"Kramers-Wannier dualities via symmetries"

Ruelle, Philippe

Abstract
Kramers-Wannier dualities in lattice models are intimately connected with symmetries. We show that they can be found directly and explicitly from the symmetry transformations of the boundary states in the underlying conformal field theory. Intriguingly the only models with a self-duality transformation turn out to be those with an auto-orbifold property. Comment: 4 pages, no figure

Document type: Preprint

Référence bibliographique
Kramers-Wannier dualities via symmetries

Philippe Ruelle

Université catholique de Louvain
Institut de Physique Théorique
B-1348 Louvain-la-Neuve, Belgium

(Dated: February 2, 2008)

Kramers-Wannier dualities in lattice models are intimately connected with symmetries. We show that they can be found directly and explicitly from the symmetry transformations of the boundary states in the underlying conformal field theory. Intriguingly the only models with a self-duality transformation turn out to be those with an auto-orbifold property.

PACS numbers: 05.50.+q,64.60.Cn,11.25.Hf

Universal properties of two-dimensional critical phenomena, when conformally invariant, are described by conformal field theories. Virtually all their aspects can be understood within an appropriate conformal theory. Duality transformations, which have played a prominent role in the history of critical phenomena, have however so far largely remained outside the conformal description.

The notion of duality is inspired by the duality discovered by Kramers-Wannier in the Ising model \cite{1}, and more specifically its interpretation in terms of disorder lines given by Kadanoff and Ceva \cite{2}: a duality exchanges order with disorder fields, where the disorder fields are associated with defects lines, non-local perturbations concentrated on lines. The Kramers-Wannier duality had the surprising and important feature of relating the high and low temperature regimes. Dualities were subsequently discovered in many other models, and also generalized to relate different models \cite{3}.

Recently the dualities in self-dual models have been reconsidered from a conformal theory point of view \cite{4}. It was shown there that the information about the dualities is in fact contained in the fusion algebra of special fields associated with conformal defects.

In this letter, we provide an alternative point of view by connecting the dualities directly with the symmetries of the underlying conformal theory. We give a simple and completely explicit way to find out when there is a duality, and to compute the duality transformations. Although the method is general, we will mostly deal with the unitary Virasoro minimal theories. We will conclude that only six of them have a non trivial duality.

The exact way the present work relates to \cite{4} is not clear to us. The dualities discussed in \cite{4} are generated by specific duality defects which are not group-like. In contrast, our approach is based on the transformation laws of the boundary states under internal symmetries. The results nonetheless seem to fully agree, which indicates that the two approaches are complementary.

The dualities discussed here are derived in the conformal setting, namely at criticality. They however extend at least to the perturbative neighbourhood of the critical point, where the conformal picture holds. When the critical point is self-dual, the duality is a true symmetry which establishes relations among correlators and operator product coefficients. The duality is also useful to connect different relevant perturbations. If the perturbing term is $\lambda \int \phi(x)$ for a duality odd field, the duality establishes a correspondence between the $\lambda > 0$ and the $\lambda < 0$ regimes, and between the corresponding renormalization group flows, generalizing the duality between the low and high temperature phases of the Ising model.

The Ising model. - It is instructive to reconsider the Ising model at zero magnetic field. On an $L \times T$ cylindrical lattice with free boundary conditions along the bottom and top boundaries, the contour representation of the partition functions is

$$Z_{P,ff}^{L\times T}(\beta) = 2^{LT} (\sinh 2\beta)^{L(T-1)} \sum_{C} (\tanh \beta)^{|C|},$$

where the sum runs over closed contours. A contour is a set of loops drawn on the lattice bonds, where each bond is used at most once ($|C|$ is the number of bonds of $C$).

A contour $C$ defines the variations of a dual spin configuration $s^*$ on the dual lattice \cite{5}. The dual spins are constant inside the loops of $C$, and change sign when one crosses a contour line. If the dual spin at a single dual site is fixed, the whole dual configuration is unambiguously fixed by $C$. By relating the energy of the dual configuration to the value of $|C|$, one can write the above partition function in terms of a partition function for the dual spins. This dual partition function is evaluated at the dual temperature $\beta^*$ defined by $\tanh \beta^* = e^{-2\beta}$, while the dual spins are subjected to dual boundary conditions, usually distinct from the original ones. In the present case, the relation reads

$$Z_{P,ff}^{L\times T}(\beta) = 2^{L/2} (\sinh 2\beta)^{LT-L/2} \times \left[ Z_{P,++}^{L\times (T+1)}(\beta^*) + Z_{P,+-}^{L\times (T+1)}(\beta^*) \right],$$

where $Z_{P,++}$ (resp. $Z_{P,+-}$) is the cylinder partition function for fixed and equal (resp. opposite) spins on the two boundaries.

The same calculation can be repeated for antiperiodic boundary condition along the perimeter of the cylinder.
In this case, the energy is computed as before except that two neighbouring spin columns are coupled antiferromagnetically. The duality relation reads in this case
\[ \mathcal{Z}_{A,P}(\beta) = \frac{1}{2} (\sinh 2\beta)^{LT} \times \left[ \mathcal{Z}_{PP}(\beta^*) + \mathcal{Z}_{PA}(\beta^*) + \mathcal{Z}_{AP}(\beta^*) + \mathcal{Z}_{AA}(\beta^*) \right]. \]

At the critical point, the relations show that the Ising states coincides with its action on the diagram \( G \), respectively (up to the symmetry of the Kac table). Here \( G \) is exchanged with \( A \), and \( E \) is the required data are encoded in the Dynkin diagrams of \( A_{2m} \) and \( G \).

\( \Psi \) and \( F \) are the set of all the scalar bulk fields of the theory, including those contained in sectors twisted by elements of the symmetry group \( G \). We will see that the Ishibashi states from the twisted sectors play a central role in our derivation of the dualities.

Again the case of the Ising model is particularly instructive. There are three physical conformal boundary conditions, the free condition \(|f\rangle\), and the fixed ones \(|+\rangle\) and \(|-\rangle\). The periodic sector contains three scalar fields, the identity field \( \sigma \), and the energy density \( g \), of chiral conformal weight 0, \( \frac{1}{16} \), and \( \frac{1}{2} \) respectively. They lead to three Ishibashi states \( |0\rangle_P \), \( |\frac{1}{16}\rangle_P \), \( |\frac{1}{2}\rangle_P \). The second, antiperiodic sector contains a single scalar field, the disorder field \( \mu \), with the same conformal weight \( \frac{1}{16} \) as the spin field, and gives rise to one Ishibashi state \( |\frac{1}{16}\rangle_A \).

Inserting the values of the coefficients \( c_{i}^a \), we obtain the following suggestive expansions,
\[ |+\rangle = \frac{1}{\sqrt{2}} \left[ |0\rangle_P + \sqrt{2} |\frac{1}{16}\rangle_P + |\frac{1}{2}\rangle_P \right]. \]

These equations reflect the transformation laws under the symmetry group \( Z_2 \), but also reveal the duality transformations. The duality exchanges \( \sigma \) and \( H \), and therefore the states \( |\frac{1}{16}\rangle_P \) and \( |\frac{1}{16}\rangle_A \) which are built on them. This results in the exchange of \( |f\rangle \) and \(|+\rangle \) or \(|-\rangle \). The conformal de-
Unitary minimal conformal models and relatives. - The boundary
The boundary conditions are parametrized by a in
\{1, 2, \ldots, m\} and b in \{1, 2, \ldots, q - 1\}, among which
those with \(b = \frac{q}{2}\) are \(Z_2\) invariant. We choose \(\mathcal{E}_c = \{(r, s) \in [1, 2m] \times [1, q - 1] : r \text{ odd}\}\) to
label the scalar fields of the periodic sector. For \(i = (r, s) \in \mathcal{E}_c\), the coefficients are
\[c_{i(a,b)} = \sin^2 \frac{\pi n r}{2} \sin^2 \frac{\pi n s}{2}\]
up to a non-zero factor. There are two sectors, periodic and antiperiodic. Re-
quirement (i) means that some \(|j\rangle_A\) are exchanged with some
\(|j\rangle\), and therefore implies that a boundary state
\(|\alpha\rangle\) which is not invariant under the \(Z_2\) symmetry and
which has a non-zero projection on all the states \(|i\rangle\), is
necessarily exchanged with a boundary state \(|\alpha^*\rangle\) which
expands on Ishibashi states from the \(A\) sector, that is,
with a \(Z_2\) invariant boundary state. Consistency with
the symmetry \(Z_2\) actually requires that \(|\alpha^*\rangle\) be dual to the
combination \(|\alpha\rangle + g|\alpha\rangle\)/\(\sqrt{2}\).

Since the duality must be invertible, the number of
such pairs \(|\alpha\rangle, g|\alpha\rangle\) must be smaller or equal to the
number of invariant boundary states, equal to \(m\). From
the formula given above, the coefficients \(c_{i(a,b)}\) are
different from zero for all \(i\) in the periodic sector, if and
only if \(a\) is coprime with \(2m + 1\) and \(b\) is coprime with
\(q\). The total number of pairs \(|\alpha\rangle, g|\alpha\rangle\) is thus equal to
\(\frac{1}{2} (\phi(n) \phi(2m + 1), with \phi(n) the Euler
totient function, so that the aforementioned inequality reads \(\phi(n) / \phi(2m + 1) \leq 4m\). The crude lower bound \(\phi(n) \geq n^{3/5}\) if \(n\) is odd, and
\(\phi(n) \geq (\frac{2}{3})^{3/5}\) if \(n\) is even, is enough to show that the
inequality is violated for all \(m > 128\). Checking the
finite number of remaining cases leaves only four cases:
\((m, q) = (1, 4), (2, 4), (2, 6), (3, 6)\).

By looking at the expansions of the boundary con-
ditions in terms of the Ishibashi states, the two cases
\((m, q) = (2, 6), (3, 6)\) are easily ruled out for not having a
consistent duality. The last two cases \((m, q) = (1, 4)\) and
\((2, 4)\) correspond to the Ising model, discussed above, and
the tricritical Ising model, both self-dual at the critical
point.

The tricritical Ising model \((A_4, A_3)\) has six confor-
mal boundary conditions. Four of them expand as
\[|(1, 1)\rangle = C \bigg[(|0\rangle + \eta |\frac{3}{2}\rangle + \eta |\frac{5}{2}\rangle + |\frac{3}{2}\rangle) + \frac{1}{\sqrt{2}} \bigg( |\frac{1}{10}\rangle + |\frac{3}{10}\rangle + |\frac{5}{10}\rangle + |\frac{7}{10}\rangle\bigg) \bigg], \]
\[|(1, 2)\rangle = \sqrt{2}C \bigg[|0\rangle - \eta |\frac{1}{10}\rangle + \eta |\frac{3}{2}\rangle - |\frac{3}{2}\rangle\bigg] + \frac{1}{\sqrt{2}} \bigg( |\frac{1}{10}\rangle + |\frac{3}{10}\rangle + |\frac{5}{10}\rangle + |\frac{7}{10}\rangle\bigg) A, \]
\[|(2, 1)\rangle = C \bigg[|\eta^2|0\rangle - \eta^{-1} |\frac{1}{10}\rangle - \eta^{-1} |\frac{3}{2}\rangle + \eta^2 |\frac{3}{2}\rangle \bigg] + \frac{1}{\sqrt{2}} \bigg( |\frac{1}{10}\rangle + |\frac{3}{10}\rangle + |\frac{5}{10}\rangle + |\frac{7}{10}\rangle\bigg) A, \]
\[|(2, 2)\rangle = \sqrt{2}C \bigg[|\eta^2|0\rangle + \eta^{-1} |\frac{1}{10}\rangle - \eta^{-1} |\frac{3}{2}\rangle - \eta^2 |\frac{3}{2}\rangle\bigg] + \frac{1}{\sqrt{2}} \bigg( |\frac{1}{10}\rangle + |\frac{3}{10}\rangle + |\frac{5}{10}\rangle + |\frac{7}{10}\rangle\bigg) A, \]
where \(C = \sqrt{\frac{3}{8}}\) and \(\eta = \sqrt{\frac{2}{3}}\). The last two,
\(|(1, 3)\rangle\) and \(|(2, 3)\rangle\), are the \(Z_2\) transforms of \(|(1, 1)\rangle\) and
\(|(2, 1)\rangle\) respectively, and are simply obtained from them
by changing the sign of the coefficients of \(|\frac{1}{10}\rangle\) \(p\) and
\(|\frac{3}{10}\rangle\) \(p\), since the corresponding bulk fields are odd un-
der the \(Z_2\).

The duality transformations are easily read off from
these expansions. The two bulk fields \(\varepsilon = (\frac{1}{10}, \frac{1}{10})\)
and \(\varepsilon'^* = (\frac{3}{10}, \frac{1}{10})\) are odd, whereas \(\varepsilon" = (\frac{3}{10}, \frac{3}{10})\) is even:
\[\sigma = (\frac{3}{80}, \frac{3}{80}) p\] is exchanged with \(\mu = (\frac{3}{80}, \frac{3}{80}) A\), and
\(\sigma' = (\frac{3}{80}, \frac{3}{80}) p\) with \(\mu' = (\frac{3}{80}, \frac{3}{80}) A\); finally the duality
between the boundary conditions reads
\[|(1, 2)\rangle_{p,A} \leftrightarrow \frac{1}{\sqrt{2}}(|(1, 1)\rangle \pm |(1, 3)\rangle), \]
\[|(2, 2)\rangle_{p,A} \leftrightarrow \frac{1}{\sqrt{2}}(|(2, 1)\rangle \pm |(2, 3)\rangle). \]

The complementary models \((A_{2m}, D_{3+1})\) have a \(Z_2\)
internal symmetry for \(q \geq 8\), and an \(S_3\) symmetry if
\(q = 6\).

Assume first \(q \geq 8\). There are again two sectors, \(P\) and
\(A\), and correspondingly two sets of exponents,
\[\mathcal{E}_c = \{(r, s) \in [1, 2m] \times [1, q - 1] : r, s \text{ odd}\}\]
\[\cup \{(r, \frac{q}{2}) : r \text{ odd } \in [1, 2m]\}, \]
\[\mathcal{E}_g = \{(r, s) \in [1, 2m] \times [1, q - 1] : r \text{ odd, } s \text{ even}\}. \]

As before an essential requirement is that the duality
exchanges Ishibashi states from the two sectors, \(|i\rangle\rangle \leftrightarrow
\langle\langle j\rangle\rangle\). As it must also preserve the scaling dimensions,
this actually requires \(i = j\) and so \(\mathcal{E}_c \cap \mathcal{E}_g\) must be non-
empty. That alone excludes the cases \(q = 2 \mod 4\).

In addition, the inclusion \(\mathcal{E}_g \subset \mathcal{E}_c\) should hold for
a duality to exist. Assume the contrary. The states \(|j\rangle\rangle\)
for those \(j\) which are not in \(\mathcal{E}_c\) should be left fixed by the
duality, and so the \(Z_2\) invariant physical boundary states
which have a non-zero projection on them would be per-
muted among themselves. The values of the coefficients
show that this is the case of all invariant states. The
duality would then mix the \(Z_2\) invariant states among
themselves, and being invertible, would also mix the non-
invariant states among themselves, contradicting the ex-
change of some \(|i\rangle\rangle\) with some \(|\langle\langle j\rangle\rangle\rangle\).

A simple look at the sets \(\mathcal{E}_c\) and \(\mathcal{E}_g\) shows that
no value of \(q \geq 8\) satisfies the condition \(\mathcal{E}_g \subset \mathcal{E}_c\). The
remaining two models \(q = 6\), namely the critical and
tricritical 3-Potts models, are the only ones to qualify
for a non-trivial duality, and easily checked to have one.

The nature of these dualities is somewhat different from
that in the Ising model, as they are induced by the \(Z_3\)
subgroup of a larger symmetry group \(S_4\).

In order to treat the two models simultaneously, and
also to lighten the expressions, we will compute their du-
ality in terms of their parent theory through the coset
construction, namely the \(D_4\) model with affine symme-
try \(su(2)\) at level \(k = 4\). The way this model de-
sends onto the two 3-Potts models \((A_4, D_4)\) (critical)
and \((A_6, D_4)\) (tricritical) is standard and yields the results in a straightforward fashion.

The \(D_4\) model has an \(S_3\) internal symmetry, for which all twisted torus and cylinder partition functions have been computed in \([10]\). It has four physical boundary conditions, labelled by the nodes of the \(D_4\) Dynkin diagram, with the state \(|4\rangle\) being fully invariant under \(S_3\). The other three are cyclically rotated by the \(Z_2\) subgroup, and each one is left invariant by a \(Z_2\) subgroup. We call \(r\) the generator of the \(Z_2\) subgroup, and \(g, g^r, g^{r^2}\) the generators of the three conjugate \(Z_2\) subgroups.

The full expansions in terms of the affine Ishibashi states read \([10]\)

\[
|1\rangle = \frac{1}{\sqrt{3}} \left\{ |1\rangle_e + |3\rangle_e + |3^r\rangle_e + |5\rangle_e \right. \\
= \frac{1}{\sqrt{3}} (|2\rangle_g + |4\rangle_g) \right\},
\]

\[
|2\rangle = \frac{1}{\sqrt{3}} \left\{ |1\rangle_e + \omega |3\rangle_e + \omega^2 |3^r\rangle_e + |5\rangle_e \right. \\
= \frac{1}{\sqrt{3}} (|2\rangle_{g^r} + |4\rangle_{g^r}) \right\},
\]

\[
|3\rangle = \frac{1}{\sqrt{3}} \left\{ |1\rangle_e + \omega^2 |3\rangle_e + \omega |3^r\rangle_e + |5\rangle_e \right. \\
= \frac{1}{\sqrt{3}} (|2\rangle_g + |4\rangle_g) \right\},
\]

\[
|4\rangle = \frac{1}{\sqrt{3}} \left\{ |1\rangle_e + |3\rangle_r + |3^r\rangle_r - |5\rangle_e \right. \\
= \frac{1}{\sqrt{3}} (|2\rangle_g + |2\rangle_{g^r} + |2\rangle_{g^{r^2}} - |4\rangle_g - |4\rangle_{g^r} - |4\rangle_{g^{r^2}}) \right\},
\]

where \(\omega \neq 1\) is a third root of unity.

One deduces the following duality transformations: the periodic fields \(3_e\) and \(3^r_e\) are exchanged with the twist fields \(3_r\) and \(3^r_r\), whereas \(5_e\) is odd; in the three pairs \((2_g, 4_g)\), \((2_{g^r}, 4_{g^r})\) and \((2_{g^{r^2}}, 4_{g^{r^2}})\), the fields with label 2 are even and those with label 4 are odd; for the boundary conditions, one has

\[
|4\rangle_e \leftrightarrow (|1\rangle_e + |2\rangle_e + |3\rangle_e)/\sqrt{3}, \quad |4\rangle_g \leftrightarrow |1\rangle_g, \quad \quad (21)
\]

\[
|4\rangle_r \leftrightarrow (|1\rangle_e + \omega |2\rangle_e + \omega^2 |3\rangle_e)/\sqrt{3}, \quad \quad (22)
\]

up to symmetries.

The critical Potts model is simply obtained by juxtaposing an \(r = 1, 2\) index to the above \(su(2)\) labels to form a Kac label. The known duality transformations are then recovered \([15]\). The duality of the tricritical 3-Potts model is similar.

The last unitary Virasoro minimal models with a symmetry \((Z_2)\) are the two models \((A_{10}, E_6)\) and \((A_{12}, E_6)\). They both possess a non trivial duality which comes directly from the duality of the \(E_6\) model with affine symmetry \(\hat{su}(2)\) at level \(k = 10\).

To conclude, only six unitary minimal models have a duality, and they are precisely the only six models to be their own orbifold. They all inherit their duality from that of the parent theories, the \(A_3, D_4\) and \(E_6\) models with an \(\hat{su}(2)\) affine symmetry, which again are the only three \(\hat{su}(2)\) affine theories to be self-orbifold.

The reference \([1]\), which generalizes the torus identity \([13]\) to all unitary minimal models, strongly points to a general relation between a duality and the orbifold construction. The above observations can be repeated for pairs of models which are orbifold of each other, and lead to duality relations. For instance, the orbifold of the \(D_4\) model with affine symmetry \(\hat{su}(2)_4\) under one of its \(Z_2\) subgroup is the \(A_5\) model (same level). One finds the duality relations

\[
\frac{1}{\sqrt{2}}(|1\rangle_A + |5\rangle_A) \leftrightarrow |1\rangle_D, \quad \frac{1}{\sqrt{2}}(|2\rangle_A + |4\rangle_A) \leftrightarrow |4\rangle_D, \\
|3\rangle_A \leftrightarrow \frac{1}{\sqrt{2}}(|2\rangle_D + |3\rangle_D),
\]

between the boundary conditions of the two models.

I would like to thank the authors of \([13]\) for useful discussions, and for communicating as yet unpublished results, and Paul Fendley for drawing my attention to the reference \([13]\). I thank the Belgian Fonds National de la Recherche Scientifique for financial support.

---