0}. \hspace{1cm} (3.125)$$

where we have used the notations: $g_{\sigma_{0j'}} \equiv \frac{\partial g_{\sigma_{0j'}}}{\partial \sigma_{0j'}} \mid (t_{M+1}, \tau_{j'})$ and $g_{p_{0j'}} \equiv \frac{\partial g_{p_{0j'}}}{\partial p_{0j'}} \mid (t_{M+1}, \tau_{j'})$. 

Modeling and algorithms for two-scale time homogenization of VE-VP solids
3.4. Computational algorithm

3.4.4.2 One point integration

In this subsection we study the case of one point integration. We assume that the averages can be approximated by the value of the function at the midpoint of the integration interval, which corresponds to $\tau = \frac{1}{2}$.

Consider a $J_2$ VE-VP model written in its multiaxial expression. Combining equations (3.119) and (3.27) and for $\tau = \frac{1}{2}$, it follows that:

$$
\frac{2}{3} N_0 \left( t_{M_{n+1}}, \frac{1}{2} \right) \sigma_{0eq} \left( t_{M_{n+1}}, \frac{1}{2} \right) = -2 \hat{G}(\Delta t_M) N_0 \left( t_{M_{n+1}}, \frac{1}{2} \right) \Delta p_0
$$

$$
\frac{2}{3} N_0^{\text{pred}} \left( t_{M_{n+1}}, \frac{1}{2} \right) \sigma_{0eq}^{\text{pred}} \left( t_{M_{n+1}}, \frac{1}{2} \right) .
$$

(3.126)

We arrive to a radial return algorithm defined by the equations below:

$$
\begin{cases}
N_0 \left( t_{M_{n+1}}, \frac{1}{2} \right) = N_0^{\text{pred}} \left( t_{M_{n+1}}, \frac{1}{2} \right), \\
\sigma_{0eq} \left( t_{M_{n+1}}, \frac{1}{2} \right) + 3 \hat{G}(\Delta t_M) \Delta p_0 = \sigma_{0eq}^{\text{pred}} \left( t_{M_{n+1}}, \frac{1}{2} \right).
\end{cases}
$$

(3.127)

The use of the discretized form (3.116) to write $\Delta p_0$, implies two scalar equations which must be solved for the two unknowns $p_0(t_{M_{n+1}})$ and $\sigma_{0eq}(t_{M_{n+1}}, \frac{1}{2})$.

$$
\begin{cases}
\Delta p_0 - g_v (\sigma_{0eq}, p_0) \Delta t_M = 0, \\
\sigma_{0eq} \left( t_{M_{n+1}}, \frac{1}{2} \right) + 3 \hat{G}(\Delta t_M) \Delta p_0 - \sigma_{0eq}^{\text{pred}} \left( t_{M_{n+1}}, \frac{1}{2} \right) = 0.
\end{cases}
$$

(3.128)

These two equations are nonlinear and can be solved iteratively using Newton-Raphson method. The solution at $t_{M_{n+1}}$ and $\tau = \frac{1}{2}$ can then be completely updated:

$$
\begin{cases}
N_0 \left( t_{M_{n+1}}, \frac{1}{2} \right) = N_0^{\text{pred}} \left( t_{M_{n+1}}, \frac{1}{2} \right), \\
s_0 \left( t_{M_{n+1}}, \frac{1}{2} \right) = \frac{2}{3} \sigma_{0eq} \left( t_{M_{n+1}}, \frac{1}{2} \right) N_0 \left( t_{M_{n+1}}, \frac{1}{2} \right), \\
\sigma_0 \left( t_{M_{n+1}}, \frac{1}{2} \right) = s_0 \left( t_{M_{n+1}}, \frac{1}{2} \right) + \frac{1}{3} \left( \text{tr} \sigma_0^{\text{pred}} \left( t_{M_{n+1}}, \frac{1}{2} \right) \right) 1, \\
\varepsilon_0^{vp} \left( t_{M_{n+1}} \right) = \varepsilon_0^{vp} \left( t_{M_{n}} \right) + N_0 \left( t_{M_{n+1}}, \frac{1}{2} \right) \Delta p_0.
\end{cases}
$$

(3.129)
3.4.5 Numerical implementation

In this subsection we give an overview of the algorithm. Consider a macro-time interval \([t_{Mn}, t_{Mn+1}]\), for which we assume that the macro- and micro-total strains \(\varepsilon_M(t_{Mn})\) and \(\tilde{\varepsilon}(t_{Mn}, \tau)\) and macro- and micro-strain increments \(\Delta \varepsilon_M\) and \(\Delta \tilde{\varepsilon}\) and all zeroth-order macro- and micro-history variables at \(t_{Mn}\) and \(\tau\) are known. The problem is to compute the zeroth-order stress \(\sigma_0(t_{Mn+1}, \tau)\). The algorithm is presented in figure 3.3 and described hereafter:

1. Initialization: \(\sigma_{0M} = \tilde{\sigma}_0 = 0; \ p_0 = 0; \ \varepsilon_{v0} = 0\).
2. Resolution of the VE micro-chronological problem at \(t_{Mn+1}\) and \(\tau = \tau_j\).
3. Call VE-VP constitutive box with \(\tilde{\sigma}_0(t_{Mn+1}, \tau = \tau_j), \ v_M(t_{Mn})\) and \(\Delta \varepsilon_M\) as an input.
4. After convergence, the macro-time is incremented.

3.5 Numerical simulations and their verification

The time-homogenization method was implemented and tested for several loading cases using one-point integration for the micro-chronological averages. A comparison with multi-point integration is also illustrated in one loading case. All the simulations are performed using a PC with an Intel Core i5 processor (2.4 GHz) and 4 GB of RAM.

3.5.1 Uniaxial loading using one point micro-chronological time integration

Consider a one-dimensional cylindrical bar of length \(L\) clamped at one end \((x = 0)\) and subjected at the other end \((x = L)\) to a displacement which is linear at first and then sinusoidal with period \(T^* = 0.1s\), and amplitude \(U = 0.05L\):

\[
\begin{cases}
    u_{b_T^*} (x = L, t^*) = \alpha L t^*, & \text{if } t^* \leq T^* \\
    u_{b_T^*} (x = L, t^*) = U \left(0.45 \sin \left(\frac{2\pi}{T^*} t^*\right) + 0.55\right), & \text{otherwise},
\end{cases}
\]

(3.130)

where \(\alpha = 0.275 \text{s}^{-1}\).

The body forces \(f\) are neglected and \(3.10^4\) cycles are applied. The identified VE-VP material parameters are listed in table 3.1.
3.5. Numerical simulations and their verification

Macro time
$t_M = 0, t_M, \ldots, t_M, t_M, \ldots, t_M, t_M$, $t_M + T^*$

Resolution of the micro-chronological problem at $t_M$ and $\tau = \tau_j$

Resolution of the macro-chronological problem at $t_M$ and $\tau = \tau_j$

$\triangleright\triangleright\triangleright\triangleright\triangleright\triangleright$

Converges?

End

Figure 3.3: Summary of the time-homogenization method for coupled VE-VP solids. Time stepping is performed with respect to macro time $t_M$
Table 3.1: Constitutive model parameters for polyamide (PA) at 40°C (Miled, 2011) identified from experimental measurements of Baquet, 2011

<table>
<thead>
<tr>
<th>Viscoelastic parameters</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial shear modulus</td>
<td>$G_0 = 1074$ MPa</td>
</tr>
<tr>
<td>Initial bulk modulus</td>
<td>$K_0 = 3222$ MPa</td>
</tr>
<tr>
<td>$G_i$ (MPa)</td>
<td>$g_i (s)$</td>
</tr>
<tr>
<td>158</td>
<td>0.021</td>
</tr>
<tr>
<td>80</td>
<td>0.378</td>
</tr>
<tr>
<td>37</td>
<td>0.648</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Viscoplastic parameters</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Hardening function</td>
<td>$k = 79$ MPa</td>
</tr>
<tr>
<td>Viscoplastic function</td>
<td>$\zeta = 305$ MPa.s</td>
</tr>
<tr>
<td>Yield stress</td>
<td>$\sigma_y = 40$ MPa</td>
</tr>
</tbody>
</table>

The prescribed displacement can be rewritten using equation (3.35) relating $t^*$ to $t_M^*$, $\tau$ and $T^*$. The loading ratio $R$, which corresponds to the ratio between the minimum and the maximum of the loading: $R = \frac{\mu_{\min}}{\mu_{\max}}$, is equal to 0.1.

Comparisons between the reference non-homogenized (full-time) calculation and the two-scale one are given in figures (3.4)-(3.7). The reference numerical solution is obtained using a very fine time step $\Delta t^* = \frac{T^*}{20}$, whereas the time-homogenized computation is started by a full calculation until time $t^* = \frac{T^*}{3}$ and then the calculations are continued using $\Delta t_M^* = T^*$. In figure 3.4 are represented the stress-strain results for some selected cycles. The solid lines show the reference results, while the black crosses depict the time homogenization predictions. It is noted that due to the repeated action of the rather small applied strain, the stress level decreases significantly from a maximum of 107 MPa in the first cycle to about 80 MPa in the following ones. Furthermore the stress-strain hysteresis loop “shrinks” rapidly with increasing number of cycles. It is seen in figures (3.5) and (3.6) that for the first cycles the crosses do not superpose with the peaks of stress, however, for the

---

With respect to the equations of the model, the values of $g_i$ and $k_j$ should be divided by $T_F^*$ as to become dimensionless.
3.5. Numerical simulations and their verification

Figure 3.4: Hysteresis loops: First hundred cycles, loading period: $T^* = 0.1 \text{s}$, loading ratio: $R = 0.1$.

Figure 3.5: Hysteresis loops: Last twenty cycles, loading period: $T^* = 0.1 \text{s}$, loading ratio: $R = 0.1$. 
last cycles they nearly perfectly coincide with the stress peaks. There are two reasons for the discrepancy between time homogenized and reference solutions in the first cycles oscillatory stress field for the first twenty cycles (figure 3.6 (a)). First, only the zero-order in the asymptotic expansion is considered. Second, for the first twenty cycles the load period is not sufficiently small compared to the observation time which is equal in this case to $20 T^*$. Relative differences between time homogenized and full-scale results are studied to check the accuracy of time homogenization approach. A relative difference is defined as:

$$\text{Relative difference} = \frac{\text{Reference result} - \text{Time-homogenized result}}{\text{Reference result}}.$$ 

Figure 3.7 presents the evolution of the error on the accumulated plastic strain as a function of the number of cycles. The relative difference between the zeroth-order homogenized solution $p_0$ and the reference one $p$ decreases in absolute value with the number of cycles; in the beginning it is about 40%, but at the end of the simulation it is about 1.6%. For the reference calculation which corresponds to the full-scale computation of 30,000 cycles, the CPU time is 153.4 s, while with the time homogenization method the CPU time is 5.3 s. The relative gain in computation time is then about 97%, in this simple example, which gives a hint about the potential of the time homogenization method.

Time homogenization theory allows to reconstruct all the solution. Figure 3.8 shows the zero-order stress $\sigma_0$ for different points of the load cycle. The red squares represent the values of the reference solution at the same time where homogenized solution is calculated. In this figure only solutions corresponding to three points of the cycle ($\frac{T^*}{4}$, $\frac{T^*}{2}$, $\frac{3T^*}{4}$), are shown. The other solutions may also be found using the macro- and micro-chronological decomposition: $\sigma_0(t_M, \tau) = \sigma_0(t_M) + \tilde{\sigma}_0(t_M, \tau)$.

3.5.1.1 Influence of the load period $T^*$ and a comparison with some characteristic times of the material

The small scaling parameter $T, (0 < T << 1)$, plays an important role in the accuracy of the time homogenization approach and it should be taken into consideration. In this work $T$ was defined as the ratio between the load period $T^*$ and the observation time $T_F^*$. In this section we study the influence of this parameter, which can depend on the material properties, on the accuracy of the time homogenization method.
3.5. Numerical simulations and their verification

Figure 3.6: Evolution of the stress: homogenized solution in comparison with reference results, loading period: $T^* = 0.1 \, s$, loading ratio: $R = 0.1$. (a) The first twenty cycles. (b) The last twenty cycles.

Figure 3.7: Evolution of the error on the accumulated plastic strain. Loading period: $T^* = 0.1 \, s$, loading ratio: $R = 0.1$. 
Figure 3.8: Evolution of the stress: homogenized solution in comparison with reference results, loading period: $T^* = 0.1 \text{s}$, loading ratio: $R = 0.1$. (b) The last five cycles.

Figure 3.9: Evolution of the error on the accumulated plastic strain for the last cycle as a function of the load period $T^*$. Loading ratio $R = 0.1$. 

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Figure 3.10: Evolution of the error on the accumulated plastic strain for the last cycle as a function of the VP modulus $\zeta$. Loading period: $T^* = 0.1$ s, loading ratio: $R = 0.1$.

Figure 3.9 presents a comparative study of the full-time and temporal homogenization methods in terms of the load period $T^*$. It illustrates the evolution of the relative error (in absolute value) on the accumulated plastic strain for the last cycle, for different loading period $T^* \in [0.01, 5]$ s. The observation time $T_F^*$ is fixed to 3000 s. It is noticed that the error does not exceed 4% and it decreases when the load period $T^*$ decreases to reach the value 1.2% when $T^* = 0.01$ s. The accuracy of temporal homogenization improves for high frequencies.

A VP characteristic time $t_c$ of the material is defined as the ratio between the viscosity $\zeta$ and the initial stiffness constant $E_0 = G_0(1 + \nu)$. Assuming that Poisson’s ratio $\nu$ is constant and equal to 0.3, the VP characteristic time is then calculated: $t_c = \frac{\zeta}{E_0} = 0.1$ s. The relative error is less than 1.5% when $T^* < t_c$ but it increases otherwise (figures 3.9).

When the load period $T^*$ is between the smallest and the largest relaxation times $0.007$ (s) $< T^* < 0.648$ (s), the relative error is in $[1\%, 2.5\%]$. If $T^* > 0.648$ (s), the relative error increases to reach 3.6% for $T^* = 5$s.

### 3.5.1.2 Influence of the viscoplastic modulus

Several simulations for the viscoplastic modulus $\zeta \in [0.5, 400]$ MPa.s are done. The observation time $T_F^*$ is fixed to 3000 s. The effect of $\zeta$ on the evolution of the error on the accumulated plastic strain is illustrated in figure 3.10 for the last cycle, the other parameters in table 3.1 being kept fixed. It is noticed that the error does not exceed 2%. It is minimum for $\zeta \simeq 50$ MPa.s. The homogenization prediction overestimates
the solutions for $\zeta \geq 50$ MPa.s, and underestimates it otherwise.
When $\zeta \in [0.5, 300]$ MPa.s, the VP characteristic time $t_c = \frac{\zeta}{E_0} \in [10^{-4}, 0.1]$ s, the relative error (in absolute value), in this case, does not exceed 1.5% and the ratio $T^*/t_c \in [1, 10^3]$.

### 3.5.1.3 Influence of the loading ratio $R$

A comparative study of the full-time and temporal homogenization methods in terms of the ratio $R$ is done. Figure 3.11 presents the error on the accumulated plastic strain between the two methods, for different ratios $R \in [-0.5, 0.8]$. The observation time $T_F^*$ is fixed to 3000 s. It is noticed that the error does not exceed 2.5% for $R \in [-0.4, 0.8]$, but it increases dramatically for $R < -0.4$. In subsection 3.5.3, we will see that this problem can be resolved if we increase the number of integration points, in the micro-time averaging.

### 3.5.1.4 Influence of the macro-time step $\Delta t_M^*$

In this part we study the influence of the macro-time step $\Delta t_M^*$ on the accuracy of the time-homogenization calculation. Several simulations are done for the material of table 3.1 for $\Delta t_M^* \in [T^*, 50T^*]$, $T^* = 0.1$ s being the load period. The observation time $T_F^*$ is 3000 s, the frequency $f = 10$ Hz and the load ratio $R = 0.1$. For $\Delta t_M^* \in [T^*, 20T^*]$ the error on the accumulated plastic strain is almost 1.7%, and for $\Delta t_M^* > 20T^*$, it increases dramatically to reach the value of 20% when $\Delta t_M^* = 50T^*$ (figure 3.12).

This can be explained by the fact that the system response is relatively smooth in macro-time when $t_M^* \in [T^*, 20T^*]$, thus the macro-time step $\Delta t_M^*$ can be increased so that the computational cost is reduced. Whereas the system response is rapidly changing in macro-time when $t_M^* > 20T^*$, and the macro-chronological time step $\Delta t_M^*$ must be relatively small.

In figure 3.13, the computational efficiency, defined as the ratio between the CPU time of the full-scale method and that of the time-homogenized calculation, is depicted as a function of the ratio $\frac{\Delta t_M^*}{T^*}$. It can be observed that when $\Delta t_M^*$ increases, an enormous gain in computational efficiency (by a factor which can reach 400) occurs with almost no loss of accuracy for $\Delta t_M^* < 20T^*$. 
3.5. Numerical simulations and their verification

Figure 3.11: Evolution of the error on the accumulated plastic strain for the last cycle as a function of loading ratio $R$. Loading period $T^* = 0.1$ s.

Figure 3.12: Evolution of the error on the accumulated plastic strain for the last cycle as a function of the ratio $\frac{\Delta t^*}{T^*}$. Loading period: $T^* = 0.1$ s, loading ratio: $R = 0.1$.

Figure 3.13: Evolution of the computational efficiency as a function of the ratio $\frac{\Delta t^*}{T^*}$. Loading period: $T^* = 0.1$ s, loading ratio: $R = 0.1$. 
3.5.2 Multiaxial loading using one point micro-chronological time integration

A simulation with torsion and tension/compression is performed in order to evaluate the efficiency of the time-homogenization method for a multiaxial and non-proportional loading history. Consider the same bar of section 3.5.1 which is still clamped at one end ($x = 0$) but submitted at the other end ($x = L$) to an axial load and a torque. This entails applying axial $\varepsilon_{11}$ and shear $\varepsilon_{12} = \varepsilon_{21}$ and $\varepsilon_{13} = \varepsilon_{31}$ strains. They evolve as a function of physical time $t$ as follows:

\[
\begin{align*}
\varepsilon_{11} &= \alpha_0 t^*, \quad \text{if } t^* < T^* \\
\varepsilon_{11} &= 0.05 \left( 0.45 \cos \left( \frac{2\pi}{T^*} t^* \right) + 0.55 \right), \quad \text{otherwise.} 
\end{align*}
\] (3.131)

\[
\begin{align*}
\varepsilon_{12} = \varepsilon_{13} &= \alpha_1 t^*, \quad \text{if } t^* < T^* \\
\varepsilon_{12} = \varepsilon_{13} &= 0.01 \left( 0.45 \sin \left( \frac{2\pi}{T^*} t^* \right) + 0.55 \right), \quad \text{otherwise.} 
\end{align*}
\] (3.132)

where $T^* = 0.05 \text{s}$ is the axial strain period and $T^* = 2T$ is the shear strain period, $\alpha_0 = 1 \text{s}^{-1}$ and $\alpha_1 = 1.1 \text{s}^{-1}$.

The representation of the loading shape in $(\varepsilon_{11}, \varepsilon_{12})$ plane are given in figure 3.14.

Since $T^* < T^*$ the asymptotic expansions will be done according to $T^*$, then the loadings are rewritten as follows:

\[
\begin{align*}
\varepsilon_{11} &= 0.05 \left( 0.45 \cos \left( \frac{2\pi}{T^*} (t^* M + T^* \tau) \right) + 0.55 \right), \\
\varepsilon_{12} = \varepsilon_{13} &= 0.01 \left( 0.45 \sin \left( \frac{\pi}{T^*} (t^* M + T^* \tau) \right) + 0.55 \right),
\end{align*}
\] (3.133)

The observation time is $T_F^* = 3000 \text{s}$ which corresponds to 60,000 cycles. The material obeys the VE-VP model in its multiaxial form. The identified material parameters are listed in table 3.1.

Comparisons between the reference calculation and the homogenized one are given in figures (3.15)-(3.17). The full-scale solution is obtained using a very fine time step $\Delta t^* = T^*_{20}$, whereas the time-homogenized computation was started by a full computation until $t^* = 0.5 T^*$ and the remaining calculations are done with a macro-time step of $\Delta t^*_{M} = T^*$. 

Figure 3.15 gives a comparison between the evolution of the zeroth-order homogenized axial stress $\sigma_{011}$ and the reference solution $\sigma_{11}$. Figure 3.15 (a) gives the oscillatory axial stress at the beginning of the loading during $[0, 0.5s]$ which corresponds to the first ten cycles. As in section 3.5.1 for the first cycles the homogenized solution does not coincide with the reference solution in figure 3.15 (a). However, in figure 3.15 (b), for the last ten cycles the homogenized solution is almost superposed with the reference one. Figure 3.16 shows the evolution of the the zeroth-order homogenized shear stress $\sigma_{012}$ and the reference solution $\sigma_{12}$. The red squares represent the values of the reference solution at the same time where homogenized solution is calculated. Figure 3.16 (a) shows that the black crosses don’t coincide with the red squares, however, for the last five cycles shown in figure 3.16 (a), the black crosses and the red squares are nearly perfectly superposed.

Figure 3.17 presents the evolution of the relative difference on the accumulated plastic strain as a function of the number of cycles. The error between the zeroth-order homogenized solution $p_0$ and the reference one $p$ decreases when the number of cycles increases: in the beginning of loading it is about 47%, nevertheless at the end of the simulation it is about 1.4%. For the reference calculation the CPU time is 136.17 s, while with the time homogenization method the CPU time is 5.92 s. The relative gain in computation time is about 95%.

### 3.5.3 Comparison between one-point and multi-point micro-chronological integration methods

Given that the one-point micro-chronological integration does not give a good approximation when the loading ratio $R$ tends towards $-1$ (figure 3.11), the multi-point algorithm of subsection 3.4.4.1 has been tested and implemented in the case of $R = -1$.

Consider the same problem as in section 3.5.1 but the bar is now submitted to a sinusoidal displacement with a zero mean:

$$u_{b_{T^*}}(x = L, t^*) = U \sin \left(\frac{2\pi}{T^*} t^*\right),$$

with a period $T^* = 0.1$ s, and an amplitude $U = 0.05 L$.

Comparisons between the reference calculation (solid line) and the homogenized one using two integration methods are presented in figure 3.18. For the reference calculation a very fine time step is used.
\[ \varepsilon_{11} = 0.05 \left( 0.45 \cos \left( \frac{2\pi}{T} t^* \right) + 0.55 \right) \]
\[ \varepsilon_{12} = 0.01 \left( 0.45 \sin \left( \frac{2\pi}{T'} t^* \right) + 0.55 \right) \]

Figure 3.14: Description of the loading path shape in the \((\varepsilon_{11}, \varepsilon_{12})\) plane.

Figure 3.15: Multiaxial load. Evolution of the axial stress: homogenized prediction in comparison with reference solution. (a) The first ten cycles. (b) The last ten cycles.
3.5. Numerical simulations and their verification

Figure 3.16: Multiaxial load. Evolution of the shear stress: homogenized solution in comparison with the reference one. (a) The first five cycles. (b) The last five cycles.

Figure 3.17: Multiaxial load. Evolution of the error on the accumulated plastic strain. For the axial and shear loadings, the loading ratio is: $R = 0.1$. 
Figure 3.18: Evolution of the stress: homogenized solution in comparison with the reference one for the last twenty cycles. Loading period: $T^* = 0.1 \text{s}$, loading ratio: $R = -1$. 
3.5. Numerical simulations and their verification

\[ \Delta t^* = \frac{T^*}{20}, \] whereas for the homogenized one \[ \Delta t^*_H = T^*. \] For multi-point integration, the trapezoidal rule is used with \( N = 4. \)

Figure 3.18 presents the evolution of the oscillatory stress for the last twenty cycles. Only peak stresses are presented for the homogenized solutions. The black crosses show the results obtained by one-point integration, whereas the blue dots show the results obtained by multi-point integration \((N = 4)\). It is noticed that the homogenized solution using multi-point integration gives a better approximation than the one-point integration. Using four-point integration, the relative error on the stress at the end of the simulation is about 3\%. Whereas, if one uses the one-point integration, the relative error is about 27\%.

3.5.4 Comparison with experimental results

In this section we compare the VE-VP model to experimental data of an ultra-high molecular weight polyethylene (UHMWPE). Experimental data, for uniaxial tensile tests, is gathered from the works of [Avanzini, 2011] and [Kurtz et al., 2002].

Material parameters describing the VE and VP responses should be identified based on separate experiments. The VE response is usually analyzed based on a dynamic mechanical analysis (DMA) in which a sample is subjected to a variable sinusoidal excitation. This technique enables us to determine several material parameters including the relaxation times and moduli involved in the Prony series. Once the VE parameters are identified, the VP parameters may be determined by loading the material at different strain rates above the yield stress. Given that, in the literature, DMA tests are not always available, an uniaxial tension-compression test under strain control applied to UHMWPE with a strain rate of 0.096 s\(^{-1}\) at different temperatures ([Avanzini, 2011]), was used to identify the VE parameters.

In uniaxial tension the viscoelastic Young’s modulus may be described by a Prony series:

\[
E(t^*) = E_\infty + \sum_i E_i \exp\left(\frac{-t^*}{\lambda_i}\right).
\] (3.136)

\(E_i, \lambda_i,\) and \(E_\infty\) are determined together with the VE parameters based on an inverse analysis. Poisson’s ratio can be determined by measuring the transverse strain during transverse uniaxial tension or compression tests. Because of the lack of sufficient experimental data, we consider in this work that Poisson’s ratio of UHMWPE is constant \((\nu = 0.46)\).
Table 3.2: Constitutive model parameters for UHMWPE identified from experimental data of [Avanzini, 2011] and [Kurtz et al., 2002]

<table>
<thead>
<tr>
<th>Viscoelastic parameters</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial shear modulus</td>
<td>$G_0 = 507 \text{ MPa}$</td>
</tr>
<tr>
<td>Initial bulk modulus</td>
<td>$K_0 = 6174 \text{ MPa}$</td>
</tr>
<tr>
<td>$G_i$ (MPa)</td>
<td>$g_i(\text{s})$</td>
</tr>
<tr>
<td>323.5</td>
<td>0.0292</td>
</tr>
<tr>
<td>70</td>
<td>0.4</td>
</tr>
<tr>
<td>57</td>
<td>265,136</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Viscoplastic parameters</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Hardening function</td>
<td>$k = 28 \text{ MPa}$</td>
</tr>
<tr>
<td>Viscoplastic function</td>
<td>$\zeta = 198 \text{ MPa.s}$</td>
</tr>
<tr>
<td>Yield stress</td>
<td>$\sigma_y = 10 \text{ MPa}$</td>
</tr>
</tbody>
</table>

(Kurtz, 2004). Similar assumptions were made by Miled et al., 2011 and Kästner et al., 2012 for High density polyethylene and Polypropylene, respectively. Then the shear and bulk relaxation moduli can be derived as follows:

$$G_i = \frac{E_i}{2(1 + \nu)}, \quad K_i = \frac{E_i}{3(1 - 2\nu)}, \quad \text{(no sum)},$$

(3.137)

The shear and bulk relaxation times are given by these expressions:

$$g_i = \frac{\lambda_i E_i}{G_i}, \quad k_i = \frac{\lambda_i E_i}{K_i}, \quad \text{(no sum)}.$$

(3.138)

Power-law expressions for isotropic hardening and Norton’s VP function are considered. The yield stress ($\sigma_y$) is considered rate-independent. To identify the VP parameters, monotonic uniaxial tension tests at different strain rates are used from the work of Kurtz et al., 2002. The fitted material parameters are listed in table 3.2.

Figure (3.19) illustrates the VE-VP response in uniaxial tension at strain rates of 0.02 s$^{-1}$, 0.05 s$^{-1}$ and 0.1 s$^{-1}$. The behavior of the material is inelastic and strain rate dependent above and below the yield stress. An acceptable correlation between numerical and experimental results is found. Figure (3.20) compares experiments and predictions in case of UHMWPE subjected to a pulsating test ($R = 0$) at a strain rate of
3.5. Numerical simulations and their verification

Figure 3.19: Prediction of uniaxial tension of UHMWPE at 37°C at different strain rates. Experimental data is obtained from [Kurtz et al., 2002]. Identified material parameters are listed in table 3.2.

0.096 s\(^{-1}\) and with a maximum elongation of 10%. The loading period is \(T^* = 2\) s. The cyclic loading has the following form:

\[
\varepsilon_{11}(t^*) = 0.1 \left( -0.5 \cos \left( \frac{2\pi t^*}{T^*} \right) + 0.5 \right) \tag{3.139}
\]

During the test the temperature is controlled and stabilized to 23°C. Two cycles are reported (\(N = 1\) and 2000). Figure 3.20 shows an acceptable correlation between numerical and experimental results. During the test a cyclic softening of the material is observed which can also be observed using numerical simulation. The figure confirms the ability of the constitutive model to describe and to reasonably predict the mechanical behavior of UHMWPE under isothermal conditions.

In reality, when a polymer material is subjected to cyclic loadings, additional phenomena such as self-heating and damage are involved, which lead to the degradation of the material’s properties. [Avanzini, 2011] studied the influence of self-heating on the cyclic behavior of UHMWPE. He found that the evolution of surface temperature, when the material is subjected to cyclic loading, increases the material cyclic softening. The peak stress reaches the value of 22 MPa when the temperature of the material was stabilized to 23°C (figure (3.20)), whereas
Figure 3.20: Prediction of uniaxial loading-unloading cycles performed on UHMWPE at 23°C at a strain rate of 0.096 s⁻¹. Experimental data is obtained from [Avanzini, 2011]. Identified material parameters are listed in table 3.2.
the self-heating is taking into account the peak stress value reaches 16 MPa. The constitutive model presented in this work does not take into account neither self-heating nor damage evolution under cyclic loadings.

Figure (3.21) presents the relative difference on the peak stresses between the homogenized and the reference (full time) calculations as a function of the number of cycles. The homogenized computation is performed using one point integration method. For the reference calculation a fine time step is used $\Delta t^* = 0.05s$, whereas for the homogenized one $\Delta t^*_M = 2s$. The observation time is fixed to $T^{*F}_F = 2 \cdot 10^5$ s which corresponds to $10^5$ cycles. The error between the zeroth-order homogenized solution and the reference one decreases when the number of cycles increases: for the first 500 cycles the error is about 4%, nevertheless at the end of the simulation it is about 0.12%. The CPU time for the full-time calculations is about 1300 s, whereas for the homogenized one is about 50 s, the relative gain in computation time is about 96%.
3.6 Finite element modeling of micro- and macro-chronological problems

In this section we give a suggestion for the implementation of the temporal homogenization approach to simulate the behavior of real structures (e.g., figure 3.22) subjected to cyclic loading ($\mathbf{u}_{b_T^*}$) with large numbers of cycles. The method proposed hereafter, is an approximate method. First the initial boundary value problem is solved using a FE software for the first cycles. Appropriate user supplied subroutines (e.g. UMAT in ABAQUS) should be provided to integrate the constitutive model of the problem. Then critical points, in the structure, are identified. We assume that the total deformation $\varepsilon_{b_T^*}(x, t^*)$, calculated at each critical point, will be repeated for the remaining cycles. Then for each critical point, we use the algorithm proposed in figure (3.23).

The loading $\varepsilon_{b_T^*}(x, t^*)$ is decomposed into macro- and micro-chronological parts. For each macro-time step and for each critical point:

- Knowing micro-chronological state variables at the beginning of the macro-time step, and the micro-chronological strain increment, the micro-chronological boundary problem can be solved at each (critical) integration point of the mesh. Information are printed into an external transfer file.

- Next, knowing macro-chronological state variables at the beginning of the macro-time step, the macro-chronological strain increment and information about micro-chronological state variables, the macro-chronological boundary problem can be solved. An algorithmic tangent operator is calculated (Appendix C).

The macro-chronological time step $\Delta t_M^*$ can be optimized so that the computational cost is reduced. It must be relatively small when the system response is rapidly changing in macro-time and can be increased when the response is relatively smooth. It can be evaluated by controlling the error of certain fields of interest (e.g. accumulated plastic strain) at each integration point.
3.6. Finite element modeling of micro- and macro-chronological problems

Figure 3.22: 3D bending beam ([Cavin, 2006]).

Figure 3.23: Program architecture of the two-scale temporal analysis of a real structure.
3.7 Conclusions

The first contribution in this chapter is the formulation of a coupled viscoelastic-viscoplastic (VE-VP) model based on thermodynamics of irreversible processes. Expressions of the Helmholtz free energy function and the dissipation potential are proposed. Both VE and VP contributions are accounted for.

A two-scale time homogenization formulation for solids and structures made of VE-VP materials and subjected to large numbers of cycles was developed. The formulation extends the theory proposed by [Guenouni, 1988] from elasto-viscoplasticity to coupled VE-VP. Two time scales are defined: a macro-chronological (slow variation) time and a micro-chronological time (for rapid evolution). Asymptotic time expansions are supposed for the unknown variables, the small parameter being the loading period. The initial boundary value problem is decomposed in two: a purely VE micro-time problem, and a nonlinear VP macro-time one. Numerical implicit time integration algorithms are proposed. The temporal homogenization approach was implemented and tested for two types of cyclic loadings. The first one is uniaxial tension/compression and the second is multiaxial, with combined torsion and tension/compression. A significant reduction in the amount of computation time (about 94%) in addition to a small error (not exceeding 4% for the last cycles) between time homogenized and full-scale predictions were obtained.

The influence of loading frequency, the viscoplastic modulus, the loading ratio and the macro time increment on the accuracy of the time homogenized predictions was studied. The two-scale results were found to be in good agreement with the reference ones as long as the scaling parameter $T^*$ remains small. For micro-time averaging, two numerical schemes were developed: one-point and multi-point integration. Simulations using one-point integration do not give a good approximation when the loading ratio $R$ tends towards $-1$ (the error then exceeds 15%). However it was found that for $R = -1$ increasing the number of integration points improves the homogenized predictions: the error on the stress decreases from 27% to 3%.

The reference calculations were also compared with experimental data for UHMWPE obtained by [Avanzini, 2011] and [Kurtz et al., 2002] and a good agreement is obtained under isothermal conditions.

In the present work, a modeling framework was applied to a VE-VP
Conclusions

model with coupling linear isotropic VE and $J2$ VP, under isothermal conditions and a small deformation hypothesis. In order to be able to predict the high cycle fatigue of thermoplastics polymers, the current VE-VP model need to be enriched to include pressure-dependent yield, self heating and fatigue damage.

Although the present work does not include fatigue, the multiscale approach in time opens up the possibility of solving fatigue problems for the entire fatigue life presumably with additional assumptions ([Os- kay and Fish, 2004] and [Fish et al., 2012]). In chapter 5 the present temporal homogenization scheme is extended to VE-VP coupled with a fatigue damage model. Based on chapter 4 and 5 the method could also be generalized to fatigue analysis of heterogeneous solids, which are characterized by multiple temporal and spatial scales.
CHAPTER 4

Space and time homogenization for viscoelastic-viscoplastic undamaged composites under large numbers of cycles

4.1 Introduction

In this chapter a new multiscale computational strategy is proposed for the analysis of composite materials subjected to large numbers of cycles. This multiscale computational strategy is introduced for the evolution of VE-VP problems that includes a homogenization procedure in both time and space levels. The first point of this approach is the multi-time-scale phenomena: an asymptotic time homogenization approach is used (Aubry and Puel, 2010, Oskay and Fish, 2004, Yu and Fish, 2001). This technique was introduced by Guennouni (1988) for elasto-viscoplastic (E-VP) homogeneous materials and then extended to VE-VP materials by Haouala and Doghri, 2015. For the multiscale phenomena in space the mean-field space homogenization approach for inelastic materials is used to determine the effective behavior of the heterogeneous material.

4.2 Anisotropic viscoelastic-viscoplastic solid: constitutive equations

In this section, we give the basic equations for anisotropic VE-VP solids. For this model the total strain \( \varepsilon \) is assumed to be subdivided into VE (\( \varepsilon^{ve} \)) and VP (\( \varepsilon^{vp} \)) parts and the Cauchy stress is related to the history
Space and time homogenization for viscoelastic-viscoplastic undamaged composites under large numbers of cycles of VE strains via a linear VE model written as a Boltzmann integral:

\[ \varepsilon = \varepsilon^{ve} + \varepsilon^{vp}, \]

\[ \sigma(t) = \int_{-\infty}^{t} C^{ve}(t - \xi) : \frac{\partial \varepsilon^{ve}(\xi)}{\partial \xi} d\xi. \]  

(4.1)

Where \( C^{ve} \) is a fourth-order anisotropic relaxation tensor which can be described by a Prony series as a function of the long-term elastic Hooke’s operator, constant stiffness tensors and relaxation times which are denoted by \( C_{\infty}, E_i \) and \( \lambda_i, (i = 1 \ldots I) \):

\[ C^{ve}(t) = C_{\infty} + \sum_{i=1}^{I} E_i \exp \left( -\frac{t}{\lambda_i} \right), \]  

(4.2)

Combining equation (4.2) and (4.1-b), the Cauchy stress can be rewritten as follows:

\[ \sigma(t) = C_{\infty} : \varepsilon^{ve}(t) + \sum_{i=1}^{I} \sigma_i(t), \]  

(4.3)

where the viscous stresses \( \sigma_i \) are given by the following expression:

\[ \sigma_i(t) = E_i : \int_{-\infty}^{t} \exp \left( -\frac{t - \xi}{\lambda_i} \right) \frac{\partial \varepsilon^{ve}(\xi)}{\partial \xi} d\xi \]  

(4.4)

For the viscoplastic part we consider the same general viscoplastic model studied by [Doghri et al., 2010b]:

\[ \dot{\varepsilon}^{vp}(t) = B(\sigma(t), V(t)), \quad \dot{V}(t) = C(\sigma(t), V(t)). \]  

(4.5)

The equations above are similar to those of Section 3.3 except that we allow here for a general set of scalar and/or tensor internal variables \( V \). For instance, one might have \( V = (p, X_{ij}) \), where \( p \) is a scalar variable for isotropic hardening, and \( X \) is a second order tensor which represents the kinematic hardening. \( B \) and \( C \) are given functions defining the evolution of the VP strain and the internal variables.
4.3 Space and time homogenization of VE-VP anisotropic materials subjected to large numbers of cycles

In this section a multiscale computational strategy is proposed for the analysis of structures, which are described at a refined level both in space and in time. The proposal is applied to two-phase VE-VP composite materials subjected to large numbers of cycles.

4.3.1 Definition of multiple spatial and temporal scales

Consider a VE-VP anisotropic solid occupying a spatial domain $\Omega'$ to which we associate the global coordinate system $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$. This domain is subjected to body forces $\bar{f}_T^*$. $\Gamma_u$ and $\Gamma_f$ correspond to the boundary where $T^*$-periodic displacements $\bar{u}_{bT^*}$ and tractions $\bar{g}_{T^*}$ are prescribed, respectively. We denote by $\bar{n}$ the outer unit vector normal to the boundary and by $\bar{u}^I$ and $\sigma^I$ the initial displacement and stress, respectively. The load period $T^*$ is supposed to be very small compared to the observation time $T^*_F$.

The microstructure of the material under consideration is basically taken into account by representative volume elements (RVE) having a local coordinate system $(x_1, x_2, x_3)$. The overall properties of each RVE represent the overall properties of the composite material.

At each material point $\bar{x}$ at the macroscopic scale is associated an RVE $(\Omega)$, which at smaller micro scale contains a finite number of constituents (figure 4.1). Each mechanical variable $\Psi_T^* (\bar{x}, t^*)$ is assumed to depend on both spatial scales $\bar{x}$ and $\bar{x}$

$$\Psi_T^* (\bar{x}, t^*) \equiv \Psi_T^* (\bar{x}, \bar{x}, t^*). \quad (4.6)$$

On the other hand this RVE is subjected to rapidly cyclic loading with small loading period ($T^*$).

The physical time $t^*$ can be defined as the sum of the macro time $t_M^*$, and a fraction $\tau$ of the period $T^*$, $\tau \in [0, 1]$; $t^* = t_M^* + T^*\tau$.

Let us define a small scaling parameter $T$ as follows:

$$T = \frac{T^*}{T^*_F}. \quad (4.7)$$

A new dimensionless relation between both time scales is obtained:

$$t = t_M + T\tau, \quad t = \frac{t^*}{T^*_F} \in [0, 1], \quad t_M^* = \frac{t^*_M}{T^*_F} \in [0, 1], \quad \tau \in [0, 1]. \quad (4.8)$$
The following function substitution is made:

\[ \phi_T (\vec{x}, \vec{x}, t) = \Psi_T^* (\vec{x}, \vec{x}, t^*), \quad (4.9) \]

Each variable \( \phi_T (\vec{x}, \vec{x}, t) \) is then assumed to depend on both dimensionless time scales \( t_M \) and \( \tau \) and to be periodic with the dimensionless period \( T \) with respect to the fast variable \( \tau \):

\[ \phi_T (\vec{x}, \vec{x}, t) = \phi (\vec{x}, \vec{x}, t_M, \tau). \quad (4.10) \]

Using the chain rule, the time differentiation in the two dimensionless time scales is given as:

\[ \dot{\phi}_T (\vec{x}, t) = \frac{\partial \phi (\vec{x}, \vec{x}, t_M, \tau)}{\partial t_M} + \frac{1}{T} \frac{\partial \phi (\vec{x}, \vec{x}, t_M, \tau)}{\partial \tau}, \quad (4.11) \]

where the superposed dot denotes the total derivative with respect to physical dimensionless time \( t \).

It is implicit that all variables are functions of positions \( \vec{x} \) and \( \vec{x} \), so from now on and without loss of generality, the dependence of all variables on \( \vec{x} \) and \( \vec{x} \) is omitted for simplicity.

The initial-boundary VE-VP anisotropic problem is then expressed using the two dimensionless time scales:

**Initial conditions:**

\[ \vec{u} (\vec{x}, t_M = 0, \tau = 0) = \vec{u}^I (\vec{x}), \quad \text{on } \Omega \quad (4.12) \]

\[ \sigma (\vec{x}, t_M = 0, \tau = 0) = \sigma^I (\vec{x}), \quad \text{on } \Omega \quad (4.13) \]

**Boundary conditions:**

\[ \vec{u} (t_M, \tau) = \vec{u}_b (t_M, \tau), \quad \text{on } \Gamma_u \times [0, 1] \times [0, 1] \quad (4.14) \]

\[ ^t \sigma \cdot \vec{n} = \tilde{g} (t_M, \tau), \quad \text{on } \Gamma_f \times [0, 1] \times [0, 1] \quad (4.15) \]

**Equilibrium equations:**

\[ \nabla \cdot \sigma (t_M, \tau) + \vec{f} (t_M, \tau) = \vec{0}, \quad \text{on } \Omega \times [0, 1] \times [0, 1] \quad (4.16) \]

**Kinematic compatibility:**

\[ \varepsilon (\vec{u}) = \frac{1}{2} \left( \nabla \vec{u} + ^t \nabla \vec{u} \right), \quad \text{on } \Omega \times [0, 1] \times [0, 1] \quad (4.17) \]
4.3. Space and time homogenization of VE-VP anisotropic materials subjected to large numbers of cycles

Figure 4.1: Composite medium subjected to large numbers of cycles: illustration of the multiscale phenomenon in both spatial and temporal levels.
Space and time homogenization for viscoelastic-viscoplastic undamaged composites under large numbers of cycles

Constitutive equations:

\[ \varepsilon (\vec{u}) = \varepsilon^{ve} (t_M, \tau) + \varepsilon^{vp} (t_M, \tau), \quad \text{on } \Omega \times [0, 1] \times [0, 1] \]  \hspace{1cm} (4.18)

\[ \sigma (t_M, \tau) = C_\infty : \varepsilon^{ve} (t_M, \tau) + \sum_{i=1}^{I} \sigma_i (t_M, \tau), \quad \text{on } \Omega \times [0, 1] \times [0, 1] \]  \hspace{1cm} (4.19)

\[ \dot{\varepsilon}^{vp} = B (\sigma, \nabla V), \quad \text{on } \Omega \times [0, 1] \times [0, 1] \]  \hspace{1cm} (4.20)

\[ \dot{V} = C (\sigma, V), \quad \text{on } \Omega \times [0, 1] \times [0, 1] \]  \hspace{1cm} (4.21)

Where \( \nabla \cdot \) and \( \nabla \) denote the divergence and gradient operators, respectively, and upper script "**" denotes a transpose. B and C are viscoplastic operators.

4.3.2 Two scale time analysis of VE-VP anisotropic materials

A general decomposition of all variables \( \phi (\vec{x}, t_M, \tau) \) into macro- and micro- chronological fields is proposed in the form:

\[ \phi (t_M, \tau) = \phi_M (t_M) + \tilde{\phi} (t_M, \tau), \]  \hspace{1cm} (4.22)

The average value \( \phi_M (t_M) \) of the function \( \phi (t_M, \tau) \) with respect to \( \tau \) and at time \( t_M \) is defined as:

\[ \phi_M (t_M) = \int_0^1 \phi (t_M, \tau) \, d\tau \equiv \langle \phi (t_M, \tau) \rangle. \]  \hspace{1cm} (4.23)

Each mechanical variable \( \phi (t_M, \tau) \) is supposed to be regular enough so that it can be expanded into an asymptotic series of powers of \( T \). Given that \( \frac{T^*}{T_F} \ll 1 \), we can assume that:

\[ \phi (t_M, \tau) = \sum_{i=0}^{\infty} T^i \phi_i (t_M, \tau), \]  \hspace{1cm} (4.24)

where \( \phi_i (t_M, \tau) \) are \( \tau \)-periodic functions and \( i \) denotes the order of the terms in the expansion.
4.3. Space and time homogenization of VE-VP anisotropic materials subjected to large numbers of cycles

Only the constitutive equations [4.19]-[4.21] of the initial-boundary anisotropic VE-VP problem [4.12]-[4.21] are changed, compared to the initial-boundary isotropic VE-VP problem presented in Section 3.3.2. The asymptotic expansions of each variable are injected into equations [4.18] to [4.21]. Using the chain rule and the total differentiation [4.11] and gathering terms of equal order, the constitutive equations can then be rewritten at the orders ($-1$) and (0) as follows:

- **Order $-1$ problem:**
  \[
  \frac{\partial}{\partial \tau} \varepsilon_{vp}^0(t_M, \tau) = 0, \quad \text{on } \Omega \times [0, 1] \times [0, 1] \quad (4.25)
  \]
  \[
  \frac{\partial}{\partial \tau} V_0(t_M, \tau) = 0, \quad \text{on } \Omega \times [0, 1] \times [0, 1] \quad (4.26)
  \]

- **Order 0 problem:**
  \[
  \varepsilon(\bar{u}_0) = \varepsilon_{ve}^0(t_M, \tau) + \varepsilon_{vp}^0(t_M, \tau), \quad \text{on } \Omega \times [0, 1] \times [0, 1] \quad (4.27)
  \]
  \[
  \sigma_0(t_M, \tau) = C_\infty : \varepsilon_{ve}^0(t_M, \tau) + \sum_{i=1}^{I} \sigma_{i0}(t_M, \tau), \quad \text{on } \Omega \times [0, 1] \times [0, 1] \quad (4.28)
  \]
  \[
  \frac{\partial}{\partial t_M} \varepsilon_{vp}^0(t_M, \tau) + \frac{\partial}{\partial \tau} \varepsilon_{vp}^1(t_M, \tau) = B(\sigma_0, V_0), \quad \text{on } \Omega \times [0, 1] \times [0, 1] \quad (4.29)
  \]
  \[
  \frac{\partial}{\partial t_M} V_0(t_M, \tau) + \frac{\partial}{\partial \tau} V_1(t_M, \tau) = C(\sigma_0, V_0), \quad \text{on } \Omega \times [0, 1] \times [0, 1] \quad (4.30)
  \]

As for isotropic VE-VP homogeneous materials in Section [3.3.3], the VP strain and the internal variables $V_0$ are a function of the slow time variable ($t_M$) only:

\[
\varepsilon_{vp}^0(t_M, \tau) = \varepsilon_{vp}^0(t_M), \quad (4.31)
\]
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\[ V_0(t_M, \tau) = V_0(t_M), \quad (4.32) \]

and at the zero-order the rapid evolution of the total deformation is equal to that of the VE part:

\[ \frac{\partial}{\partial \tau} \varepsilon_0(t_M, \tau) = \frac{\partial}{\partial \tau} \varepsilon_{0}^{ve}(t_M, \tau), \quad \text{on } \Omega \times [0,1] \times [0,1] \quad (4.33) \]

Following the decomposition defined in equation (4.22) and taking the average as defined by equation (4.23) of equations (4.27)-(4.30) we obtain a decomposition of the constitutive equations into macro- and micro-chronological problems:

- **Macro-chronological problem:**
  \[ \varepsilon_M(\bar{u}_0) = \varepsilon_{0-M}^{ve}(t_M) + \varepsilon_{0}^{vp}(t_M) \quad \text{on } \Omega \times [0,1] \quad (4.34) \]

  \[ \sigma_{0-M}(t_M) = C_{\infty} : \varepsilon_{0-M}^{ve}(t_M) + \sum_{i=1}^{I} \sigma_{i0-M}(t_M), \quad \text{on } \Omega \times [0,1] \quad (4.35) \]

  \[ \frac{d}{dt_M} \varepsilon_{0}^{vp}(t_M) = B_M((\sigma_{0-M} + \bar{\sigma}_0), V_0), \quad \text{on } \Omega \times [0,1] \quad (4.36) \]

  \[ \frac{d}{dt_M} V_0(t_M) = C_M((\sigma_{0-M} + \bar{\sigma}_0), V_0), \quad \text{on } \Omega \times [0,1] \quad (4.37) \]

- **Micro-chronological problem:**
  \[ \bar{\varepsilon}(\bar{u}_0) = \bar{\varepsilon}_0^{ve}(t_M, \tau), \quad \text{on } \Omega \times [0,1] \times [0,1] \quad (4.38) \]

  \[ \bar{\sigma}_0(t_M, \tau) = C_{\infty} : \bar{\varepsilon}_0(t_M, \tau) + \sum_{i=1}^{I} \bar{\sigma}_{i0}(t_M, \tau), \quad \text{on } \Omega \times [0,1] \times [0,1] \quad (4.39) \]

The macro-chronological problem (4.34)-(4.37) is nonlinear and corresponds to the resolution of VE-VP anisotropic composite materials problem.

Assuming the fast time periodicity of \( \varepsilon_1^{vp}(t_M, \tau) \) and \( V_1(t_M, \tau) \), the averages of \( \frac{\partial}{\partial \tau} \varepsilon_1^{vp}(t_M, \tau) \) and \( \frac{\partial}{\partial \tau} V_1(t_M, \tau) \) vanish and the micro-chronological (4.38)-(4.39) problem corresponds to the resolution of anisotropic VE
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composites problem only, given that \( \tilde{\varepsilon}_{vp}^0(t_M, \tau) = 0 \) and \( \tilde{V}_0(t_M, \tau) = 0 \)
from equations (4.25) and (4.26).

The initial-boundary VE-VP anisotropic problem (4.12)-(4.20), which corresponds to the macroscopic behaviour in space of the composite, is written in a formal manner to check the application of the time homogenization approach for anisotropic materials. In this work the VP operator \( V \) will not be defined explicitly, and the effective response of the composite will be achieved using MFH.

4.4 Micromechanical modeling of coupled anisotropic and heterogeneous VE-VP macro- and micro-chronological problems

Up to now, the emphasis has been on the multiscale phenomena in time. the main aim of this section is to predict the effective properties of the composite based on mean-field homogenization approaches. Consider a two phase VE-VP anisotropic composite. The inclusions are elastic and the behavior of the matrix is supposed to be VE coupled with isotropic \( J_2 \) VP. Then the macro- and micro-chronological problems (4.34)-(4.37) and (4.38)-(4.39) correspond to coupled VE-VP and VE two phase composites, respectively. The macro-chronological problem is similar to the anisotropic VE-VP two phase composite studied in Miled et al., 2013 (VE-VP isotropic matrix reinforced with elastic inclusion).

To determine the effective response of the composite by MFH a linearization of the matrix’s behaviour is required (Miled et al., 2013). First of all, we start by giving a review of the two time scale numerical algorithms proposed in Chapter 3 for the resolution of coupled homogeneous and isotropic VE-VP homogeneous materials. Second, an incrementally affine linearization method developed by Miled et al., 2013 for the homogenization of coupled VE-VP composites is used. The latter leads to an affine relation between stress and strain increments via an algorithmic tangent operator. This formulation is form-similar to linear thermoelasticity. Thus homogenization models for linear thermoelastic composites can be applied.
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4.4.1 Two time scale computational algorithm for homogeneous and isotropic VE-VP problem

The matrix phase of the composite is supposed to be VE coupled with the classical $J_2$ rate dependent VP model with isotropic hardening presented in Chapter 3.

As the linearization technique sustaining our MFH approach makes use of algorithmic tangent operators, the two time scale numerical algorithm proposed in Chapter 3 for VE-VP isotropic material is summarized in this subsection.

Consider the $J_2$ rate-dependent VE-VP problem with isotropic hardening presented in Chapter 3, with the following yield criterion:

$$f = \sigma_{eq} - \sigma_y - R(p), \quad \sigma_{eq} = \sqrt{\frac{3}{2} s : s}.$$  \hspace{1cm} (4.40)

Where $\sigma_y$ is the initial yield stress (which may depend on the strain rate), $R(p)$ the hardening stress and $s$ is the deviatoric part of the stress tensor.

The VE-VP model was decomposed into macro- and micro-chronological problems and written in the discretized form in Section 3.4, over a time interval $[t_{Mn}, t_{Mn+1}]$. The micro-chronological problem is only VE and solved only once. The macro-chronological problem is VE-VP and solved iteratively based on a return mapping algorithm on two-steps, VE predictor followed by VP corrector.

Knowing all history variables at $t_{Mn}$ and the total zero-order macro-strain increment $\Delta \varepsilon_0^M$, the values of the zero-order variables at $t_{Mn+1}$ are found. If the trial stress $\sigma_{0{\text{eq}}}(t_{Mn+1}, \tau)$ satisfies the yield condition:

$$f_0^{{\text{pred}}} (t_{Mn+1}, \tau) = \sigma_{0{\text{eq}}}^{{\text{pred}}} (t_{Mn+1}, \tau) - \sigma_y - R(p_0(t_{Mn})) \leq 0, \hspace{1cm} (4.41)$$

then the basic assumption is valid and the zeroth-order VE predictor is the solution at $t_{Mn+1}$:

$$\sigma_0(t_{Mn+1}, \tau) = \sigma_{0{\text{eq}}}^{{\text{pred}}} (t_{Mn+1}, \tau), \hspace{1cm} (4.42)$$

$$\varepsilon_{0{\text{vp}}}^{{\text{vp}}} (t_{Mn+1}) = \varepsilon_0^{{\text{vp}}} (t_{Mn}), \quad p_0(t_{Mn+1}) = p_0(t_{Mn}). \hspace{1cm} (4.43)$$

If $f_0^{{\text{pred}}} (t_{Mn+1}, \tau) > 0$ then the assumption is incorrect and a VP corrector is needed. In this latter case, the solution at $t_{Mn+1}$ and $\tau$ has to
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satisfy the following equation:

\[ \sigma_0(t_{Mn+1}, \tau) = \sigma_0^{\text{pred}}(t_{Mn+1}, \tau) - \hat{E}(\Delta t_M) : \Delta \varepsilon_0^{vp}, \] (4.44)

together with equations (4.45) and (4.46).

\[ \Delta \varepsilon_0^{vp} = F_{\varepsilon M}(\sigma_0^M + \tilde{\sigma}_0, p_0)|_{(t_{Mn+1})} \Delta t_M, \] (4.45)

\[ \Delta p_0 = F_{p M}(\sigma_0^M + \tilde{\sigma}_0, p_0)|_{(t_{Mn+1})} \Delta t_M. \] (4.46)

Here \( F_{\varepsilon M} \) and \( F_{p M} \) are two operators describing the VP evolution within the matrix phase.

The zero order trial stress is given by the following expression:

\[ \sigma_0^{\text{pred}}(t_{Mn+1}, \tau) = \tilde{\sigma}_0(t_{Mn+1}, \tau) + C_\infty : \varepsilon_0^{ve}(t_M) + \hat{E}(\Delta t_M) : \Delta \varepsilon_0^M \]

\[ + \sum_{i=1}^I \exp \left( \frac{-\Delta t_M}{g_i} \right) s_{0M}(t_M) + \sum_{j=1}^J \exp \left( \frac{-\Delta t_M}{k_j} \right) \sigma_{H_{0M}}(t_M) 1, \] (4.47)

here \( s_{0M} \) and \( \sigma_{H_{0M}} \) are the macro-chronological zero order deviatoric and dilatational viscous components, respectively and \( \tilde{\sigma}_0(t_{Mn+1}, \tau) \) is the zero-order stress fluctuation defined by the following expression:

\[ \tilde{\sigma}_0(t_{Mn+1}, \tau) = C_\infty : \tilde{\varepsilon}_0(t_M, \tau) + \hat{E}(\Delta t_M) : \Delta \tilde{\varepsilon}_0 \]

\[ + \sum_{i=1}^I \exp \left( \frac{-\Delta t_M}{g_i} \right) \tilde{s}_{0}(t_M, \tau) + \sum_{j=1}^J \exp \left( \frac{-\Delta t_M}{k_j} \right) \tilde{\sigma}_{H_{0}}(t_M, \tau) 1. \] (4.48)

where \( \tilde{s}_{0} \) and \( \tilde{\sigma}_{H_{0}} \) are the micro-chronological zero order deviatoric and dilatational viscous components, respectively.

\[ C_\infty = 2G_\infty I^{\text{dev}} + 3K_\infty I^{\text{vol}} \] is the long-term elastic Hooke’s operator and \( \hat{E} = 2G I^{\text{dev}} + 3K I^{\text{vol}} \).

Here, \( G_\infty \) and \( K_\infty \) are the elastic shear and bulk long-term moduli, respectively; \( g_i \) \(( i = 1 \ldots I) \) and \( k_j \) \(( j = 1 \ldots J) \) are shear and bulk relaxation times respectively; and \( G_i \) \(( i = 1 \ldots I) \) and \( K_j \) \(( j = 1 \ldots J) \) are shear and bulk weights respectively.

For isotropic VE-VP materials the total zero-order stress tensor can be
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written at \( t_{M_n} \) and \( \tau \) as follows:

\[
\sigma_0(t_{M_n}, \tau) = C_\infty : \varepsilon_0^{ve}(t_{M_n}, \tau) + \sum_{i=1}^{I} s_{i0}(t_{M_n}, \tau) + \sum_{j=1}^{J} \sigma_{Hj0}(t_{M_n}, \tau) \mathbf{1}
\]

\[
= \tilde{\sigma}_0(t_{M_n}, \tau) + C_\infty : \varepsilon_{0M}^{ve}(t_{M_n}) + \sum_{i=1}^{I} s_{i0M}(t_{M_n}) + \sum_{j=1}^{J} \sigma_{Hj0M}(t_{M_n}) \mathbf{1}
\]

From equations (4.44) and (4.49), the following expression of the zero-order stress increment is obtained:

\[
\Delta \sigma_0 = \hat{E}(\Delta t_M) : (\Delta \varepsilon_{0M}^{ve} - \Delta \varepsilon_0^{vp}) + \Delta \tilde{\sigma}_0 + a_{0M}(t_{M_n}). \tag{4.50}
\]

Equation (4.50) represents the local stress increment within the matrix phase. This local equation will be used later for the incrementally affine formulation presented in the following section.

The macro- and micro- chronological zero-order stress increments are given by the following expressions:

\[
\Delta \sigma_{0M} = \hat{E}(\Delta t_M) : (\Delta \varepsilon_{0M}^{ve} - \Delta \varepsilon_0^{vp}) + a_{0M}(t_{M_n}). \tag{4.51}
\]

\[
\Delta \tilde{\sigma}_0 = \hat{E}(\Delta t_M) : \Delta \tilde{\varepsilon}_0 + \tilde{a}_0(t_{M_n}, \tau). \tag{4.52}
\]

Where \( a_{0M}(t_{M_n}) \) and \( \tilde{a}_0(t_{M_n}, \tau) \) are the following zero-order macro- and micro- chronological second order tensors:

\[
\begin{cases}
    a_{0M}(t_{M_n}) = - \sum_{i}^{I} \left[ 1 - \exp \left( -\frac{\Delta t_M}{g_i} \right) \right] s_{i0M}(t_{M_n}) \\
    - \sum_{j}^{J} \left[ 1 - \exp \left( -\frac{\Delta t_M}{k_j} \right) \right] \sigma_{Hj0M}(t_{M_n}) \mathbf{1}, \\
    \tilde{a}_0(t_{M_n}, \tau) = - \sum_{i}^{I} \left[ 1 - \exp \left( -\frac{\Delta t_M}{g_i} \right) \right] \tilde{s}_{i0}(t_{M_n}, \tau) \\
    - \sum_{j}^{J} \left[ 1 - \exp \left( -\frac{\Delta t_M}{k_j} \right) \right] \tilde{\sigma}_{Hj0}(t_{M_n}, \tau) \mathbf{1}.
\end{cases}
\]

\[
(4.53)
\]
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The macro- and micro-chronological zero order stress-strain incremental relations in equations (4.51) and (4.52) are affine relations. The former is similar to the EVP case studied in [Doghri et al., 2010a], except that in this work we use a VE tangent operator \( \hat{E}(\Delta t_M) \) instead of a constant elastic stiffness and that we are using the zero-order macro-chronological part of the deformation instead of the total one. A new second order tensor \( a_M^0 \) is also introduced due to viscoelasticity, this tensor is nil in the EVP case. The latter corresponds to a VE micro-chronological evolution, and it relates the zero-order stress fluctuation increment \( \Delta \tilde{\sigma}_0 \) to the zero-order strain fluctuation increment \( \Delta \tilde{\varepsilon}_0 \).

Contrary to EP, a continuum tangent operator relating the zero-order stress and strain rates, does not exist in rate-dependent VE-VP. However, a zero-order algorithmic tangent operator \( C_{\text{algo}}^0 \) relating finite zero-order stress and strain increments, can be derived by consistent linearization of the time discretized constitutive equations around the solution at \( t_{Mn+1} \) and \( \tau \):

\[ \delta \sigma_0 (t_{Mn+1}, \tau) = C_{\text{algo}}^0 : \delta \varepsilon_0 (t_{Mn+1}, \tau), \]  

(4.54)

where \( \delta \) denotes a total variation at \( t_{Mn+1} \) and \( \tau \).

For the micro-scale time averaging, consider the case of one point integration:

\[ \phi_M(t_{Mn+1}) = \int_0^1 \phi (t_{Mn+1}, \tau) \, d\tau \simeq \phi (t_{Mn+1}, \tau = \frac{1}{2}). \]  

(4.55)

After some mathematical developments (see Appendix [3]), and using the classical \( J_2 \) rate-dependent VP model, the following expression of the sought-after zero-order algorithmic tangent operator \( C_{\text{algo}}^0 \) is obtained:

\[ C_{\text{algo}}^0 (t_{Mn+1}, \tau = \frac{1}{2}) = \hat{E}(\Delta t_M) - \frac{(2\hat{G})^2}{h_v} N_0 (t_{Mn+1}, \tau = \frac{1}{2}) \otimes \]

\[ N_0 (t_{Mn+1}, \tau = \frac{1}{2}) - \frac{(2\hat{G})^2}{h_v g_{\sigma_0}} \sigma_{eq}(t_{Mn+1}, \tau = \frac{1}{2}) \Delta p_0 \left( \frac{\partial N_0}{\partial \sigma_0} \right) (t_{Mn+1}, \tau = \frac{1}{2}) - \frac{2\hat{G}}{h_v g_{\sigma_0}} N_0 (t_{Mn+1}, \tau = \frac{1}{2}) \otimes g_v \sigma_0. \]  

(4.56)

Here, \( g_v \) designates the VP function and \( N_0 = \frac{3}{2} \sigma_{eq}. \)

For the VP function \( g_v \) we consider in this work the Norton’s power law:
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\[ g_v = g_v(f) = \frac{\sigma_y}{\zeta} \left( \frac{f}{\sigma_y} \right)^n, \text{ if } f > 0; \quad g_v = 0, \text{ if } f \leq 0, \quad (4.57) \]

Where: \( g_{\sigma_0} \equiv \frac{\partial g_v}{\partial \sigma_0}; \quad g_{p_0} \equiv \frac{\partial g_v}{\partial p_0}; \quad g_{\varepsilon_0} \equiv \frac{1}{\Delta t_M} \frac{\partial g_v}{\partial \varepsilon_0}, \) and the denominator \( h_v \) is defined by this expression:

\[ h_v \equiv \frac{1}{\Delta t_M g_{\sigma_0}} + 3\dot{\gamma} - \frac{g_{p_0}}{g_{\sigma_0}}, \quad (4.58) \]

4.4.2 Incrementally affine linearization method

As discussed in Chapter 2.3 Section 2.3.5, the homogenization of non-linear two phase composites is based on the linearization of the local constitutive laws and the definition of uniform reference properties for each phase, valid for a given stage of deformation. According to the choice of the linearization procedure, the problem becomes similar to an elastic or thermo-elastic homogenization problem. Several linearization strategies were proposed, among them one can cite the secant ([Berveiller and Zaoui, 1979], [Tandon and Weng, 1988]), incremental ([Hill, 1965]), affine ([Pierard and Doghri, 2006]) and incrementally affine ([Doghri et al., 2010b], [Miled et al., 2013]) approaches. The latter one is the chosen method for this work. It is valid for multi-axial, non-monotonic and non-proportional loading histories.

Consider the same general macro-chronological constitutive model in equations (4.34) - (4.37) studied in Section 4.3. Linearization at time \( t_M \) close to a time \( \xi \) of equations (4.36) and (4.37) leads to:

\[ \tilde{\varepsilon}_0^{np}(t_M) \approx \tilde{\varepsilon}_0^{np}(\xi) + B_{M,\sigma_0}(\xi) : (\sigma_0(t_M, \tau) - \sigma_0(\xi, \tau)) + B_{M,V_0}(\xi) \bullet (V_0(t_M) - V_0(\xi)), \quad (4.59) \]

\[ \tilde{V}_0(t_M) \approx \tilde{V}_0(\xi) + C_{M,\sigma_0}(\xi) : (\sigma_0(t_M, \tau) - \sigma_0(\xi, \tau)) + C_{M,V_0}(\xi) \bullet (V_0(t_M) - V_0(\xi)), \quad (4.60) \]

Notations: \( B_{M,\sigma_0} \equiv \frac{\partial B_M}{\partial \sigma_0}; \quad B_{M,V_0} \equiv \frac{\partial B_M}{\partial V_0}; \quad C_{M,\sigma_0} \equiv \frac{\partial C_M}{\partial \sigma_0}; \quad C_{M,V_0} \equiv \frac{\partial C_M}{\partial V_0}. \)
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The bullet symbol designates a sum of appropriate inner products. For instance, if \( \mathbf{V}_0 = (p_0, \mathbf{X}_0) \) then \( \mathbf{B}_M \mathbf{V}_0 \cdot \delta \mathbf{V}_0 = \mathbf{B}_M p_0 \delta p_0 + \mathbf{B}_M \mathbf{X}_0 : \delta \mathbf{X}_0 \).

Consider a time interval \([t_{M_n}, t_{M_{n+1}}]\) such that: the numerical solution at \( t_{M_n} \) is known, time and total strain increments \( \Delta t_M \) and \( \Delta \varepsilon_0 \) are given. Linearization times are chosen as follows:

\[
t_M = t_{M_n}, \quad \xi = t_{M_{n+1}},
\]

the time discrete form of equations (4.59) and (4.60) using a fully implicit backward Euler time integration is:

\[
\begin{align*}
\frac{1}{\Delta t_M} \Delta \varepsilon^{vp}_0 &= \dot{\varepsilon}^{vp}_0(t_{M_n}) + \mathbf{B}_M \sigma_0(t_{M_{n+1}}) : \Delta \sigma_0 + \mathbf{B}_M \mathbf{V}_0(t_{M_{n+1}}) \cdot \Delta \mathbf{V}_0, \\
\frac{1}{\Delta t_M} \Delta \mathbf{V}_0 &= \dot{\mathbf{V}}_0(t_{M_n}) + \mathbf{C}_M \sigma_0(t_{M_{n+1}}) : \Delta \sigma_0 + \mathbf{C}_M \mathbf{V}_0(t_{M_{n+1}}) \cdot \Delta \mathbf{V}_0.
\end{align*}
\]

Equation (4.61-b) can be rewritten as follows:

\[
\left[ \frac{1}{\Delta t_M} \mathbf{I} - \mathbf{C}_M \mathbf{V}_0(t_{M_{n+1}}) \right] \cdot \Delta \mathbf{V}_0 = \dot{\mathbf{V}}_0(t_{M_n}) + \mathbf{C}_M \sigma_0(t_{M_{n+1}}) : \Delta \sigma_0.
\]

Combining equation (4.61-a) and (4.62) we have:

\[
\Delta \varepsilon^{vp}_0 = \left[ \dot{\varepsilon}^{vp}_0(t_{M_n}) + \mathbf{B}_M \mathbf{V}_0(t_{M_{n+1}}) \cdot \left[ \cdots \right]^{-1} \cdot \dot{\mathbf{V}}_0(t_{M_n}) + \left\{ \cdots \right\} : \Delta \sigma_0 \right] \times \Delta t_M,
\]

where,

\[
\left[ \cdots \right] = \frac{1}{\Delta t_M} \mathbf{I} - \mathbf{C}_M \mathbf{V}_0(t_{M_{n+1}}),
\]

\[
\left\{ \cdots \right\} = \mathbf{B}_M \sigma_0(t_{M_{n+1}}) + \mathbf{B}_M \mathbf{V}_0(t_{M_{n+1}}) \cdot \left[ \cdots \right]^{-1} \cdot \mathbf{C}_M \sigma_0(t_{M_{n+1}})
\]

Substituting equations (4.63) in equation (4.50) we have:

\[
\Delta \sigma_0 = \mathbf{C}_0^{algo}(t_{M_{n+1}}) \cdot \left[ \left( \Delta \varepsilon_0 - \Delta \varepsilon_{af}^{0M} \right) + \mathbf{E}^{-1}(\Delta t_M) : \Delta \tilde{\sigma}_0 \right].
\]

Replacing \( \Delta \tilde{\sigma}_0 \) by its expression in equation (4.52) we have:

\[
\Delta \sigma_0 = \mathbf{C}_0^{algo}(t_{M_{n+1}}) \cdot \left[ \Delta \varepsilon_0 - \left( \Delta \varepsilon_{af}^{0M} - \mathbf{E}^{-1}(\Delta t_M) : \mathbf{a}_0(t_{M_n}, \tau) \right) \right].
\]
Where the zero order algorithmic tangent operator $C_{0}^{algo}(t_{M_{n+1}})$ and the macro-chronological zero order affine strain increment $\Delta \varepsilon_{0}^{af}$ are defined by the following equations:

$$
\begin{align*}
C_{0}^{algo}(t_{M_{n+1}}) &= \left[ \hat{E}^{-1}(\Delta t_{M}) + \Delta t_{M}\{\cdots}\right]^{-1}, \\
\Delta \varepsilon_{0}^{af} &= \Delta \varepsilon_{0}^{af}_{evp} - \hat{E}^{-1}(\Delta t_{M}) : \tilde{a}_{0}(t_{M_{n}}).
\end{align*}
$$

(4.68)

Here $\Delta \varepsilon_{0}^{af}_{evp}$ is given by this expression:

$$
\Delta \varepsilon_{0}^{af}_{evp} = \left[ \tilde{\varepsilon}_{0}^{vp}(t_{M_{n}}) + B_{M}V_{0}(t_{M_{n}+1}) \cdot [\cdots]^{-1} \cdot \tilde{V}_{0}(t_{M_{n}}) \right] \Delta t_{M}.
$$

(4.69)

If one chooses $g_{i} \to +\infty$ and $k_{j} \to +\infty$, the second order tensor $a_{0M}$ vanishes, and $\Delta \varepsilon_{0}^{af} = \Delta \varepsilon_{0}^{af}_{evp}$, which is the zero order macro-chronological EVP affine strain increment given by [Doghri et al., 2010b]. Using equation (4.67) and stress and strain macro-and micro-chronological decomposition, the macro-chronological zero-order stress increment can be written as follows:

$$
\Delta \sigma_{0} = C_{0}^{algo}(t_{M_{n+1}}) : \left[ \Delta \varepsilon_{0} - \Delta \varepsilon_{0}^{af} \right] - \Delta \tilde{\sigma}_{0},
$$

(4.70)

Where the zero order affine strain increment $\Delta \varepsilon_{0}^{af}$ is given, using equation (4.52) by this expression:

$$
\Delta \varepsilon_{0}^{af} = \left( \Delta \varepsilon_{0}^{af} - \hat{E}^{-1}(\Delta t_{M}) : \tilde{a}_{0}(t_{M_{n}}, \tau) - \Delta \tilde{\varepsilon}_{0} \right).
$$

(4.71)

The zero order stress-strain incremental relation in equation (4.70) is an affine relation in the time domain of the form $y = ax + b$. It can be applied for any VE-VP model whose inelastic behavior is described by a set of scalar and/or tensor variables $V$. Based on the local VE-VP model of the matrix phase. The form is similar to that proposed by [Miled et al., 2013] for VE-VP anisotropic materials, except that here we are providing a convenient way to find an approximation of the solution to the zero order, instead of the exact one.

4.4.3 Application to $J_{2}$ viscoplasticity

In order to illustrate the incrementally affine method, we apply it to the classical rate-dependent $J_{2}$ VP model of the matrix phase presented in
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Chapter 3, Section 3.2.4 and assuming an isotropic hardening only, the macro-chronological zero order evolution equations (4.36) and (4.37) of the VP strain $\varepsilon^{vp}$ and the scalar internal variable $V \equiv p$ are given by the following expressions:

$$\dot{\varepsilon}^{vp}_0 = \mathbf{F}_{\varepsilon M}((\sigma_0, p_0) = <\dot{p}_0 N_0 >), \quad (4.72)$$

$$\dot{p}_0 = F_{p M}((\sigma_0, p_0) = <g_v (\sigma_0_{eq}, p_0) >). \quad (4.73)$$

Here $\mathbf{F}_{\varepsilon M}$ and $F_{p M}$ are two operators describing the VP evolution within the matrix phase.

In this subsection we study only the case of one point integration. We assume that the time averages can be approximated by the value of the function at the midpoint of the integration interval, which corresponds to $\tau = \frac{1}{2}$.

The partial derivatives are found by simple use of the chain rule:

$$C_{M,\sigma_0} (t_{Mn+1}) = \frac{\partial g_v}{\partial \sigma_0_{eq}} (t_{Mn+1}, \tau = \frac{1}{2}) N_0 (t_{Mn+1}, \tau = \frac{1}{2}),$$

$$C_{M,p_0} (t_{Mn+1}) = \frac{\partial g_v}{\partial p_0} (t_{Mn+1}, \tau = \frac{1}{2}),$$

$$B_{M,\sigma_0} (t_{Mn+1}) = C_M \frac{\partial N_0}{\partial \sigma_0} (t_{Mn+1}, \tau = \frac{1}{2})$$

$$+ N_0 (t_{Mn+1}, \tau = \frac{1}{2}) \otimes C_{M,\sigma_0},$$

$$B_{M,p_0} (t_{Mn+1}) = C_{M,p_0} N_0 (t_{Mn+1}, \tau = \frac{1}{2}). \quad (4.74)$$

Then the macro zero-order affine strain increment $\Delta \varepsilon^{af}_{0\text{evp} M}$ is found by replacing the expressions of partial derivatives (4.74) in equation (4.69):

$$\Delta \varepsilon^{af}_{0\text{evp} M} = \dot{p}_0 (t_{Mn}) \Delta t_M \left[ N_0 \left( t_{Mn}, \tau = \frac{1}{2} \right) \right.$$

$$+ N_0 \left( t_{Mn+1}, \tau = \frac{1}{2} \right) \frac{g_v (t_{Mn+1}, \tau = \frac{1}{2}) \Delta t_M}{1 - g_v (t_{Mn+1}, \tau = \frac{1}{2}) \Delta t_M} \right]. \quad (4.75)$$
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4.5 Mean-field homogenization (MFH)

4.5.1 Mean-field homogenization (MFH) of the heterogeneous VE-VP macro-chronological problems

4.5.1.1 Analogy with thermoelasticity

As the zero order response of a VE-VP anisotropic material can be expressed in incrementally affine form, equation (4.70), which is quiet similar to linear thermoelasticity (Section 2.3.4), existing homogenization models for thermoelastic composites can therefore be used at each time step via a simple analogy. In fact, if one consider a two phase linear thermoelastic composite subjected to cyclic loading, the local constitutive response is written as:

\[
\begin{align*}
\sigma_{\vec{x},t} &= C_{el}^{el}(\vec{x}) : (\varepsilon(\vec{x},t) - \varepsilon^{th}(\vec{x},t)), \\
\varepsilon^{th} &= \alpha(\vec{x},t) \Delta \theta, \\
\beta(\vec{x},t) &= -C_{el}^{el}(\vec{x}) : \alpha(\vec{x},t) \Delta \theta.
\end{align*}
\]

(4.76)

Where \(C_{el}^{el}(\vec{x})\) is the elastic stiffness, \(\Delta \theta\) is the increment of temperature and \(\alpha(\vec{x},t)\) the thermal expansion.

Consider a two phase VE-VP, representative volume element (RVE) occupying a domain \(\Omega\) and containing a finite number of inclusions occupying a volume \(\Omega_I\) with a volume fraction \(\upsilon_I\). The matrix of volume fraction \(\upsilon_m\) occupies a volume \(\Omega_m\). The subscripts \(m\) and \(I\) denote matrix and inclusion phases, respectively. We denote by \(< \phi >_{\Omega_r}\) the volume average of \(\phi\) in a given phase \(r = (I,m)\), and by \(\bar{\phi} = < \phi >_{\Omega}\) the volume average of \(\phi\) over the RVE. Recalling equation (4.70):

\[
\Delta \sigma_{0M} = C_{0}^{algo}(t_{M_{n+1}}) : \left[ \Delta \varepsilon_{0M} - \Delta \varepsilon_{0}^{af} \right] - \Delta \tilde{\sigma}_0.
\]

(4.77)

Using the decomposition \(\Delta \sigma_0 = \Delta \sigma_{0M} + \Delta \tilde{\sigma}_0\), the following substitutions are made:

\[
\sigma \rightarrow \Delta \sigma_0, \quad C^{el} \rightarrow C_0^{algo}, \quad \varepsilon \rightarrow \Delta \varepsilon_{0M}, \quad \beta \rightarrow -C_0^{algo} : \Delta \varepsilon_{0}^{af}.
\]

(4.78)

Using the substitutions in equation (4.78) and equations (2.55 - 2.56) the
4.5. Mean-field homogenization (MFH)

Macroscopic response of the VE-VP composite can be written as follows:

\[
\Delta \sigma_0 = \bar{C}_0 : \Delta \varepsilon_{0M} - \nu m C_{m0}^{algo} : \Delta \varepsilon_{af}^{m0} - \nu I C_{I0}^{algo} : \Delta \varepsilon_{af}^{I0} - \nu I (C_{I0}^{algo} - C_{m0}^{algo}) \\
: (A^e - I) : \left( C_{I0}^{algo} - C_{m0}^{algo} \right)^{-1} : \left( C_{I0}^{algo} : \Delta \varepsilon_{af}^{I0} - C_{m0}^{algo} : \Delta \varepsilon_{af}^{m0} \right).
\]  

(4.79)

Here \( \bar{C}_0 \) is the zero-order effective instantaneous stiffness of the composite as computed in the isothermal case:

\[
\bar{C}_0 = C_{m0}^{algo} + \nu I \left( C_{I0}^{algo} - C_{m0}^{algo} \right) : A^e.
\]

(4.80)

\( A^e \) is the strain concentration tensor, expressed by the following equation in the case of MT scheme:

\[
A^e = \left( I + (1 - \nu I) S : \left( C_{I0}^{algo} \right)^{-1} : C_{m0}^{algo} - I \right)^{-1}.
\]

(4.81)

\( S \) is Eshelby’s tensor, which depends only on the properties of the matrix and the inclusion shape.

4.5.1.2 Isotropisation of the tangent operator

The zero-order algorithmic tangent operators for the matrix \( C_{m0}^{algo} \) and inclusion \( C_{I0}^{algo} \) phases are uniform and anisotropic. Numerical experience has shown that predictions of incrementally affine (or incremental) formulation, using anisotropic moduli are too stiff. Then, to obtain good numerical predictions, Eshelby’s tensor and Hill’s tensor are computed from the isotropic part of the tangent moduli and not from the anisotropic tensors \( (C_{ani}^{algo}) \), ([Pierard and Doghri, 2006](#pierard2006), [Doghri and Friebel, 2005](#doghri2005), [Doghri and Ouaar, 2003](#doghri2003)). Several techniques or isotropisation methods have been proposed to extract the isotropic part of an anisotropic tensor among them we may cite the general method and the spectral method. The key idea is to find two scalars \( \mu_t \) and \( k_t \) from \( C_{ani}^{algo} \) so that the isotropic fourth-order tensor \( C_{iso}^{algo} \) can be expressed as follows:

\[
C_{iso}^{iso} = 3k_t \Gamma_{vol} + 2\mu_t \Gamma_{dev}.
\]

(4.82)
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4.5.2 Mean-field homogenization (MFH) of the heterogeneous VE micro-chronological problem

To obtain the global properties of a VE composite homogenization schemes in the Laplace-Carson space can be applied (For more details see the PhD thesis of [Friebel, 2007]). Another alternative (used in this work) is to obtain the effective behavior of the micro-chronological problem using the same technique as for the macro problem when the initial yield stress tends to $+\infty$. In this case the VE predictor is always the solution.

4.6 Numerical implementation

In order to compute the effective behavior of the composite material subjected to large numbers of cycles, two cases are studied. First a VE-VP matrix reinforced with aligned inclusions. Second a composite containing misaligned fibers. The numerical procedure is described in the following subsections.

4.6.1 Composites with aligned fibers

In this subsection we give an overview of the algorithm. Consider a macro-dimensionless time interval $[t_{M_n}, t_{M_{n+1}}]$, for which we assume that the global macro- and micro-chronological total strains $\overline{\varepsilon_0}(t_{M_n})$ and $\overline{\tilde{\varepsilon}}(t_{M_n}, \tau)$ and the global macro- and micro-chronological strain increments $\Delta \overline{\varepsilon_0}$ and $\Delta \overline{\tilde{\varepsilon}}$ and all global and per phase zeroth-order macro- and micro-chronological history variables $\overline{\phi_0}$, $< \phi_0 > \Omega$, $\phi_0$ and $< \tilde{\phi}_0 > \Omega$, at $t_{M_n}$ and $\tau$ are known. The problem is to compute the global zeroth-order stress $\overline{\sigma_0}(t_{M_{n+1}}, \tau)$ and the zeroth-order effective instantaneous stiffness $\overline{C_0}(t_{M_{n+1}}, \tau)$. The algorithm is presented in figure 1 and described hereafter:

1. Initialization of all variables to zero.

2. Resolution of the VE micro-chronological problem (VE matrix reinforced with aligned elastic inclusions) at $t_{M_n+1}$ and $\tau = \tau_j$, using the numerical algorithm proposed by [Miled et al., 2013]. The input consists of: the micro-chronological zero-order total strain $\overline{\varepsilon_0}(t_{M_{n+1}}, \tau = \tau_j)$, the micro-chronological zero-order total strain increment $\Delta \overline{\varepsilon_0}(t_{M_{n+1}})$, and the per-phase averaged zero-order micro-chronological history variables at $t_{M_n}$ and $\tau_j$. The al-
4.6. Numerical implementation

Figure 4.2: Summary of the coupled space and time-homogenization method for coupled VE-VP composites. Time stepping is performed with respect to macro dimensionless time \( t_M \).
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Algorithm proposed by [Miled et al., 2013] enables the computation of the zero-order effective micro-chronological response of the composite: \( \tilde{\sigma}_0(t_{M_n+1}, \tau = \tau_j) \) and \( \tilde{C}_0(t_{M_n+1}, \tau = \tau_j) \). It determines the the zero-order micro-chronological strain and stress averages per-phase.

The output needed for the homogenization of the macro-chronological problem are: the per-phase zero-order micro-chronological stresses \( < \tilde{\sigma}_0(t_{M_n+1}, \tau = \tau_j) > \Omega_r \) and strains \( < \tilde{\varepsilon}_0(t_{M_n+1}, \tau = \tau_j) > \Omega_r \).

3. Resolution of the VE-VP macro-chronological problem (VE-VP matrix reinforced with aligned elastic inclusions), using the numerical algorithm proposed by [Miled et al., 2013]. The input consists of: the macro-chronological zero-order total strain \( \varepsilon_0(t_{M_n}) \), the macro-chronological zero-order total strain increment \( \Delta \varepsilon_0(t_{M_n}) \), the per-phase averaged zero-order macro-chronological history variables at \( t_{M_n} \), and the per-phase zero-order micro-chronological stresses \( < \tilde{\sigma}_0(t_{M_n+1}, \tau = \tau_j) > \Omega_r \) and strains \( < \tilde{\varepsilon}_0(t_{M_n+1}, \tau = \tau_j) > \Omega_r \).

Call VE-VP constitutive box of the matrix with \( < \tilde{\sigma}_0(t_{M_n+1}, \tau = \tau_j) > \Omega_r \), \( < \tilde{\varepsilon}_0(t_{M_n+1}, \tau = \tau_j) > \Omega_r \), \( \varepsilon_0(t_{M_n}) \), \( \Delta \varepsilon_0(t_{M_n}) \), \( \bar{\varepsilon}_0(t_{M_n}) \), and \( \bar{\sigma}_0(t_{M_n}) \) as an input.

4. The macro-time is incremented.

4.6.2 Composites with misaligned fibers

Consider a two phase composite made of VE-VP matrix and misaligned elastic fibers, subjected to large numbers of cycles. In order to obtain the effective properties of this kind of composite a two step homogenization method (detailed in Section 2.3.6), is performed (figure 4.3).

1. Resolution of the micro-chronological problem (VE matrix reinforced with misaligned elastic inclusions) at \( t_{M_n+1} \) and \( \tau = \tau_j \).

- Mori-Tanaka inside the pseudo-grains:
  For each pseudo-grain MFH model is integrated using the algorithm proposed in Section 4.6.1. The algorithm enables the computation of the effective micro-chronological zero-order response of each pseudo-grain: \( \tilde{C}_0(t_{M_n+1}, \tau = \tau_j) \) and \( \tilde{\sigma}_0(t_{M_n+1}, \tau = \tau_j) \).
4.6. Numerical implementation

Figure 4.3: Illustration of the pseudo-grain numerical implementation.

- Homogenization over all pseudo-grains using the Voigt model:
  The homogenized response of the micro-chronological RVE is computed based on the response of individual pseudo-grains, by ODF-weighted averaging over all pseudo-grains (see Section 2.3.6).

2. Resolution of the VE-VP macro-chronological problem (VE-VP matrix reinforced with misaligned elastic inclusions).

- Mori-Tanaka inside the pseudo-grains:
  For each pseudo-grain MFH model is integrated using the algorithm proposed in Section 4.6.1. Knowing the effective micro-chronological zero-order response of each pseudo-grain: \( \tilde{C}_{0,p.g.}(t_{M_n+1}, \tau = \tau_j) \) and \( \tilde{\sigma}_{0,p.g.}(t_{M_n+1}, \tau = \tau_j) \), the algorithm enables the computation of the effective zero-order response of each pseudo-grain: \( C_{0,p.g.}(t_{M_n+1}) \) and \( \sigma_{0,p.g.}(t_{M_n+1}) \).

Call VE-VP constitutive box with \( \langle \tilde{\sigma}_0(t_{M_n+1}, \tau = \tau_j) \rangle_{\Omega_{r,p.g.}}, \langle \tilde{\varepsilon}_0(t_{M_n+1}, \tau = \tau_j) \rangle_{\Omega_{r,p.g.}}, \tilde{\varepsilon}_{0_M}(t_{M_n}) \) and \( \Delta \tilde{\varepsilon}_{0_M} \) as an input.

- Homogenization over all pseudo-grains using the Voigt model:
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The homogenized response of the macro-chronological VE-VP RVE is computed based on the response of individual pseudo-grains, by ODF-weighted averaging over all pseudo-grains.

3. After convergence, the macro-time is incremented.

4.7 Numerical simulations and their verification

The coupled space and time-homogenization method was implemented and tested for an uniaxial loading case using one-point integration for the micro-chronological averages, and supposing the periodicity of the accumulated plastic strain variable $p$ with respect to the fast time coordinate $\tau$.

Kinematic hardening is neglected and the hardening function considered in the following is of power-law type:

$$ R(p) = kp^n, $$

where, $k$ [MPa] is the hardening modulus and $n$ the hardening exponent. The matrix’s overstress due to the rate-dependence also obeys a power-law (equation (4.57)).

All the simulations are performed using a machine with 6 cores and 32 GB of RAM.

4.7.1 Composites with aligned fibers

In this section we consider a two phase composite made of a polyamide matrix reinforced with 30% of volume fraction of aligned glass fibers with an angle 45° relative to the injection flow direction (IFD) and with same aspect ratio $A = 23$. The VE-VP material parameters of the matrix and the elastic properties of the short glass fibers are listed in table 4.1.

Consider a one-dimensional cylindrical bar of length $L$ clamped at one end ($x = 0$) and subjected at the other end ($x = L$) to a displacement which is linear at first and then sinusoidal with period $T^* = 0.1s$, and amplitude $U = 0.05L$:

$$ u_k^T (x = L, t^*) = \alpha L t^*, \quad \text{if} \quad t^* \leq T^* $$

$$ u_k^T (x = L, t^*) = U \left( 0.45 \sin \left( \frac{2\pi}{T^*} t^* \right) + 0.55 \right) \quad \text{otherwise}, \quad (4.84) $$
4.7. Numerical simulations and their verification

Table 4.1: Constitutive model parameters for polyamide (PA) matrix at 40°C ([Miled, 2011]) identified from experimental measurements of [Baquet, 2011], reinforced with 30% of volume fraction of aligned short glass fiber elastic inclusions

<table>
<thead>
<tr>
<th>VE-VP matrix</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Viscoelastic parameters</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Initial shear modulus</td>
<td>$G_0 = 1074$ MPa</td>
<td></td>
</tr>
<tr>
<td>Initial bulk modulus</td>
<td>$K_0 = 3222$ MPa</td>
<td></td>
</tr>
<tr>
<td>$G_i$(MPa)</td>
<td>$g_i(s)$</td>
<td>$K_i$ (MPa)</td>
</tr>
<tr>
<td>158</td>
<td>0.021</td>
<td>472</td>
</tr>
<tr>
<td>80</td>
<td>0.378</td>
<td>242</td>
</tr>
<tr>
<td>37</td>
<td>0.648</td>
<td>111</td>
</tr>
<tr>
<td>Viscoelastic parameters</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Hardening function</td>
<td>$k = 79$ MPa</td>
<td>$n = 0.15$</td>
</tr>
<tr>
<td>Viscoplastic function</td>
<td>$\zeta = 305$ MPa.s</td>
<td>$m = 4.02$</td>
</tr>
<tr>
<td>Yield stress</td>
<td>$\sigma_y = 10$ MPa</td>
<td></td>
</tr>
<tr>
<td>Elastic inclusions</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Young’s modulus</td>
<td>$E_I = 76$ GPa</td>
<td></td>
</tr>
<tr>
<td>Poisson’s ratio</td>
<td>$\nu_I = 0.22$</td>
<td></td>
</tr>
</tbody>
</table>
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where \( \alpha = 0.275 \text{s}^{-1} \).

The body forces \( \vec{f} \) are neglected and \( 10^5 \) cycles are applied. The prescribed displacement can be rewritten using equation relating \( t^* \) to \( t^M, \tau \) and \( T^* \):

\[
t^* = t^*_M + T^*\tau, \quad t^*_M \in [0, T^*_F] \text{ and } \tau \in [0, 1].
\]

The loading ratio \( R \), which corresponds to the ratio between the minimum and the maximum of the loading: \( R = \frac{U_{\text{min}}}{U_{\text{max}}} \), is equal to 0.1.

Comparisons between the reference non-homogenized (full-time) calculation and the two-scale one are given in figures (4.4)-(4.10). The reference numerical solution is obtained using a very fine time step \( \Delta t^* = T^*_F \), whereas the time-homogenized computation is started by a full calculation until time \( t^* = \frac{2}{3} T^* \) and then the calculations are continued using \( \Delta t^*_M = T^* \).

In figure 4.4 are represented the effective stress-strain results for some selected cycles. The solid lines show the reference results, while the black crosses depict the time homogenization predictions. It is noted that due to the repeated action of the rather small applied strain, the stress level decreases significantly from a maximum of 200 MPa in the first cycle to about 150 MPa in the following ones. Furthermore the stress-strain hysteresis loop "shrinks" rapidly with increasing number of cycles. It is seen in figures (4.5) and (4.6) that for the first cycles the crosses do not superpose with the peaks of stress, however, for the last cycles they converge towards the stress peaks.

Relative differences between time homogenized and of full-scale results are studied to check the accuracy of the coupled time and space homogenization approach. A relative difference is defined as:

\[
\text{Relative difference} = \frac{\text{Reference result} - \text{Time-homogenized result}}{\text{Reference result}}.
\]

Homogenized calculations underestimate the solution. The relative difference, in absolute value, between the effective zero order homogenized solution \( \bar{\sigma}_0 \) and the effective reference solution \( \bar{\sigma} \) for the first twenty cycles is about 19% and about 9% for the last twenty cycles.

In figure (4.7) are represented the zero-order stress averages within the matrix phase. The relative difference, in absolute value, between the
zero order homogenized stress within the matrix \(<\sigma_0>_{\Omega_m}\) and the reference solution \(<\sigma>_{\Omega_m}\) decreases when the number of cycles increases, to reach the value of 7%. The evolution of the relative error on the stress average in the matrix phase is similar to that observed in Chapter 3 for homogeneous PA, but its value is still important when compared to the homogeneous case (it is about 1%).

This can be explained by several reasons. First, only the zero-order in the asymptotic expansion is considered. Second, for the first twenty cycles the load period is not sufficiently small compared to the observation time which is equal in this case to 20 \(T^*\). Third, the increase of the value of the relative error on the average stress in the matrix phase compared to the homogeneous case can be explained by the anisotropy of the material. In fact the accuracy of the time homogenization depends on the time scales in the VE and VP parts. This dependency was discussed in Chapter 3. In the case of anisotropic materials, VE and VP properties are directionally dependent and this can influence the accuracy of the time homogenization.

Figure (4.8) shows the evolution of the macro-chronological zero order strain average within the matrix phase \(<\varepsilon_0>_{\Omega_m}\) as function of the number of cycles. One can note that \(<\varepsilon_0>_{\Omega_m}\) is not constant and it varies according to the number of cycles. This leads to a variation of the imposed maximum strain value within the matrix \(<\varepsilon_0>_{\Omega_m} = <\varepsilon_0>_{\Omega_m} + <\tilde{\varepsilon}_0>_{\Omega_m}\). In this case the asymptotic expansion is re-done each time when the imposed strain value \(<\varepsilon_0>_{\Omega_m}\) changes. This variation is expected to have an influence on the accuracy of the asymptotic expansion although figure (4.8) shows that the range of strain variation is limited to 0.004%.

Figure 4.9 presents the evolution of the accumulated plastic strain in the matrix at the end of the simulation. Coupled time and space homogenization underestimates the accumulated plastic strain within the matrix. The evolution of the relative difference, in absolute value, between the zero order homogenized solution \(p_0\) and the reference solution \(p\) as function of the number of cycles is shown in figure 4.10. The relative difference, in absolute value, between the zero-order homogenized solution \(p_0\) and the reference solution \(p\) for the last twenty cycles is about 40%. This can be explained by the variation of the imposed strain value to the matrix phase.

For the reference calculation which corresponds to the full-scale computation of \(10^5\) cycles, the CPU time is 3242 s, while with the time
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Figure 4.4: Uniaxial load. Hysteresis loops: First hundred cycles, loading period: $T^* = 0.1$ s, loading ratio: $R = 0.1$. The RVE is composed of a PA matrix at 40°C and 30% volume fraction of aligned glass fibers (45° relative to the IFD).

The homogenization method the CPU time is 244 s. The relative gain in computation time is then about 92%.

4.7.2 Composites with misaligned fibers

In this section we consider a two phase composite made of a polyamide matrix reinforced with 30% of misaligned glass fibers with same aspect ratio $A = 23$. The VE-VP material parameters of the matrix and the elastic properties of the glass particles are listed in table 4.1. A total of 36 pseudo-grains is considered representing different fiber orientations relative to the IFD. Their weights are presented in figure 4.11. All experiments were carried out by Solvay Engineering Plastics company on dry specimens with 0% of relative humidity (RH). Consider a one-dimensional cylindrical bar of length $L$ clamped at one end ($x = 0$) and subjected at the other end ($x = L$) to a displacement which is linear at first and then sinusoidal with period $T^* = 0.1$s, and amplitude $U = 0.02L$:

\[
\begin{cases}
    u_b^T (x = L, t^*) = \alpha L t^* , & \text{if } t^* \leq T^* \\
    u_b^T (x = L, t^*) = U \left( 0.45 \sin \left( \frac{2\pi}{T^*} t^* \right) + 0.55 \right) & \text{otherwise}
\end{cases}
\]

(4.86)

where $\alpha = 0.11 \text{s}^{-1}$. 

4.7. Numerical simulations and their verification

Figure 4.5: Uniaxial load. Hysteresis loops: Last twenty cycles, loading period: $T^* = 0.1$ s, loading ratio: $R = 0.1$. The RVE is composed of a PA matrix at 40°C and 30% volume fraction of aligned glass fibers (45° relative to the IFD).

Figure 4.6: Uniaxial load. Evolution of the effective stress: homogenized solution in comparison with reference results, loading period: $T^* = 0.1$ s, loading ratio: $R = 0.1$. (a) The first twenty cycles. (b) The last twenty cycles. The RVE is composed of a PA matrix at 40°C and 30% volume fraction of aligned glass fibers (45° relative to the IFD).
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Figure 4.7: Uniaxial load. Evolution of the zero-order stress average in matrix phase: homogenized solution in comparison with reference results, loading period: $T^* = 0.1 \, \text{s}$, loading ratio: $R = 0.1$. (a) The first twenty cycles. (b) The last twenty cycles. The RVE is composed of a PA matrix at $40^\circ \text{C}$ and 30\% volume fraction of aligned glass fibers ($45^\circ$ relative to the IFD).

Figure 4.8: Evolution of the macro-chronological strain average within the matrix $\langle \varepsilon_{0_M} \rangle_{\Omega_m}$. Loading period: $T^* = 0.1 \, \text{s}$, loading ratio: $R = 0.1$. 
4.7. Numerical simulations and their verification

Figure 4.9: Evolution of the accumulated plastic strain within the matrix phase: homogenized solution in comparison with reference results. Loading period: $T^* = 0.1 \, \text{s}$, loading ratio: $R = 0.1$. (a) The first twenty cycles. (b) The last twenty cycles.
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Figure 4.10: Evolution of the error on the accumulated plastic strain within the matrix phase. Loading period: \( T^* = 0.1 \) s, loading ratio: \( R = 0.1 \).

Figure 4.11: Weights of 36 pseudo-grains representing different fiber orientations relative to the injection flow direction (IFD). Weights are used differently with respect to the fiber mass fractions: 20%, 30% and 50% (or 10%, 16% and 30% volume fractions), (Kammoun, 2011).
4.8. Conclusions

Figure 4.12: Uniaxial load. Evolution of the effective stress: homogenized solution in comparison with reference results. The last twenty cycles, loading period: $T^* = 0.1 \, s$, loading ratio: $R = 0.1$. The RVE is composed of a PA matrix at $40^\circ C$ and 30% volume fraction of misaligned glass fibers with same aspect ratio $A = 23$.

The body forces $\vec{f}$ are neglected and $2 \times 10^5$ cycles are applied. The prescribed displacement can be rewritten using equation (4.85) relating $t^*$ to $t^*_M$, $\tau$ and $T^*$. The loading ratio $R$, is equal to 0.1.

Comparisons between the reference calculation (solid line) and the homogenized in the case of PA66 reinforced with misaligned short glass fibers are presented in figure 4.12. For the reference calculation a very fine time step is used $\Delta t = \frac{T}{2T}$, whereas for the homogenized one $\Delta t^*_M = T$.

Figure 4.12 presents the evolution of the oscillatory stress for the last twenty cycles. Only peak stresses are presented for the homogenized solutions. The black crosses show the results obtained by time homogenization using one-point integration. The relative error on the stress at the end of the simulation is about 13%.

4.8 Conclusions

In this chapter, a coupled spatial and temporal homogenization formulation for a VE-VP anisotropic constitutive model has been presented. The initial boundary value problem is decomposed into coupled micro-
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chronological (fast time scale $\tau$) and macro-chronological (slow time-scale ($t_M$)) problems. The former corresponds to a purely VE composite, whereas the latter problem is nonlinear and corresponds to a VE-VP composite. Mean-field space homogenization is used for both micro- and macro-chronological problems to determine the effective behavior of the composite material.

The coupled space and time homogenization approach has been implemented and tested in an uniaxial test, for nonlinear PA matrix reinforce with aligned and misaligned short glass fibers and has been found to be in good agreement with the reference solution while estimating the effective and the per-phase stresses. Nevertheless, the method does not give a good approximation of the accumulated plastic strain in the matrix phase. This, is explained by the anisotropy of the material and the variation of the imposed zero-order strain value to the matrix phase.

A significant reduction in the amount of computation time is obtained (about 92%).

In Chapter 5, the temporal homogenization scheme presented in Chapter 3 is extended to fatigue analysis of homogeneous solids, in order to resolve later the problem of fatigue of heterogeneous materials.
Fatigue life prediction of a viscoelastic-viscoplastic homogeneous material with ductile damage

5.1 Introduction

Fatigue analysis and lifetime evaluation are very important in the design of compliant mechanisms to ensure their safety and reliability. Time-varying cyclic loads result in failure of components at stress values below the yield or ultimate strength of the material. There are two commonly recognized forms of fatigue: high cycle fatigue (HCF) which consists on applying relatively small stress amplitudes which induce large numbers of cycles to failure (larger than $10^5$) and low cycle fatigue (LCF) which consists on applying stress amplitudes above the yield stress which induce lower numbers of cycles to failure (smaller than $10^4$). To describe the fatigue of a material one should determine the stress required to cause the fatigue failure for some number of cycles. This can be shown in the S-N curve in figure 5.1 which corresponds to a plot of the applied stress amplitude $\sigma_a$ against the number of cycles to failure $N_f$ for the high density polyethylene (HDPE) under different loading cases ([Berrehili, 2010]). If the stress is below the fatigue limit (or endurance limit) $\sigma_e$, the component has effectively infinite life (figure 5.1). Fatigue failure of components takes place by the initiation and propagation of a crack until it becomes unstable and then propagates to sudden failure. The goal is then to avoid or to predict the failure of the material especially under conditions of cyclic loading. This can be done by studying the evolution of internal damage before macro-cracks become visible. In this work we are interested in fatigue modeling of materials by damage accumulation.

Two approaches are mainly used to model damage. First, meso-damage mechanics ([Feng et al., 2004], [Gurson, 1977]), which takes into consideration the influence of microstructures on the failure be-
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Figure 5.1: S-N curve for HDPE for different loading cases, ([Berrehili, 2010](#)).

Damage and the damage mechanisms of materials. Second, continuum damage mechanics (CDM) ([Lemaitre, 1985; Lemaitre, 1992](#)), which studies the damage behavior on the basis of continuum thermodynamics and continuum mechanics. The objective is to propose a continuum mechanics based framework to characterize at the macroscopic scale the effects of distributed defects on the material behavior. CDM approach is based on the concept of damage variable \( D \) and effective stress \( \sigma^{ef} \).

Consider a finite volume of a damaged solid (figure 5.2) loaded by a force \( F \). Then the usual uniaxial stress is \( \sigma = \frac{F}{S} \), where \( S \) is a cross section of the volume element. The damage is obtained by measuring the effective area \( S^{ef} \) of the intersection of all defects with that plane: \( S^{ef} = S - S_D \). \( S_D \) represents the defects trace in the considered plane. The following positive scalar \( D \) is then commonly considered as a damage variable, in this simple one-dimensional case of figure 5.2:

\[
D = \frac{S_D}{S}. \tag{5.1}
\]

The damage \( D \) related to the growth of defects, is then bounded by 0 which corresponds to an undamaged material and 1 which corresponds to an entirely damaged material and the breaking of the volume element,
It is then convenient to introduce for the damaged material the effective stress $\sigma^{ef}$ which is given in uniaxial case by the following expression:

$$\sigma^{ef} = \frac{F}{S^{ef}} = \frac{F}{S(1 - \frac{S_D}{S})} = \frac{\sigma}{1 - D}. \quad (5.2)$$

In this work we will limit ourselves to the case of isotropic damage, and we suppose that cracks and cavities are uniformly distributed in all directions. Then the damaged state is characterized by a scalar $D$. For anisotropic damage, the variable is no longer a scalar, and it depends on the orientation.

For metals, damage can be directly measured by evaluating the area $S_D$ of the intersection of all micro-cracks and micro-voids which lie in $S$ using micrographic pictures ([Lemaitre and Dufailly, 1987]). In the case of pure ductile damage, the defects are assumed to be spherical and the damage can be evaluated by measuring the decrease of the density with apparatuses based on the Archimedean principle, ([Lemaitre and Dufailly, 1987]). If $\rho^{ef}$ and $\rho$ designate the densities of the damaged and undamaged state, respectively, by means of micromechanics and
assuming no residual micro-stress, the damage $D$ can be evaluated as follows:

$$ D = \left(1 - \frac{\rho_{ef}}{\rho}\right)^{\frac{2}{3}}. $$

(5.3)

Damage can also be measured indirectly by evaluating its effects on material properties (e.g. loss of stiffness, figure 5.3). In uniaxial tensile loading, [Lemaitre and Chaboche, 1990] give the expression of the damage in function of the loss of stiffness in the case of elastic law coupled with damage:

$$ D = 1 - \left(\frac{E_{ef}}{E}\right), $$

(5.4)

where $E_{ef}$ and $E$ are Young modulus of the damaged and undamaged material, respectively. For this kind of measurement, [Lemaitre and Chaboche, 1990] recommend the following procedure:

- Use of specimens with weakened central section, in order to localize damage (figure 5.3).
- Measurement of strains with small gauges.
- In order to eliminate the nonlinearities on the stress-strain curve, the elasticity modulus is evaluated during the elastic unloading, between a lower and upper value of stress.
5.2. A damage coupled constitutive model for thermoplastic polymers

Another alternative based also on the variation of the stiffness, consists in measuring the speed of ultrasonic waves. Other techniques and a detailed description of them are available in (Lemaitre and Dufailly, 1987; Lemaitre and Chaboche, 1990).

For polymeric materials, the behavior is viscoelastic (viscoplastic), the load and unload phase are not linear. The material loses energy when a load is applied, then removed. Hysteresis is observed in the stress-strain curve. In this case, the stiffness is taken between the upper and lower point of the hysteresis. This technique is not really rigorous, because the loss of stiffness, can be caused by other phenomena, other than damage, such as selfheating.

Moreover ultrasonic waves are not very effective in detecting internal defects in polymeric materials. Indeed, ultrasonic energy is attenuated very fast in these materials. Another alternative to detect damage in polymers is the THz electromagnetic radiation. Rahani et al., 2011 have used this technique to detect damage in polymer tiles. In fact, its wavelength is small enough to detect internal defects and it can penetrate deep inside the material.

Fluorescence microscopy was also used by Samuel et al., 2007 to detect nanoscale deformation and damage in polymeric materials. This technique borrows from fluorescence dye based imaging, which is commonly used in cell and molecular biology.

5.2 A damage coupled constitutive model for thermoplastic polymers

A thermodynamically-based constitutive model coupling viscoelasticity, viscoplasticity and ductile damage (VE-VP-D) was proposed by Krairi and Doghri, 2014 for isotropic homogeneous thermoplastic polymers. Its first main assumption is that the total strain is decomposed into viscoelastic ($\varepsilon^{ve}$) and viscoplastic ($\varepsilon^{vp}$) parts:

$$\varepsilon = \varepsilon^{ve} + \varepsilon^{vp}. \quad (5.5)$$

5.2.1 Linear viscoelastic part

The Cauchy stress is related to the damage variable $D(t)$ and the history of VE strains through Boltzmann’s integral:

$$\sigma(t) = (1 - D(t)) \int_{-\infty}^{t} C^{ve}(t - \xi) : \frac{\partial \varepsilon^{ve}(\xi)}{\partial \xi} d\xi, \quad (5.6)$$
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where \( \mathbf{C}^{ve} \) is a fourth-order relaxation tensor.

A new second-order tensor \( \mathbf{\sigma}^{ef}(t) \) called effective stress is then introduced:

\[
\mathbf{\sigma}^{ef}(t) = \mathbf{\sigma}(t) \left( 1 - D(t) \right) = \int_{-\infty}^{t} \mathbf{C}^{ve}(t - \xi) : \frac{\partial \mathbf{\varepsilon}^{ve}(\xi)}{\partial \xi} d\xi.
\] (5.7)

Using the thermodynamics of irreversible processes a damage thermodynamic force \( Y \) associated with the damage variable \( D \) is defined by [Krairi and Doghri, 2014] as follows:

\[
Y(t) = \frac{1}{2} \int_{-\infty}^{t} \int_{-\infty}^{t} \frac{\partial \mathbf{\varepsilon}^{ve}(\xi)}{\partial \xi} : \mathbf{C}^{ve}(2t - \xi - \eta) \frac{\partial \mathbf{\varepsilon}^{ve}(\eta)}{\partial \eta} d\xi d\eta.
\] (5.8)

For an isotropic material, the fourth rank relaxation tensor takes the following form:

\[
\mathbf{C}^{ve}(t) = 2G(t)\mathbf{I}^{dev} + 3K(t)\mathbf{I}^{vol},
\] (5.9)

\( G(t) \) and \( K(t) \) are the shear and bulk relaxation moduli, respectively, which can be expressed in the form of Prony series:

\[
G(t) = G_\infty + \sum_{i=1}^{I} G_i \exp \left( -\frac{t}{g_i} \right); \quad K(t) = K_\infty + \sum_{j=1}^{J} K_j \exp \left( -\frac{t}{k_j} \right).
\] (5.10)

Here, \( G_\infty \) and \( K_\infty \) are the elastic shear and bulk long-term moduli, respectively; \( g_i \) \( (i = 1 \ldots I) \) and \( k_j \) \( (j = 1 \ldots J) \) are shear and bulk relaxation times respectively; and \( G_i \) \( (i = 1 \ldots I) \) and \( K_j \) \( (j = 1 \ldots J) \) are shear and bulk weights respectively.

By substituting equations (5.9) and (5.10) into equation (5.7), a decomposition of the effective stress tensor into deviatoric \( (\mathbf{s}^{ef}(t)) \) and dilatational \( (\mathbf{\sigma}^{ef}(t)) \) parts, and the strain tensor into deviatoric \( (\mathbf{\varepsilon}^{dev}(t)) \) and dilatational \( (\mathbf{\varepsilon}^{vol}(t)) \) parts is obtained:

\[
\begin{cases}
\mathbf{s}^{ef}(t) = \mathbf{s}_{\infty}^{ef}(t) + \sum_{i=1}^{I} \mathbf{s}_i^{ef}(t), \\
\mathbf{\sigma}^{ef}(t) = \mathbf{\sigma}_{\infty}^{ef}(t) + \sum_{j=1}^{J} \mathbf{\sigma}_{H,j}^{ef}(t),
\end{cases}
\] (5.11)
5.2. A damage coupled constitutive model for thermoplastic polymers

where:

\[
\begin{align*}
\sigma_{\text{ef}}(t) & = 2G \int_{-\infty}^{t} \exp \left( \frac{\eta - t}{g_i} \right) \frac{\partial \varepsilon_{\text{ve}}(\eta)}{\partial \eta} d\eta, \\
\sigma_{H_{\infty}}(t) & = 3K \int_{-\infty}^{t} \exp \left( \frac{\eta - t}{k_j} \right) \frac{\partial \varepsilon_{H}(\eta)}{\partial \eta} d\eta.
\end{align*}
\]  

(5.12)

Then, using equation (5.11), the effective stress tensor can be written as:

\[
\sigma^{\text{ef}}(t) = C_{\infty} : \varepsilon_{\text{ve}}(t) + \sum_{i=1}^{I} \sigma_{H_{i}}^{\text{ef}}(t) + \sum_{j=1}^{J} \sigma_{H_{j}}^{\text{ef}}(t) 1.
\]  

(5.13)

A simple expression of the damage thermodynamic force is obtained using equations (5.11) and (5.12) ([Krairi and Doghri, 2014]):

\[
Y(t) = \frac{s_{\text{ef}}^{\text{ef}} : s_{\text{ef}}^{\text{ef}}}{4G_{\infty}} + \frac{(\sigma_{H_{\infty}}^{\text{ef}})^2}{2K_{\infty}} + \sum_{i=1}^{I} \frac{s_{i}^{\text{ef}} : s_{i}^{\text{ef}}}{4G_{i}} + \sum_{j=1}^{J} \frac{(\sigma_{H_{j}}^{\text{ef}})^2}{2K_{j}}.
\]  

(5.14)

5.2.2 Viscoplastic part

[Krairi and Doghri, 2014] used the classical $J_2$ rate-dependent model with isotropic and kinematic hardening to depict the VP behavior. In this work only isotropic hardening is considered. The yield criterion is given as follows:

\[
f(\sigma, R; D) = \sigma_{eq} - \sigma_{y} - R(r).
\]  

(5.15)

Here, $\sigma_{eq}$ is the von Mises measure of $\sigma$:

\[
\sigma_{eq} = \left( \frac{3}{2} \sigma^{\text{ef}} : \sigma^{\text{ef}} \right)^{1/2}.
\]  

(5.16)

Here, $\sigma_{y}$ is the initial yield stress (which may depend on the strain rate) and $R(r)$ is the hardening stress. $r$ is an internal variable which models isotropic hardening and is related to the accumulated plastic strain $p$ through the following expression:

\[
\dot{r} = (1 - D(t)) \dot{p}; \quad \dot{p} = \sqrt{2 \dot{\varepsilon}^{\text{vp}} : \dot{\varepsilon}^{\text{vp}}}.
\]  

(5.17)

It is defined by a VP function $g_{\text{v}}$:

\[
\begin{align*}
\dot{r} & = 0 \quad \text{if } f \leq 0, \\
\dot{r} & = g_{\text{v}}(\sigma_{eq}, r) > 0 \quad \text{if } f > 0.
\end{align*}
\]  

(5.18)
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e.g. Norton power law: \( g_v = \frac{\sigma_v}{\xi} \left( \frac{\dot{f}}{\sigma_v} \right)^m \), where the two parameters \( \xi \) and \( m \) represent the VP modulus and exponent, respectively.

The VP strain rate follows a plastic flow rule:

\[
\dot{\varepsilon}^{vp} = \dot{p} N^{ef}, \quad N^{ef} = \frac{3}{2} \frac{\sigma^{ef}}{\sigma_{eq}}.
\]  

(5.19)

For the damage evolution, the Chaboche-Lemaitre evolution law [Lemaitre, 1985] is used:

\[
\begin{align*}
\dot{D} &= \left( \frac{Y}{S} \right)^s \dot{p} \geq 0, \text{ if } p > p_D, \\
D &= D_c \rightarrow \text{crack initiation}.
\end{align*}
\]  

(5.20)

Here \( s \) and \( S \) are material parameters, \( p_D \) is a damage threshold and \( D_c \) is a critical damage value.

5.3 Problem statement for the fatigue of thermoplastic polymers

In this chapter, two-scale time homogenization approach is carried out to model the degradation of material properties due to fatigue of a solid \( \Omega \) subjected to cyclic loading with a small period \( T^* \) over the interval \([0, T^*_F]\). Two time scales are defined. We suppose that the cyclic loading and all response fields are made of two parts: a slow variation, represented by the slow or macro time \( t^* M \) and a rapid variation, represented by the fast or micro scale \( \tau \). The physical time \( t^* \) can be defined as the sum of the macro time \( t^*_M \) and a fraction \( \tau \) of the period \( T^* \), \( \tau \in [0, 1] \\
\begin{align*}
t^* &= t^*_M + T^* \tau \quad \text{(figure 3.1)}.
\end{align*}

(5.21)

Each mechanical variable \( \Psi_{T^*} (\vec{x}, t^*) \), at a given spatial location \( \vec{x} \) is then supposed to be \( T^* \)-periodic (with respect to \( t^* \)) and to depend on macro and micro scales \( t^*_M \) and \( \tau \):

\[
\Psi_{T^*} (\vec{x}, t^*) = \Psi (\vec{x}, t^*_M, \tau).
\]  

(5.21)

A local 1-periodicity assumption with respect to \( \tau \) is then made for field variables \( \Psi_{T^*} (\vec{x}, t^*) \) (unless otherwise stated).

Let us define a dimensionless time variable \( t \):

\[
t = \frac{t^*}{T^*_F}, \quad t = \in [0, 1],
\]  

(5.22)
5.3. Problem statement for the fatigue of thermoplastic polymers

a dimensionless macro time variable \( t_M \):

\[
t_M = \frac{t_M^*}{T_F}, \quad t_M \in [0, 1],
\]

and a small scaling parameter \( T \) as follows:

\[
T = \frac{T^*}{T_F},
\]

Then the following function substitution is made:

\[
\phi_T (\vec{x}, t) = \Psi_{T^*} (\vec{x}, t^*),
\]

then \( \phi_T (\vec{x}, t) \) is \( T \)-periodic with respect to the dimensionless time \( t \). \[1\]

Subscripts \( T^* \) or \( T \) associated with a variable denote its association with the two scales. A new dimensionless relation between both time scales is obtained:

\[
t = t_M + T \tau, \quad t \in [0, 1]; \quad t_M = \in [0, 1]; \quad \tau \in [0, 1].
\]

Consequently, a local periodicity assumption, with respect to \( \tau \) is made for field variables \( \phi_T (\vec{x}, t) \) (unless otherwise stated), and each periodic variable \( \phi_T (\vec{x}, t) \) is assumed to depend on both dimensionless time scales \( t_M \) and \( \tau \):

\[
\phi_T (\vec{x}, t) = \phi (\vec{x}, t_M, \tau).
\]

Using the chain rule, the time differentiation in the two time scales is given as:

\[
\dot{\phi}_T (\vec{x}, t) = \frac{\partial \phi (\vec{x}, t_M, \tau)}{\partial t_M} + \frac{1}{T} \frac{\partial \phi (\vec{x}, t_M, \tau)}{\partial \tau},
\]

where the superposed dot denotes the total derivative with respect to the dimensionless physical time \( t \).

It is implicit that all variables are functions of position \( \vec{x} \), so from now on and without loss of generality, the dependence of all variables on \( \vec{x} \) is omitted for simplicity.

\[
\phi_T (\vec{x}, t + T) = \phi_T \left( \vec{x}, \frac{t^* + T^*}{T_F} \right) = \Psi_{T^*} \left( \vec{x}, \frac{(t^* + T^*)}{T_F} T_F \right),
\]

\[
= \Psi_{T^*} (\vec{x}, t^* + T^*) = \Psi_{T^*} (\vec{x}, t^*),
\]

\[
= \phi_T (\vec{x}, t).
\]
The proposed VE-VP-D model can be described by the following set of equations using the two time scales:

\[
\begin{aligned}
\varepsilon(t_M, \tau) &= \varepsilon_{ve}(t_M, \tau) + \varepsilon_{vp}(t_M, \tau), \quad \text{in } \Omega \times [0, 1] \times [0, 1], \\
\sigma(t_M, \tau) &= (1 - D(t_M, \tau))\sigma^{ef}(t_M, \tau), \quad \text{in } \Omega \times [0, 1] \times [0, 1], \\
\dot{\varepsilon}_{vp} &= F_{\varepsilon}(\sigma^{ef}, r, D), \quad \text{in } \Omega \times [0, 1] \times [0, 1], \\
\dot{r} &= F_{r}(\sigma^{ef}, r) > 0, \quad \text{if } f > 0; \quad \dot{r} = 0, \quad \text{if } f \leq 0, \\
\dot{p} &= \frac{F_{r}(\sigma^{ef}, r)}{(1 - D(t_M, \tau))}, \quad \text{in } \Omega \times [0, 1] \times [0, 1], \\
\dot{D} &= F_{D}(Y, \sigma^{ef}, r, D) \geq 0, \quad \text{if } p > p_D, \\
\text{until } D &= D_c (\text{crack initiation}), \quad \text{in } \Omega \times [0, 1] \times [0, 1], \\
Y(t_M, \tau) &= \frac{s^{ef}_\infty}{4G_\infty} + \frac{(\sigma^{ef}_i H_i)^2}{2K_i} + \sum_{i=1}^{I} \frac{s^{ef}_i}{4G_i} + \sum_{j=1}^{J} \frac{(\sigma^{ef}_j H_j)^2}{2K_j}, \quad \text{in } \Omega \times [0, 1] \times [0, 1].
\end{aligned}
\]  

(5.29)

Here, \( F_{\varepsilon} \) and \( F_{r} \) are VP operators defined in the context of \( J_2 \) viscoplasticity by the following equations:

\[
F_{\varepsilon}(\sigma^{ef}, r, D) \equiv \frac{\dot{r}}{1 - D(t)} N^{ef}, \quad F_{r}(\sigma^{ef}, r) \equiv g_v = \frac{\sigma_y}{\zeta} \left( \frac{f}{\sigma_y} \right)^m. \quad (5.30)
\]

Where \( f \) and \( N^{ef} \) are defined by the following equations:

\[
f(\sigma, R; D) = \sigma_{eq} - \sigma_y - R(r); \quad N^{ef} = \frac{3 \sigma^{ef}}{2 \sigma_{eq}}. \quad (5.31)
\]

And \( F_{D} \) is a function describing the damage evolution:

\[
F_{D}(Y, \sigma^{ef}, r, D) = \left( \frac{Y}{S} \right)^{s} \dot{p}. \quad (5.32)
\]

By comparison to the VE-VP constitutive model presented in Chapter 3, one scalar variable \( D \) is added to describe ductile damage.

In practice, for isotropic hardening, a power law or an exponential law (with saturation) are often used:

\[
R(r) = kp^n; \quad \text{or } \quad R(r) = R_\infty [1 - \exp(nr)], \quad (5.33)
\]

where, \( k(\geq 0), n(\geq 0), \) and \( R_\infty \geq 0 \) are material parameters.
5.3. Problem statement for the fatigue of thermoplastic polymers

5.3.1 Two-scale time fatigue analysis

Each variable is expanded according to the following asymptotic expansion regarding \( T \):

\[
\phi (t_M, \tau) = \sum_{i=0}^{\infty} T^i \phi_i (t_M, \tau),
\]

In this Section we proceed in the same way as in Chapter 3. Here, the functions \( \phi_i (t_M, \tau) \) are supposed to be 1-periodic with respect to the variable \( \tau \), except for certain variables such as damage and accumulated plastic strain which are cumulative and non-periodic in the time domain. This will be explained in detail in Section 5.3.2.

If a nonlinear operator \( F_{\varepsilon}(\sigma^{ef}, r, D) \) admits a gradient at a point, its first order asymptotic expansion can be written as follows:

\[
F_{\varepsilon}(\sigma^{ef}, r, D) = F_{\varepsilon}(\sigma^{ef}_0, r_0, D_0) + TDF_{\varepsilon}(\sigma^{ef}_0, r_0, D_0) \cdot (\sigma^{ef}_1, r_1, D_1) + O(T^2).
\]

where \( O \) is the Landau notation and \( D F_{\varepsilon}(\sigma^{ef}_0, r_0, D_0) \) is the gradient of \( F_{\varepsilon} \) expressed in \( (\sigma^{ef}_0, r_0, D_0) \).

Using the total differentiation rule in equation (5.28) and following the above approximation in equation (5.34), each variable is replaced by its asymptotic expansion into equations (5.29 a-h) together with equation (5.13). Then, gathering terms of equal order (i.e. equal powers of \( T \)), the problem can be rewritten at the orders \((-1)\) and \((0)\) as follows:

- Order \((-1)\) problem:

\[
\frac{\partial}{\partial \tau} \varepsilon^{vp}_0 (\bar{x}, t_M, \tau) = 0, \quad \text{in } [0, 1] \times [0, 1]
\]

\[
\frac{\partial}{\partial \tau} r_0 (\bar{x}, t_M, \tau) = 0, \quad \text{in } [0, 1] \times [0, 1]
\]

\[
\frac{\partial}{\partial \tau} p_0 (\bar{x}, t_M, \tau) = 0, \quad \text{in } [0, 1] \times [0, 1]
\]

\[
\frac{\partial}{\partial \tau} D_0 (\bar{x}, t_M, \tau) = 0, \quad \text{in } [0, 1] \times [0, 1]
\]
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- Order 0 problem:

  \[ \tilde{u}_0 (\tilde{x}, t_M = 0, \tau = 0) = \tilde{u}^f (\tilde{x}), \quad \text{in } \Omega \]  

  \[ \sigma_0 (\tilde{x}, t_M = 0, \tau = 0) = \sigma^f (\tilde{x}), \quad \text{in } \Omega \]  

  \[ \tilde{u}_0 (\tilde{x}, t_M, \tau) = \tilde{u}_b (\tilde{x}, t_M, \tau), \quad \text{in } [0, 1] \times [0, 1] \]  

  \[ '\sigma_0 (\tilde{x}, t_M, \tau).\tilde{n} = \tilde{g} (\tilde{x}, t_M, \tau), \quad \text{in } [0, 1] \times [0, 1] \]  

  \[ \nabla \cdot \sigma_0 (\tilde{x}, t_M, \tau) + f (\tilde{x}, t_M, \tau) = \tilde{0}, \quad \text{in } [0, 1] \times [0, 1] \]  

  \[ \varepsilon (\tilde{u}_0) = \frac{1}{2} (\nabla \tilde{u}_0 + \tilde{u}_0^t \nabla \tilde{u}_0), \quad \text{in } [0, 1] \times [0, 1] \]  

  \[ \varepsilon (\tilde{u}_0) = \varepsilon_0^{ve} (\tilde{x}, t_M, \tau) + \varepsilon_0^{vp} (\tilde{x}, t_M, \tau), \quad \text{in } [0, 1] \times [0, 1] \]  

  \[ \sigma_0 (\tilde{x}, t_M, \tau) = (1 - D_0 (\tilde{x}, t_M, \tau)) \sigma_0^{ef} (\tilde{x}, t_M, \tau), \quad \text{in } [0, 1] \times [0, 1] \]  

  \[ \sigma_0^{ef} (\tilde{x}, t_M, \tau) = C_\infty : \varepsilon_0^{ve} (\tilde{x}, t_M, \tau) + \sum_{i=1}^{I} s_{i0}^{ef} (\tilde{x}, t_M, \tau) \]

  \[ + \sum_{j=1}^{J} \sigma_{H_{ij0}}^{ef} (\tilde{x}, t_M, \tau) 1, \quad \text{in } [0, 1] \times [0, 1] \]  

  \[ \frac{\partial \varepsilon_0^{vp}}{\partial t_M} (\tilde{x}, t_M, \tau) + \frac{\partial \varepsilon_0^{vp}}{\partial \tau} (\tilde{x}, t_M, \tau) = F_\varepsilon (\sigma_0^{ef}, r_0, D_0), \quad \text{in } [0, 1] \times [0, 1] \]  

  \[ \frac{\partial r_0}{\partial t_M} (\tilde{x}, t_M, \tau) + \frac{\partial r_1}{\partial \tau} (\tilde{x}, t_M, \tau) = F_r (\sigma_0^{ef}, r_0), \quad \text{in } [0, 1] \times [0, 1] \]
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\[
\frac{\partial p_0}{\partial t_M}(\vec{x}, t_M, \tau) + \frac{\partial p_1}{\partial \tau}(\vec{x}, t_M, \tau) = \frac{F_r(\sigma_0^{ef}, r_0)}{(1 - D_0)}, \quad \text{in } [0, 1] \times [0, 1] \tag{5.51}
\]

\[
\frac{\partial D_0}{\partial t_M}(\vec{x}, t_M, \tau) + \frac{\partial D_1}{\partial \tau}(\vec{x}, t_M, \tau) = \left(\frac{Y_0(\vec{x}, t_M, \tau)}{S}\right)^s \left(\frac{\partial p_0}{\partial t_M}(\vec{x}, t_M, \tau) + \frac{\partial p_1}{\partial \tau}(\vec{x}, t_M, \tau)\right),
\]

\text{in } [0, 1] \times [0, 1] \tag{5.52}

\[
Y_0(\vec{x}, t_M, \tau) = \frac{\sigma_0^{ef}}{4G_\infty} + \frac{(\sigma_{H_0}^{ef})^2}{2K_\infty} + \sum_{i=1}^L \frac{s_{i0}^{ef}}{4G_i} + \sum_{j=1}^J \frac{(\sigma_{H_0}^{ef})^2}{2K_j},
\]

\text{in } [0, 1] \times [0, 1]. \tag{5.53}

Equations (5.36) to (5.39) show that at order 0, \(\varepsilon_0^{vp}, r_0, p_0\), and \(D_0\) are function of the slow time variable \((t_M)\) only, which means that at the order 0 the rapid evolution of VP deformation and damage are blocked:

\[
\varepsilon_0^{vp}(\vec{x}, t_M, \tau) = \varepsilon_0^{vp}(\vec{x}, t_M), \tag{5.54}
\]

\[
r_0(\vec{x}, t_M, \tau) = r_0(\vec{x}, t_M). \tag{5.55}
\]

\[
p_0(\vec{x}, t_M, \tau) = p_0(\vec{x}, t_M). \tag{5.56}
\]

\[
D_0(\vec{x}, t_M, \tau) = D_0(\vec{x}, t_M). \tag{5.57}
\]

Using equations (5.36) and (5.46), we have:

\[
\frac{\partial}{\partial t_M}\varepsilon_0(t_M, \tau) = \frac{\partial}{\partial \tau}\varepsilon_0^{vp}(t_M, \tau), \quad \text{in } \Omega \times [0, 1] \times [0, 1] \tag{5.58}
\]

Equation (5.58) shows that at the zero-order the rapid evolution of total deformation is equal to that of its VE part.
5.3.2 Macro and micro chronological problems

First we could proceed as in Chapter 3. A general decomposition of all variables \( \phi_i(x, t_M, \tau) \) into macro and micro chronological fields is proposed in the form:

\[
\phi_i(t_M, \tau) = \phi_{iM}(t_M) + \tilde{\phi}_i(t_M, \tau). \tag{5.59}
\]

The average value \( \phi_{iM}(t_M) \) of the function \( \phi_i(t_M, \tau) \) with respect to \( \tau \) and at time \( t_M \) is defined as:

\[
\phi_{iM}(t_M) = \int_0^1 \phi_i(t_M, \tau) d\tau \equiv \langle \phi_i(t_M, \tau) \rangle, \tag{5.60}
\]

where \( \phi_i(t_M, \tau) \) are \( \tau \)-periodic functions.

However, damage, accumulated plastic strain and isotropic hardening variables \( D, p \) and \( r \) are cumulative, therefore they are not periodic with respect to the fast time coordinate \( \tau \) (\(< \frac{\partial D_i}{\partial \tau} > \neq 0, < \frac{\partial p_i}{\partial \tau} > \neq 0, \) and \(< \frac{\partial r_i}{\partial \tau} > \neq 0 \)). Thus another decomposition of these variables is proposed:

\[
\begin{align*}
D_i(t_M, \tau) &= D_{iM}(t_M) + \tilde{D}_i(t_M, \tau), \\
p_i(t_M, \tau) &= p_{iM}(t_M) + \tilde{p}_i(t_M, \tau) , \\
r_i(t_M, \tau) &= r_{iM}(t_M) + \tilde{r}_i(t_M, \tau),
\end{align*} \tag{5.61}
\]

in which \( D_{iM}, p_{iM}, \) and \( r_{iM} \) are macro-chronological variables at the order \( i \) that depend only on the macro time scale \( t_M \). And \( \tilde{D}_i, \tilde{p}_i \) and \( \tilde{r}_i \) represent the micro-chronological variables at the order \( i \) that depend on both time scales \( t_M \) and \( \tau \).

In the following we try to find an expression of the first order functions \( r_1, p_1 \) and \( D_1 \), in the asymptotic expansion of the isotropic hardening \( r \), accumulated plastic strain \( p \) and damage \( D \), respectively.

We have:

\[
\dot{r}(t) = g_{v}(\sigma_{eq}, r), \quad \text{if } f > 0. \tag{5.62}
\]

Given that \( t = t_M + \tau T \), by integrating \( \dot{r} \) and using Chasles relation, we have:

\[
r(t) \equiv r(t_M, \tau) = r(t_M) + \int_{t_M}^{t_M+\tau T} g_{v}(\sigma_{eq}(\rho), r(\rho)) d\rho. \tag{5.63}
\]

\( \rho \) being the integration variable.

A first order Taylor expansion of the above equation, with respect to \( T \),
5.3. Problem statement for the fatigue of thermoplastic polymers

in the neighborhood of zero gives:

\[ r(t_M, \tau) = r(t_M) + \tau \lim_{T \to 0} \frac{1}{\tau T} \int_{t_M}^{t_M+\tau T} g_v(\sigma_{eq}(\rho), r(\rho)) d\rho T + O(T^2). \] (5.64)

Using the intermediate value theorem, there is a \( c_T \in [t_M, t_M + \tau T] \) such that:

\[ r(t_M, \tau) = r(t_M) + \tau \lim_{T \to 0} g_v(\sigma_{eq}(c_T), r(c_T)) T + O(T^2). \] (5.65)

Then, given that when \( T \) tends towards zero \( c_T \) tends towards \( t_M \), we have:

\[ r(t_M, \tau) = r(t_M) + \tau g_v(\sigma_{eq}(t_M), r(t_M)) T + O(T^2). \] (5.66)

where \( O \) is the Landau notation.

Given the uniqueness of the asymptotic expansion we have:

\[ r_0(t_M, \tau) = r(t_M); \quad \text{and} \quad r_1(t_M, \tau) = \tau g_v(\sigma_{eq}(t_M), r_0). \] (5.67)

Following the same procedure, the first order asymptotic expansion of the accumulated plastic strain \( p \) with respect to \( T \), in the neighborhood of zero is:

\[
\begin{cases}
  p(t_M, \tau) = p(t_M) + \tau \frac{g_v(\sigma_{eq}(t_M), r(t_M))}{1 - D(t_M)} T + O(T^2), \\
  = p_0(t_M, \tau) + p_1(t_M, \tau) T + O(T^2),
\end{cases}
\] (5.68)

For the damage variable \( D \) we have:

\[ \dot{D} = \left( \frac{Y}{S} \right)^s \dot{\rho} \geq 0, \text{ if } p > p_D, \]
\[ = \left( \frac{Y}{S} \right)^s \frac{\dot{\rho}}{1 - D}. \] (5.69)

By integrating \( \dot{D}(1 - D) \) we have:

\[ D(t) = \frac{D^2(t)}{2} = \int_0^t \left( \frac{Y(\rho)}{S} \right)^s \dot{\rho} d\rho. \] (5.70)

Given that \( D < 1 \) and assuming that \( \int_0^t \left( \frac{Y}{S} \right)^s \dot{\rho} d\rho < 0.5 \), we have:

\[ D(t) = 1 - \sqrt{1 - 2 \int_0^t \left( \frac{Y(\rho)}{S} \right)^s \dot{\rho} d\rho}. \] (5.71)
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The first order Taylor expansion of the damage variable $D$, with respect to $T$, in the neighborhood of zero is:

$$
\begin{align*}
D(t_M, \tau) &= D(t_M) + \frac{dD(t_M, \tau)}{dT}T + O(T^2), \\
&= D(t_M) + \tau \frac{dD(t_M, \tau)}{dT}T + O(T^2), \\
&= D(t_M) + \tau \left( \frac{Y(t_M)}{S} \right)^s \frac{g_v(\sigma_{eq}(t_M), r(t_M))}{1 - D(t_M)} T + O(T^2), \\
&= D_0(t_M, \tau) + D_1(t_M, \tau) T + O(T^2).
\end{align*}
$$

(5.72)

Assuming that $\sigma_{eq}(t_M) \simeq \sigma_{eq}(t_M, \tau = 0)$ and $Y(t_M) \simeq Y_0(t_M, \tau = 0)$, the following expressions of the first order terms of the isotropic hardening, accumulated plastic strain and damage variables are found:

$$
\begin{align*}
\frac{\partial r_1(t_M, \tau)}{\partial t_M}(\vec{x}, t_M) &= \tau g_v(\sigma_{eq}(t_M, \tau = 0), r_0(t_M)), \\
\frac{\partial p_1(t_M, \tau)}{\partial t_M}(\vec{x}, t_M) &= \frac{r_1(t_M, \tau)}{1 - D_0(t_M)}, \\
\frac{\partial D_1(t_M, \tau)}{\partial t_M}(\vec{x}, t_M) &= \left( \frac{Y_0(t_M, \tau = 0)}{S} \right)^s p_1(t_M, \tau).
\end{align*}
$$

(5.73)

Substituting equations (5.73 (a)-(b)-(c)) into equations (5.50), (5.51) and (5.52), the derivatives of the zero order terms $r_0$, $p_0$ and $D_0$ with respect to the macro time $t_M$ become:

$$
\begin{align*}
\frac{\partial r_0}{\partial t_M}(\vec{x}, t_M) &= F_r(\sigma^0_{eq}, r_0) - g_v(\sigma_{eq}(\vec{x}, t_M, \tau = 0), r_0(\vec{x}, t_M)), \\
&\text{in } [0, 1] \times [0, 1] \quad (5.74)
\end{align*}
$$

$$
\begin{align*}
\frac{\partial p_0}{\partial t_M}(\vec{x}, t_M) &= \frac{1}{(1 - D_0(\vec{x}, t_M))} \frac{\partial r_0}{\partial t_M}(\vec{x}, t_M), \\
&\text{in } [0, 1] \times [0, 1] \quad (5.75)
\end{align*}
$$

$$
\begin{align*}
\frac{\partial D_0}{\partial t_M}(\vec{x}, t_M) &= \frac{1}{(1 - D_0(\vec{x}, t_M))} \left[ \left( \frac{Y_0(\vec{x}, t_M, \tau = 0)}{S} \right)^s F_r(\sigma^0_{eq}, r_0) \\
- \left( \frac{Y_0(\vec{x}, t_M, \tau = 0)}{S} \right)^s g_v(\sigma_{eq}(\vec{x}, t_M, \tau = 0), r_0(\vec{x}, t_M)) \right], \\
&\text{in } [0, 1] \times [0, 1] \quad (5.76)
\end{align*}
$$
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In order to solve the zero-order problem of equations (5.40) to (5.53), we follow the additive decomposition into mean and fluctuation parts defined in equations (5.59) and (5.61). Taking the micro-time average as defined by equation (5.60) of equations (5.40) to (5.48), (5.53) and (5.74) to (5.76), we obtain a decomposition of the original problem into macro- and micro-chronological problems:

- **Macro-chronological problem:**

\[
\vec{u}_0(x, t_M) = \vec{u}_0(x, t_M), \quad \text{in } [0, 1] \tag{5.77}
\]

\[
\vec{v}_0(x, t_M) = \vec{v}_0(x, t_M), \quad \text{in } [0, 1] \tag{5.78}
\]

\[
\nabla \cdot \sigma_0(x, t_M) + f_M(x, t_M) = 0 \quad \text{in } [0, 1] \tag{5.79}
\]

\[
\varepsilon_M(\vec{u}_0) = \frac{1}{2} \left( \nabla \vec{u}_0 + (\nabla \vec{u}_0)^T \right), \quad \text{in } [0, 1] \tag{5.80}
\]

\[
\varepsilon_M(\vec{u}_0) = \varepsilon_{0M}(x, t_M) + \varepsilon_{\text{eq}}(x, t_M) \quad \text{in } [0, 1] \tag{5.81}
\]

\[
\sigma_0(x, t_M) = (1 - D_0(x, t_M)) \sigma_{0M}(x, t_M), \quad \text{in } [0, 1] \tag{5.82}
\]

\[
\sigma_{0M}(x, t_M) = C_{\infty} \varepsilon_{\text{eq}}(x, t_M) + \sum_{i=1}^{L} s_{0iM}(x, t_M) + \sum_{j=1}^{J} \sigma_{H_{ijM}}(t_M) 1, \quad \text{in } [0, 1] \tag{5.83}
\]

\[
\frac{d\varepsilon_{\text{eq}}}{dt_M}(x, t_M) = F_{\varepsilon}(\sigma_{0M} + \sigma_{\text{eq}}), r_0, D_0) \tag{5.84}
\]

\[
\frac{dr_0}{dt_M}(x, t_M) = F_{r_M}(\sigma_{0M} + \sigma_{\text{eq}}), r_0, g_0(\sigma_0, t_M, \tau = 0), r_0(t_M)), \quad \text{in } [0, 1] \tag{5.85}
\]

\[
\frac{dp_0(t_M)}{dt_M} = \frac{1}{(1 - D_0(x, t_M))} \frac{dr_0}{dt_M}(x, t_M), \quad \text{in } [0, 1] \tag{5.86}
\]

\[\text{Given the fast time periodicity assumption, } \int_0^1 \frac{\partial}{\partial \tau} \varepsilon_{\text{eq}}^p(x, t_M, \tau) d\tau = \varepsilon_{\text{eq}}^p(t_M, 1) - \varepsilon_{\text{eq}}^p(t_M, 0) = 0.\]
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\[
\frac{dD_0(\vec{x}, t_M)}{dt_M} = \frac{1}{1 - D_0(\vec{x}, t_M)} \left[ \left( \frac{Y_0}{S} \right)^s F_r(\sigma_{0M}^e + \sigma_{eff}(0), r_0) \right] \\
- \left( \frac{Y_0(\vec{x}, t_M, \tau = 0)}{S} \right)^s g_v(\sigma_{0eq}(\vec{x}, t_M, \tau = 0), r_0(\vec{x}, t_M)) ,
\] in \([0, 1]\) (5.87)

\[
Y_{0M}(\vec{x}, t_M) = \frac{<s_{\infty}^e : s_{\infty}^e>}{4G_\infty} + \frac{(s_{H_\infty}^e)^2}{3K_\infty} + \sum_{i=1}^{I} \frac{<s_{i0}^e : s_{i0}^e>}{4G_i} \\
+ \sum_{j=1}^{J} \frac{(s_{H_j0}^e)^2}{2K_j} , \text{ in } [0, 1]. \ (5.88)
\]

Given that \(\tilde{\varepsilon}_{\infty}^{vp}(\vec{x}, t_M, \tau) = 0, \tilde{p}_0(\vec{x}, t_M, \tau) = 0, \tilde{r}_0(\vec{x}, t_M, \tau) = 0\) and \(\tilde{D}_0(\vec{x}, t_M, \tau) = 0\) from equations (5.36)-(5.39), the micro-chronological problem is VE coupled with damage and it is defined as follows:

- **Micro-chronological problem:**
  \[
  \tilde{u}_0(\vec{x}, t_M, \tau) = \vec{u}_b(\vec{x}, t_M, \tau) , \text{ in } [0, 1] \times [0, 1] \ (5.89)
  \]
  \[
  t' \tilde{\sigma}_{0}(\vec{x}, t_M, \tau) . \vec{n} = \tilde{g}(\vec{x}, t_M, \tau) , \text{ in } [0, 1] \times [0, 1] \ (5.90)
  \]
  \[
  \nabla \cdot \tilde{\sigma}_{0}(\vec{x}, t_M, \tau) + \tilde{f}(\vec{x}, t_M, \tau) = 0 , \text{ in } [0, 1] \times [0, 1] \ (5.91)
  \]
  \[
  \tilde{\varepsilon}(\vec{u}_0) = \frac{1}{2} \left( \nabla \vec{u}_0 + \nabla \vec{u}_0^T \right) , \text{ in } [0, 1] \times [0, 1] \ (5.92)
  \]
  \[
  \tilde{\varepsilon}(\vec{u}_0) = \tilde{\varepsilon}_{0}^{ve}(\vec{x}, t_M, \tau) , \text{ in } \Omega \times [0, 1] \times [0, 1] \ (5.93)
  \]
  \[
  \tilde{\sigma}_{0}(\vec{x}, t_M, \tau) = (1 - D_0(\vec{x}, t_M)) \tilde{\sigma}_{0}^{ef}(\vec{x}, t_M, \tau) , \text{ in } [0, 1] \times [0, 1] \ (5.94)
  \]
  \[
  \tilde{\sigma}_{0}^{ef}(\vec{x}, t_M, \tau) = C_{\infty} : \tilde{\varepsilon}_{0}(\vec{x}, t_M, \tau) + \sum_{i=1}^{I} \tilde{s}_{i0}^{ef}(\vec{x}, t_M, \tau) + \\
  \sum_{j=1}^{J} \tilde{\sigma}_{H_{j0}}^{ef}(\vec{x}, t_M, \tau) \mathbf{1} , \text{ in } [0, 1] \times [0, 1] \ (5.95)
  \]
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Equations (5.89) to (5.95) correspond to the resolution of a VE problem only, since the VP flow rule does not depend on the fast time variable explicitly.

The structure of the two problems shows that in practice, for each \( t_M \), one solves the micro-time problem (5.89) to (5.95). Next the macro-chronological problem (5.77) to (5.88) completed by the following initial conditions can be solved:

\[
\begin{align*}
\tilde{u}_{0M}(\tilde{x}, t_M = 0) + \tilde{u}_0(\tilde{x}, t_M = 0, \tau = 0) &= \tilde{u}^f(\tilde{x}), \quad \text{in } \Omega \\
\sigma_{0M}(\tilde{x}, t_M = 0) + \tilde{\sigma}_0(\tilde{x}, t_M = 0, \tau = 0) &= \sigma^f(\tilde{x}), \quad \text{in } \Omega
\end{align*}
\]

In addition, to solve the macro-chronological problem and to determine the damage evolution in equation (5.87), the zero-order damage thermodynamic force \( \tilde{Y}_0 \) needs to be calculated. Where the macro-chronological damage thermodynamic force \( Y_{0M} \) is calculated from equation (5.88) and the micro-chronological damage thermodynamic force \( \tilde{Y}_0 \) from the following equation:

\[
\tilde{Y}_0(\tilde{x}, t_M, \tau) = \frac{s_{\infty}^f : s_{\infty}^f}{4G_{\infty}} + \frac{(\sigma_{H_{\infty}}^f)^2}{2K_{\infty}} + \sum_{i=1}^{I} \frac{s_{0i}^f : s_{0i}^f}{4G_i}
\]

\[
+ \sum_{j=1}^{J} \frac{(\sigma_{H_{j0}}^f)^2}{2K_j}, \quad \text{in } [0, 1] \times [0, 1], \quad (5.98)
\]

If one knows the fluctuations \( \tilde{s}_{0i}^f \) and \( \tilde{\sigma}_{H_{j0}}^f \) and the averages \( s_{i0}^f \) and \( \sigma_{H_{j0}}^f \) of the effective viscous components, the fluctuations \( s_{i0}^f : s_{i0}^f \) and \( (\sigma_{H_{j0}}^f)^2 \) can be calculated as follows:

\[
\begin{align*}
\tilde{s}_{i0}^f : s_{i0}^f &= 2s_{i0}^f : s_{i0}^f + s_{0i}^f : s_{0i}^f - \langle s_{i0}^f : s_{i0}^f \rangle^3 \quad (5.99) \\
(\tilde{\sigma}_{H_{j0}}^f)^2 &= 2\sigma_{H_{j0}}^f : \sigma_{H_{j0}}^f + (\tilde{\sigma}_{H_{j0}}^f)^2 - \langle (\tilde{\sigma}_{H_{j0}}^f)^2 \rangle^2. \quad (5.100)
\end{align*}
\]

\(^3\text{Let } f \text{ and } g \text{ be two functions decomposed into macro- and micro-chronological parts:}

\( f = f_M + \tilde{f} \) and \( g = g_M + \tilde{g} \). The product \( fg \) is also decomposed into average and fluctuation: \( fg = \langle fg \rangle + f_M \tilde{g} + g_M \tilde{f} - \langle \tilde{f} \tilde{g} \rangle \).}
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After solving the macro-chronological problem (5.77) to (5.88) and knowing the micro-chronological zero-order effective stress \( \tilde{\sigma}_{ef}^0 \) from the micro-chronological problem (5.89) to (5.95), the micro-chronological zero-order stress \( \tilde{\sigma}_0 \) can be found using the following equation:

\[
\tilde{\sigma}_0 (\tilde{x}, t_M, \tau) = (1 - D_0(\tilde{x}, t_M)) \tilde{\sigma}_{ef}^0 (\tilde{x}, t_M, \tau), \text{ in } [0, 1] \times [0, 1] \quad (5.101)
\]

The macro-chronological problem (5.77) to (5.88) is VE-VP-D, it is different from the one presented in Chapter 3, not only because of the presence of damage \( D \) variable, but also by the addition of new terms due to the non-periodic assumption of the isotropic hardening, accumulated plastic strain and damage variables with respect to the fast time coordinate \( \tau \).

5.4 Macro- and micro- chronological problem with damage and accumulated plastic strain periodicity assumption

In this Section, we assume that damage, accumulated plastic strain and isotropic hardening variables \( D, p \) and \( r \) are periodic with respect to the fast time coordinate \( \tau \):

\[
< \frac{\partial D_i}{\partial \tau} > = 0, \quad < \frac{\partial p_i}{\partial \tau} > = 0, \quad \text{and} \quad < \frac{\partial r_i}{\partial \tau} > = 0. \quad (5.102)
\]

For micro- and macro- chronological decomposition we follow the same notation as in equation (5.61).

According to the previous assumption in equation (5.102) and following the same procedure as in the previous Section, the macro- and micro-chronological decomposition of the zero-order problem (5.40) to (5.53) is as follows:

- **Macro-chronological problem:**

  \[
  \tilde{u}_{0M} (\tilde{x}, t_M) = \tilde{u}_{bM} (\tilde{x}, t_M), \quad \text{in } [0, 1] \quad (5.103)
  \]

  \[
  l^t \sigma_{0M} (\tilde{x}, t_M) . \tilde{n} = \tilde{g}_M (\tilde{x}, t_M), \quad \text{in } [0, 1] \quad (5.104)
  \]

  \[
  \nabla \cdot \sigma_{0M} (\tilde{x}, t_M) + \tilde{f}^M (\tilde{x}, t_M) = 0 \quad \text{in } [0, 1] \quad (5.105)
  \]
5.4. Macro- and micro-chronological problem with damage and accumulated plastic strain periodicity assumption

\[
\varepsilon_M(\bar{u}_0) = \frac{1}{2} \left( \nabla \bar{u}_0 + \nabla^\top \bar{u}_0 \right), \quad \text{in } [0, 1]
\]

(5.106)

\[
\varepsilon_M(\bar{u}_0) = \varepsilon^{\text{ve}}_{M, (\bar{x}, t_M)} + \varepsilon^{\text{vp}}_{M, (\bar{x}, t_M)} \quad \text{in } \Omega \times [0, 1]
\]

(5.107)

\[
\sigma_{0M}(t_M) = (1 - D_0(\bar{x}, t_M))\sigma^{\text{ef}}_{0M} (\bar{x}, t_M), \quad \text{in } [0, 1]
\]

(5.108)

\[
\sigma^{\text{ef}}_{0M} (\bar{x}, t_M) = \mathbf{C}_\infty : \varepsilon^{\text{ve}}_{0M} (\bar{x}, t_M) + \sum_{i=1}^{I} s^{\text{ef}}_{i0M} (\bar{x}, t_M) + \sum_{j=1}^{J} \sigma^{\text{Hj0}}_{0M} (\bar{x}, t_M) \mathbf{1}, \quad \text{in } [0, 1]
\]

(5.109)

\[
\frac{d\varepsilon^{\text{vp}}_{M}}{dt_M}(\bar{x}, t_M) = \mathbf{F}_M^{\text{M}} ((\sigma^{\text{ef}}_{0M} + \tilde{\sigma}^{\text{ef}}_{0}), r_0, D_0), \quad \text{in } [0, 1]
\]

(5.110)

\[
\frac{dr_0}{dt_M}(\bar{x}, t_M) = F_r M ((\sigma^{\text{ef}}_{0M} + \tilde{\sigma}^{\text{ef}}_{0}), r_0), \quad \text{in } [0, 1]
\]

(5.111)

\[
\frac{dp_0(\bar{x}, t_M)}{dt_M} = \frac{F_r M ((\sigma^{\text{ef}}_{0M} + \tilde{\sigma}^{\text{ef}}_{0}), r_0)}{(1 - D_0(\bar{x}, t_M))}, \quad \text{in } [0, 1]
\]

(5.112)

\[
\frac{dD_0(t_M)}{dt_M} = \frac{1}{1 - D_0(\bar{x}, t_M)} < \left( \frac{Y_0}{S} \right)^s F_r (\sigma^{\text{ef}}_{0M} + \tilde{\sigma}^{\text{ef}}_{0}), r_0 >, \quad \text{in } [0, 1]
\]

(5.113)

\[
Y_0^M (\bar{x}, t_M) = \frac{\mathbf{s}^{\text{ef}}_\infty : \mathbf{s}^{\text{ef}}_\infty}{4G_\infty} + \frac{< (\sigma^{\text{H0}}_\infty)^2 >}{2K_\infty} + \sum_{i=1}^{I} \frac{< \mathbf{s}^{\text{ef}}_{i0} : \mathbf{s}^{\text{ef}}_{i0} >}{4G_i} + \sum_{j=1}^{J} \frac{< (\sigma^{\text{Hj0}}_{j0})^2 >}{2K_j}, \quad \text{in } [0, 1].
\]

(5.114)

• Micro-chronological problem:
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\[ \tilde{u}_0 (\vec{x}, t_M, \tau) = \tilde{u}_b (\vec{x}, t_M, \tau), \quad \text{in } [0, 1] \times [0, 1] \] (5.115)

\[ \tilde{t} \tilde{\sigma}_0 (\vec{x}, t_M, \tau). \hat{n} = \tilde{g} (\vec{x}, t_M, \tau), \quad \text{in } \Gamma_f \times [0, 1] \times [0, 1] \] (5.116)

\[ \nabla \cdot \tilde{\sigma}_0 (\vec{x}, t_M, \tau) + \tilde{f} (\vec{x}, t_M, \tau) = 0, \quad \text{in } [0, 1] \times [0, 1] \] (5.117)

\[ \tilde{\varepsilon} (\tilde{u}_0) = \frac{1}{2} \left( \nabla \tilde{u}_0 + t \nabla \tilde{u}_0 \right), \quad \text{in } [0, 1] \times [0, 1] \] (5.118)

\[ \tilde{\varepsilon} (\tilde{u}_0) = \tilde{\varepsilon}_0^\infty (\vec{x}, t_M, \tau), \quad \text{in } \Omega \times [0, 1] \times [0, 1] \] (5.119)

\[ \tilde{\sigma}_0 (\vec{x}, t_M, \tau) = (1 - D_0 (\vec{x}, t_M)) \tilde{\sigma}_0^{ef} (\vec{x}, t_M, \tau), \quad \text{in } [0, 1] \times [0, 1] \] (5.120)

\[ \tilde{\sigma}_0^{ef} (\vec{x}, t_M, \tau) = C_\infty \cdot \tilde{\varepsilon}_0 (\vec{x}, t_M, \tau) + \sum_{i=1}^{I} \tilde{s}_0^{ef} (\vec{x}, t_M, \tau) + \sum_{j=1}^{J} \tilde{\sigma}_0^{ef} H_0 (\vec{x}, t_M, \tau) \mathbf{1}, \quad \text{in } [0, 1] \times [0, 1] \] (5.121)

\[ \tilde{Y}_0 (\vec{x}, t_M, \tau) = \frac{s_0^{ef}}{4G_\infty} \tilde{\sigma}_0^{ef} + \frac{(\tilde{\sigma}_0^{ef})^2}{2K_\infty} + \sum_{i=1}^{I} \frac{s_i^{ef}}{4G_i} + \sum_{j=1}^{J} \frac{(\tilde{\sigma}_0^{ef})^2}{2K_j}, \quad \text{in } [0, 1] \times [0, 1] \] (5.122)

Equations (5.115) to (5.122) correspond to the resolution of a VE problem only, since the VP flow rule does not depend on the fast time variable explicitly.

The structure of the two problems shows that in practice, for each \( t_M \), one solves the micro-time problem (5.115) to (5.122). Next the macro-chronological problem (5.103) to (5.114) completed by the following initial conditions can be solved:

\[ \tilde{u}_{0,M} (\vec{x}, t_M = 0) + \tilde{u}_0 (\vec{x}, t_M = 0, \tau = 0) = \tilde{u}^I (\vec{x}), \quad \text{in } \Omega \] (5.123)

\[ \sigma_{0,M} (\vec{x}, t_M = 0) + \tilde{\sigma}_0 (\vec{x}, t_M = 0, \tau = 0) = \sigma^I (\vec{x}), \quad \text{in } \Omega \] (5.124)
5.5. Computational algorithm

In this Section we propose a computational algorithm to resolve the VE-VP-D problem presented in Section 5.4. A numerical time discretization algorithm is proposed in this work, for the time homogenization of a VE-VP-D material subjected to large number of cycles. The algorithm is based on the technique proposed by [Haouala and Doghri, 2015] for the time homogenization of coupled VE-VP materials.

The time discrete form of the effective stress and the thermodynamic force update in equations (5.13) and (5.14) over a time interval \([t_n, t_{n+1}]\) are obtained using a numerical integration scheme of the effective viscous components as in Section 3.4.1. They are given by the following equations:

\[
\sigma^{ef}(t_{n+1}) = C_\infty : \varepsilon^{ve}(t_n) + \hat{E}(\Delta t) : \Delta \varepsilon^{ve} + \sum_{i=1}^I \exp \left( \frac{-\Delta t}{g_i} \right) \sigma^{ef}_i(t_n) \\
+ \sum_{j=1}^J \exp \left( \frac{-\Delta t}{k_j} \right) \sigma^{ef}_{H_j}(t_n) \mathbf{1}. \tag{5.125}
\]

\[
Y(t_{n+1}) = \frac{s^{ef}_\infty(t_{n+1}) : s^{ef}_\infty(t_{n+1})}{4G_\infty} + \frac{(\sigma^{ef}_{H_\infty}(t_{n+1}))^2}{2K_\infty} + \\
\sum_{i=1}^I \frac{s^{ef}_i(t_{n+1}) : s^{ef}_i(t_{n+1})}{4G_i} + \sum_{j=1}^J \frac{(\sigma^{ef}_{H_j}(t_{n+1}))^2}{2K_j}. \tag{5.126}
\]

Where the following incremental relaxation moduli are defined:

\[
\hat{G}(\Delta t) = G_\infty + \sum_{i=1}^I \hat{G}_i(\Delta t); \tag{5.127}
\]

\[
\hat{K}(\Delta t) = K_\infty + \sum_{j=1}^J \hat{K}_j(\Delta t); \tag{5.128}
\]

\[
\hat{E} = 2\hat{G}I^{dev} + 3\hat{K}T^{vol}. \tag{5.129}
\]

Here \(\Delta\) designate an increment of a variable: \(\Delta (\cdot) = (\cdot)_{n+1} - (\cdot)_n\).

Equation (5.125) is identical to equation except that here we are talking about the effective stress.
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The discrete form of the effective viscous components $s_{\infty}^{ef}$ and $\sigma_{H_j}^{ef}$ over a time interval $[t_n, t_{n+1}]$, using a mid-point time integration rule is given by:

\[
\begin{align*}
\begin{cases}
s_{\infty}^{ef}(t_{n+1}) &= 2G_\infty \xi^{ve}(t_{n+1}); \\
s_i^{ef}(t_{n+1}) &= \exp\left(\frac{-\Delta t}{g_i}\right) s_i^{ef}(t_n) + 2\hat{G}_i(\Delta t) \Delta \xi^{ve}; \\
\sigma_{H_\infty}^{ef}(t_{n+1}) &= 3K_\infty \varepsilon_{H_\infty}^{ve}(t_{n+1}); \\
\sigma_{H_j}^{ef}(t_{n+1}) &= \exp\left(\frac{-\Delta t}{k_j}\right) \sigma_{H_j}^{ef}(t_n) + 3\hat{K}_j(\Delta t) \Delta \varepsilon_{H_j}^{ve},
\end{cases}
\end{align*}
\] (5.130)

The expressions of $\hat{G}_i(\Delta t)$ and $\hat{K}_j(\Delta t)$ are given by:

\[
\begin{align*}
\begin{cases}
\hat{G}_i(\Delta t) &= G_i \exp\left(-\frac{\Delta t}{2g_i}\right); \\
\hat{K}_j(\Delta t) &= K_j \exp\left(-\frac{\Delta t}{k_j}\right).
\end{cases}
\end{align*}
\] (5.131)

As the effective stress and the strain energy release ($Y$) depend on both time scales $t_M$ and $\tau$, their discrete form in equations (5.125) and (5.126), respectively, depend also on these two scales. If one calculates all the terms of equations (5.125) and (5.126) at the same micro-time $\tau$ we have $\Delta t = \Delta t_M$. Consequently, the zero order effective stress and the strain energy release ($Y$) update are now given by the following equations:

\[
\sigma_{0}^{ef}(t_{M_{n+1}}, \tau) = C_\infty : \varepsilon_{0}^{ve}(t_{M_n}, \tau) + \sum_{i=1}^{I} \exp\left(-\frac{\Delta t_M}{g_i}\right) s_i^{ef}(t_{M_n}, \tau)
\]
\[
\quad + \sum_{j=1}^{J} \exp\left(-\frac{\Delta t_M}{k_j}\right) \sigma_{H_0}^{ef}(t_{M_n}, \tau) + \tilde{E}(\Delta t_M) : \Delta \varepsilon_{0}^{ve},
\] (5.132)

\[
Y_{0}(t_{M_{n+1}}, \tau) = \frac{s_{\infty}^{ef}(t_{M_{n+1}}, \tau) : s_{\infty}^{ef}(t_{M_{n+1}}, \tau)}{4G_\infty} + \frac{(\sigma_{H_\infty}^{ef}(t_{M_{n+1}}, \tau))^2}{2K_\infty}
\]
\[
\quad + \sum_{i=1}^{I} \frac{s_i^{ef}(t_{M_{n+1}}, \tau) : s_i^{ef}(t_{M_{n+1}}, \tau)}{4G_i} + \sum_{j=1}^{J} \frac{(\sigma_{H_j}^{ef}(t_{M_{n+1}}, \tau))^2}{2K_j}.
\] (5.133)
with the zero-order terms of these viscous stresses over \([t_{M_n}, t_{M_{n+1}}]\):

\[
\begin{align*}
{\tilde{\sigma}}_{\infty}^{e f}(t_{M_{n+1}}, \tau) &= 2G_{\infty} \xi_0^{ve}(t_{M_{n+1}}, \tau), \\
{s}_{i0}^{e f}(t_{M_{n+1}}, \tau) &= \exp \left( \frac{-\Delta t_M}{g_i} \right) s_{i0}^{e f}(t_{M_n}, \tau) + 2G_i (\Delta t_M) \Delta \xi_i^{ve}, \\
\sigma_{H_{in}}^{e f}(t_{M_{n+1}}, \tau) &= 3K_{\infty} \varepsilon_{H_0}^{ve}(t_{M_{n+1}}, \tau), \\
\sigma_{H_{jn}}^{e f}(t_{M_{n+1}}, \tau) &= \exp \left( \frac{-\Delta t_M}{k_j} \right) \sigma_{H_{jn}}^{e f}(t_{M_n}, \tau) + 3K_j (\Delta t_M) \Delta \varepsilon_{H_0}^{ve}.
\end{align*}
\]

Macro and micro-chronological decompositions of discretized problem are obtained and presented in the following subsections.

5.5.1 Micro-chronological problem

The micro-chronological problem of equations (5.115) to (5.122) is VE coupled with damage. For each macro-time interval \([t_{M_n}, t_{M_{n+1}}]\), the micro-time problem is solved only once, and the effective stresses and the strain energy release rate \((Y)\) are updated explicitly (no iterations required) according to the following discrete form of equations (5.135) and (5.136), respectively:

\[
\begin{align*}
{\tilde{\sigma}}_0^{e f}(t_{M_{n+1}}, \tau) &= C_{\infty} : \tilde{\varepsilon}_0(t_{M_n}, \tau) + \tilde{E}(\Delta t_M) : \Delta \tilde{\varepsilon}_0 \\
+ \sum_{i=1}^I \exp \left( \frac{-\Delta t_M}{g_i} \right) s_{i0}^{e f}(t_{M_n}, \tau) + \sum_{j=1}^J \exp \left( \frac{-\Delta t_M}{k_j} \right) \tilde{\sigma}_{H_{jn}}^{e f}(t_{M_n}, \tau) \mathbf{1},
\end{align*}
\]

\[
\begin{align*}
\tilde{Y}_0(t_{M_{n+1}}, \tau) &= \frac{s_{\infty 0}^{e f}(t_{M_{n+1}}, \tau) : \tilde{s}_{\infty 0}^{e f}(t_{M_{n+1}}, \tau)}{4G_{\infty}} + \frac{(\sigma_{\infty}^{e f}(t_{M_{n+1}}, \tau))^2}{2K_{\infty}} \\
+ \sum_{i=1}^I \frac{s_{i0}^{e f}(t_{M_{n+1}}, \tau) : \tilde{s}_{i0}^{e f}(t_{M_{n+1}}, \tau)}{4G_i} + \sum_{j=1}^J \frac{(\sigma_{H_{jn}}^{e f}(t_{M_{n+1}}, \tau))^2}{2K_j}.
\end{align*}
\]

5.5.2 Macro-chronological problem with damage and accumulated plastic strain periodicity assumption

The rate macro zero order equations (5.103)- (5.114) are discretized in time according to a fully implicit (backward Euler) integration scheme.
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The discrete form of equations (5.103)-(5.114) are given by the following set of equations:

\[
\Delta \varepsilon_{0M} = \Delta \varepsilon_{0M}^{ve} + \Delta \varepsilon_{0M}^{vp},
\]

\[
\sigma_{0M}(t_{Mn+1}) = (1 - D_0(t_{Mn+1})) \sigma_{0M}^{ef}(t_{Mn+1}),
\]

\[
\sigma_{0M}^{ef}(t_{Mn+1}) = C_{\infty} : \varepsilon_{0M}^{ve}(t_{Mn}) + \tilde{E}(\Delta t_M) : \Delta \varepsilon_{0M} + \sum_{i=1}^{I} \exp \left(-\frac{\Delta t_M}{g_i}\right) s_{0M}^{ef}(t_{Mn}) + \sum_{j=1}^{J} \exp \left(-\frac{\Delta t_M}{k_j}\right) \sigma_{0M}^{ef}(t_{Mn})^{1},
\]

\[
\Delta \varepsilon_{0M}^{vp} = F_{\varepsilon_M}(\sigma_{0M}^{*} + \tilde{\sigma}_0, r_0, D_0) \Delta t_M,
\]

\[
\Delta r_0 = F_{r_M}(\sigma_{0M}^{*} + \tilde{\sigma}_0, r_0) \Delta t_M,
\]

\[
\Delta p_0 = \frac{\Delta r_0}{1 - D_0(t_M)},
\]

\[
\Delta D_0 = \left(\frac{Y_0}{S}\right)^s > \Delta p_0.
\]

\[
Y_{0M}(t_{Mn+1}) = \frac{<s_{0M}^{ef}(t_{Mn+1}, \tau) : \sigma_{0M}^{ef}(t_{Mn+1}, \tau)>}{4G_{\infty}} + \frac{<(\sigma_{0M}^{ef}(t_{Mn+1}, \tau))^2>}{2K_{\infty}} + \sum_{i=1}^{I} \frac{<s_{0M}^{ef}(t_{Mn+1}, \tau) : \sigma_{0M}^{ef}(t_{Mn+1}, \tau)>}{4G_{i}} + \sum_{j=1}^{J} \frac{<(\sigma_{0M}^{ef}(t_{Mn+1}, \tau))^2>}{2K_{j}},
\]

The macro-chronological problem (5.137)-(5.144) is nonlinear and solved iteratively at each macro time step using the Newton Raphson method. A return mapping algorithm which was proposed first by Doghri, 1995 in the case of plasticity models coupled with ductile damage is used in order to solve the macro-chronological problem. The VE predictor is defined by:

\[
\sigma_{0M}^{pred}(t_{Mn+1}, \tau) = (1 - D_0(t_{Mn})) \sigma_{0M}^{ef, pred}(t_{Mn+1}, \tau),
\]
5.5. Computational algorithm

where,

\[ \sigma_{0}^{\text{pred}}(t_{Mn+1}, \tau) = \tilde{\sigma}_{0}^{\text{ef}}(t_{Mn+1}, \tau) + C_{\infty} : \varepsilon_{0M}^{\text{eq}}(t_{Mn}) + \hat{E}(\Delta t_{M}) : \Delta \varepsilon_{0M} \]

\[ + \sum_{i=1}^{I} \exp \left( -\frac{\Delta t_{M}}{g_{i}} \right) s_{0M}^{\text{ef}}(t_{Mn}) + \sum_{j=1}^{J} \exp \left( -\frac{\Delta t_{M}}{k_{j}} \right) \sigma_{H0M}^{\text{ef}}(t_{Mn})1, \]

\[ (5.146) \]

where the stress fluctuation \( \tilde{\sigma}_{0}^{\text{ef}}(t_{Mn+1}, \tau) \) was computed in the micro-time problem, equation (5.135).

If this trial stress satisfies the yield condition:

\[ f_{0}^{\text{pred}}(t_{Mn+1}, \tau) = \sigma_{0eq}^{\text{pred}}(t_{Mn+1}, \tau) - \sigma_{y} - R(r_{0}(t_{Mn})) \leq 0. \]  \[ (5.147) \]

then all the VP variables are equal to their values at \( t_{Mn} \).

If the trial state does not satisfy the yield condition, then the solution at \( t_{Mn+1} \) has to satisfy the following equation:

\[ \sigma_{0}(t_{Mn+1}, \tau) = (1 - D_{0}(t_{Mn+1})) \sigma_{0}^{\text{ef}}(t_{Mn+1}, \tau), \]  \[ (5.148) \]

\[ \sigma_{0}^{\text{ef}}(t_{Mn+1}, \tau) = \sigma_{0}^{\text{pred}}(t_{Mn+1}, \tau) - \hat{E}(\Delta t_{M}) : \Delta \varepsilon_{0}^{\text{eq}}, \]  \[ (5.149) \]

together with equations (5.140)-(5.144).

5.5.3 Application to J_2 viscoplasticity

For the rate-dependent J_2 VP model we have:

\[ \Delta \varepsilon_{0}^{\text{eq}} \approx \Delta p_{0} < N_{0}^{\text{ef}}(t_{Mn+1}, \tau) >, \]

\[ \Delta r_{0} \approx g_{v}(\beta_{0eq}(t_{Mn+1}, \tau), r_{0}(t_{Mn+1})) > \Delta t_{M}; \]

\[ \Delta p_{0} = \frac{\Delta r_{0}}{(1 - D_{0}(t_{M}))}, \]

\[ (5.151) \]

\[ (5.152) \]

then, the following relation between \( \sigma_{0}^{\text{ef}} \) and \( \sigma_{0}^{\text{pred}} \):

\[ \sigma_{0}^{\text{ef}}(t_{Mn+1}, \tau) = \sigma_{0}^{\text{pred}}(t_{Mn+1}, \tau) - 2G(\Delta t_{M}) < N_{0}^{\text{ef}}(t_{Mn+1}, \tau) > \Delta p_{0}. \]  \[ (5.153) \]
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Noting that:

\[ \text{tr } \sigma^0_{ef} (t_{M_n+1}, \tau) = \text{tr } \sigma^0_{pred} (t_{M_n+1}, \tau). \]  

\[ (5.154) \]

Only the deviatoric part of equation (5.153) needs to be considered.

\[ s^0_{ef} (t_{M_n+1}, \tau) = s^0_{ef}^{pred} (t_{M_n+1}, \tau) - 2 \dot{G}(\Delta t_M) < N^0_{ef} (t_{M_n+1}, \tau) > \Delta p_0. \]

\[ (5.155) \]

We assume that the average \(< N^0_{ef} (t_{M_n+1}, \tau) >\) can be approximated by the value of the function at the midpoint of the integration interval, which corresponds to \( \tau = \frac{1}{2}. \)

Combining equations (5.19) and (5.155) and for \( \tau = \frac{1}{2}, \) it follows that:

\[ \frac{2}{3} N^0_{ef} \left( t_{M_n+1}, \frac{1}{2} \right) \sigma_{0eq} \left( t_{M_n+1}, \frac{1}{2} \right) = -2 \dot{G}(\Delta t_M) N^0_{ef} \left( t_{M_n+1}, \frac{1}{2} \right) \Delta p_0 \]

\[ + \frac{2}{3} N^0_{ef}^{pred} \left( t_{M_n+1}, \frac{1}{2} \right) \sigma_{0eq}^{pred} \left( t_{M_n+1}, \frac{1}{2} \right). \]

\[ (5.156) \]

We arrive to a radial return algorithm defined by the equations below:

\[
\begin{cases}
N^0_{ef} \left( t_{M_n+1}, \frac{1}{2} \right) = N^0_{ef}^{pred} \left( t_{M_n+1}, \frac{1}{2} \right), \\
\sigma_{0eq} \left( t_{M_n+1}, \frac{1}{2} \right) + 3 \dot{G}(\Delta t_M) \Delta p_0 = \sigma_{0eq}^{pred} \left( t_{M_n+1}, \frac{1}{2} \right).
\end{cases}
\]

\[ (5.157) \]

The problem is reduced to finding the three unknowns \( \sigma_{0eq}(t_{M_n+1}, \tau = \frac{1}{2}), r_0(t_{M_n+1}) \) and \( D_0(t_{M_n+1}) \) which satisfy the following system of four equations:

\[
\begin{cases}
k_\sigma \equiv \sigma_{0eq} \left( t_{M_n+1}, \frac{1}{2} \right) + 3 \dot{G}(\Delta t_M) \frac{\Delta r_0}{(1 - D_0(t_{M_n+1}))} - \sigma_{0eq}^{pred} \left( t_{M_n+1}, \frac{1}{2} \right) = 0, \\
k_\tau \equiv \Delta r_0 - g_0 (\sigma_{0eq}, r_0) \Delta t_M = 0, \\
k_D \equiv \Delta D_0 - \left( \frac{Y_0}{S} \right)^s \frac{\Delta r_0}{(1 - D_0(t_{M_n+1}))} = 0.
\end{cases}
\]

\[ (5.158) \]

These equations are nonlinear and solved iteratively using Newton-Raphson method.
5.6 Numerical simulation and their verification

The time-homogenization method was implemented and tested for an uniaxial loading case using one-point integration for the micro-chronological averages, and periodicity assumption of isotropic hardening, accumulated plastic strain and damage variables with respect to the fast time coordinate $\tau$. All the simulations are performed using a machine with 6 cores and 32 GB of RAM.

Consider a one-dimensional cylindrical bar of length $L$ clamped at one end ($x = 0$) and subjected at the other end ($x = L$) to a displacement which is linear at first and then sinusoidal with period $T^\ast = 0.1s$, and amplitude $U = 0.0075L$:

\[
\begin{cases}
  u^T_b (x = L, t^\ast) = \alpha L t^\ast, & \text{if } t^\ast \leq T^\ast \\
  u^T_b (x = L, t^\ast) = U \left( 0.45 \sin \left( \frac{2\pi}{T^\ast} t^\ast \right) + 0.55 \right) & \text{otherwise,}
\end{cases}
\]  

(5.159)

where $\alpha = 0.04125 s^{-1}$.

The body forces $\vec{f}$ are neglected and $10^5$ cycles are applied. The VE-VP-D material parameters are listed in table 5.1. The material parameters were identified by [Krairi, 2015] while neglecting kinematic and isotropic hardening.

The prescribed displacement can be rewritten using equation relating $t^\ast$ to $t_M^\ast$, $\tau$ and $T^\ast$:

\[
t^\ast = t_M^\ast + T^\ast \tau, \quad t_M^\ast \in [0, T_F^\ast] \text{ and } \tau \in [0, 1].
\]  

(5.160)

The loading ratio $R$, which corresponds to the ratio between the minimum and the maximum of the loading: $R = \frac{U_{\min}}{U_{\max}}$, is equal to 0.1.

Full time calculations were performed until the fracture of the specimen, when the damage $D$ reaches its critical value $D_c$. The number of cycles to failure $N_r$ was found to be $N_r = 82312$. Figure (5.4) shows the evolution of damage $D$ and accumulated plastic strain $p$ for the first twenty cycles. The damage and the accumulated plastic strains increase faster, especially at the beginning. It is clear that the variables are cumulative, which still proves it is worth taking the rapid evolution into account by assuming the non-periodicity in the asymptotic expansion functions (Section 5.3.2).
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Table 5.1: Constitutive model parameters for high density polyethylene (HDPE) at 23°C (Krairi, 2015) identified from experimental measurements of Berrehili, 2010

<table>
<thead>
<tr>
<th>Viscoelastic parameters</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial shear modulus</td>
<td>$G_0 = 1153.8$ MPa</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Initial bulk modulus</td>
<td>$K_0 = 2500$ MPa</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$G_i$(MPa)</td>
<td>$g_i(s)$</td>
<td>$K_i$ (MPa)</td>
<td>$k_i(s)$</td>
</tr>
<tr>
<td>377.5</td>
<td>0.026</td>
<td>817.9</td>
<td>0.012</td>
</tr>
<tr>
<td>235</td>
<td>0.936</td>
<td>909.1</td>
<td>0.143</td>
</tr>
<tr>
<td>192.3</td>
<td>208</td>
<td>416.66</td>
<td>96</td>
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</table>

<table>
<thead>
<tr>
<th>Viscoplastic parameters</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Viscoplastic function</td>
<td>$\zeta = 85$ MPa.s</td>
</tr>
<tr>
<td>Yield stress</td>
<td>$\sigma_y = 7$ MPa</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Damage parameters</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$S = 0.02$</td>
<td></td>
</tr>
<tr>
<td>$s = 2.3$</td>
<td></td>
</tr>
<tr>
<td>$D_c = 25%$</td>
<td></td>
</tr>
<tr>
<td>$p_D = 0.00$</td>
<td></td>
</tr>
</tbody>
</table>
5.6. Numerical simulation and their verification

Figure 5.4: Evolution of the damage (a) and the accumulated plastic strain (b). Full time calculation for the first twenty cycles. Loading period: $T^* = 0.1 \, s$, loading ratio: $R = 0.1$. 
Comparisons between the reference non-homogenized (full-time) calculation and the two-scale one are given in figures (5.5)-(5.8). The reference numerical solution is obtained using a very fine time step $\Delta t^* = \frac{T^*}{20}$, whereas the time-homogenized computation is started by a full calculation until time $t = \frac{23}{4} T^*$ and then the calculations are continued using $\Delta t^*_M = T^*$. Relative differences between time homogenized and of full-scale results are studied to check the accuracy of time homogenization approach. A relative difference is defined as:

$$\text{Relative difference} = \frac{\text{Reference result} - \text{Time-homogenized result}}{\text{Reference result}}.$$ 

In figure 5.5 are represented the stress-strain results for some selected cycles. The solid lines show the reference results, while the black crosses depict the time homogenization predictions. It is noted that due to the repeated action of the rather small applied strain, the stress level decreases significantly from a maximum of 13 MPa in the first cycle to about 7 MPa in the following ones. Furthermore the stress-strain hysteresis loop "shrinks" rapidly with increasing number of cycles. The figure shows the decay of the stress-carrying capability of the material as the damage increases.

It is seen in figures 5.5 and 5.7-(a) that for the first cycles the crosses do not superpose with the peaks of stress, the relative error between the zeroth-order homogenized solution $\sigma_0$ and the reference one $\sigma$ is about 33%. However, for the last cycles (figure 5.7-(b)) the error decreases in absolute value but it persists and it is about 27%. The fact that the error exists and that it decreases when the number of cycles increases was already noted in Chapter 3 in the case of VE-VP materials. It was explained by two reasons. First, only the zero-order in the asymptotic expansion is considered. Second, for the first twenty cycles the load period is not sufficiently small compared to the observation time which is equal in this case to $20 T^*$. Nevertheless in VE-VP-D materials the error persists at the end of the simulation and the homogenized solution $\sigma_0$ does not really converge towards the reference one. Indeed several hypotheses were made in our development. First of all, in this work we assumed that the response resulting from a cyclic loading of period $T^*$ is periodic with the same load-period. This assumption is not really correct, and the response fields are non-periodic but rather almost-periodic. This almost-periodicity is a byproduct of irreversible processes, such as damage accumulation, which naturally violates the condition of temporal periodicity. Second the one-point micro-chronological integration
5.7. Conclusions

The first contribution in this work is the formulation of a two-scale time homogenization formulation for solids and structures made of viscoelastic-viscoplastic materials coupled with ductile damage (VE-VP-D) and subjected to large numbers of cycles. The formulation extends the theory proposed by [Haouala and Doghri, 2015] for VE-VP homogeneous materials to VE-VP-D.

Two time scales are defined: a macro-chronological (slow variation) time and a micro-chronological time (for rapid evolution). Asymptotic time expansions are supposed for the unknown variables, the small parameter being the loading period. The initial boundary value problem is
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Figure 5.5: Hysteresis loops: First hundred cycles, loading period: $T^* = 0.1 \, \text{s}$, loading ratio: $R = 0.1$.

Figure 5.6: Hysteresis loops: Last twenty cycles, loading period: $T^* = 0.1 \, \text{s}$, loading ratio: $R = 0.1$. 
Figure 5.7: Evolution of the stress: homogenized solution in comparison with reference results, loading period: \( T^* = 0.1 \) s, loading ratio: \( R = 0.1 \).
(a) The first twenty cycles. (b) The last twenty cycles.
 decomposed in two: a purely VE micro-time problem which consists on
the determination of the effective stress, and a nonlinear VP macro-time
one coupled with ductile damage.

Compared to Chapter 3, we propose in this chapter a new multiscale
methodology which leads to a new formulation of the macro-chronological
problem and allows to take into account the local character of damage
and plasticity. Indeed, damage variable is irreversible and cumulative
in nature, for each unload-reload cycle its value increases, thus it can
not be supposed to be periodic with respect to the fast time coordi-
nate. This is considered also for the accumulated plastic strain. In this
chapter we propose a new paradigm which consists in a new multiscale
decomposition of the damage and plastic variables without supposing
the local periodicity.

Nevertheless, in order to evaluate the impact of the periodicity assump-
tion, numerical implicit time integration algorithms are proposed, while
assuming the periodicity of damage, accumulated plastic strain and
isotropic hardening variables $D$, $p$ and $r$ with respect to the fast time
coordinate $\tau$.

The temporal homogenization approach was implemented and tested for
an uniaxial tension/compression while assuming the damage and accu-
mulated plastic strain periodicity. For micro-time averaging, one-point
integration numerical schema was used. A significant loss of accuracy was noted and the time homogenized model assuming damage and accumulated plastic strain periodicity is unable to predict the life failure events.

A number of challenges remain to be investigated. The temporal homogenization formulation using the new assumptions of non-periodicity should be implemented and tested.
CHAPTER 6

Conclusion

The principal objective of this thesis is to predict the response under a large number of cycles, for homogeneous and reinforced thermoplastic polymers, while simulating a much smaller number.

We have first in Chapter 3 investigated a two-scale time homogenization approach for coupled viscoelastic-viscoplastic (VE-VP) homogeneous solids and structures subjected to large numbers of cycles. The main aim is to give a description of the long time behaviour, by calculating the evolution of internal variables within the structure, while reducing the computational overhead. Based on asymptotic expansions, and assuming that all the functions of the asymptotic expansions are periodic with respect to the fast time coordinate, together with the decomposition of all mechanical variables into macro- and micro-chronological parts, the original VE-VP initial-boundary problem is decomposed into coupled micro-chronological (fast time scale) and macro-chronological (slow time-scale) problems. The evolution of the viscoplastic strain and accumulated plastic strain is found to be blocked at the zero-order, with respect to the fast time scale. The proposed methodology was implemented and studied for $J_2$ VP coupled with VE using fully implicit time integration and a return-mapping algorithm. An illustration of the time homogenization on several cases was presented and a good agreement with the reference solution was observed. A significant reduction in the amount of computation time (about 94%) in addition to a small error (not exceeding 4% for the last cycles) between time homogenized and full-scale predictions were obtained.

In the second part of this work, Chapter 4, a multiscale computational strategy is proposed for the analysis of structures, which are described at a refined level both in space and in time. The proposal is applied to two-phase viscoelastic-viscoplastic (VE-VP) short glass fiber reinforced thermoplastics (SGFRTP) subjected to large numbers of cycles. The main aim is to predict the effective long time response while reducing the computational cost considerably. The proposed computational framework is a combination of the mean-field space homogenization based on the generalized incrementally affine formulation, pro-
posed by Miled et al., 2013 for VE-VP composites, and the asymptotic time homogenization approach for coupled isotropic VE-VP homogeneous solids under large numbers of cycles, proposed by Haouala and Doghri, 2015. The time homogenization method is based on the definition of micro- and macro-chronological time scales, and on asymptotic expansions of the unknown variables. All the functions of the asymptotic expansions are supposed to be periodic with respect to the fast time coordinate, and the evolution, with respect to the fast time scale, of the viscoplastic strain and accumulated plastic strain within the matrix is found to be blocked at the zero-order. Using the micro- and macro-chronological decomposition, the original anisotropic VE-VP initial-boundary value problem of the composite material is firstly decomposed into coupled micro-chronological (fast time scale $\tau$) and macro-chronological (slow time-scale $t_M$) problems. The former corresponds to a purely VE composite, whereas the latter problem is nonlinear and corresponds to a VE-VP composite. Second, mean-field space homogenization is used for both micro- and macro-chronological problems to determine the effective behavior of the composite material. The proposal was implemented for an extended Mori-Tanaka scheme and tested in the case of an uniaxial test, for a PA matrix reinforced with 30% of aligned and misaligned short glass fibers. The coupled space and time homogenization approach has been found to be in good agreement with the reference solution while estimating the effective and the per-phase stresses. Nevertheless, it does not give a good approximation of the accumulated plastic strain in the matrix phase and the error is about (40%).

In the final part of this work, Chapter 5, we proposed a new two-time scale analysis for viscoelastic-viscoplastic homogeneous materials coupled with ductile damage (VE-VP-D), under large numbers of cycles. This approach attempts to resolve the scale disparity between the time span of the period of loading and life span of a structural component. The method is based on an asymptotic expansion technique and supposes that both the response fields and the applied loads depend on both rapid and slow time scales $\tau$ and $t_M$, respectively. The proposed methodology is based on the generalization of the mathematical time homogenization theory proposed in Chapter 3 to account for irreversible inelastic deformation, which gives rise to non periodic fields in time domain (i.e. damage and accumulated plastic strain variables). Using the time homogenization approach, the boundary problem is de-
Conclusion

composed into macro-chronological and micro-chronological problems. The approach was developed while making two assumptions. The first one consists in a new multiscale decomposition of the damage and plastic variables without supposing the local periodicity. And the second one, consists in assuming the periodicity of damage, accumulated plastic strain and isotropic hardening variables, with respect to the fast time coordinate $\tau$. The proposed life prediction was implemented and verified against full time calculation in the case of local periodicity assumption of damage and plastic variables. A significant loss of accuracy was noted and the time homogenized model is unable to predict the life failure events.

Among the future work, some ideas that might be investigated are summarized below:

- The comparison between time homogenized solution with full time calculations in the case of homogeneous materials showed that, for the micro-scale time averaging, the multi-point integration algorithm gives better predictions, in uniaxial case, than the one-point one. To take into account the loading shape (triangular, sinusoidal...) and for better predictions, the multi-point integration algorithm is recommended and needs to be extended to multiaxial case. Moreover, further investigation consist in taking into account higher order terms in the asymptotic expansions, which will affect the accuracy of the time homogenized solutions.

- Our numerical experience so far (including the results shown in this work as well as other ones) seems to suggest that the coupled VE-VP model exhibits plastic shakedown under a large number of cycles. This is the case with both complete analysis or the time homogenization procedure. However the issue of the asymptotic behavior of a structure subjected to cyclic loadings was not studied in detail yet neither theoretically nor numerically. There are three types of asymptotic response:

  - Elastic shakedown in which the structure can accumulate plasticity during the first cycles but the steady state is perfectly elastic. Which means that the total dissipated energy during the whole load, remains bounded when time tends to infinity.
– Plastic shakedown in which the steady state is a closed elastic-plastic loop. If one thinks in terms of dissipated energy, the dissipation value during each cycle tends to a constant.

– And ratcheting in which the loop doesn’t close or stabilize and the steady state is an open elastic-plastic loop.

Given that, knowing the steady state of a structure is a fundamental element for the fatigue design, another research direction concerning the study of the asymptotic behavior of a structure subjected to a cyclic loading can be addressed. And the ability of the time homogenization approach to predict these kinds of steady states, which may occur in polymeric materials under cyclic loadings, must be verified.

• In this work the fatigue of structures is treated as a multiscale phenomenon in time domain. In this context the response fields were assumed to be locally periodic (1-periodicity with respect to the fast time scale \( \tau \)) whereas the state variables such as damage and accumulated plastic strain were assumed to be non periodic. The non periodic contribution was modeled by a new macro- and micro-chronological decomposition of the local fields. However, supposing that the response fields are periodic with a period equal to the load period \( T^* \) would not necessarily be true, especially in the case of localization problems (i.e., damage). Another research direction is to suppose that all the fields are not necessarily periodic and could be modeled as almost periodic. The concept of almost periodicity can take into account the perturbation, caused by the inelastic deformation, to the periodic field. The fatigue problem can then be formulated using a variant time homogenization approach developed for almost periodic fields.

• To resolve the problem of High Cycle Fatigue (HCF) of non-reinforced thermoplastics, [Krairi, 2015] proposed, within the framework of continuum damage mechanics (CDM), a multiscale approach to predict the failure of the material. The damaged material is regarded as a matrix containing a small volume fraction (2%) of spherically shaped micro heterogeneities, called also weak spots. The approach supposes that the matrix remains VE and the evolution of damage takes place only in the weak spots. The matrix
Conclusion

is then supposed VE, whereas the weak spots are VE-VP-D. MFH homogenization techniques are then used to predict the fracture of the material. The material fails when the damage, within the weak spots, reaches a critical value.

For the HCF of short glass fiber reinforced thermoplastics, the damaged material is regarded as a three-phase composite (matrix, fibers and weak spots). In this case and to treat the HCF of composite materials, [Krairi, 2015] proposed the so-called decoupled approach. Two problems are considered and treated separately. The first one consists in a VE-VP matrix reinforced with elastic inclusions subjected to cyclic loading. And the second one consists in the weak spots which are considered as VE-VP-D.

The first step consists in applying a cyclic loading with large numbers of cycles to the matrix reinforced with short glass fibers. The matrix is VE-VP and the fibers are elastic. Using MFH the strain averages within the matrix can be obtained and they serves as loadings for the (VE-VP-D) weak spots. The composite fails when the damage within the weak spots reaches its critical value. This approach is justified by the fact that the volume fraction of the weak spots is too small and does not have a big influence on the macro scale.

Although the decoupled approach proposed by [Krairi, 2015] helps to reduce the computation time, it still remains costly to use. Predicting the behavior of short glass fiber reinforced thermoplastics, in an economic way, is an objective of DURAFIP project. The time homogenization of VE-VP materials coupled with ductile damage presented in Chapter 5 needs to be extended and implemented to treat the problems of HCF of composite materials.

- In order to improve predictions and to better account for the heterogeneity of microscopic field, extending the proposed incrementally affine linearization method from first- to higher- order homogenization would be interesting. In this approach the incremental formulation is coupled with second statistical moments of per-phase strain increment fields (Doghri et al., 2011).

- The incrementally affine linearization needs to be extended to the case of coupled thermo-viscoelastic-viscoplastic materials which is an objective of DURAFIP project and MFH needs to be generalized to this case. Then a double homogenization approach with respect to both space and time must be applied to the thermosen-
sitive heterogeneous media subjected to harmonic excitations of long duration.

- For polymer materials, the degree of crystallinity is important since crystallinity influences many of the polymer properties. For instance [Riddell et al., 1966] showed that an increased degree of crystallinity improves the fatigue resistance of the material. The degree of crystallinity influences also the cyclic deformation, for instance [Drozdov et al., 2013] studied the mechanical behavior of low density polyethylene (LDPE) under cyclic loadings and showed that the maximum strains can reach 80%. For this case, the constitutive model presented in this work is not able to predict the mechanical behavior of the material and a finite strain formulation of the model is required and should give theoretically more justified and better predictions (e.g. [Holmes and Loughran, 2010] and [Peric and Dettmer, 2003]).
Asymptotic expansions

An asymptotic expansion describes the asymptotic behavior of a function in terms of a sequence of reference functions. Let’s define a comparison scale, that is a set of functions $\mathcal{E}$ of the following types:

1. $x^\alpha (\alpha \neq 0)$, $(\ln x)^\beta (\beta \neq 0)$, $e^{cx^\gamma} (c \neq 0, \gamma > 0)$,

verifying: the product of two functions of $\mathcal{E}$ is in $\mathcal{E}$, if $f \in \mathcal{E}$ then it exists $\lambda > 0$ such as $f$ is strictly positive in $]0, \lambda[$ and non constant functions tend towards 0 or $+\infty$ when $x$ tends towards $0+$.

Let $f$ be a real (or complex) function. The principal part of $f$ with respect to $\mathcal{E}$ is a function $c_1g_1(x)$, where $g_1 \in \mathcal{E}$ such as:

$$\lim_{x \to 0^+} \frac{f(x)}{g_1(x)} = c_1.$$ \hfill (A.1)

The principal part, if it exists, it is unique and we can write:

$$f(x) = c_1g_1(x) + o(g_1(x)).$$ \hfill (A.2)

To obtain a better approximation of $f(x)$ using functions of $\mathcal{E}$, we are led to compare the function $f(x) - c_1g_1(x)$ to functions of $\mathcal{E}$; if this function has a principal part $c_2g_2(x)$, $g_2(x) = o(g_1(x))$, we can write:

$$f(x) = c_1g_1(x) + c_2g_2(x) + o(g_2(x)).$$ \hfill (A.3)

Generally, we designate by asymptotic expansions to $k$ terms of the function $f(x)$ (at $0+$) with respect to $\mathcal{E}$, a sum $\sum_{j=1}^k c_jg_j(x)$ where $c_j$ are non-zero constants and $g_j(x)$ are functions of $\mathcal{E}$ such as $g_{j+1}(x) = o(g_j(x))$ and we have:

$$f(x) = \sum_{j=1}^k c_jg_j(x) + o(g_k(x)).$$ \hfill (A.4)

The difference $f(x) - \sum_{j=1}^k c_jg_j(x)$ is called rest of the expansion.
If the asymptotic expansion exists, it is unique since for any \(1 \leq j \leq k\), \(c_j g_j(x)\) is the principal part of \(f(x) - \sum_{i=1}^{j-1} c_i g_i(x)\).

When restricting an asymptotic expansion to its first \(j\) terms, we obtain the asymptotic expansion of \(f\) with the accuracy \(g_j\).

If we replace \(E\) by a larger scale \(E'\), the asymptotic expansion to \(k\) terms (if it exists) of \(f\) with respect to \(E\) is its asymptotic expansion to \(k\) terms with respect to \(E'\).

When we know the asymptotic expansions of two functions \(f_1, f_2\) with respect to the same scale \(E\),
\[
f_1(x) = \sum_{j=1}^{k} c_j g_j + o(g_k), \quad f_2(x) = \sum_{j=1}^{k} c_j h_j + o(h_k),
\]
we obtain the asymptotic expansion of the sum \(f_1 + f_2\) by adding coefficients of similar terms then by gathering terms with respect to the order \((g_j = h_j\) or \(g_j = o(h_j))\); finally, we remove all terms in \(g_i\) such as \(g_j = o(h_i)\) and all terms in \(h_r\) such as \(h_r = o(g_s)\). The accuracy will be the smaller of the two accuracies.

**Asymptotic expansion of viscous stresses**

In this section we give the asymptotic expansion of the viscous stresses \(s_i(t):\)
\[
s_i(t_M + T) = 2G_i \exp\left(-\frac{(t_M + T)}{g_i}\right) \int_{-\infty}^{t_M + T} \exp\left(\frac{\eta}{g_i}\right) \frac{\partial \xi(t, \eta)}{\partial \eta} d\eta,
\]  
(A.6)

Equation (A.6) is the product of two functions \(2G_i \exp\left(-\frac{(t_M + T)}{g_i}\right)\) and \(\int_{-\infty}^{t_M + T} \exp\left(\frac{\eta}{g_i}\right) \frac{\partial \xi(t, \eta)}{\partial \eta} d\eta\). Its asymptotic expansion is the product of the asymptotic expansions of these two functions at the neighborhood of \(T = 0\).

We have:
\[
\exp\left(-\frac{(t_M + T)}{g_i}\right) = \exp\left(-\frac{t_M}{g_i}\right) - \frac{T}{g_i} \exp\left(-\frac{t_M}{g_i}\right) T + O(T^2). \quad (A.7)
\]

Using Chasles relation we have:
\[
\int_{-\infty}^{t_M + T} \exp\left(\frac{\eta}{g_i}\right) \frac{\partial \xi(t, \eta)}{\partial \eta} d\eta = \int_{-\infty}^{t_M} \exp\left(\frac{\eta}{g_i}\right) \frac{\partial \xi(t, \eta)}{\partial \eta} d\eta
\]
\[
+ \int_{t_M}^{t_M + T} \exp\left(\frac{\eta}{g_i}\right) \frac{\partial \xi(t, \eta)}{\partial \eta} d\eta \quad (A.8)
\]
Let $f$ and $g$ be the functions defined by the following expressions:

$$f(t_M + T\tau) = \int_{t_M}^{t_M+T\tau} \exp\left(\frac{\eta}{g_i}\right) \frac{\partial \xi^{ve}(\eta)}{\partial \eta} d\eta,$$  \hspace{1cm} (A.9)

$$g(t_M + T\tau) = \exp\left(\frac{t_M}{g_i}\right) \left(1 + \frac{T}{g_i}\right) (\xi^{ve}(t_M + T\tau) - \xi^{ve}(t_M)).$$  \hspace{1cm} (A.10)

We can easily demonstrate, using the mean value theorem, that $f = g + O(T^2)$. Then the asymptotic expansion of $f$ is the same as that for $g$ and we have:

$$f(t_M + T\tau) = \exp\left(\frac{t_M}{g_i}\right) \left[(\xi^{ve}_0(t_M + T\tau) - \xi^{ve}_0(t_M))
\right.
\left.+ \left(\xi^{ve}_1(t_M + T\tau) + \frac{1}{g_i}(\xi^{ve}_0(t_M + T\tau) - \xi^{ve}_0(t_M))\right)\right].$$  \hspace{1cm} (A.11)

Then the asymptotic expansion of $s_i(t_M + T\tau)$ is:

$$s_i(t_M + T\tau) = s_{i_0}(t_M, \tau) + s_{i_1}(t_M, \tau) T + O(T^2),$$  \hspace{1cm} (A.12)

where $s_{i_0}(t_M, \tau)$ and $s_{i_1}(t_M, \tau)$ are given by the following expressions:

$$s_{i_0}(t_M, \tau) = s_i(t_M) + 2G_i (\xi^{ve}_0(t_M, \tau) - \xi^{ve}_0(t_M, \tau = 0)),$$  \hspace{1cm} (A.13)

$$s_{i_1}(t_M, \tau) = 2G_i \xi^{ve}_1(t_M, \tau) + \frac{2G_i}{g_i} \left[-\tau \int_{-\infty}^{t_M} \exp\left(\frac{\eta}{g_i}\right) \xi^{ve}(\eta) d\eta + (1 - \tau) (\xi^{ve}_0(t_M, \tau) - \xi^{ve}_0(t_M, \tau = 0))\right].$$  \hspace{1cm} (A.14)
B.0.1 Linear VE part

According to the Stone-Weierstrass theorem, every real continuous scalar-valued or (tensor-valued) functional can be uniformly approximated as closely as desired by a polynomial in a set of real continuous linear functionals, and using the Riesz’s representation theorem, these linear functionals may be further expressed in terms of Stieltjes integrals. This method was used by [Green and Rivlin, 1959] to give an approximation of the stress in the form of sums of multiple integrals, and this approximation was later discussed by [Chacon and Rivlin, 1964]. [Christensen and Naghdi, 1967] made use of these results to find a suitable representation for the VE free energy in terms of linear and quadratic functionals of $\varepsilon_{ij}$ and temperature under non-isothermal conditions. Using the same methodology, and considering isothermal deformations of a linear VE solid, we assume that the free energy can be represented as a functional of $\varepsilon_{ij}^{ve}$ in the following manner:

$$
\rho \psi_{ve} = \frac{1}{2} \int_{-\infty}^{t} \int_{-\infty}^{t} G_{ijkl}(t - \xi, t - \tau) \frac{\partial \varepsilon_{ij}^{ve}(\xi)}{\partial \xi} \frac{\partial \varepsilon_{kl}^{ve}(\tau)}{\partial \tau} d\xi d\tau. \quad (B.1)
$$

First, the integral in equation (B.1) is rewritten in the following form:

$$
I(t) = \int_{-\infty}^{t} \int_{-\infty}^{t} F(t, \xi, \tau) d\xi d\tau. \quad (B.2)
$$

Assuming the continuity and differentiability of the function $F$, the derivative of the integral $I$ is:

$$
\frac{dI}{dt} = \int_{-\infty}^{t} \int_{-\infty}^{t} \frac{\partial F(t, \xi, \tau)}{\partial t} d\xi d\tau + \int_{-\infty}^{t} F(t, \xi, t) d\xi + \int_{-\infty}^{t} F(t, t, \tau) d\tau. \quad (B.3)
$$
According to [Christensen and Naghdi, 1967], the integrating function has the following properties:

\[
\begin{align*}
G_{ijkl}(\xi, \tau) &= 0 \quad \text{for} \quad \xi < 0, \tau < 0, \\
G_{ijkl}(\xi, \tau) &= G_{jikl}(\xi, \tau) = G_{ijlk}(\xi, \tau), \\
G_{ijkl}(\xi, \tau) &= G_{klij}(\tau, \xi), \\
G_{ijkl}(\xi, \tau) &= C_{ijkl}^{ve}(\xi + \tau).
\end{align*}
\] (B.4)

Using the rule in equation (B.3), and the previous properties, the time derivative of \( \rho \psi_{ve} \) is:

\[
\dot{\rho} \psi_{ve} = \left( \int_{-\infty}^{t} C_{ijkl}^{ve}(t - \tau) \frac{\partial \varepsilon_{kl}^{ve}(\tau)}{\partial \tau} d\tau \right) \dot{\varepsilon}_{ij}^{ve} \\
+ \frac{1}{2} \int_{-\infty}^{t} \int_{-\infty}^{t} \frac{\partial}{\partial \tau} C_{ijkl}^{ve}(2t - \xi - \tau) \frac{\partial \varepsilon_{ij}^{ve}(\xi)}{\partial \xi} \frac{\partial \varepsilon_{kl}^{ve}(\tau)}{\partial \tau} d\xi d\tau
\] (B.5)

**B.0.2 Viscoplastic part**

The viscoplastic part of the free energy is defined by:

\[
\rho \psi_{vp} = \int_{0}^{r(t)} R(\xi) d\xi
\] (B.6)

Its derivative is:

\[
\dot{\rho} \psi_{vp} = R \dot{r}.
\] (B.7)

**B.0.3 Dissipation inequality**

Using equations (3.2) and (3.3) and replacing the derivative of \( \rho \psi \) in the Clausius-Duhem inequality (3.1) under isothermal conditions, we get:

\[
\left( \sigma_{ij} - \int_{-\infty}^{t} C_{ijkl}^{ve}(t - \tau) \frac{\partial \varepsilon_{kl}^{ve}(\tau)}{\partial \tau} d\tau \right) \dot{\varepsilon}_{ij}^{ve} + \sigma_{ij} \dot{\varepsilon}_{ij}^{vp} \\
- \frac{1}{2} \int_{-\infty}^{t} \int_{-\infty}^{t} \frac{\partial}{\partial \tau} C_{ijkl}^{ve}(2t - \xi - \tau) \frac{\partial \varepsilon_{ij}^{ve}(\xi)}{\partial \xi} \frac{\partial \varepsilon_{kl}^{ve}(\tau)}{\partial \tau} d\xi d\tau - R \dot{r} \geq 0.
\] (B.8)

The fact that inequality (B.8) holds for arbitrary transformation implies that the coefficient of \( \dot{\varepsilon}_{ij}^{ve} \) vanishes. Thus:

\[
\sigma_{ij} = \int_{-\infty}^{t} C_{ijkl}^{ve}(t - \tau) \frac{\partial \varepsilon_{kl}^{ve}(\tau)}{\partial \tau} d\tau,
\] (B.9)
and consequently the Clausius-Duhem inequality requires the dissipation \( \Phi \) to be necessarily non-negative:

\[
\Phi = \sigma_{ij} \dot{\varepsilon}_{ij} - \frac{1}{2} \int_{-\infty}^{t} \int_{-\infty}^{t} \frac{\partial}{\partial t} C_{ijkl}^{ve}(2t - \xi - \tau) \frac{\partial \varepsilon_{ij}^{ve}(\xi)}{\partial \xi} \frac{\partial \varepsilon_{kl}^{ve}(\tau)}{\partial \tau} d\xi d\tau - R \dot{\varepsilon} \geq 0. \quad (B.10)
\]

The dissipation \( \Phi \) is decomposed into viscoplastic (\( \Phi_{vp} \)) and viscoelastic (\( \Phi_{ve} \)) components, and each can be required to be non-negative:

\[
\Phi_{vp} = \sigma_{ij} \dot{\varepsilon}_{ij}^{vp} - R \dot{\varepsilon} \geq 0. \quad (B.11)
\]

\[
\Phi_{ve} = -\frac{1}{2} \int_{-\infty}^{t} \int_{-\infty}^{t} \frac{\partial}{\partial t} C_{ijkl}^{ve}(2t - \xi - \tau) \frac{\partial \varepsilon_{ij}^{ve}(\xi)}{\partial \xi} \frac{\partial \varepsilon_{kl}^{ve}(\tau)}{\partial \tau} d\xi d\tau \geq 0. \quad (B.12)
\]
Algorithmic tangent operator

In this appendix we develop the expression of the zero-order algorithmic tangent operator $C^{algo}_{0}$, in the case of VE-VP materials. Indeed, contrary to EP, a continuum tangent operator relating the zero-order stress and strain rates, does not exist in rate-dependent VE-VP. However, a zero-order algorithmic tangent operator $C^{algo}_{0}$ relating finite zero-order stress and strain increments, can be derived by consistent linearization of the time discretized constitutive equations around the solution at $t_{M_{n+1}}$ and $\tau$:

Linearization of relations (3.117), (3.115) and (3.116) gives:

$$
\begin{align*}
\delta \sigma_{0M}(t_{M_{n+1}}) &= \hat{E} : \delta \varepsilon_{0M}(t_{M_{n+1}}) - 2\hat{G} \delta \varepsilon_{0vp}^{v}(t_{M_{n+1}}), \\
\delta \varepsilon_{0vp}^{v}(t_{M_{n+1}}) &= \left( \frac{\partial N_{0}}{\partial \sigma_{0}} |_{(t_{M_{n+1}}, \tau = \frac{1}{2})} : \delta \sigma_{0}(t_{M_{n+1}}, \tau = \frac{1}{2}) \right) \Delta p_{0} \\
&+ N_{0}(t_{M_{n+1}}, \tau = \frac{1}{2}) \delta p_{0}(t_{M_{n+1}}), \\
\delta p_{0}(t_{M_{n+1}}) &= g_{,\sigma_{0}} \left[ N_{0}(t_{M_{n+1}}, \tau = \frac{1}{2}) : \delta \sigma_{0}(t_{M_{n+1}}, \tau = \frac{1}{2}) \right] \Delta t_{M} \\
&+ \left[ g_{,p_{0}} \delta p_{0}(t_{M_{n+1}}) + g_{,\varepsilon_{0}} : \delta \varepsilon_{0}(t_{M_{n+1}}, \tau = \frac{1}{2}) \right] \Delta t_{M}.
\end{align*}
$$

(C.1)

Where: $g_{,\sigma_{0}} \equiv \frac{\partial g_{,\sigma}}{\partial \sigma_{0}}$; $g_{,p_{0}} \equiv \frac{\partial g_{,p}}{\partial p_{0}}$; $g_{,\varepsilon_{0}} \equiv \frac{1}{\Delta \varepsilon_{0}} \frac{\partial g_{,\varepsilon}}{\partial \varepsilon_{0}}$.

Combining equation (C.1 a) with (C.1 b) leads to:

$$
\delta \sigma_{0M}(t_{M_{n+1}}) = \hat{E} : \delta \varepsilon_{0M}(t_{M_{n+1}}) - 2\hat{G} \left[ N_{0}(t_{M_{n+1}}, \tau = \frac{1}{2}) \delta p_{0}(t_{M_{n+1}}) \\
+ \left( \frac{\partial N_{0}}{\partial \sigma_{0}} |_{(t_{M_{n+1}}, \tau = \frac{1}{2})} : \delta \sigma_{0M}(t_{M_{n+1}}) \right) \Delta p_{0} \right].
$$

(C.2)

Based on the flow rule equation (3.22), the zeroth-order of the second-order tensor $N$ at $t_{M_{n+1}}$ and $\tau = \frac{1}{2}$ is:

$$
N_{0}(t_{M_{n+1}}, \tau = \frac{1}{2}) = \frac{3}{2} \frac{s_{0}(t_{M_{n+1}}, \tau = \frac{1}{2})}{\sigma_{0eq}(t_{M_{n+1}}, \tau = \frac{1}{2})}.
$$

(C.3)
After some mathematical developments, we have:

\[
\frac{\partial N_0}{\partial \sigma_0} \bigg|_{(t_{M_{n+1}}, \tau = \frac{1}{2})} = \frac{1}{\sigma_{0eq}(t_{M_{n+1}}, \tau = \frac{1}{2})} \times 
\left( \frac{3}{2} I^{dev} - N_0(t_{M_{n+1}}, \tau = \frac{1}{2}) \otimes N_0(t_{M_{n+1}}, \tau = \frac{1}{2}) \right). \tag{C.4}
\]

From which it is deduced that:

\[
N_0 : \frac{\partial N_0}{\partial \sigma_0} \bigg|_{(t_{M_{n+1}}, \tau = \frac{1}{2})} = 0. \tag{C.5}
\]

and

\[
\frac{\partial N_0}{\partial \sigma_0} \cdot \frac{\partial N_0}{\partial \sigma_0} \bigg|_{(t_{M_{n+1}}, \tau = \frac{1}{2})} = \frac{3}{2\sigma_{0eq}(t_{M_{n+1}}, \tau = \frac{1}{2})} \frac{\partial N_0}{\partial \sigma_0} \bigg|_{(t_{M_{n+1}}, \tau = \frac{1}{2})}. \tag{C.6}
\]

Hence, if one permultiplies equation (C.2) by \( N_0(t_{M_{n+1}}, \tau = \frac{1}{2}) \), the following relation holds:

\[
N_0(t_{M_{n+1}}, \tau = \frac{1}{2}) : \delta \sigma_0(t_{M_{n+1}}) = N_0(t_{M_{n+1}}, \tau = \frac{1}{2}) : \hat{E} : \delta \varepsilon_0(t_{M_{n+1}}) - 3 \hat{G} \delta p_0(t_{M_{n+1}}). \tag{C.7}
\]

Rewriting equation (C.1 c) using equation (C.7) leads to:

\[
\delta p_0(t_{M_{n+1}}) = \left[ g, \sigma_0 \left(N_0(t_{M_{n+1}}, \tau = \frac{1}{2}) : \hat{E} : \delta \varepsilon_0(t_{M_{n+1}}) - 3 \hat{G} \delta p_0 \right) \right.
+ \left. g, \sigma_0 \left(N_0(t_{M_{n+1}}, \tau = \frac{1}{2}) : \delta \tilde{\sigma}_0(t_{M_{n+1}}, \tau = \frac{1}{2}) \right) + g, p_0 \delta p_0(t_{M_{n+1}}) 
+ g, \varepsilon_0 : \delta \varepsilon_0(t_{M_{n+1}}, \tau = \frac{1}{2}) \right] \Delta t_M. \tag{C.8}
\]

Which implies that:

\[
\delta p_0(t_{M_{n+1}}) = \frac{1}{h_v} \left[ \left( N_0(t_{M_{n+1}}, \tau = \frac{1}{2}) : \hat{E} + \frac{g, \varepsilon_0}{g, \sigma_0} \right) : \delta \varepsilon_0(t_{M_{n+1}}) 
+ N_0(t_{M_{n+1}}, \tau = \frac{1}{2}) : \delta \tilde{\sigma}_0(t_{M_{n+1}}) \right. 
+ \left. \frac{g, \varepsilon_0}{g, \sigma_0} : \delta \varepsilon_0(t_{M_{n+1}}, \tau = \frac{1}{2}) \right], \tag{C.9}
\]

where the denominator \( h_v \) is defined by this expression:

\[
h_v = \frac{1}{\Delta t_M g, \sigma_0} + 3 \hat{G} - \frac{g, p_0}{g, \sigma_0}. \tag{C.10}
\]
Substituting the latter expression of $\delta p_0$ into equation (C.2) leads to:

$$
\delta \sigma_{0,M}(t_{M+1}) = \mathbf{E} : \delta \varepsilon_{0}^{M}(t_{M+1}) - 2\hat{G} \left[ \frac{1}{h_v} \mathbf{N}_0(t_{M+1}, \tau = \frac{1}{2}) \otimes \left( \mathbf{N}_0(t_{M+1}, \tau = \frac{1}{2}) : \mathbf{E} + \frac{g \varepsilon_0}{g \sigma_0} : \delta \varepsilon_0(t_{M+1}) \right) + \mathbf{N}_0(t_{M+1}, \tau = \frac{1}{2}) : \delta \tilde{\sigma}_0(t_{M+1}, \tau = \frac{1}{2}) \right) + \left( \frac{\partial \mathbf{N}_0}{\partial \sigma_0} \right)_{(t_{M+1}, \tau = \frac{1}{2})} : \delta \sigma_{0,M}(t_{M+1}) \right] \Delta p_0. \tag{C.11}
$$

Using equation (3.101), we have:

$$
\delta \tilde{\sigma}_0(t_{M+1}, \tau = \frac{1}{2}) = \mathbf{E} : \delta \tilde{\varepsilon}_0(t_{M+1}, \tau = \frac{1}{2}) \tag{C.12}
$$

Finally, the consistent or algorithmic tangent operator is found to be:

$$
C_{0}^{algo} \equiv \left[ \mathbf{I} + 2\hat{G} \Delta p_0 \frac{\partial \mathbf{N}_0}{\partial \sigma_0} \big|_{(t_{M+1}, \tau = \frac{1}{2})} \right]^{-1} : \left[ \mathbf{I} - 2\hat{G} \frac{\mathbf{N}_0(t_{M+1}, \tau = \frac{1}{2}) \otimes \mathbf{N}_0(t_{M+1}, \tau = \frac{1}{2})}{h_v g \sigma_0} : \mathbf{E} - \frac{2\hat{G}}{h_v g \sigma_0} \mathbf{N}_0(t_{M+1}, \tau = \frac{1}{2}) \otimes g \varepsilon_0 \right]. \tag{C.13}
$$

Making use of equation (C.4), one may check that:

$$
\left[ \mathbf{I} + 2\hat{G} \Delta p_0 \frac{\partial \mathbf{N}_0}{\partial \sigma_0} \big|_{(t_{M+1}, \tau = \frac{1}{2})} \right]^{-1} = \mathbf{I} - 2\hat{G} \frac{\sigma_{0,eq}(t_{M+1}, \tau = \frac{1}{2}) \Delta p_0}{\sigma_{0,eq}(t_{M+1}, \tau = \frac{1}{2}) + 3\hat{G} \Delta p_0 \frac{\partial \mathbf{N}_0}{\partial \sigma_0} \big|_{(t_{M+1}, \tau = \frac{1}{2})}} \tag{C.14}
$$

Hence, one finally gets:

$$
C_{0}^{algo} = \hat{\mathbf{E}} - \frac{(2\hat{G})^2}{h_v} \mathbf{N}_0(t_{M+1}, \tau = \frac{1}{2}) \otimes \mathbf{N}_0(t_{M+1}, \tau = \frac{1}{2}) + \frac{2\hat{G}}{h_v g \sigma_0} \mathbf{N}_0(t_{M+1}, \tau = \frac{1}{2}) \otimes g \varepsilon_0
- \frac{(2\hat{G})^2}{\sigma_{0,eq}(t_{M+1}, \tau = \frac{1}{2}) \Delta p_0} \frac{\partial \mathbf{N}_0}{\partial \sigma_0} \big|_{(t_{M+1}, \tau = \frac{1}{2})} \tag{C.15}
$$
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