"On the indices of zeros of Nash fields"

Demichelis, Stefano ; Germano, Fabrizio

Abstract
Given a game and a dynamics on the space of strategies it is possible to associate to any component of Nash equilibria, an integer, this is the index, see Ritzberger (1994). This number gives useful information on the equilibrium set and in particular on its stability properties under the given dynamics. We prove that indices of components always coincide with their local degrees for the projection map from the Nash equilibrium correspondence to the underlying space of games, so that essentially all dynamics have the same indices. This implies that in many cases the asymptotic properties of equilibria do not depend on the choice of dynamics, a question often debated in recent literature. In particular many equilibria are asymptotically unstable for any dynamics. Thus the result establishes a further link between the theory of learning and evolutionary dynamics, the theory of equilibrium refinements and the geometry of Nash equilibria. The proof holds for very general situations that...


Référence bibliographique

On the Indices of Zeros of Nash Fields

Stefano DeMichelis† and Fabrizio Germano‡

this version: January 23, 2000
(first version: September 1996)

Abstract

Given a game and a dynamics on the space of strategies it is possible to associate to any component of Nash equilibria, an integer, this is the index, see Ritzberger (1994). This number gives useful information on the equilibrium set and in particular on its stability properties under the given dynamics.

We prove that indices of components always coincide with their local degrees for the projection map from the Nash equilibrium correspondence to the underlying space of games, so that essentially all dynamics have the same indices. This implies that in many cases the asymptotic properties of equilibria do not depend on the choice of dynamics, a question often debated in recent literature. In particular many equilibria are asymptotically unstable for any dynamics. Thus the result establishes a further link between the theory of learning and evolutionary dynamics, the theory of equilibrium refinements and the geometry of Nash equilibria.

∗This paper is a revised version of a UCSD preprint that appeared in 1996, the main proof is essentially the same apart from substituting singular homology to De Rham cohomology; we wish to thank Michel Le Breton, Francesco De Sinopoli, Peter Doyle, Zheng-Xu He, Plato, Klaus Ritzberger, Joel Sobel, Jorgen Weibull an associate editor, a referee, and seminar participants in Antalya, Mannheim, Paris, Tel Aviv, Toulouse, Vienna, and at CORE and UCSD whose valuable comments and conversations have helped us to improve (hopefully) the paper. All errors are ours.

†Dipartimento di Matematica, Università degli Studi di Pavia, 27100 Pavia, Italy.
‡Eitan Berglas School of Economics, Tel Aviv University, Tel Aviv, 69978, Israel.
The proof holds for very general situations that include not only any number of players and strategies but also general equilibrium settings and games with a continuum of pure strategies such as Shapley-Shubik type games, this case will be studied in a forthcoming paper.
1 Introduction

Given a finite game in normal form, Ritzberger (1994) showed how to associate to any component of Nash equilibria an integer, namely the index of a vector field representing the replicator dynamics; he further showed how this notion of index can be profitably used to derive statements on general properties of Nash equilibria, in particular concerning refinements and robustness of equilibria. More generally, however, the index can be defined for arbitrary vector fields, and, in each case, it conveys useful information about the dynamic behavior of the vector field around the given component of zeros, in particular about its stability properties (see e.g. Corollaries 1 and 2 below and Demichelis (2000)). As Ritzberger and Weibull (1995) point out, the exact quantitative details of dynamics are typically unknown, and the only information one can use involves qualitative features such as monotonicity, so it is often important to have results on the stability of zeros that do not depend on variations of the dynamics from classes as large as possible; a typical example is Corollary 4 of Ritzberger and Weibull (1995) where they show that, for any sign preserving selection dynamics, asymptotically stable faces must contain an essential component of equilibria. (It will be seen below (Corollary 4) that this relation between stability and essentiality (and more strongly degree of a component) is far from being a coincidence). Given such a requirement for “robustness” one is lead to ask, how much can dynamics on the space of strategies differ from each other? or, how much of the stability properties are intrinsic to the equilibria rather than to the dynamics? We address this by asking, to what extent does the index depend on the choice of dynamics? In Corollary 2 it is shown that indeed many components of Nash equilibria are unstable for any dynamics (without even assuming monotonicity or the sign preserving property), implying that instability is in many cases an intrinsic game-theoretic property of equilibria which does not really depend on the dynamics chosen.

On a different account, Kohlberg and Mertens (1986) gave, among other things, a careful description of the geometry of the Nash equilibrium correspondence and of its projection on the space of games, in particular they proved that this projection is homotopic to the identity and so has global degree one. The local degree, and more generally the homological nontriviality of the projection map, is an essential ingredient in the definition of stable sets of equilibria (see Kohlberg and Mertens (1986), Mertens (1989, 1991)); Govindan and Wilson (1996) gave further applications in the spirit

---

1Güll et al. (1993) also defined a notion of (fixed point) index of a certain map, which they used to obtain a lower bound on the number of mixed Nash equilibria of generic normal form games.
of Mertens’ and Ritzberger’s work.

Although both notions of index and local degree, by associating an integer to each component of Nash equilibria, provide a way of characterizing and therefore also of classifying components of Nash equilibria, they are conceptually quite distinct objects: the index is defined starting from a given game and a given dynamics; very roughly it measures how the dynamics varies when players change their strategies around a given component while payoffs stay fixed; a priori it depends on the dynamics chosen. The local degree, on the other hand, measures, still very roughly, how the graph of the Nash equilibrium correspondence varies at the given component as one varies the payoffs of the underlying game; its definition does not require a dynamics; rather it can be expressed in purely game-theoretic terms.

Given that refinements of Nash equilibria typically involve considerations concerning the Nash equilibrium correspondence in a neighborhood of the underlying game, (e.g. when doing payoff perturbations), it is in some sense natural (especially in light of Mertens’ work) to derive statements concerning equilibrium refinements from the notion of degree. On the other hand, the fact that one can derive analogous statements using Ritzberger’s index, which is defined from a vector field on the space of unperturbed strategies for the same fixed game, comes somewhat as a surprise.

In the present paper, we give a technical explanation of this fact showing that for any dynamics in a very wide class (essentially any continuous dynamics consistently defined for all games) its index on a component is equal to the degree of that component. The proof is quite technical but the intuition behind it very simple: even if the set of Nash equilibria of a fixed game can be complicated, (multiple equilibria possibly clustered in several connected components), the global geometry of all Nash equilibria can be disentangled (see already the structure theorem of Kohlberg and Mertens (1986)), and in a related paper, DeMichelis and Germano (2000) show how the unknottedness of the Nash correspondence (in a precise topological sense) can be exploited to give directly an elementary proof of a weaker version of some of the results presented here. A particular instance of this was conjectured by Govindan and Wilson (1997a) where it is proved for a specific map (a map used in Gül et al. (1993)); the authors also point out that their result may extend to other maps, and in Govindan and Wilson (1997b) they show this to be the case for finite two player games and for whose fixed points are exactly the Nash equilibria. Their proof however relies crucially on the linearity of payoffs for two player games and does not solve the question for more than two players. This question is solved in Theorem 1 in the present paper. Moreover, since many standard dynamics especially in evolutionary game theory vanish also outside the set of Nash equilibria, e.g. the replicator dynamics, we deal also
with dynamics of such a type, which we call Nash fields.

More importantly, Theorem 1 can be used to relate the local degree of components to their asymptotic stability: in Corollary 2 it is shown that for a component to be asymptotically stable, its local degree must equal its Euler characteristic, (i.e., +1 for points, convex or contractible sets). As a consequence it is enough for a point or a convex component to be asymptotically stable for some dynamics in order for it to be essential and hence to contain a strategically stable set (this result should be compared with Swinkels (1992, 1993) and Ritzberger and Weibull (1995)). On the other hand, points and components with the “wrong” degree cannot be asymptotically stable for any dynamics. As an example consider a generic normal form game: it has \(2n + 1\) Nash equilibria of which \(n + 1\) have degree +1 and \(n\) have degree −1, the latter always have index −1 and so are unstable in any dynamics. Remark that since the theorem does assume very little on the dynamics, it has a very strong cutting power in eliminating equilibria which can never be stable. Remark also that equilibria of degree −1 are always mixed strategy equilibria (see G"ul et al. (1993), Ritzberger (1994)), so that this complements literature on learning mixed equilibria like Fudenberg and Kreps (1993), Jordan (1993), Kaniovski and Young (1995), Oechssler (1997) Hopkins (1999) and Benaim and Hirsch (1999). In fact, our instability result also carries over to discrete time dynamics approximating the continuous time dynamics via a stochastic approximation theorem of Pemantle (1990), see also Fudenberg and Levine (1998).

Example 2 of Section 2.2 shows that in the remaining cases, say mixed equilibria of degree +1, stability depends on the dynamics chosen; in order to obtain conclusions about asymptotic or Lyapunov stability here, further assumptions on monotonicity or the sign preserving property are necessary, as Ritzberger and Weibull (1995) show; this is in line with the “mixed” results obtained in the literature on learning mixed equilibria; Hopkins (1999) provides further clarification of the stability properties of these equilibria for two player games.

The proof given here applies to more general settings: an obvious case is the one of single-population dynamics on symmetric games, less immediate are the applications to Walrasian equilibria of Arrow-Debreu exchange economies and especially to Shapley-Shubik games, where finitely many players can choose among a continuum of uncountably many (pure) strategies. Here our theorem provides (again through the indices) a link between “out of equilibrium” adjustment processes such as the tatonnement dynamics on one hand and “in equilibrium” processes obtained from the approximating Shapley-Shubik games on the other.

Computationally, Theorem 1 can be used to calculate degrees of equi-
librium components via indices of appropriately chosen vector fields or vice versa; often one computation is much easier and transparent than the other; typically reducing to the evaluation Jacobian matrices of nicely behaved vector fields.

As for the techniques involved: Theorem 1 is based on the Topological Lemma proved in the appendix; the proof of the lemma consists essentially in identifying both index and degree with the algebraic number of intersections of the Nash equilibrium correspondence with a copy of the strategy space. These two identifications become very clear, and almost tautological, if one uses some homology theory, this also allows us to treat at once the case of isolated components, that of course is unavoidable when dealing with the normal form of games that are generic only in their extensive form.

The paper is organized as follows: Section 2 introduces some notation and definitions; the notions of Nash field, degree and index are given and the relation of index and dynamics is illustrated with some examples. Section 3 contains the main results, and Section 4 some applications to specific economic situations. The prerequisites to read these sections do not go beyond the standard notions of point set and basic differential topology that are common in much economic theory literature. The appendix contains the more technical proofs and requires a working knowledge of basic algebraic topology; the reader who is interested mainly in the applications can safely skip the proofs without prejudice to the understanding of the main results of the paper.

2 Preliminary Notions

This section consists mostly of definitions and examples. To avoid using too much topology, many concepts are not defined in the most general or mathematically natural form; for this the reader is referred to the appendix.

We will consider games $\Gamma$ involving finitely many players denoted by $i = 1, \ldots, I$, the strategies available to player $i$ will be denoted by $\Sigma_i$, they could be the mixed strategies of a finite game, see below, or some more general set such as the quantities of goods player $i$ chooses to sell or produce, such an example is given in the section on the Shapley-Shubik game. We will assume that $\Sigma_i$ is a smooth manifold with boundary $\partial \Sigma_i$, $\Sigma_{i,\epsilon}$ will be a collared copy of $\Sigma_i$, i.e., $\Sigma_{i,\epsilon} = \Sigma_i \cup \partial \Sigma_i \times [0, \epsilon]$; A more explicit definition for finite games is given below. The product $\times_{i \in I} \Sigma_i$ is denoted by $\Sigma$ and the product of $\Sigma_{i,\epsilon}$ will be $\Sigma_{\epsilon}$. $\Sigma^\circ$ will denote the interior of $\Sigma$ i.e. $\Sigma \setminus \partial \Sigma$. Given a game $\Gamma$ and a strategy profile $\sigma \in \Sigma$, the corresponding payoff to player $i$ will be denoted by $U_i(\Gamma, \sigma)$ and we will assume that the functions $U_i(\Gamma, \sigma)$
are smooth in their arguments.

We will consider families of games parametrized by a smooth manifold $N$; in applications $N$ could be a space of payoffs in a normal or extensive form, as is the case for finite games (see below), or the space of endowments of players of a Shapley-Shubik game. Most of our analysis will be done on the space $N \times \Sigma$ of couples $(\Gamma, \sigma)$ where $\Gamma \in N$ is a game identified with its parameters and $\sigma$ a strategy profile. In $N \times \Sigma$ there is a closed set denoted by $\eta_N$, or simply $\eta$ when no ambiguity is possible, that is the set of Nash equilibria defined in the usual way:

$$\eta_N = \{ (\Gamma, \sigma) \in N \times \Sigma : U_i(\Gamma, \sigma) \geq U_i(\Gamma, \tau^i, \sigma^{-i}), \forall \tau^i \in \Sigma_i, \forall i \in I \},$$

where as usual $\sigma^{-i}$ is the strategy profile of all players but $i$. Also, we will denote by $\pi_1$ and $\pi_2$ the projections of $N \times \Sigma$ onto its factors $N$ and $\Sigma$ respectively.

If we consider the family of all normal form games, $N$ becomes the Euclidean space $\mathbb{R}^{\kappa I}$ where $\kappa = \prod_i K_i$. By a theorem of Kohlberg and Mertens (1986), the set of Nash equilibria of a finite normal form game can always be written as the disjoint union of a finite number of nonempty, compact, and connected components of Nash equilibria, which we refer to as the components of the game. Let $\eta \subset \mathbb{R}^{\kappa I} \times \Sigma$ denote the set of Nash equilibria as before. Kohlberg and Mertens (1986), also show that $\eta$ is a manifold of dimension $\kappa I$ homeomorphic to $\mathbb{R}^{\kappa I}$, and that its projection $\pi_1$ onto $\mathbb{R}^{\kappa I}$ is a proper map of degree 1.

### 2.1 Dynamics, Nash fields and normal extensions

The notion of dynamics or vector field that is central to our paper is essentially the one of adjustment process given in Samuelson and Zhang (1992) extended to a family of games. We consider a family of adjustment processes that depends continuously on the underlying parameters, e.g., the payoffs of games. Consider first the case of finite normal form games. For any game $\Gamma \in \mathbb{R}^{\kappa I}$ a dynamics is defined at each strategy profile $\sigma \in \Sigma$ by differential equations:

$$\frac{d\sigma_i^j(t)}{dt} = F_i^j(\Gamma, \sigma),$$

An important special case is the one of finite games in normal form. If $S_i$ denotes player $i$’s pure strategies and $K_i = \# S_i$, $\Sigma_i$ will be the set of probability measures on $S_i$, i.e., the $K_i - 1$ dimensional simplex $\Sigma_i = \{ \sigma_i^k | \sum_{k=1}^{K_i} \sigma_i^k = 1, \sigma_i^k \geq 0 \ \forall k \}$; note that in this case the tangent space $T\Sigma_i$ to $\Sigma_i$ or to $\Sigma_i, \epsilon$ can be identified with the Euclidean space $\mathbb{R}^{K_i - 1}$ given by: $T\Sigma_i = \{ \sigma_i^k | \sum_{k=1}^{K_i} \sigma_i^k = 1 \}$. In the same way we will think of the polyhedra $\Sigma$ and $\Sigma, \epsilon$ as full dimensional submanifolds of $\mathbb{R}^{K_i - 1}$ and of $T\Sigma$ as $\mathbb{R}^{K_i - 1}$ with $K = \Sigma_i K_i$. 

---

2 An important special case is the one of finite games in normal form. If $S_i$ denotes player $i$’s pure strategies and $K_i = \# S_i$, $\Sigma_i$ will be the set of probability measures on $S_i$, i.e., the $K_i - 1$ dimensional simplex $\Sigma_i = \{ \sigma_i^k | \sum_{k=1}^{K_i} \sigma_i^k = 1, \sigma_i^k \geq 0 \ \forall k \}$; note that in this case the tangent space $T\Sigma_i$ to $\Sigma_i$ or to $\Sigma_i, \epsilon$ can be identified with the Euclidean space $\mathbb{R}^{K_i - 1}$ given by: $T\Sigma_i = \{ \sigma_i^k | \sum_{k=1}^{K_i} \sigma_i^k = 1 \}$. In the same way we will think of the polyhedra $\Sigma$ and $\Sigma, \epsilon$ as full dimensional submanifolds of $\mathbb{R}^{K_i - 1}$ and of $T\Sigma$ as $\mathbb{R}^{K_i - 1}$ with $K = \Sigma_i K_i$. 

---

6
where the maps $F^i_k$ must satisfy certain conditions in order for the solution to exist and to be contained in $\Sigma$. For instance, we want $\sum_{k=1}^{K_i} \sigma^i_k(t) = 1$ for all $t$, this implies that $\sum_{k=1}^{K_i} F^i_k(\Gamma, \sigma) = 0$ for any $i \in I$, in other words the vector $F(\Gamma, \sigma) = ((F^i_k)_{k,i})$ must be contained in $T\Sigma$, the tangent space of $\Sigma$. We obtain the following as our working definition of a dynamics:

**Definition 1** A dynamics on the space of mixed strategies parameterized by games is a map $F: \mathbb{R}^{\kappa I} \times \Sigma \rightarrow T\Sigma$, $(\Gamma, \sigma) \mapsto F(\Gamma, \sigma)$ such that:

(i) $F$ is continuous in $\Gamma$, (Lipschitz) continuous in $\sigma$,

(ii) $F^i_k(\Gamma, \sigma) \geq 0$ whenever $\sigma^i_k = 0$.

Condition (i) ensures that solutions to the differential equation (1) exist (and are unique if it is Lipschitz), condition (ii) says that the flow defined by (1) will not escape from the space of mixed strategies, i.e., it is weakly inward pointing along the boundary of $\Sigma$. An identical definition works in the more general case of a manifold of parameters $N$ and for a general strategy manifold, the only difference is that, since we do not have coordinates condition (ii) must be phrased as: (ii) $F^i_k(\Gamma, \sigma)$ is weakly interior pointing.

This definition is too broad, in the sense that we would like Nash equilibria to be rest points of the dynamics. The natural condition $F^{-1}(0) = \eta$, which in some instances is fine, is too restrictive in the present game-theoretic setting: many natural selection dynamics such as the (multipopulation) replicator dynamics vanish on Nash equilibria but contain further zeros that are not Nash. In many cases, it can be shown that these “unwanted” zeros can be eliminated after a small deformation of the dynamics, which justifies the following definition:

**Definition 2** A dynamics $F$ is a Nash field if it can be embedded in a family of dynamics $F_s: \mathbb{R}^{\kappa I} \times \Sigma \rightarrow T\Sigma$, $s \in [0, 1]$, such that:

(i) $F_0(\Gamma, \sigma) = F(\Gamma, \sigma)$,

(ii) $F^{-1}_s(0) \supset \eta$, for all $s \in [0, 1]$,

(iii) $F^{-1}_s(0) = \eta$ for $s > 0$.

In all applications known to the authors, the perturbation $F_s$ consists in “pushing in” the zeros that are not Nash equilibria, this definition is very close in spirit to the concept of “interior approximation” of Ritzberger, the main difference is that, in our case, the zeros that are actual Nash equilibria are not displaced. Also, our use of the term Nash field differs from Ritzberger’s (1994), where it is used for a particular case of our notion of Nash field. Note that the Nash fields constitute a very wide class in fact any monotonic

---

3For instance in the Walrasian, general equilibrium case studied in DeMichelis and Germano (1999)
or sign-preserving dynamics as defined in Ritzberger and Weibull (1995) is a Nash field. For further reference, we note that the (multipopulation) replicator dynamics defined by:

\[
\frac{d\sigma_i^k(t)}{dt} = (U_i(\sigma_{-i}, s^i_k) - U_i(\sigma))\sigma^k_i
\]

(4)

and its generalization, the aggregate monotonic dynamics, defined by:

\[
\frac{d\sigma_k(t)}{dt} = \omega_i(\sigma)(U_i(\sigma_{-i}, s^i_k) - U_i(\sigma))\sigma^k_i
\]

(5)

where \( \omega_i, i \in I \), is a smooth and strictly positive function, are examples of both monotonic and sign preserving dynamics and so are of Nash fields.

There are interesting dynamics that vanish on Nash equilibria only; one example is the dynamics derived from Nash’s fixed point map and given in Weibull (1996) as an example of an innovative adaptation dynamics:

\[
\frac{d\sigma_k(t)}{dt} = U_{i,k}^+(\sigma)(\sigma_{-i}, s^i_k) - U_i(\sigma); 0] - \sigma^k_i \cdot \sum_{k'} U_{i,k'}^+(\sigma)
\]

(6)

where \( U_{i,k}^+(\sigma) = \max[U_i(\sigma_{-i}, s^i_k) - U_i(\sigma); 0] \). An example is the dynamics taken from a \( \Sigma_i \) and expected payoffs. Throughout the paper, we shall assume that \( F \) is semialgebraic or subanalytic to avoid pathological sets of zeros.

In order to define the notion of index for components of zeros touching the boundary of \( \Sigma \), we will need the following concept of a normal extension.

**Definition 3** Given a dynamics \( F \) on \( \mathbb{R}^{\kappa I} \times \Sigma \), its normal extension \( F_\epsilon \) is a dynamics on \( \mathbb{R}^{\kappa I} \times \Sigma_\epsilon \), where \( \Sigma_\epsilon = \{ \sigma \in \mathbb{R}^\kappa | \sigma^i_k \geq -\epsilon \text{ and } \sum_k \sigma^i_k = 1 \} \), \( \epsilon > 0 \), such that:

4These dynamics are defined by an equation of the type:

\[
\frac{d\sigma^i_k(t)}{dt} = f^i_k(\sigma)\sigma^i_k
\]

(2)

where the functions \( f^i_k(\sigma) \) satisfy the conditions given in Ritzberger and Weibull (1995) p. 1377; it is easy to check that both in the monotonic and in the sign preserving case the appropriate family of dynamics \( F_\epsilon \) is given by:

\[
\frac{d\sigma^i_k(t)}{dt} = f^i_k(\sigma)(\sigma^i_k + \epsilon(\sigma)\max[0; f^i_k])
\]

(3)

where \( \epsilon(\sigma) \) is an appropriate small positive function vanishing outside a neighborhood of the boundary of the strategy space. This perturbation consists roughly in adding to the dynamics a small “push in” as given by the dynamics of Weibull quoted below.

5We are grateful to J. Weibull for having pointed out this example to us.
\begin{itemize}
    \item[(i)] $F_\epsilon$ is continuous in $\Gamma$, (Lipschitz) continuous in $\sigma$,
    \item[(ii)] $(F_\epsilon)|_{\Sigma} = F$,
    \item[(iii)] $(F_\epsilon)^{-1}(0) = F^{-1}(0)$,
    \item[(iv)] $(F_\epsilon)^k(\Gamma, \sigma) > 0$ whenever $\sigma^k = -\epsilon$.
\end{itemize}

A normal extension is a dynamics on the enlarged space $\mathbb{R}^{\kappa I} \times \Sigma_\epsilon$ that is strictly inward pointing along the boundary of $\Sigma_\epsilon$ and that has the same zeros as the original dynamics; in particular it has no zeros on the boundary of $\Sigma_\epsilon$.

### 2.2 Degrees and indices

Next, we recall the definition of index (see also Hirsch (1976)). Let $V \subset \mathbb{R}^n$ be an open set, let $F : V \to TV = \mathbb{R}^n$ be a dynamics on $V$, and let $x \in V$ be a regular zero of $F$, i.e., $x \in F^{-1}(0)$ such that $|D_x F| \neq 0$, where $D_x F$ denotes the Jacobian matrix at $x$, and $|\cdot|$ denotes the determinant. Then the index of $F$ at $x$ is defined by:

$$i_F(x) = \text{sign}(-D_x F).$$

If $C \subset V$ is an isolated compact component of zeros of $F$ and $U \subset V$ is a relatively compact neighborhood containing $C$, and no other zeros of $F$, let $\tilde{F} : V \to \mathbb{R}^n$ be a smooth map close to $F$ that has only regular zeros (such a map exists by Sard’s theorem), then the index of $F$ at $C$ is defined by:

$$i_F(C) = \sum_{x \in F^{-1}(0), x \in U} i_{\tilde{F}}(x).$$

(7)

It can be shown that such a definition does not depend on the choice of $\tilde{F}$ provided $\tilde{F}$ is close enough to $F$.

Consider now a dynamics $F(\Gamma; \sigma)$, $\sigma \in \Sigma$ with a normal extension $F_\epsilon(\Gamma; \sigma)$, $\sigma \in \Sigma_\epsilon$, keep the game $\Gamma \in \mathbb{R}^{\kappa I}$ fixed, then we can define the index of an isolated component of zeros of $F$ as follows: if $\Gamma$ is fixed, $F_\epsilon$ is a map

$$F_\epsilon : \Sigma_\epsilon \subset \mathbb{R}^{K-I} \to T\Sigma_\epsilon \approx \mathbb{R}^{K-I}$$

and $C$ is an isolated component of its zeros. Hence $i_{F_\epsilon}(C)$ is well-defined. Formally we have:

**Definition 4** Let $F$ be a dynamics and let $C \subset \Sigma$ be an isolated component of zeros of $F$ at the game $\Gamma \in \mathbb{R}^{\kappa I}$, then the index of the dynamics $F$ at $C$ is defined as the integer:

$$i_F(C) = i_{F_\epsilon}(C),$$

6It is not hard to see that any dynamics admits a normal extension and that any two normal extensions can be deformed to one another within the class of normal extensions; also there is no problem in extending the definition to general parameters $N$ and strategy spaces.

9
where $F_\varepsilon$ is a normal extension of $F$ and $i_{F_\varepsilon}$ is computed from (7).

The homotopy invariance of the index (see appendix) tells us that the definition does not depend on the normal extension chosen.

Note that in defining the index we do not need to make a choice of orientation, this is true also in the more general case of a vector field on a manifold that is not $\mathbb{R}^n$, as can be seen in the appendix.

To provide some intuition, we give some examples of games, dynamics, and indices that also illustrate the relation between indices and the asymptotic stability of (components of) zeros under alternative dynamics.

**Example 1.** Consider the following coordination game:

$$
\Gamma_C = \begin{pmatrix}
1, 1 & 0, 0 \\
0, 0 & 2, 2
\end{pmatrix}.
$$

This game has three Nash equilibria, two of which are in pure strategies and one of which is in mixed strategies. For concreteness, we consider the aggregate monotonic dynamics defined by (5) where the functions $\omega_i, i = 1, 2,$ are defined as:

- a) $\omega_i(\sigma) = U_i(\sigma)$,
- b) $\omega_i(\sigma) = 1$,
- c) $\omega_i(\sigma) = (4 - U_i(\sigma))$,

for $\sigma \in \Sigma$. In particular, case b) corresponds to the replicator dynamics. Applied to the game $\Gamma_C$ the three dynamics lead to three vector fields that all behave qualitatively as the vector field depicted in Figure 1.

Figure 1 about here.

Notice that all the Nash equilibria of $\Gamma_C$ correspond to zeros of these vector fields. One can check (by explicit computation) that in all three cases, the index of the two pure strategy equilibria is $+1$; the mixed strategy equilibrium has index $-1$, and the two zeros that are not Nash equilibria (the remaining two pure strategy profiles) have index $0$. The only two asymptotically stable points are those with index $+1$; Corollary 2 in the next section shows that this is a general fact that holds for any dynamics. The equilibrium with index $-1$ is unstable; again this is an instance of a general fact. The zeros that are not Nash equilibria can be removed with a small perturbation, pushing inside; we know this must be so since aggregate monotonic dynamics are Nash fields, since they are removable they must have index $=0$. We shall see that this is a general fact: it follows from the equation proved in Theorem 1 once one remarks that the sum on the right hand side is over the empty set, so it is zero.
Relabelling the players’ strategies one obtains the anti-coordination game:

\[ \Gamma_{AC} = \begin{pmatrix} 0,0 & 2,2 \\ 1,1 & 0,0 \end{pmatrix}, \]

which can be studied in a similar way, but which exhibits significant differences when analyzed with single-population dynamics, as we will discuss later. △

**Example 2.** As it is true that asymptotic stability of a zero implies that its index is +1, the converse may be false without further assumptions, as can be seen by considering the following matching pennies game:

\[ \Gamma_{MP} = \begin{pmatrix} 3,1 & 1,3 \\ 1,3 & 3,1 \end{pmatrix}. \]

This game has a unique Nash equilibrium in mixed strategies. For the three dynamics of the previous example the vector fields are depicted in Figures 2a-2c.

Figures 2a-2c about here.

Again, the unique Nash equilibrium corresponds to a zero of the three vector fields, which moreover has index +1 in all three cases. However, unlike the previous example, the dynamic behavior at this zero is quite different across dynamics. In particular, it is a unstable in 1a) and it is stable in 1c). △

**Example 3.** Consider the following ultimatum game:

\[ \Gamma_U = \begin{pmatrix} 1,4 & 1,4 \\ 0,0 & 2,2 \end{pmatrix}. \]

Player 1 can choose between entering a market, a strategy we shall call E, or staying out (strategy O). If he enters, Player 2 can choose whether to fight (strategy F) or to accommodate (strategy A).

The game in normal form is not generic, it has an isolated equilibrium at (E; A) and a component of equilibria \((O; p \cdot F + (1 - p) \cdot A, \ p \in [1/2, 1])\); see Figure 3, where the flow for the replicator dynamics is also depicted. △

Figure 3 about here.

Once the zeros that are not Nash are removed, the dynamics has a zero of index +1 corresponding to the isolated equilibrium, while the index of the component is zero. Again, Corollary 2 implies that this component cannot

---

7This example is taken from Weibull (1995)
be asymptotically stable for any dynamics that is a Nash field; this is evident from the picture for the case of the replicator dynamics.

Note that $C$ has many weakly (Lyapunov) stable points, but as a set it is unstable: a sequence of small disturbances will move any point of it towards the left endpoint until it will be eventually drained by the flow to $(E; A)$. △

The definition of degree is formally the same as that of index. Note however, that the map $f$ does not come from a dynamics and that its domain and its source, althought of the same dimension, are different spaces , in our case the Nash equilibria and the space of games. Given a map $f : U \subset \mathbb{R}^n \to \mathbb{R}^n$, and points $y \in \mathbb{R}^n$, $x \in f^{-1}(y) \subset U$. If $|D_x f| \neq 0$, we define the degree of the map $f$ at the point $x$ as:

$$d_f(x) = \text{sign}|D_x f|.$$ (8)

Again, we can use perturbations to define the degree for non-regular points and components of continuous maps. Unlike the index, the degree depends on the orientation chosen for $U$ and $\mathbb{R}^n$; it is possible to change them independently, so that the sign of the determinant is changed; more on this point can be found in the appendix.

Now, given that $\eta \subset \mathbb{R}^{\kappa I} \times \Sigma$, the space of all Nash equilibria, is homeomorphic to the space of games $\mathbb{R}^{\kappa I}$, and that $\pi_1$ restricts to a map $\pi: \eta \approx \mathbb{R}^{\kappa I} \to \mathbb{R}^{\kappa I}$, for a fixed $\Gamma \in \mathbb{R}^{\kappa I}$, the set $\pi^{-1}(\Gamma)$ is the set of Nash equilibria corresponding to the game $\Gamma$. We can define:

**Definition 5** Let $\Gamma \in \mathbb{R}^{\kappa I}$ be a game in normal form, let $C \subset \pi^{-1}(\Gamma)$ be a component of Nash equilibria of $\Gamma$, then we define the degree of the component $C$ as the degree of $C$ under the projection map $\pi$:

$$d(C) = d_\pi(C),$$

where $d_\pi(C)$ is computed via (8).

In order to define the degree in the general case of games parameterized by a manifold with strategies also taken from a manifold, it is necessary to assume that the space of Nash equilibria has an “orientation class” in dimension $n$, e.g., it is an oriented manifold.

Again, observe that the maps involved in the definition of index and degree are very different. The index involves dynamics while the degree does not. In particular, it makes sense to define the index of a component without any reference to nearby games, while this is not possible for the degree. On the other hand, the degree is an intrinsically game-theoretic object that does not depend on and “a priori” seems completely unrelated to any dynamics.

---

8(Kohlberg and Mertens (1986)

9see the appendix for the formal treatment
3 Main Results

We are now ready to state the main results of the paper.

**Theorem 1** Let $F$ be a Nash field, let $C \subset \Sigma$ be a component of zeros of $F$ for a fixed game $\Gamma \in \mathbb{R}^{m}$, let $C_1, \ldots, C_m \subset \Sigma$ be the components of Nash equilibria contained in $C$, then:

$$i_F(C) = \sum_{C_i \subset C} d(C_i).$$

**Proof.** Let us first choose a family of dynamics $(F_t)_{t \in [0,1]}$ from Definition 2. Then we have:

Claim: $i_F(C) = \sum_{C_i \subset C} i_{F_t}(C_i)$ for $t > 0$.

Note first that given a small enough neighborhood $U$ of $C$, $F_t^{-1}(0) \cap U = \cup_{C_i \subset C} C_i$ for $t > 0$. This follows immediately from the fact that $F_t^{-1}(0)$ consists only of Nash equilibria for $t > 0$. Now let $(F_{t,\epsilon})_{t \in [0,1]}$ be a family of normal extensions of $F_t$, $\epsilon > 0$. Applying Definition 4, we have:

$$i_F(C) = i_{F,\epsilon}(C) \quad \text{and} \quad i_{F_t}(C_i) = i_{F_{t,\epsilon}}(C_i) \quad \text{for} \quad t > 0.$$

For $t > 0$, let $\tilde{F}_{t,\epsilon}$ be a small perturbation of $F_{t,\epsilon}$ that has regular zeroes inside the $U_i$ and is equal to $F_{t,\epsilon}$ outside the union of the neighborhoods $U_i$ of the components $C_i$ as in (7). Let the regular zeros of $\tilde{F}_{t,\epsilon}$ in $U_i$ be $\{\sigma_{d,i}\}_{d=1}^{d_i}$. By (8) we have that:

$$i_{F_t}(C_i) = i_{F_{t,\epsilon}}(C_i) = \sum_{d=1}^{d_i} i_{\tilde{F}_{t,\epsilon}}(\sigma_{d,i})$$

for any $i = 1, \ldots, m$. Now, if $t$ is small, $\tilde{F}_{t,\epsilon}$ is also a perturbation of $F_{t,\epsilon}$ and its zeros contained in $U$ are all the $\{\sigma_{d,i}\}_{d=1}^{d_i}$, so we have:

$$i_F = \sum_{i=1}^{m} \sum_{d=1}^{d_i} i_{\tilde{F}_{t,\epsilon}}(\sigma_{d,i}) = \sum_{i=1}^{m} i_{F_{t,\epsilon}}(C_i).$$

This proves the claim.

Given the claim, we can assume that $F$ vanishes only on Nash equilibria, and we are in a position to apply Lemma 3 of the appendix to a normal extension of $F$ to get the result. △

The theorem implies that the sufficient condition for essentiality of components of Nash equilibria of Govindan and Wilson (1996) is equivalent to
the one given in Ritzberger (1994). In fact Govindan and Wilson (1996) require \( d(C) \neq 0 \), which by Theorem 1 is equivalent to requiring \( i_F(C) \neq 0 \) for any dynamics whose zeros coincide with Nash equilibria; which in the special case of the perturbed replicator dynamics gives Ritzberger’s (1994) condition. The next corollary is a special case of the one that follows it, we isolate it because its proof is completely elementary and has already some interesting applications.

**Corollary 1** If \( \sigma \in \Sigma \) is a Nash equilibrium of degree \(-1\) of a given game \( \Gamma \in \mathbb{R}^{\kappa I} \), then \( \sigma \) must be linearly unstable under any Nash field for which it is a regular zero.

**Proof.** The proof is by contradiction: if \( \sigma \) were regular and linearly stable all the eigenvalues of the Jacobian of the Nash field would have negative real part, but then, with our normalization the index would be \(+1\). \( \triangle \)

Since Nash equilibria of degree \(-1\) are always in mixed strategies, this relates to the literature mentioned in the introduction on learning mixed equilibria. In particular, it allows to discriminate against a subset of equilibria, such as the mixed equilibria of the coordination and the anticoordination games given in Example 1, for a wide class of dynamics. A theorem of Pemantle (1990) shows that the instability of such equilibria will persist for discrete time dynamics that approximate the Nash fields with some noise. Example 2 on the other hand, shows that mixed equilibria of degree \(+1\) can be either stable or unstable depending on the dynamics.

The generalization to not necessarily regular equilibria or components is the following:

**Corollary 2** For a component of zeros \( C \subset \Sigma \) of any given Nash field at a game \( \Gamma \in \mathbb{R}^{\kappa I} \) to be asymptotically stable\(^{10}\) it is necessary that:

\[
\chi(C) = \sum_{C_i \subset C} d(C_i),
\]

where \( C_1, ..., C_m \) denote the components of Nash equilibria at \( \Gamma \) contained in \( C \) and \( \chi(C) \) is the Euler\(^{11}\) characteristic of \( C \).

\(^{10}\)Recall the definition of asymptotic stability (see e.g. Ritzberger and Weibull (1995)): A closed invariant set \( C \) is **asymptotically stable** if for every neighborhood \( V \) of \( C \) there exists a neighborhood \( U \subset V \) such that \( \forall t \geq 0, \forall x \in U, \sigma(t; x) \in V \) and \( d(\sigma(t; x); C) \to 0 \) as \( t \to \infty \), where \( \sigma(t; x) \in \Sigma \) denotes the state at time \( t \) for initial state \( x \).

\(^{11}\)The definition of Euler characteristic can be found in Spanier (1966) or also Hirsch (1976); here we only need the fact that it is a topological invariant, that is \(+1\) for convex (or contractible) sets and zero for the circle.
Proof. In Demichelis (2000) it is shown that a necessary condition for asymptotic stability is that $i_F(C) = \chi(C)$; using Theorem 1 the result follows. △

A good illustration of this fact is given by the component giving the “Out” option in the entry game of Example 3 in Section 2.2.

A similar necessary condition holds with a requirement slightly stronger than Lyapunov stability (see Ritzberger and Weibull (1995) or Weibull (1995) for a definition of Lyapunov stability).

Corollary 3 If a component $C$ admits a neighborhood $U$ that is invariant under a Nash field and such that $\bar{U}$ is a manifold with boundary, then:

$$\chi(U) = \sum_{C_i \subset C} d(C_i).$$

Proof. See the appendix. △

Note that this implies that indifferent equilibrium points as in Example 2 of Section 2.2 (the matching pennies game) must have index $+1$ since they have an obvious invariant manifold neighborhood.

We now state a generalization of Theorem 1 to the case of a general space of strategies and parameters.

Theorem 2 Let $N$ be a manifold parameterizing games with strategy space $\Sigma$, let $F$ be a dynamics vanishing only on the Nash equilibria, and let the set of all Nash equilibria $\eta$ be a connected, orientable manifold. Then there exists an integer $m$ (that depends only on the geometry of $\eta \subset N \times \Sigma$ and not on the game nor the choice of the dynamics) such that:

$$i_F(C) = m \cdot d(C)$$

for any game and component $C$ of Nash equilibria of the game. Moreover, if there exists an $F$ vanishing transversely on $\eta$, then the orientation of $\eta$ can be chosen such that $m$ is equal to $+1$.

This theorem, whose proof is given in the appendix, should be seen as a member of a family of theorems that can be deduced from the Lemmas 2 and 3 there (or adaptations thereof). We chose a form that, although not the most general, can be easily adapted to the examples of the next section.
4 Examples and Applications

4.1 A class of $2 \times 2$ games

We first give a family of games that contains some of the examples already quoted in Section 2, and where the application of Theorem 1 is straightforward. An important remark here is that we can deduce the stability properties and the values for the degree and the index from the geometry of Nash equilibria without resorting to any calculations. Note also that the following family of games provides an example of a “cusp-like” singularity of the projection map $\pi$. The family of games is given by:

$$\Gamma(x, y, z) = \begin{pmatrix} x, y & x, z \\ 0, 0 & 2, 2 \end{pmatrix},$$

where $x > 0$. Their equilibrium sets are illustrated in Figure 4. Game I is a coordination game (as $\Gamma_C$ in Example 1 above) and Game VIII is the ultimatum game (as $\Gamma_U$ in Example 3 above).

The reader is invited to provide details to the following remarks. Game III has a unique strict equilibrium in strictly dominant strategies. This is asymptotically stable and so has index $+1$ and hence also degree $+1$.

Games IV to VII all exhibit the same properties. The equilibrium point moves around or explodes into a component, but its index and degree are always equal to $+1$ by the invariance under homotopy deformations.

In Game II, there are exactly two equilibrium points, one of which (namely the equilibrium in the lower right corner) disappears as one moves in direction of Game III. Because of this it must have degree 0, and therefore also index 0 for all dynamics, which implies that it cannot be a limit point of any of our dynamics. (This is also in agreement with its “bad” refinements properties; it is weakly dominated and hence not perfect.)

On the other side, moving from Game VII to Game VIII, a component appears. It must have degree zero and, as in the discussion before, it is not asymptotically stable under any dynamics.

To compute the index and the degree for Game I, note again that we do not have to use a parameterization nor a dynamics. Moving from Game III towards Game I through Game II shows that the upper-left equilibrium must have index $+1$, while moving from Game VII towards Game I through Game VIII shows that the lower-right equilibrium must also have degree $+1$. Now, the mixed strategy equilibrium of Game I collapses together with the lower-right equilibrium to a point that is a degree 0 equilibrium point.
in Game II. Since the lower-right equilibrium in Game I has index +1 and
the definition of index implies that it is additive, the mixed strategy equi-
librium must have index and hence degree equal to −1, of course this also
follows from the formula for the Poincare’ Hopf formula for the sum of the
indices of a vector field. This means that it can never be a stable point
for any dynamics (we knew this for the aggregate monotonic dynamics of
Example 1). Moreover, it satisfies all possible refinement tests and is a KM
and a Mertens-stable set; this shows one important difference between con-
cepts of evolutionary stability and strategic stability. Moreover the variance
is very robust in the sense that does not really depend on the choice of the
underlying dynamics.

4.2 Single-population games

By Lemma 4 in the appendix the equivalence of degree of a game and index
of a dynamics holds also in the situation of symmetric two person $k \times k$
games and single-population dynamics, see Weibull (1995) or Fudenberg and
Levine (1998) for the definitions. We give an example to illustrate how
the degree and index of equilibria in the symmetric situation are different
from degree and index of the same equilibria but considered in the set of all
(not necessarily symmetric) strategies; this reflects the fact that, when the
condition that equilibria must be symmetric is imposed, previously unstable
equilibria may become stable.

Consider the anticoordination game $\Gamma_{AC}$ of Example 1, inside the space
of all $2 \times 2$ games , i.e. $\mathbb{R}^8$. It has two pure strategy equilibria and a mixed
equilibrium. The latter has index $-1$ and is thus unstable for all Nash fields.
Moreover, the index and the degree do not change if the parameter space is
restricted from $\mathbb{R}^8$ to the space of symmetric games, in this case $\mathbb{R}^4$, keeping
the space of strategies unrestricted.

However, if we restrict the space of strategies to symmetric strategies, and
consider a dynamics leaving it invariant such as single-population dynamics,
then the situation changes and the index and degree of the mixed strategy
equilibrium becomes +1, i.e., the equilibrium becomes potentially stable, and
it is easy to see that it is asymptotically stable for the single-population rep-
locator dynamics. Notice also that since ESS equilibria of symmetric games
are isolated and asymptotically stable for the (single-population) replicator
dynamics, by Corollary 2, they must always have degree +1.
4.3 Asymptotic stability and strategic stability

Corollary 2 says that asymptotically stable components must have degree equal to their Euler characteristic. Since this is +1 for points and convex sets it follows from Theorem 1 that asymptotically stable points or convex sets must have degree=index=1, and so they contain strategically stable sets in the sense of Mertens (1989); note that even if essentiality of a set of equilibria as is usually defined does not imply that the set contains stable sets, nontriviality of the degree does. It is not hard to see that the same conclusion must hold for asymptotically stable invariant convex sets such as faces (the detailed proof is in Demichelis (2000)). Explicitly:

**Corollary 4** Let $C$ be a closed invariant convex (or more generally contractible) set, e.g., a face, in the strategy space that is asymptotically stable (or is Lyapunov stable and satisfies the conditions of Corollary 3) under some dynamics, then it must contain a stable set in the sense of Mertens (1989).

Note that no assumption of monotonicity or sign preserving is made on the dynamics so that our result extends the well known result of Ritzberger and Weibull (1995); see also the related results in Swinkels (1992, 1993). If the set of equilibria is not contractible the index may be zero: Hofbauer and Planck (1996) exhibit an asymptotically stable set of Nash equilibria homeomorphic to the circle. They show that it has index zero (as is predicted by Corollary 2) and that it is not essential. This may be a hint that, for non-contractible sets of equilibria, evolutionary stability may not imply strategic stability. On the other hand, the mixed equilibrium of Example 2 in Section 2.2 already clearly shows that strategic stability does not imply evolutionary stability.

5 Appendix

The background required for this appendix is a solid knowledge of basic algebraic topology, we refer to Hirsch (1976) for differential topology, Spanier (1966) for algebraic topology, and Milnor and Stasheff (1972) for additional information about the properties of zeros of sections of vector bundles.

We first recall the homological definitions of index and degree, which are equivalent to or generalize the ones given in Section 2, but which for our purposes are much easier to work with; since homology is an homotopically invariant functor it is immediate that index and degree are invariant under deformations. All homology groups will be with rational coefficients.

**Index:** Let $M$ be an orientable manifold (possibly with boundary) of dimension $t$, let $F: M \to TM$ be a vector field, $C \subset M$ a compact component of
zeros of $F$ not touching the boundary, and $U$ an open neighborhood of $C$ isolating it. If $\tau \in H^n(TM, TM \setminus \{0\})$ is the Thom class and $\varphi \in H^n(U, U \setminus C)$ is the orientation class along $C$, then we define the index $i_F(C)$ of $F$ at $C$ as the integer $\xi$ such that:

$$F^* \tau = (-1)^i \xi \cdot \varphi,$$

where $F^*$ is the induced map in cohomology $F^* : H^n(TM, TM \setminus \{0\}) \to H^n(U, U \setminus C)$.

Note that if we change the orientation of $M$ both the Thom class and the orientation class change their sign so the index does not change.

**Degree:** Let $f : M \to N$ be a proper map from the oriented manifold $M$ to the oriented manifold $N$ (both possibly with boundary), let $n \in N$ be a point in the interior of $N$, and let $C \subset f^{-1}(n)$ be a component contained in the interior of $N$, let $\sigma_U \in H_n^c(U)$ and $\sigma_N \in H_n^c(N)$ be the fundamental classes induced by the orientation, we also refer to these as “orientation classes,” (see Spanier, Ch. 6, Sect. 3). We define the degree $d_f(C)$ of $f$ at $C$ as the integer $\xi$ such that:

$$f_* \sigma_U = \xi \cdot \sigma_N,$$

where $f_* : H_n^c(U) \to H_n^c(N)$ is the map induced on homology with closed supports.

Note that our definition of degree makes sense even if $M$ is simply a topological space $\eta$ with a distinguished homology class $z_\eta \in H_n^c(\eta)$, which we will call too a “fundamental class.”

The degree obviously depends on the choices of the fundamental classes. In the following discussion the target $N$ will be kept fixed so we can assume an orientation for it has been chosen from the beginning; as for the domain $M$, when there is ambiguity about the choice of the fundamental class, we will make the dependence on the class $z_U$ explicit by writing $d_{f,z_U}(C)$. When $z_U$ is not explicitly written it will be understood that we are dealing with an oriented manifold and that the fundamental class is the one induced by the orientation; in this case the definition of degree agrees with the one given in Section 2.

Recall that the group $H_n^c(\eta)$ defined in Spanier (1966) is the inverse limit $\lim \leftarrow H_n(\eta, \eta \setminus C)$, where $C$ runs over compact subsets of $\eta$; it is often called “homology with closed supports”, the little $c$ standing for “closed”; if $\eta$ has a non pathological end this is also the same as the homology relative to the point at infinity of the one point (Alexandroff) compactification of $\eta$.  

19
**Topological Lemmas:** These lemmas apply to the following situation: $\Sigma$ is a manifold of dimension $l$ usually with boundary, $N$ is another manifold of dimension $n$, and $T\Sigma$ denotes the tangent manifold to $\Sigma$ over $N \times \Sigma$ (with slight abuse of notation, since more precisely we should denote it by $\pi_2^*T\Sigma$, where $\pi_2 : N \times \Sigma \to \Sigma$ is projection map). Also $F : N \times \Sigma \to T\Sigma$ is a section (a vertical vector field) that is strictly inward pointing on the boundary $\partial \Sigma \times N$ and such that $\eta = F^{-1}(0) \subset \Sigma \times N$ does not touch the boundary, remember that we assume that $F$ is semialgebraic or subanalytic so that $\eta$, although not necessarily a smooth submanifold, is a neighborhood retract, this is the case arising in game theory. On the other side for our computations we do not need to assume any kind of differentiability. $\pi_1 : N \times \Sigma \to N$ is the projection map. In the main body of the paper the calculation of indexes and degree as been reduced to this case via normal extensions, the topological facts we need to complete the proofs are summarized in the following two lemmas:

**Lemma 1** There exists a fundamental class $\eta \in H^*_c(\eta)$ such that if $C$ is a compact component of $F^{-1}(0) \cap (\{n\} \times \Sigma)$, $n \in N$, then:

$$i_F(C) = d_{\pi_1, \eta}(C).$$

Moreover, if $\eta$ is a connected orientable manifold inside $N \times \Sigma$, then $\eta$ is a multiple of the orientation class $\sigma_\eta$, i.e., $\eta = m \cdot \sigma_\eta$, and if $F$ vanishes transversely on $\eta$, $\eta$ becomes the usual orientation class of $\eta$ as the zero set of a section so that the degree becomes the usual degree and we can simply write

$$i_F(C) = d_{\pi_1}(C).$$

**Proof.** If $\tau \in H^i(T\Sigma; T\Sigma \setminus \{0\})$ is the Thom class of $T\Sigma$, let $e = F^*\tau \in H^i(\Sigma \times N; (\Sigma \times N) \setminus \eta)$, let furthermore $\sigma_\Sigma \in H^i(\Sigma)$ and $\sigma_N \in H^i_N(N)$ be orientation classes so that $\sigma_N \times \sigma_\Sigma \in H^i_{n+t}(N \times \Sigma)$ is the orientation class of $N \times \Sigma$.

One can check that the cap product:

$$H^i_{n+t}(N \times \Sigma) \times H^i(N \times \Sigma; (N \times \Sigma) \setminus \eta) \to H^i_\eta(\eta)$$

is well-defined,$^{12}$ and we will denote $(-1)^i(\sigma_N \times \sigma_\Sigma) \cap e$ by $z_\eta$, this will be our fundamental class of $\eta$.

Let now $C$ be a compact component of $F^{-1}(0) \cap (\{y\} \times \Sigma)$ and let $U$ be a neighborhood of $n \in N$ and $V$ an isolating neighborhood of $C$ in $\Sigma$; we

---

$^{12}$Here we need the fact that $\eta$ is a neighborhood retract; apart from this to check that the map is well defined one has only to go through the definition of $\lim$.
will choose them so that \((U; \partial U)\) and \((V; \partial V)\) are manifolds with boundary; 
\((U; \partial U)\) is homeomorphic to a Euclidean ball and \(\eta \cap \partial(U \times V) \subset \partial U \times V\). 
\(\sigma_N\) and \(\sigma_\Sigma\) induce well-defined relative homology classes in \(\sigma_U \in H_n(N; N \setminus \bar{U})\) and \(\sigma_V \in H_l(\Sigma; \Sigma \setminus V)\); also remark that by excision \(H_n(N; N \setminus \bar{U}) \approx H_n(U; \partial U)\) and \(H_n(\Sigma; \Sigma \setminus V) \approx H_n(V; \partial V)\). Similarly, we will denote by \(z_C\) the image of \(z_\eta\) under the composition of maps:

\[ H_n^c(\eta) \to H_n(\eta; \eta \setminus (\bar{U} \times \bar{V})) \xrightarrow{\simeq} H_n(\eta \cap (U \times V); \eta \cap \partial(U \times V)) \to H_n(U \times V; \partial U \times V). \]

The first map comes from the definition of \(H_n^c\) as an inverse limit, the second is the excision isomorphism, the last is induced by inclusion.

If we denote by \(e_C\) the image of \(e\) under the map \(H^l(N \times \Sigma; (N \times \Sigma) \setminus \eta) \to H^l(U \times V; U \times \partial V)\) induced by the inclusion \((U \times V; U \times \partial V) \to (\Sigma \times N; (\Sigma \times N) \setminus \eta)\). The functoriality of cap products implies that:

\[ (-1)^l(\sigma_U \times \sigma_V) \cap e_C = z_C. \]

Also note that if \(\varphi_U \in H^n(U; \partial U)\) is the cohomology class such that \([\sigma_U; \varphi_U] = 1\) in the algebraic duality between homology and cohomology and \(\{n\}\) is the class of a point in \(H_0(U)\) we have:

\[ (\sigma_U \times \sigma_V) \cap \pi_1^*\varphi_U = (-1)^n(\sigma_V \times \sigma_U) \cap \pi_1^*\varphi_U = (-1)^n\sigma_V \times \{n\}, \]

under the cap product

\[ H_{n+l}(U \times V; \partial(U \times V)) \times H^n(U \times V; \partial U \times V) \to H_l(U \times V; U \times \partial V). \]

Note that by definition of degree:

\[ \pi_1 z_C = \xi \cdot \sigma_U \text{ with } \xi = d_{\pi_1}(C), \]

thus:

\[ \xi = [\xi \sigma_U; \varphi_U] = [\pi_1 z_C; \varphi_U] = [z_C; \pi_1^* \varphi_U]. \]

On the other hand, by the definition of the index, we have:

\[ i_C(F) = (-1)^l[\sigma_V; h^*e_C] = (-1)^l[\sigma_V \times \{n\}; e_C] \]

with the inclusion \(h : (V; \partial V) \to (V \times U; \partial V \times U)\). Equipped with all this notation, the computation runs as follows:

\[
\begin{align*}
\xi &= [z_C; \pi_1^* \varphi_U] = (-1)^l[(\sigma_U \times \sigma_V) \cap e_C; \pi_1^* \varphi_U] \\
&= (-1)^l[\sigma_U \times \sigma_V; e_C \cup \pi_1^* \varphi_U] = (-1)^l[\sigma_U \times \sigma_V; (-1)^n \pi_1^* \varphi_U \cup e_C] \\
&= (-1)^l[(\sigma_U \times \sigma_V) \cap \pi_1^* \varphi_U; (-1)^n e_C] \\
&= (-1)^l[(-1)^n \sigma_V \times \{n\}; (-1)^n e_C] = i_C(F).
\end{align*}
\]
As for the last two statements, the first one is obvious because on a manifold $\mathcal{H}_c^n(\eta^n) \approx \mathbb{Z}$ generated by $\sigma_\eta$. The fact that $z_C$ becomes the usual orientation class if $F$ is transversal is standard, see Milnor (1972). △

This theorem can be used to give a proof also of the analogous equivalence of degree and index in the Walrasian setting proved in Demichelis and Germano (1999). In the case of finite games we have:

**Lemma 2** Let $N = \mathbb{R}^n$, $n = \kappa I$ be the space of finite normal form games, let $\Sigma$ be the space of mixed strategies, and, as before let $F$ vanish only on $\eta$ then the class $z_\eta$ defined above coincides with the generator of $\mathcal{H}_c^n(\eta) = \mathcal{H}_c^n(\mathbb{R}^n) \approx \mathbb{Z}$ so in this case too for any component $C$ of $F^{-1}(0) \cap \pi_1^{-1}(n)$ we have $i_F(C) = d_{\pi_1}(C)$.

**Proof.** The statement does not follow immediately from the previous lemma since $\eta$ is not necessarily the zero set of a transversely vanishing function, so we can say only that $z = m \cdot [1]$, where $m$ is an integer and $[1]$ is the generator of $\mathcal{H}_c^n(\mathbb{R}^n) \approx \mathbb{Z}$ such that $\pi_1 : \eta \approx \mathbb{R}^n \to N = \mathbb{R}^n$ has total degree one. In this case, we would have $i_F(C) = m \cdot d(C)$. To prove our result, note that if $C$ runs over all components of equilibria of any given game, we have:

$$\sum_C d(C) = 1 \text{ and } \sum_C i_F(C) = 1$$

and therefore $m$ must equal 1. △

In the same way one proves the following:

**Lemma 3** Let $N_s$ be the space of symmetric $k \times k$ games, $\Sigma_s$ be the space of symmetric strategies and let $C$ be a component of symmetric Nash equilibria, see Weibull (1995) for the definitions, let $F$ and $\pi_1$ be a dynamics and the projection as before, then:

$$i_F(C) = d_{\pi_1}(C)$$

Note however that the degree (index) of a symmetric Nash equilibrium is not necessarily its degree (index) as a Nash equilibrium even if $F$ extends to non-symmetric strategies.

**Proof of Theorem 2:** The set $\eta$ is by hypothesis a connected orientable manifold, so by Lemma 2 the class $z_\eta$ is a multiple of the orientation class:

---

\(^{13}\)It is easy to adapt the proof of Kohlberg and Mertens (1986) to show that the space $\eta_s$ of symmetric Nash equilibria is a manifold homeomorphic to $N_s$ and that it projects onto $N_s$ with a degree one map.
\( z_\eta = m \cdot \sigma_\eta \), so \( d_{\pi_1, z_\eta}(C) = m \cdot d_{\pi_1}(C) \), again by Lemma 2 \( i_F(C) = d_{\pi_1, z_\eta}(C) \). Thus finally \( i_F(C) = m \cdot d_{\pi_1}(C) \). Note also that if \( \chi(\Sigma) \) is the Euler characteristic of \( \Sigma \) and \( d \) is the global degree of the projection \( \pi_1 : \eta \subset N \times \Sigma \rightarrow N \) the Poincaré-Hopf theorem implies that \( \chi(\Sigma) = m \cdot d \), from which the value of \( m \) can be deduced independently of the dynamics. \( \Delta \)

**Proof of Corollary 3:** Consider the Nash field \( F : U \rightarrow TU \) on the manifold with boundary \( (U, \partial U) \); since \( U \) is invariant \( F \) is weakly inward pointing, let us take perturbation \( \tilde{F} \) that has only regular zeros in the interior of \( U \), by (7) we have \( i_F(C) = \sum_{x \in \tilde{F}^{-1}(0), x \in U} i_{\tilde{F}}(x) \); on the other hand, by the Poincaré-Hopf theorem for manifolds with boundary, \( \sum_{x \in \tilde{F}^{-1}(0), x \in U} i_{\tilde{F}}(x) = \chi(U) \), and by Theorem 2 \( \sum_{C_i \subset C} d(C_i) = i_F(C) \) so the result follows. \( \Delta \)

**References**


Figure 3.

Figure 4.

EQUILIBRIA OF THE GAME

\[
\begin{array}{c|cc}
 & L & R \\
\hline
T & (x, y) & x, z \\
B & 0, 0 & 2, 2 \\
\end{array}
\]