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Référence bibliographique

Price Symmetry in a Duopoly with Congestion

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Key Words: duopolistic pricing, congestion, symmetric equilibria, covered market.

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1 Introduction

A product is said to be subject to congestion - or, more formally, negative network externalities - if the utility to each customer purchasing the product decreases with an increase in the total number of customers purchasing the product. Congestion is most characteristic of services, since often there is an increase in waiting time, or a reduction in quality of service at the point of delivery, resulting from an increase in the number of customers utilizing the service. Examples of congested products include Internet service provision (e.g., Mackie-Mason and Varian (1995)), medical care (discussed in Gibbens, Mason and Steinberg (2000)), and electricity supply (Wilson (1989)), among many others. Typically, customers will be heterogeneous with respect to their disutility from congestion.

In a heterogeneous market, a monopolistic firm has a natural incentive to price-discriminate by exploiting differences in customer willingness to pay (e.g., Mussa and Rosen (1978)). However, when the market is covered, as is conceivably the case in medical care and the other examples above, an unregulated monopoly will offer a single and arbitrarily high price. A straightforward way in which a regulator can increase the social benefit is by requiring the monopoly to offer a two-tier pricing structure. In the specific case of a market with congestion, customers will self-select between the higher- and lower-priced products according to their individual aversion to congestion.

In a market without congestion, competition in prices drives them downward, resulting in zero profits. By contrast, in congested markets, the downward pressure on prices is countered to some extent by the congestion effect. Due to welfare improving effects resulting from duopolistic competition, one might expect that the duopoly pricing will mimic the regulation, effectively creating a two-tier price structure.

In this paper we show, in fact, that the duopoly situation differs markedly from the regulated one. We show that, under general congestion functions and a general distribution of customer disutility from congestion, there cannot exist an asymmetric Nash equilibrium. Indeed, if an equilibrium exists in the duopoly, the firms charge identical and uniquely determined prices. We also provide simple closed-form expressions for the equilibrium prices and profits.

An important assumption behind our results is that the underlying market is covered. If the market were not covered, competing firms might indeed choose to differentiate themselves by each offering a different price, and thus
a different quality of the product (in terms of congestion). For instance, Levhari and Luski (1978) consider a congested queuing system with two servers, which can be shown to be a special case of our model without the covered market assumption. They find that, in general, only asymmetric equilibria can occur: one firm will offer a higher price than the other, appealing only to the most congestion-sensitive customers, while the majority of customers will use the service of the second, less expensive, firm.

The reason for the difference in the equilibria in the covered versus uncovered markets can be found in a comparison of the factors determining demand. The asymmetry of equilibria in the uncovered market means that, if the prices were to be equal, one of the firms would have an incentive to unilaterally change its price. If the firm raises its price, there will be both a loss of customers to the rival firm based on price comparison, and a gain of new, highly congestion-sensitive customers, who preferred not to use the service under the initial symmetric prices and undifferentiated congestion levels. Alternatively, if the firm lowers its price, it will capture some of the customers from the rival firm, which sees them partially replaced by new, highly congestion-sensitive ones. However, both price deviations are less profitable when the market is covered. Indeed, the higher price would not bring new customers to the deviant firm; the lower price would draw a smaller share of customers from the rival firm which now gains no new customers, and thus is more attractive on account of lower congestion. Thus, the incentive to separate equal prices in an uncovered market subsides when the market is covered.

Our model is akin to the one used by de Palma and Leruth (1989), who consider consumer utility functions that are additively separable in terms corresponding to the benefit from utilizing the service, a linear congestion factor, and the price of the service. However, while they assume the distribution of customer disutility from congestion to be uniform, we allow for arbitrary distribution functions. Thus our results apply to the realistic situation where consumer disutility for congestion is not uniform, but more general; even in the case where the distribution function is unknown, the result of non-existence of an asymmetric equilibrium still obtains. Further, we do not restrict ourself to a linear congestion function, but allow for general functions.
2 The Model

We assume that the customer disutility from congestion, $\theta$, is distributed along the interval $[0,1]$ with a cumulative distribution function $F$ that is continuously differentiable and strictly increasing. For ease of presentation, we assume that customers are uniquely characterized by disutility from congestion.\footnote{1}{In Section 4 we show how this assumption can be relaxed.} Thus we identify each customer with his corresponding $\theta$.

Upon choosing firm $i$ ($i = I, II$), a customer $\theta$ receives utility $U(\theta, i)$, which has three components: (i) a benefit $V(\theta)$ $\geq 0$ from the use of the product, independent of the firm at which it was purchased; (ii) a dis-benefit $\theta \cdot K(Q^i)$, where $K(Q^i)$ is a congestion factor that is a continuously differentiable and strictly increasing function of $Q^i$, the mass of customers choosing firm $i$ - its customer base; and (iii) a dis-benefit from having to pay a purchase price $p^i$. In symbols, the utility of customer $\theta$ choosing firm $i$ is

$$U(\theta, i) = V(\theta) - \theta \cdot K(Q^i) - p^i.$$  

This formulation of customer utility is due to de Palma and Leruth (1989). Note that $\theta$ can be thought of as the per-unit cost of congestion incurred by the customer $\theta$.

For convenience, we make a mild normalization assumption, $K : [0,1] \rightarrow [0,1]$, and assume that it takes all values in its range. Note that, since $K$ is strictly increasing, $K(0) = 0$ and $K(1) = 1$.

Given a pair of prices $(p^I, p^{II}) \in R^2_+$, the customer bases of the two firms, $Q^I(p^I, p^{II})$ and $Q^{II}(p^I, p^{II})$, are determined by the following conditions.

Assume that $0 < p^j < p^i$, for $i, j \in \{I, II\}$. Then:

1. the mass of customers $\theta$ for which $U(\theta, i) > U(\theta, j)$ is $Q^i(p^I, p^{II})$, according to the distribution $F$;

2. the mass of customers $\theta$ for which $U(\theta, j) > U(\theta, i)$ is $Q^j(p^I, p^{II})$, according to the distribution $F$.

Note that there is at most one $\theta$, the marginal customer, who is indifferent between the two firms. Thus $Q^I(p^I, p^{II}) + Q^{II}(p^I, p^{II}) = 1$. In the case of $p^I = p^{II}$, we assume that the firms split the market equally, that is, $Q^I(p^I, p^{II}) = Q^{II}(p^I, p^{II}) = \frac{1}{2}$.

\textbf{Lemma 1} If $p^j < p^i$, for $i, j \in \{I, II\}$, then $Q^j > Q^i$. 

$\$
Proof. If $Q^i \geq Q^j$, then the congestion at $i$ would be at least as great as the congestion at $j$, and every customer of $i$ would individually prefer to switch to firm $j$, where he suffers no greater congestion but pays a lower price. Thus $Q^i = 1 - Q^j = 0$, contradicting the assumption that $Q^i \geq Q^j$. ■

Lemma 2, below, characterizes the marginal customer. Lemma 3 shows that the customer bases of the two firms are uniquely defined, and expresses them in closed functional form.

**Lemma 2** Assume that $p^j < p^i < p^j + 1$ for $i, j \in \{I, II\}$. Then there exists a unique $\theta^* = \theta^* (p^I, p^{II}) \in [0, 1]$ such that

$$V(\theta^*) - \theta^* \cdot K(1 - F(\theta^*)) - p^i = V(\theta^*) - \theta^* \cdot K(F(\theta^*)) - p^j.$$  \hspace{1cm} (1)

Moreover, define

$$G(\theta) = [K(F(\theta)) - K(1 - F(\theta))] \theta$$ \hspace{1cm} (2)

for all $\theta \in [0, 1]$. Then

$$\theta^* (p^I, p^{II}) = G^{-1}(p^I - p^j),$$ \hspace{1cm} (3)

and $\theta^*$ is continuously differentiable at all such $(p^I, p^{II})$.

**Proof.** Observe that $G(\cdot)$ is strictly increasing on $[F^{-1}(\frac{1}{2}), 1]$, being a product of two non-negative and strictly increasing functions. Therefore $G(\cdot)$ possesses an inverse $G^{-1} : [0, 1] \rightarrow [F^{-1}(\frac{1}{2}), 1]$. From (1) it follows that $G(\theta^*) = p^i - p^j$, and so $\theta^*$ is given by (3). Since $G^{-1}$ is continuously differentiable on $[0, 1]$, then $\theta^* (p^I, p^{II})$ is continuously differentiable at every $(p^I, p^{II})$ satisfying the conditions of the lemma. ■

By continuity, we can extend the definition of $\theta^* (p^I, p^{II})$ to price vectors with equal components, and get

$$\theta^* (p^I, p^{II}) = \begin{cases} G^{-1}(p^I - p^{II}), & p^I \geq p^{II}, \\ G^{-1}(p^{II} - p^I), & p^{II} > p^I. \end{cases} \hspace{1cm} (4)$$
Lemma 3 Given a pair of prices \((p^I, p^{II})\), the customer bases of the two firms are uniquely determined by

\[
(Q^I (p^I, p^{II}), Q^{II} (p^I, p^{II})) =
\]

\[
= \begin{cases} 
(1 - F [\theta^* (p^I, p^{II})], F [\theta^* (p^I, p^{II})]), & \text{if } p^{II} < p^I < p^{II} + 1, \\
(0, 1), & \text{if } p^I + 1 \leq p^I, \\
(\frac{1}{2}, \frac{1}{2}), & \text{if } p^I = p^{II}, \\
(F [\theta^* (p^I, p^{II})], 1 - F [\theta^* (p^I, p^{II})]), & \text{if } p^I < p^{II} < p^I + 1, \\
(1, 0), & \text{if } p^I + 1 \leq p^{II}.
\end{cases}
\]

Proof. If \(p^I = p^{II}\) then the equal split is postulated. Assume now that \(p^I < p^I < p^I + 1, i, j \in \{I, II\}\). Observe that \(Q^I, Q^{II} > 0\), since in the complementary case the only possibility is \((Q^I, Q^I) = (0, 1)\), by Lemma 1. This implies that \(U ((\theta, i) \leq U ((\theta, j)\) for all \(\theta \in [0, 1]\), and in particular \(\theta = 1\). Thus, clearly, \(p^I \geq p^I + 1\), contrary to the assumption.

Since \(Q^I, Q^{II} > 0\), the sets \(\{\theta \mid U ((\theta, i) > U ((\theta, j))\}\) and \(\{\theta \mid U ((\theta, j) > U ((\theta, i))\}\) are disjoint non-vanishing intervals with total measure 1. It is easy to see that marginal customer \(\theta\), that separates these intervals, satisfies (1), and so it is equal to \(\theta^* (p^I, p^{II})\). Thus the measures of these sets, according to \(F\), are \(1 - F [\theta^* (p^I, p^{II})]\) and, respectively, \(F [\theta^* (p^I, p^{II})]\), which establishes the first and the third lines in the definition of \((Q^I (p^I, p^{II}), Q^{II} (p^I, p^{II}))\) in (5).

Now assume that \(p^I - p^I \geq 1\). In this case, \(Q^I (p^I, p^{II}) = 0\), since \(U ((\theta, i) < U ((\theta, j))\) for all \(\theta \in [0, 1]\) regardless of levels of congestion at the two firms.

Profit functions of the firms, \(\pi^I (p^I, p^{II})\) and \(\pi^{II} (p^I, p^{II})\), are given by

\[
\pi^I (p^I, p^{II}) = p^I Q^I (p^I, p^{II}) \quad \text{and} \quad \pi^{II} (p^I, p^{II}) = p^{II} Q^{II} (p^I, p^{II}).
\]

The following is a corollary to Lemma 3:

Corollary The profit functions \(\pi^I (p^I, p^{II})\) and \(\pi^{II} (p^I, p^{II})\) are continuous and given by

\[
(\pi^I (p^I, p^{II}), \pi^{II} (p^I, p^{II})) =
\]

6
In equilibrium, both firms maximize their profits, conditional on each other’s prices. That is, a pair of prices \((p^I, p^II)\) is a \textit{Nash equilibrium}, or simply \textit{equilibrium}, if

\[
\pi^I (p^I, p^II) = \max_{p^I \geq 0} \pi^I (p^I, p^II) \quad \text{and} \quad \pi^II (p^I, p^II) = \max_{p^II \geq 0} \pi^II (p^I, p^II).
\]

We say that an equilibrium \((p^I, p^II)\) is \textit{symmetric} if \(p^I = p^II\), and we call it \textit{asymmetric} otherwise.

### 3 Results

**Theorem 1** \textit{There does not exist an asymmetric equilibrium.}

**Proof.** Suppose that \((p^I, p^II)\) is an asymmetric equilibrium. Without loss of generality assume that \(p^II < p^I\), and suppose first that \(0 < p^II\). The profit functions \(\pi^I (p^I, p^II)\) and \(\pi^II (p^I, p^II)\) are continuously differentiable in a neighborhood of \((p^I, p^II)\) by the Corollary and Lemma 2. The first order conditions for profit maximization, taken at \((p^I, p^II)\), are:

\[
0 = \frac{\partial \pi^I}{\partial p^I} = 1 - F [\theta^* (p^I, p^II)] - p^I s, \quad (6)
\]

\[
0 = \frac{\partial \pi^II}{\partial p^II} = F [\theta^* (p^I, p^II)] - p^II s, \quad (7)
\]

where

\[
s = \frac{\partial F (\theta^*)}{\partial p^I} \bigg|_{(p^*, p^II)} = - \frac{\partial F (\theta^*)}{\partial p^II} \bigg|_{(p^I, p^*)}
\]
(and the last equality follows from (4) and the chain rule). From (7),
\[ F[\theta^*(\bar{p}, \bar{p}')] = \bar{p}^I. \]

However, as a derivative of a composition of increasing functions, \( s \geq 0 \). Thus
\[ F[\theta^*(\bar{p}, \bar{p}')] \leq \bar{p}^I. \]

From (6)
\[ 0 = 1 - F[\theta^*(\bar{p}, \bar{p}')] - \bar{p}^I s \leq 1 - 2F[\theta^*(\bar{p}, \bar{p}')], \]
and therefore
\[ F[\theta^*(\bar{p}, \bar{p}')] \leq \frac{1}{2}. \] \hspace{1cm} (8)

On the other hand, by (4),
\[ G(\theta^*(\bar{p}, \bar{p}')) = \bar{p}^I - \bar{p}^I > 0. \]

It follows from (2) that \( K(F[\theta^*(\bar{p}, \bar{p}')] > K(1 - F[\theta^*(\bar{p}, \bar{p}')] \)], and so \( F[\theta^*(\bar{p}, \bar{p}')] > \frac{1}{2} \) (because \( K(x) - K(1 - x) \) is an increasing function that has a zero at \( \frac{1}{2} \)), contradicting (8).

Suppose now that \( \bar{p}^I - \bar{p}^I \geq 1 \). According to (5), \( Q^I = 0 \), and thus \( \pi^I(\bar{p}, \bar{p}^I) = 0 \). In this case firm \( I \) will have an incentive to take half of the market from firm \( II \) by lowering its price to \( \bar{p}^I \) and getting positive, rather than zero, profits, contradicting the definition of equilibrium.

Finally, if \( \bar{p}^I = 0 \), then, analogously, firm \( II \) has an incentive to raise its price to \( \bar{p}^I \) and thus get positive, rather than zero, profits, a contradiction.

**Theorem 2** Suppose \( (\bar{p}^I, \bar{p}^I) = (p, p) \) is a symmetric equilibrium. Then

\[ p = \frac{1}{2(F \circ G^{-1})'(0)} \]

where \( (F \circ G^{-1})'(0) \) is the right derivative of the function \( F \circ G^{-1} \) at 0, and

\[ \pi^I(p, p) = \pi^I(p, p) = \frac{p}{2}. \]
Proof. Note that \( p > 0 \), since otherwise a small increase in price by one of the firms will yield it a positive profit. Since \( \theta^* (p^I, p^II) \) possesses right derivative with respect to \( p^I \) and left derivative with respect to \( p^II \), so do the functions \( \pi^I (p^I, p^II) \) and \( \pi^II (p^I, p^II) \). And, as both \( \pi^I (\cdot, p) \) and \( \pi^II (p, \cdot) \) are maximized at \( p \), use of the Corollary yields

\[
0 \geq \frac{\partial^+ \pi^I}{\partial p^I} = 1 - F [\theta^* (p, p)] - p \frac{\partial^+ F(\theta^*)}{\partial p^I} |_{(p,p)},
\tag{9}
\]

and

\[
0 \leq \frac{\partial^- \pi^II}{\partial p^II} = F [\theta^* (p, p)] + p \frac{\partial^- F(\theta^*)}{\partial p^II} |_{(p,p)},
\tag{10}
\]

where \( \partial^+ \) and \( \partial^- \) refer to the right and left derivatives, respectively. Note that

\[
\frac{\partial^+ F(\theta^*)}{\partial p^I} |_{(p,p)} = \frac{\partial^- F(\theta^*)}{\partial p^II} |_{(p,p)},
\tag{11}
\]

by (4) and the chain rule. By (4) and the definition of \( G \) in (2), \( F[\theta^* (p, p)] = \frac{1}{2} \), and so (9) implies

\[
p \frac{\partial^+ F(\theta^*)}{\partial p^I} |_{(p,p)} \geq \frac{1}{2}.
\]

From (10) and (11), however, it follows that

\[
p \frac{\partial^+ F(\theta^*)}{\partial p^I} |_{(p,p)} \leq \frac{1}{2},
\]

and so these two inequalities hold with equality. Therefore

\[
p = \frac{1}{2 \frac{\partial^+ F(\theta^*)}{\partial p^I} } |_{(p,p)}
\]

By the chain rule,

\[
\frac{\partial^+ F(\theta^*)}{\partial p^I} |_{(p,p)} = \frac{\partial^+}{\partial p^I} \left( F \circ G^{-1} \right) (p^I - p^II) |_{(p,p)} = \left( F \circ G^{-1} \right)'_+ (0),
\]

and so

\[
p = \frac{1}{2 (F \circ G^{-1})'_+ (0)}, \quad \text{and} \quad \pi^I (p, p) = \pi^II (p, p) = \frac{p}{2}.
\]
4 Conclusion

In this paper we have considered the case of a duopoly that operates in a congested market with consumers heterogeneous with respect to disutility from congestion. We have shown that, under general congestion functions and an arbitrary distribution of customer disutility from congestion, there cannot exist an asymmetric Nash equilibrium (Theorem 1). We have also shown that whenever an equilibrium does exist it is unique, and we have provided closed form expressions for the symmetric equilibrium prices and profits (Theorem 2).

When the distribution of consumer disutility from congestion is uniform, the symmetric equilibrium is known to exist (see de Palma and Leruth (1989) and Gibbens, Mason and Steinberg (2000)). However, existence is not guaranteed for all distributions. Extreme behavior of the distribution function around the point that separates the market into equal halves can cause instability for any pair of equal prices.

Throughout the paper, we have identified customers with their corresponding disutilities from congestion, by assuming that the value of $\theta$ uniquely determines the customer. This assumption is by no means necessary; it was made to facilitate the exposition. In general, customers can be modelled as a measurable space $C$, and their distributions described by a measure $\mu$ with total mass 1. The disutility from congestion for customer $c \in C$ is then given by $\theta_c \in [0, 1]$, and his utility function by

$$U(c, i) = V(c) - \theta_c \cdot K^i (Q^i) - p^i.$$

Our results carry over provided that the distribution function $F$ on $\theta$, induced by the measure $\mu$ on $C$, is continuously differentiable and strictly increasing.

Finally, an important assumption required for our results is that the market is covered. When the market is not covered, there exist situations, described in the literature, in which only asymmetric equilibria can occur.
References


