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Binding energy corrections in positronium decays

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Abstract

Positronium annihilation amplitudes that are computed by assuming a factorization approximation with on-shell intermediate leptons, do not exhibit good analytical behavior. We propose an \textit{ansatz} which allows to include binding energy corrections and obtain the correct analytical and gauge invariance behavior of these QED amplitudes. As a consequence of these non-perturbative corrections, the parapositronium and orthopositronium decay rates receive corrections of order $\alpha^4$ and $\alpha^2$, respectively. These new corrections for orthopositronium are relevant in view of a precise comparison between recent theoretical and experimental developments. Implications are pointed out for analogous decays of quarkonia.

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Parapositronium (p-Ps) and orthopositronium (o-Ps) are bound states of $e^+e^-$ whose lifetimes for $2\gamma$ and $3\gamma$ channels, respectively, have been measured with high precision:

$$\Gamma(\text{p-Ps} \to \gamma\gamma) = 7090.9(1.7) \mu s^{-1}[1]$$

$$\Gamma(\text{o-Ps} \to \gamma\gamma\gamma) = \begin{cases} 7.0514(14) \mu s^{-1}[2], \\ 7.0482(16) \mu s^{-1}[3], \\ 7.0398(29) \mu s^{-1}[4]. \end{cases}$$

The corresponding theoretical predictions which include perturbative QED corrections to a non-relativistic treatment of the bound state wave function, have been computed also with high accuracy [5, 6]:

$$\Gamma(\text{p-Ps} \to \gamma\gamma) = \frac{\alpha^5 m}{2} \left(1 - \frac{5 - \pi^2}{4} \frac{\alpha}{\pi} + 2\alpha^2 \ln \frac{1}{\alpha} + 1.75(30) \left(\frac{\alpha}{\pi}\right)^2 - \frac{3\alpha^3}{2\pi} \ln^2 \frac{1}{\alpha}\right)$$

$$= 7989.50(2) \mu s^{-1}$$

$$\Gamma(\text{o-Ps} \to \gamma\gamma\gamma) = \frac{2(\pi^2 - 9)\alpha^6 m}{9\pi} \left(1 - 10.28661(1) \frac{\alpha}{\pi} - \frac{\alpha^2}{3} \ln \frac{1}{\alpha} + B_0 \left(\frac{\alpha}{\pi}\right)^2 - \frac{3\alpha^3}{2\pi} \ln^2 \frac{1}{\alpha}\right)$$

$$= (7.0382 + B_0 \times 0.39 \times 10^{-4}) \mu s^{-1}.$$  

where $m$ denotes de mass of the electron and $\alpha$ is the fine structure constant. As it can be observed, predictions for parapositronium are in very good agreement with experiment, while the experimental results for orthopositronium reported in Refs. [2, 3] largely disagree with the theoretical expectation. Recent theoretical efforts have focused on a more complete evaluation of the non-logarithmic $O(\alpha^2)$ perturbative corrections for orthopositronium, with the result $B_0 = 44.52(26)$ [7] or, equivalently,

$$\Gamma(\text{o-Ps} \to \gamma\gamma\gamma) = 7.039934(10) \mu s^{-1}.$$  

This result renders even closer the theoretical prediction to the experimental measurement of Ref. [4].

Although these achievements of perturbative QED for bound states are impressive, little is known about the effects of non-perturbative corrections. In the present paper we implement a mechanism, imposed by analyticity and gauge invariance, which allows to introduce binding energy (BE) effects in the positronium decay amplitudes. These new corrections affect the $\gamma\gamma$ decay rate of parapositronium at the order $\alpha^4$, while corrections of order $\alpha^2$...
are induced in the orthopositronium decay rate and in the single photon spectrum of $p$-Dm → $e^+e^-\gamma$ and o-Ps → $\gamma\gamma\gamma$ decays (p-Dm (o-Dm) denotes the J=0(1) bound state of the dimuonium system $\mu^+\mu^-$). Moreover, the single photon energy spectrum in the last two decays previously mentioned is softened with respect to results obtained when BE effects are neglected. Our implementation of these non-perturbative corrections relies on well grounded basis of QED such as the requirement of a correct behavior of the decay amplitude in terms of the photon energies (analyticity) and on electromagnetic gauge invariance. We briefly discuss at the end, the possible implications for analogous decays of quarkonium states. Details of our calculations will be given elsewhere [8].

Let us first try to explain how problems related to analyticity of the positronium decay amplitudes appears in the usual approaches. Current calculations of the positronium rates assume a factorization approximation of the dynamics contained in the decay amplitude. The amplitudes for positronium decays into a given final state $X$ is approximated as the product of the amplitude for the bound state annihilation (described by $\Phi_0$, the $e^+e^-$ wavefunction at the origin), times the annihilation amplitude of free leptons into the final state $X$. Under this assumption, the positronium decay rates can be written as [9]:

$$\Gamma(\text{Ps}(2J+1S) \rightarrow X) = \frac{1}{2J+1} |\Phi_0|^2 \cdot (v_{rel}\sigma(e^+e^- \rightarrow X))_{v_{rel} \rightarrow 0} , \quad (6)$$

where $\sigma(e^+e^- \rightarrow X)$ denotes the total cross section for $e^+e^- \rightarrow X$ and $v_{rel}$ is the relative velocity of the constituents in their center of mass frame.

Since the leptons in the intermediate stage are taken on their mass-shell, the annihilation amplitude for $e^+e^- \rightarrow n\gamma$ is automatically gauge-invariant. However, in this approximation the positronium decay amplitudes do not exhibit the expected analytical behavior. Indeed, for definiteness let us consider the $3\gamma$ decay mode of orthopositronium where the photon energy is kinematically allowed to vanish. The behavior of the $e^+e^- \rightarrow 3\gamma$ amplitude in terms of the photon momenta is driven in the soft-photon limit by the intermediate electron propagators as follows:

$$\frac{i}{k - \not{l_i} - m} = i\frac{(k - \not{l_i} + m)}{-2l_i \cdot k} , \quad (7)$$

where $k$ is the four-momentum of the electron that emits a photon of four-momentum $l_i$. Note that in the soft-photon limit $l_i \rightarrow 0$, the positronium decay amplitude seems to diverge as $l_i^{-1}$. Actually, this is only apparent since selection rules cancels these infrared divergencies in the static limit ($k = (m,0,0,0)$), and the amplitude starts indeed at order $l_i^0$. This is
in contradiction with the fact that in the soft-photon limit the amplitude must vanish since \( o-Ps \rightarrow \gamma\gamma\gamma \) involve only neutral external bosons.

We can try to cure this bad analytical behavior by realizing that electrons in the intermediate stage are always off their mass-shell due to BE effects. Actually, the \( e^+e^- \) bound states involve two mass scales: the mass \( M \) of the bound state and the mass \( m \) of the constituent electrons. These masses differ by terms of order \( \alpha^2 \) and are related through the binding energy \( E_{\text{bind}} \),

\[
E_{\text{bind}} \equiv M - 2m = -\frac{1}{4}ma^2 . \tag{8}
\]

Thus, the relevant momentum scale for intermediate electrons is determined by \( M/2 \). In this case, the lepton propagator involved in the \( e^+e^- \rightarrow 3\gamma \) decay amplitude becomes \( (k^2 = M^2/4) \):

\[
\frac{i}{k-\xi - m} = \frac{i(k-\xi + m)}{-2i\cdot k - \gamma^2} , \tag{9}
\]

where \( \gamma^2 \equiv m^2 - M^2/4 \approx M^2\alpha^2/16 \), takes into account the bound state nature of the \( e^+e^- \) pair in the rescattering process. The presence of \( \gamma^2 \) in the denominator would provide to the decay amplitude a better analytical behavior. Unfortunately, the gauge invariance of the \( e^+e^- \rightarrow \gamma\gamma\gamma \) amplitude is spoiled due to the presence of the two mass scales \( M \) and \( m \). Thus, a more sophisticated procedure is required to restore analyticity without destroying gauge invariance.

A natural way to incorporate BE effects is by considering a model with a loop of virtual leptons for the decays of positronium at lowest order. For definiteness we consider the \( n\gamma \) decay of the positronium state \( B(2J+1S) \): \( J = 0(1) \) being the spin of the \( p\)-Ps(\( p\)-Dm) state, \( n = 2 \) for \( p\)-Ps or \( p\)-Dm, and \( n = 3 \) for \( o\)-Ps or \( o\)-Dm decays (the \( e^+e^-\gamma \) mode of paradimuonium can be reached from \( p\)-Dm\( \rightarrow \gamma\gamma\gamma^* \) where \( \gamma^* \) is a virtual photon). In this model we need to introduce a coupling \( F_B\Gamma \) to describe the \( Be^+e^- \) vertex, where \( \Gamma = \gamma_5(\hat{\eta}) \) for para(ortho)-positronium decay, \( \eta_\alpha \) is the polarization four-vector of the \( B(3S) \) state and \( F_B \) is a form factor that describes the structure of the vertex. The evaluation of the \( B(2J+1S) \rightarrow X \) decay amplitude follows standard rules (see Fig. 1).

Applying the Feynman rules to Fig. 1 the decay amplitude is given by (contraction with photon polarization vectors \( \epsilon_\mu(l_i) \cdots \) must be understood):

\[
\mathcal{M}^{\mu
u\cdots}(B \rightarrow n\gamma) = \int \frac{d^4q}{(2\pi)^4} F_B \text{Tr} \left\{ \Gamma \frac{i(\hat{\xi} - \frac{P}{2} + m)}{(q - \frac{P}{2})^2 - m^2} \Gamma^{\mu
u\cdots} \frac{i(\hat{\xi} + \frac{P}{2} + m)}{(q + \frac{P}{2})^2 - m^2} \right\} , \tag{10}
\]

where \( \Gamma^{\mu
u\cdots} \) is a properly symmetrized amplitude for annihilation of virtual lepton pairs into \( n\gamma \). It is important to emphasize that this amplitude is gauge-invariant if \( F_B \) is constant.
To clearly illustrate how this loop model reproduces, in the limit of zero BE, the lowest order amplitudes of the on-shell approximation, let us consider the following \textit{ansatz} for $F_B$:

$$F_B = iC\Phi_0 \frac{8\pi\gamma}{(q^2 + \gamma^2)^2} \cdot (q^2 + \gamma^2)$$

(11)

where $\gamma$ contains the BE, $\Phi_0 = \sqrt{\alpha^3 m^3/8\pi}$ is the ground state wavefunction of $e^+e^-$ at the origin, and $C = -2/\sqrt{M}$ is a normalization constant.

Using well known representations of Dirac-delta functions \cite{11} we can check that in the limit of zero BE ($\gamma \to 0$, $m - M/2 \to 0$) we obtain:

$$F_B \left( \frac{q - P^2}{2} - m^2 \cdot \frac{1}{q^2 - m^2} \right) \to \frac{1}{2}C\Phi_0(2\pi)^4 \delta(4)(q) \frac{\sqrt{q^2 + m^2 + M}}{q^2 - (\sqrt{q^2 + m^2 + M})^2}.$$  

(12)

Thus, upon (trivial) integration of Eq. (10) one gets:

$$M^{\mu\nu\cdots}(B \to n\gamma) = \frac{C\Phi_0}{8M} \text{Tr} \left\{ \Gamma(P - M) \Gamma^{\mu\nu\cdots}(P + M) \right\},$$

(13)

where $\Gamma^{\mu\nu\cdots}$ denotes the \textit{reduced} vertex evaluated at $q = 0$. Note that the factors $(P \pm M)$ play the role of projector operators external to the action of photon vertices.

Setting in the rest frame of positronium and using $2m = M$ (and the condition $P_\alpha \eta^\alpha = 0$ for orthopositronium) we arrive at the well known results \cite{13} of the factorization approximation, namely:

$$M^{\mu\nu\cdots}(B \to n\gamma) = -C\Phi_0 \frac{M}{2\sqrt{2}} \text{Tr} \left\{ \frac{1 + \gamma_0}{\sqrt{2}} \Gamma^{\mu\nu\cdots} \right\}.$$  

(14)

Let us return to the issues concerning gauge-invariance in the context of the present model when BE is \textit{not} neglected. Gauge invariance requires that the amplitude in Eq. (10) (contracted with photon polarizations) vanishes when $\epsilon_{\alpha}(l_i) \to l_{i\alpha}$ for any external photon. The amplitude for p-Ps\alt{\to} \gamma\gamma\gamma is always gauge-invariant, no matter the specific form of $F_B$. This follows from the fact that Lorentz covariance for the axial-\gamma vertex implies that the amplitude should corresponds to an effective operator $\sim F_{\mu\nu} \tilde{F}^{\mu\nu}$, which is automatically gauge-invariant. This is not the case for o-Ps\alt{\to} \gamma\gamma\gamma decays. In this case the amplitude of Eq. (10) is gauge invariant provided $F_B$ remains the same under the shifts $q \to q + l_i$ of the integration variable for all photons. Since this is achieved only if $F_B$ is constant, it means that other contributions should be added to the orthopositronium decay amplitude in this loop model in order to compensate for this lack of gauge invariance.

In view of these difficulties, we propose and \textit{ansatz} for positronium decay amplitudes that fulfills gauge invariance and analyticity simultaneously. Our recipe contains three steps:
(a) evaluate the expression $\Gamma_{\mu\nu\ldots}$ in Eq. (13) for constituents masses $M/2$ (this modified expression will be denoted by $\Gamma_{M/2}^{\mu\nu\ldots}$); this will ensure gauge invariance, (b) introduce BE effects by multiplying the amplitude $\mathcal{M}(B(2J+1) \to n\gamma)$ (i.e. the amplitude of Eq. (13) where we replace $m$ by $M/2$ in the argument of the Trace operator) by a factor: $A_1$ for p-Ps (p-Dm) states and $\prod_{i=1}^6 A_i$ for o-Ps (o-Dm), where we have defined $A_i \equiv P.l_i/(P.l_i + \gamma^2)$ (note that $A_1 = A_2$ for p-Ps $\to 2\gamma$). This will warrant the correct analytical properties of the amplitude in the soft-photon limit and, (c) obtain the decay rate by integration over the physical phase-space determined by the mass $M$.

Let us first note that the proposed ansatz is fulfilled automatically for the p-Ps $\to \gamma\gamma$ decay. The reduced vertex in Eq. (13) is given by (remember $q = 0$ and $P^2 = M^2$):

$$
\Gamma^{\mu\nu} = (-ie)^2 \frac{\gamma^\mu \left( -\frac{P}{2} + l_1 + m \right) \gamma^\nu + \gamma^\nu \left( \frac{P}{2} - l_1 + m \right) \gamma^\mu}{\left( \frac{P}{2} - l_1 \right)^2 - m^2} = (-ie)^2 \frac{\gamma^\mu \left( -\frac{P}{2} + l_1 + m \right) \gamma^\nu + \gamma^\nu \left( \frac{P}{2} - l_1 + m \right) \gamma^\mu}{\left( \frac{P}{2} - l_1 \right)^2 - \frac{M^2}{4}} (A_1) .
$$

In the second line of Eq. (15) we can replace $m \to M/2$ in the numerator because terms proportional to $m$ in $\Gamma^{\mu\nu}$ cancel when performing the trace in Eq. (13). Therefore, gauge invariance is preserved independently of $m$ and we have:

$$
\mathcal{M}^{\mu\nu}(p-Ps \to \gamma\gamma) = \mathcal{M}^{\mu\nu}(p-Ps \to \gamma\gamma) = \frac{C\Phi_0}{8M} \text{Tr} \left\{ \gamma_5 (P - M) \Gamma_M^{\mu\nu}(P + M) \right\} A_1 .
$$

This amplitude satisfies analyticity, as required.

Secondly, the reduced vertex for orthopositronium decay $\Gamma^{\mu\nu\rho}$ can be worked in the following way. In order to accomplish gauge invariance, we are forced to replace $m$ by $M/2$ for the mass of the constituents and simultaneously add a third analytical factor $A_i$ [8] to each of the six amplitudes contributing to $\Gamma^{\mu\nu\rho}$. This gives rise to the final gauge-invariant and analytical amplitude for orthopositronium decay:

$$
\overline{\mathcal{M}}^{\mu\nu\rho} = \frac{C\Phi_0}{8M} \text{Tr} \left\{ \delta(P - M) \Gamma_M^{\mu\nu\rho}(P + M) \right\} \prod_{i=1}^3 A_i ,
$$

where $\eta_\alpha$ represents the polarization four-vector of orthopositronium. In the soft photon limit ($l_i \to 0$), $\overline{\mathcal{M}}^{\mu\nu\rho}$ vanishes as required.

Up to now we have considered the effects of nonzero BE corrections in the dynamics of positronium decays as expressed in the decay amplitude. It is clear that the physical
phase space for these decays is determined by the masses of external particles, in particular the initial available energy $M$. Thus, we will study the effects of these non-perturbative corrections in observables associated to positronium decays, which contain the BE effects in the dynamics (amplitude) and the kinematics (phase-space).

$2\gamma$ decay of parapositronium.

The decay amplitude for $p$-Ps$\rightarrow\gamma\gamma$ obtained from Eq. (16) can be expressed as:

$$\mathcal{M}^{\mu\nu}(B(1S) \rightarrow \gamma\gamma) = 2Ce^2\Phi_0 \frac{\varepsilon^{\mu\nu\alpha\beta}l_{1\alpha}P_{\beta}}{P \cdot l_1} A_1.$$ (18)

The corresponding rate for this two-body decay where photons fly apart with energy $M/2$ in the rest frame of $p$-Ps, is given by $(2P.l_1 = M^2)$:

$$\Gamma(p$-Ps $\rightarrow \gamma\gamma) = \frac{1}{2}\alpha^5m \times \frac{4m^2}{M^2} \left[ \frac{1}{1 + 2\gamma^2/M^2} \right]^2 \approx \frac{1}{2}\alpha^5m \left( 1 - \frac{\alpha^4}{64} \right),$$ (19)

where we have used the definition of $\gamma^2$ as given after Eq. (9) and we have neglected corrections of $O(\alpha^6)$. Thus, (non-perturbative) BE corrections for $2\gamma$ decays of parapositronium are negligible small and are beyond present experimental precision.

para-dimuonium (p-Dm) decay into $e^+e^-\gamma$

The decay mechanism for this process is similar to the previous one, but where one virtual photon converts into a $e^+e^-$ pair. Therefore the corresponding amplitude is:

$$\mathcal{M}^\mu(p$-Dm $\rightarrow e^+e^-\gamma) = 2Ce^3\Phi_0 \frac{\varepsilon^{\mu\nu\alpha\beta}l_{1\alpha}P_{\beta}}{P \cdot l_1} A_1 \frac{\{\bar{u}(p)\gamma_{\nu}\nu(p')\}}{r^2},$$ (20)

where $l_1$ and $r \equiv p + p' = P - l_1$ denote, respectively, the four-momenta of the photon and the $e^+e^-$ pair.

The single photon spectrum in this case is given by $(a \equiv 4m^2/M^2, \ x = 2E_\gamma/M)$:

$$\frac{d\Gamma(p$-Dm $\rightarrow e^+e^-\gamma)}{dx} = \frac{16\alpha^3\Phi_0^2}{3M^2} \sqrt{1 - \frac{a}{1 - x}} \frac{x^3}{a + 2(1 - x)} \left( x + \frac{2\gamma^2}{M^2} \right)^{-2}.$$ (21)

Observe that this spectrum falls as $x^3$ when $x \rightarrow 0$ due to non-zero BE effects, instead of the usual behavior (proportional to $x$) expected when these effects are neglected. It is
interesting to note that when \( x \ll 2\gamma^2/M^2 \), Eq. (21) behaves as the corresponding photon energy spectrum in \( \pi^0 \to e^+e^-\gamma \) decay, for a point-like pion vertex. Thus, BE corrections affect the shape of the spectrum or, conversely, actually probes the structure of the bound state. Indeed, Eq. (21) can be seen as a result that extrapolates the spectrum between the \( e^+e^-\gamma \) decay of a pseudoscalar point particle (\( \pi^0 \) case) and the standard bound state calculations (paradimuonium decay without BE corrections).

A closed (but long) analytic expression for the decay rate can be obtained from integration of Eq. (21). A useful approximation that takes into account leading non-vanishing corrections of \( O(\alpha^2) \) is:

\[
\Gamma(p\text{-}Dm \to e^+e^-\gamma) = \frac{\alpha^6 m}{6\pi} \left[ F_0 - (1 - a)^{3/2} \frac{\alpha^2}{2} + O(\alpha^4) \right],
\]

where the function \( F_0 \equiv (4/3)\sqrt{1 - a(a - 4)} + 2\ln[(1 + \sqrt{1 - a})/(1 - \sqrt{1 - a})] \) fixes the lowest order rate.

**3\( \gamma \) decays of orthopositronium**

The squared unpolarized amplitude for \( \alpha\text{-}Ps \to \gamma\gamma\gamma \) obtained from Eq. (17), in terms of dimensionless photon energy variables \( x_i = 2E_{\gamma i}/M \) (\( x_1 + x_2 + x_3 = 2 \)) is given by:

\[
\sum_{\text{pols}} |M(3\gamma)|^2 \propto \left[ \left( \frac{1 - x_1}{x_2x_3} \right)^2 + \left( \frac{1 - x_2}{x_1x_3} \right)^2 + \left( \frac{1 - x_3}{x_1x_2} \right)^2 \right]\prod_{i=1}^3 \left( \frac{x_i}{x_i + \frac{2\gamma^2}{M^2}} \right)^2.
\]

As in the previous case, the single photon spectrum will be softened by BE corrections due to the last factor in the squared amplitude.

The phase space integration of Eq. (23) can be performed in analytic form and expressed in terms of dilogarithmic functions. However, it is more illustrative to express the decay rate in terms of an expansion in powers of \( \alpha^2 \) (or \( \gamma^2 \)). If we express the decay rate in terms of the mass \( m \) of constituents, we obtain:

\[
\Gamma(\alpha\text{-}Ps \to 3\gamma) \approx \alpha^6 m \frac{2(\pi^2 - 9)}{9\pi} \left[ 1 - \frac{5}{4} \alpha^2 \right].
\]

Thus, BE corrections affects the decay rate of orthopositronium at order \( \alpha^2 \) which are indeed relevant when confronted to accuracy of present experiments. A comparison of Eqs. (4) and (24) indicates that the net effects of BE corrections is to resize the coefficient appearing in nonlogarithmic \( O(\alpha^2) \) corrections of orthopositronium, namely:

\[
B_0 \to B_0 - \frac{5\pi^2}{4} \approx 44.52(26) - 12.34.
\]
i.e. a non-negligible 28% correction at order $\alpha^2$.

Before concluding, let us address some comments on possible implications of BE corrections in analogous decays of quarkonia. Notice that BE in quarkonia is not simply related to the masses of the constituent quarks and to the coupling $\alpha_s$ as in Eq. (8) due to confinement. If we assume, however, that BE corrections do affect quarkonium decays in a similar form as in positronium, we can address some apparent conflicts in some of their inclusive hadronic and radiative decays. First, the photon spectrum measured in $J/\psi \rightarrow gg\gamma$ decays seems to be softer than predicted by perturbative QCD [12] which indicate possible large non-perturbative effects. This is precisely the effect induced by BE corrections in the single photon spectrum of $\psi\rightarrow\gamma\gamma\gamma$. Second, different inclusive rates in quarkonium decays as discussed in [13] will, in general, get decreased by BE corrections. This may increase the relatively low values of the strong coupling constants extracted from ratios of experimental quarkonium rates [13]. Finally, these effects will also manifest in the “14% rule” observed in the ratio $BR(\psi(2S) \rightarrow X)/BR(J/\psi \rightarrow X)$ for single photon mediated decays [14], because BE are different for these two radial excitations of charmonium.

In this paper we have computed binding energy corrections to positronium decays in a specific ansatz where these non-perturbative effects are introduced as a necessity to account for analyticity and gauge invariance of the corresponding QED amplitudes. The BE corrections (of order $\alpha^2$) to orthopositronium decay are indeed relevant in view of recent efforts to achieve a precise comparison of theory and experiment. When extended to the quarkonium sector, these BE corrections may contribute [8] to solve apparent discrepancies observed between experimental data and perturbative QCD calculations of some inclusive rates of quarkonia.

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References


[11] Convenient representations of the Dirac-delta function in this case are: \( \lim_{\epsilon \to 0} \frac{8\pi \epsilon}{(x^2 + \epsilon^2)^2} = (2\pi)^3 \delta(3)(x) \) and \( \lim_{\epsilon \to 0} \frac{i\epsilon}{(x^2 - \epsilon^2)} = \pi \delta(x) \).


