"Valuation, hedging and the risk management of insurance contracts"

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Abstract
This thesis aims at contributing to the study of the valuation of insurance liabilities and the management of the assets backing these liabilities. It consists of four parts, each devoted to a specific topic. In the first part, we study the pricing of a classical single premium life insurance contract with profit, in terms of a guaranteed rate on the premium and a participation rate on the (terminal) financial surplus. We argue that, given the asset allocation of the insurer, these technical parameters should be determined by taking explicitly into account the risk management policy of the insurance company, in terms of a risk measure such as the value-at-risk or the conditional value-at-risk. We then design a methodology that allows us to fix both parameters in such a way that the contract is fairly priced and simultaneously exhibits a risk consistent with the risk management policy. In the second part, we focus on the management of the surrender option embedded in most life insuran...

Document type: Thèse (Dissertation)

Référence bibliographique

Barbarin, Jérôme. Valuation, hedging and the risk management of insurance contracts. Prom. : Devolder, Pierre
Université Catholique de Louvain
Institut des Sciences Actuarielles


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Louvain-La-Neuve
Mai 2008
A mes parents,

"On se lasse de tout sauf de comprendre",
Virgile.

A François et Louise,

"Rêve de grandes choses, cela te permettra d’en faire au moins de toutes petites",
Jules Renard.
Preface.

This dissertation has been prepared in partial fulfillment of the requirements for the Ph.D. degree at the Institute of Actuarial Sciences, of the Université Catholique de Louvain. The research was carried out in the period from September 2003 to December 2007 under the supervision of Pierre Devolder, professor and president of the Institute of Actuarial Sciences. From September 2005, it was financed by Axa Belgium under the “Axa Belgium Chair in Risk Management”.

Acknowledgment.

First of all, I would like to thank my supervisor Professor Pierre Devolder, who gave me the opportunity to start this PhD thesis. I would like to thank him for his trust, for his constant support and for his valuable suggestions and advices. I am specially grateful for the freedom he left me to complete this work.

My gratitude also goes to the members of my doctoral committee, Professors Pierre Ars, Griselda Deelstra, Michel Denuit, and Ragnar Norberg for the time they spent reading my thesis. A special thank is due to Professor Ragnar Norberg for coming from London to attend my PhD defense and for all his detailed remarks and advices on my work.

I would also like to thank Professor Thorsten Rheinlander for inviting me to present my work at the London School of Economics and for all his advices, encouragements and kind words.

During the course of this PhD, I spent two years at the “Institut d’Administration et de Gestion” (IAG) of the Louvain School of Management as a teaching assistant of Professor Philippe Grégoire. I very much enjoyed this couple of years at the IAG and Professor Gregoire deserves all my gratitude for welcoming me in the Finance Unit.

For the rest of my PhD, my work was financed by the Chair in Risk Management awarded by Axa Belgium. I would like to thank Axa Belgium for its financial support and in particular, Mr. Alfred Bouckaert and Mr. Serge Wibaut for the creation of this chair.

I would also like to acknowledge all my friends from inside the academic world, those of the Louvain School of Management, Hervé Van Oppens, Olivier Vercruysse and Tanguy de Launois as well as those of the Institute of Actuarial Sciences, Donatien Hainaut, Jean-Philippe Boucher, Aurélie Miller, Julien Trufin, Cindy Courtois and Xavier Maréchal. I would always remember with great pleasure, the passionate discussions that we had Donatien, Jean-Philippe and me about finance, economics or politics.

Monique Descamps also deserves my warmest thanks for her constant help and availability as well as for the lively atmosphere she brought to the institute.

I also wish to thank all my friends from outside the academic world: Xavier, Benoît, Laure, Christophe, Vincent and Valérie.

Last but not least, I would like to thank all my family and, in particular, my parents for their constant support in the good and the bad times, and for always having been there for me whenever I needed them. How could I ever acknowledge them enough?
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Introduction.

The past three decades have seen a tremendous development of finance theory based on the “no-arbitrage” hypothesis. This simple and natural hypothesis proved to have far-reaching implications in various areas. In particular, it profoundly influenced the actuarial academic research as well as the actuarial practice. It provided new insights and raised new deeper questions regarding the valuation of insurance liabilities and the management of the assets backing these liabilities. This thesis aims at contributing to the study of these questions in particular and to the study of the interplay between finance and actuarial science in general.

This thesis consists of four parts, each devoted to a specific topic, important for the risk management of an insurance company. In the first part, we study the impact of the risk management policy on the pricing, in terms of a guaranteed rate and a participation rate, of classical single premium life insurance contracts with profit. In the second part, we study the management, in terms of evaluation and hedging, of the surrender option. The third part is devoted to the management of life insurance contracts with systematic mortality risk. In the fourth part, we study the asset allocation problem of a non life insurance company when the inflation risk and the interest rate risk are taken into account.

One of the main tools applied in this thesis is the Risk-Minimization Theory of Föllmer and Sondermann [44]. This theory provides, for a given contingent claim, the value of the portfolio and the associated hedging strategy that minimizes a particular quadratic criterion. This theory has been applied in numerous papers in the financial literature as well as in the actuarial literature. Various arguments justify the study of this theory. First, this risk-minimization theory is interesting in itself because it extends, as shown by Møller in [69] and also in Chapter 3 of this thesis, the traditional actuarial premium principle and reserving principle based on the law of large numbers. More importantly, this theory is also interesting because it plays a key role in the solution of various and seemingly unrelated other problems. Firstly, it appears, for example, in other quadratic hedging theories. More specifically, when the financial asset prices are continuous, the local risk-minimization theory of Schweizer can be written as a risk-minimization problem under the so-called minimal martingale measure. Also, finding the mean-variance hedging strategy of a given contingent claim basically comes down, under the same condition, to finding the risk-minimizing strategy of this contingent claim under the so-called variance optimal martingale measure. Secondly, the risk-minimization theory is also a key element in the indifference pricing theory. This is not surprising when the utility function is quadratic since, in this case, the indifference pricing theory comes down to finding the mean-variance value and its hedging strategy. More surprisingly, the risk-minimization theory also appears in the indifference pricing when the utility function is exponential. Indeed, Mania
and Schweizer show in [65] that, in this case and for a small risk aversion, the indifference value of a given contingent claim and its associated hedging strategy are given by, respectively, the risk-minimizing value of the same contingent claim and its hedging strategy under the minimal entropy martingale measure. To conclude, we clearly see that this theory constitutes an important building block in many financial problems. Most of the results derived in this thesis can thus be used, mutatis mutandis, to solve these various problems.

Each chapter of this thesis can be read independently of the others. Repetitions are thus unavoidable. We apologize in advance for this inconvenience.

**Contribution of the dissertation.**

**Part I: Chapter 1.** The main motivation of this first chapter, is to study the pricing of a classical single premium life insurance contract with profit. The pricing is here considered in terms of a guaranteed rate on the premium and a participation rate on the (terminal) financial surplus.

In this chapter, we argue that these technical parameters should be determined by taking explicitly into account the risk the insurer accepts to bear. More precisely, given the asset allocation of the insurer, we should fix these parameters consistently with the risk management policy of the insurance company, in terms of a risk measure such as the value-at-risk or the conditional value-at-risk.

In order to answer these points, we suggest to divide this problem in two steps. In the first one, we consider the insurance contract without profit. Since, given the asset allocation, the financial risk comes only from the guaranteed rate, we propose to fix the guaranteed rate such that the value-at-risk or conditional value-at-risk of this modified contract, does not exceed a certain level chosen by the risk management. In the second step, we fix the participation rate according to the risk neutral valuation principle and according to the guaranteed rate found in step one. In this way, we have a contract that is simultaneously fairly priced and that exhibits, given the asset allocation, a risk consistent with the risk management policy.

We also propose in this chapter, a model for the financial portfolio that includes investment in cash, stocks and bonds. Different kinds of bond strategies are especially developed in order to model the influence of the matching policies.

Finally, we illustrate numerically our methodology and study the effects of the matching policy and the effects of the strategic asset allocation on the technical parameters.

**Part II: Chapters 2, 3 and 4.** In this second part, we focus on the management of the surrender option embedded in most life insurance contracts. More precisely, in Chapter 2, we study the valuation of insurance contracts with a surrender option and in Chapter 3 and 4, we study the hedging strategies of such insurance contracts.

In the actuarial literature, the time of surrender is usually modelled as an (optimal) stopping time with respect to the filtration generated by the financial asset prices. In such a model, if the financial market is initially complete, an insurance contract with a surrender option whose exercise depends on such a surrender time, can be perfectly hedged. The surrender risk is thus not a genuine additional risk in this framework.
In Chapter 2, we argue that, if we are to model a realistic surrender time, we should avoid following this traditional way. On the contrary, we argue we should try to model this surrender time as a random time not adapted to the filtration generated by the financial assets prices. We follow here the financial literature on the default risk and in particular, the reduced-form models. We discuss at length the pros and cons of the traditional methodology and justify the reduced-form approach by the asymmetry of information between the insurer and the policyholders. We also give a simple illustration of this approach. This second chapter should be seen as a methodological one; we mainly recall some well-known results from the financial literature on default-risk that, we think, are tailored for the modelling of the surrender time.

When the surrender time is modelled as described in Chapter 2, an insurance contract with a surrender option cannot be perfectly hedged even if the financial market is initially complete. It is thus important for an insurer to study the hedging strategies that would reduce its exposure to this surrender risk.

In Chapter 3, we study the risk-minimizing strategies of such contracts. More precisely, in a first step, we describe the risk-minimizing strategies for a single insurance contract. We show that finding these strategies basically comes down to finding the risk-minimizing strategies of properly modified purely financial claims. Then, in a second step, we use this result to describe the risk-minimizing strategies for a portfolio of $n$ insurance contracts. The important characteristic of the surrender option is that it introduces an additional systematic risk since the surrender times of the different policyholders depend on the evolution of the financial market and are thus not independent of each other. However, we show that, even though the surrender risk represents an additional systematic risk for the insurer, we can combine continuous hedging and diversification to reduce the overall risk of the portfolio (down to a limit related to the degree of incompleteness of the financial market). In this way, we generalize previous results obtained by Møller [68] and Riesner [72] when the insurance payments depend on random times (the times of death of the policyholders in these papers) independent of the financial market and independent of each others.

In Chapter 4, we continue our investigation and focus on the “locally risk-minimizing” strategies. By contrast to the risk-minimization theory described in the previous chapter, the local risk-minimization theory has only been developed for a single payment at a fixed time. The first contribution of this chapter is to extend the local risk minimization theory to payment processes. We show that we can follow the same method than Møller [68] when he extended the risk-minimization theory to payment processes. We then apply this extended theory to insurance contracts with surrender option and find the form of the locally risk-minimizing strategies. We distinguish between two cases. In the first one we assume the so-called $(H)$-hypothesis holds and in the second one, we remove this hypothesis. Intuitively, this $(H)$-hypothesis implies that the observation of the random time does not alter the dynamics of the financial market that has been initially defined. As in the previous chapter, we show that finding the locally risk-minimizing strategies comes down, in both cases, to finding the locally risk-minimizing strategies of modified purely financial claims, though, these claims are different according as the $(H)$-hypothesis holds or not. As a by-product, we also study the impact of a progressive enlargement of filtration on the so-called “minimal martingale measure".
Part III: Chapters 5 and 6. The third part is devoted to the systematic mortality risk. Due to its systematic nature, this risk cannot be diversified through increasing the portfolio. It is thus also important to study the hedging strategies an insurer should follow to mitigate its exposure to this risk.

In the first chapter, we study the risk-minimizing strategies for a life insurance contract when no mortality-linked financial assets are traded on the financial market. This problem has already been studied by Dahl and Møller [39]. Under some simplifying assumptions, they find closed-form solutions for the risk-minimizing strategies and the cost process (i.e. the unhedgeable part) of some insurance payments. In this chapter, we extend their results in at least three respects. Firstly, as far as the financial market is concerned, Dahl and Møller assume a complete market with only two traded assets, a saving account and a single zero-coupon bond with maturity $T$. The instantaneously risk-free rate is modelled as a time-homogeneous single factor affine model of the CIR type. In this chapter, we only assume the (discounted) financial assets prices follow an arbitrary $s$-dimensional local martingale. We do not assume the financial market is necessarily complete. Secondly, Dahl and Møller assume the insurance payments, given the times of death of their policyholders, are deterministic. We assume here that the insurance payments, given the time of death, can be stochastic and depend on the evolution of the financial market. Since our financial market is not necessarily complete, these payments cannot necessarily be perfectly hedged. Finally, in Dahl and Møller, the probability distribution function of the random time of death is described through its instantaneous mortality rate which is assumed to be stochastic and to follow a time-inhomogeneous single factor model of the CIR type. In this chapter, we actually start from an (almost) arbitrary stochastic process that describes the dynamics of the whole survival probability function but without giving the specific form of this probability function or the specific form of the intensity of the random time of death. This model allows us to distinguish two situations. In the first one, the $(H)$-hypothesis is assumed to hold whereas in the second situations we relax this assumption. To my knowledge, this $(H)$-hypothesis actually holds implicitly in every current stochastic mortality model.

In this general setting, we give the form of the risk-minimizing strategies and the cost process. The main contribution of this chapter is to provide a clear and intuitive explanation for the “risk-minimizing” strategies. Indeed, we show that the “risk-minimizing” strategies can be written as an average of “risk-minimizing” strategies of purely financial claims, weighted by the stochastic survival probability distribution function. The important corollary of this result is that, finding the risk-minimizing strategy of a life insurance contract can be separated in two independent problems, one purely actuarial and related to the modelling of the survival probabilities and one purely financial related to the hedging of financial claims.

In the second chapter, we introduce mortality-linked financial assets. More precisely, we want to study the application of the HJM methodology to the modelling of a longevity bonds market. The idea of using the HJM methodology for modelling the longevity bonds prices is not new in the actuarial literature. In this second chapter, we give a critical review of this literature and show that previous models actually do not define properly neither the prices nor the payoffs of the longevity bonds. Therefore, we try to tackle this problem. The first contribution of this chapter is to describe a coherent theoretical setting in which we can properly define the longevity
Introduction.

A second objective is to describe a more realistic longevity bonds market model than in cited papers. In particular, we introduce an additional effect of the actual mortality on the longevity bond prices, that does not appear in the literature. We also introduce multiple term structures of longevity bonds instead of the usual single term structure. Finally, we introduce non-homogeneous longevity bonds. In this framework, we derive the no-arbitrage condition for the longevity bonds financial market. We also discuss the links between such HJM based model and the intensity models for longevity bonds such as those of Dahl [37] (2004), Biffis [29] (2005), Luciano and Vigna [64] (2005), Schrager [74] (2006) and Hainaut and Devolder [49] (2007). Our results suggest that the standard pricing formula of these intensity models could be extended to more general settings.

Finally, as a last contribution of this chapter, we study the asset allocation problem of pure endowments and annuities portfolios. In order to solve this problem, we again study the “risk-minimizing” strategies of such portfolios, when some but not all longevity bonds are available for trading. In this way, we introduce different basis risks. As a by-product, we also found the fair values of these liabilities when such a longevity bonds market exists.

Part IV: Chapter 7. Finally, the fourth part deals with the design of ALM strategies for a non-life insurance portfolio. In particular, this chapter aims at studying the asset allocation problem of a non-life insurance company when inflation risk and interest rate risk are taken into account. The academic literature on this topic is rather scarce. There are however good arguments to study the asset allocation problem in non life insurance. Indeed, the cost of claims is highly sensitive to inflation, which, in turn, is itself correlated with the financial assets returns and in particular, with the interest rates movements. Moreover, the reserve an insurer should maintain, computed as the expected discounted claims, is subject to interest rate risk through the variation in the discounting term and indirectly through the variation in expected outstanding claims due to the inflation. The application of ALM techniques in non-life insurance is thus justified from a theoretical point of view.

Again, to study this asset allocation problem, we focus on the “risk-minimizing” strategies. We first derive the general form of these strategies when the cumulative payments of the insurer are described by an arbitrary increasing process adapted to the natural filtration of a general marked point process and when the inflation and the term structure of interest rates are simultaneously described by the HJM model of Jarrow and Yildirim [57].

We then systematically apply this result to four specific models of insurance claims. We first study two “collective” models. In the first one, we consider a portfolio with a constant number of policies. In the second one, the number of policies is stochastic and follows a point process with an inhomogeneous and stochastic intensity. In each case, the cost of claim, for a given policy, follows a compound Poisson process. We then study two “individual” models where the claims are notified at a random time and settled through time. In the first model, the claim, once notified, is settled according to a deterministic function. In the second model, the claim is settled according to a Beta-Stacy process.


Part 1

Life Insurance Contracts and the Interest Rate Risk.
CHAPTER 1

Risk Measure and Fair Valuation of an Investment Guarantee in Life Insurance.

1. Introduction.

Investment guarantees embedded in classical life insurance contracts are surely nowadays one of the most important challenges for the insurance industry. In order to cope with this risk two big paradigms seem to be available for the actuary:

1. the risk neutral approach: under classical assumptions on the market (completeness, no arbitrage,...) the price of the guarantee is associated to the price of an option and the machinery of option pricing can be used in order to valuate the product. This philosophy has been used with success for equity linked contracts with maturity guarantee (see for instance Brennan /Schwartz [34], Delbaen [40], Aase/Persson [2], Nielsen/Sandmann [70]) as well as for life insurances with profit (see for instance Bacinello [14, 15], Grosen/Jorgensen [47]). In this context computations take place under a risk neutral probability measure. We can consider these models as directly inspired by modern finance theory.

2. the risk management approach: modern tools of simulation permit more and more to generate a lot of future scenarios and to build the distribution of the final surplus of the insurer taken into account stochastic financial models. Parameters of the contracts are then fixed in relation with a certain level of solvency. Value at risk concept or more generally risk measures are the central tools (Artzner et al. [10, 11], Wirch/Hardy [83]). Of course computations are done in the real historical world. These models directly linked with DFA (Dynamic Financial Analysis developed as well in life as in non life insurance-see for instance Kaufmann et al. [59]) are in fact close to the classical risk theory and the actuarial well-known concept of probability of ruin.

These two methodologies have already been compared in terms of reserving (Boyle/Hardy [31]). The purpose of this chapter is to propose a way to combine these two methodologies in order to fix the technical parameters of a classical life insurance product with profit. In this context we suppose that the insurer guarantees a certain fixed premium rate. On top of that, participation is given at maturity if the real investment performances are good. This bonus is expressed as a rate applied to the eventual final surplus of the insurer. So the decision problems for the insurer in this kind of product consists of choosing two numbers: the guaranteed rate and the participation rate. In order to proceed we propose to decompose the problems in two steps:
• First step: in our mind the choice of a guaranteed rate is directly linked with solvency concerns. Perfect hedging for long periods as in life or pensions contracts is a utopia. Derivative instruments for very long periods as in life or in pension insurance are not common on the markets and self-hedging is too expensive. A risk neutral price could give the illusion of an absence of risk like in the Black and Scholes world but in fact risk is still there at maturity if you do not hedge perfectly as requested by the underlying pricing process. All this motivates the use of a risk management approach in order to fix the guaranteed rate for long periods.

• Second step: once this technical rate is chosen and in order to fix the second parameter - the participation rate- we can try to build a fair contract to the policyholder and the insurer. Risk neutral fairness is then used to compute the value to the participation rate.

This way, combination of the two paradigms permits to take into account at the same time solvency concerns as well as fair valuation principles.

The chapter is organized as follows. Section 2 presents the main assumptions on the financial market and the investment strategies. In this context we develop a model with cash, bonds and stocks (Cash-Bond-Stock model - CBS model hereafter) representing better than other simple models (for instance binomial or geometric Brownian) the fundamental investments used by the insurers as underlying values for life insurance products with profit. Different kinds of bond strategies are developed in order to model the influence of matching policies. Section 3 is then devoted to the risk management analysis in order to select the rate that can be guaranteed. Explicit solutions in the CBS model are given using a “value at risk” approach; not surprisingly the rate is a function of the required level of solvency. Formulas are also given if we choose as criterion the expected shortfall instead of the value at risk. In section 4 fair valuation principles are then applied in order to determine the equilibrium value of the participation rate corresponding to the “Risk management” value of the guarantee. Forward risk neutral measure is used to price the contract. Finally Section 5 presents some numerical illustrations.

2. Cash-Bond-Stock Model (CBS).

We assume a single premium is paid at time $t_0 = 0$. The insurer invests this premium in assets traded on a financial market. The value of this financial portfolio at date $t$ is denoted by $V(t)$ and changes according to the evolution of the assets values and the financial strategy chosen by the insurer. In the next section, we describe the financial market on which the insurer can trade. In the next one, we describe the financial portfolio.

2.1. The Financial Market. We consider a financial market with three different kinds of assets: A money market account, a risky asset and zero-coupon bonds of all maturities. The assumptions related to our financial market are the following:

• Assumption 1: The money market account is a financial asset with no instantaneous risk. The value of this asset, $\beta(t)$, grows at the instantaneous risk free rate $r(t)$ according to the following differential equation:

(2.1) \[ d\beta(t) = r(t)\beta(t)dt \]
with initial value $\beta(0) = 1$.

- **Assumption 2:** We assume the risk free rate follows an Ornstein-Uhlenbeck process. The risk free rate is then the solution of the following stochastic differential equation:

$$dr(t) = a(b - r(t)) \, dt + \sigma_r dW_1(t)$$

where $W_1(t)$ is a standard Brownian motion.

- **Assumption 3:** The value of the risky asset, $S(t)$, follows a geometric Brownian motion, i.e., it is the solution of the following stochastic differential equation:

$$dS(t) = S(t) \left[ \mu dt + \sigma_S \rho dW_1(t) + \sigma_S \sqrt{1 - \rho^2} dW_2(t) \right]$$

where $W_2(t)$ is a standard Brownian motion, independent of $W_1(t)$. The correlation between the value of the risky asset and the risk free rate is characterized by the parameter $\rho$.

- **Assumption 4:** We assume no arbitrage opportunity.

This last assumption together with assumptions 1 and 2 allows us to find the process followed by the value of any zero-coupon bond. It is well known that under these assumptions, the value $P(t, T)$, at time $t$ of a zero coupon bond with maturity $T$ is given by:

$$dP(t, T) = P(t, T) \left[ (r(t) - \lambda \sigma_r B(T - t)) \, dt - \sigma_r B(T - t) dW_1(t) \right]$$

where

$$B(T - t) = \frac{1}{a} \left[ 1 - e^{-a(T-t)} \right]$$

and $t \leq T$. The parameter $\lambda$ represents the price of the interest rate risk. The term $-\lambda \sigma_r B(T - t)$ represents the instantaneous risk premium for investing in a zero coupon bond with time-to-maturity $T - t$. Thus, $-\lambda$ can be understood as the price of risk for investing in the bond market.

Let us point out once more that the values of our assets are all correlated. The values of the zero-coupon bonds are all perfectly correlated with each other. They are all perfectly negatively correlated with the risk free rate and the later is also correlated with the risky asset according to the value of the parameter $\rho$.

### 2.2. The financial portfolio

Our financial portfolio is invested in the three different kinds of assets described above:

- A proportion $x_a$ is invested in the risky asset with price $S(t)$ which can be seen as a stock or a stock index.
- A proportion $x_b$ is invested in zero-coupon bonds.
- A proportion $x_c$ is invested in the money market account which can be seen as cash (or almost cash).

Additional assumptions related to the portfolio are made:

- **Assumption 5:** The portfolio is continuously rebalanced in order to keep these proportions constant over time.
Since the values of the different assets evolve over time, if the portfolio is not rebalanced, the proportions invested in these assets will fluctuate. In practice, fund managers try indeed to respect a pre-specified asset allocation.

- **Assumption 6**: The portfolio is self-financing.

We assume that once the initial premium is paid, no money comes in or comes out of the portfolio up to the term of the guarantee. One of the main purposes of this chapter is to study the impact of the financial strategy in general and the bond strategy in terms of matching of duration in particular, on the value of the portfolio at maturity. So it is important to design a fairly general framework for the bonds management that embeds different kinds of strategies.

- **Assumption 7**: At each time \( t \), Zero-coupon bonds of only one maturity can be held. We do not allow the portfolio to be invested in zero-coupon bonds of different maturities at the same time.

This assumption is only made for the sake of simplicity. At the cost of more fastidious calculations, we could have assumed our bond portfolio is invested in zero coupon bonds with several maturities.

- **Assumption 8**: Let \( s = 0, \ldots, n - 1 \) with \( t_0 = 0 \) and \( t_n = T \). We assume that the \( t_s \) are the different dates at which we can invest in new zero coupon bonds with time-to-maturity \( T_s \geq t_{s+1} - t_s \). In other words, at time \( t_s \) the insurer can sell all the zero-coupon bonds with maturity \( (t_s - 1 + T_{s-1}) \) to invest in zero-coupon bonds with maturity \( (t_s + T_s) \).

We are now ready to describe the evolution of the value of our portfolio over time and in particular the value at maturity.

2.2.1. *The value of the financial portfolio*. The value \( V(t) \) of the portfolio at time \( t \in [0, T] \), is given by:

\[
V(t) = a_t S(t) + \sum_{s=0}^{n-1} b^t_s I_{t \in [t_s, t_{s+1}]} P(t, T_s + t_s) + c_t \beta(t)
\]

where \( a_t, b^t_s, c_t \) are respectively, the number of shares of risky assets, the number of zero-coupon bonds with maturity \( (t_s + T_s) \) and finally, the number of shares of money market account. The assumption 6 (the self-financing assumption) allows us to write the evolution of the value, for \( t \in [0, T] \), of our portfolio under the following differential form:

\[
dV(t) = a_t dS(t) + \sum_{s=0}^{n-1} b^t_s I_{t \in [t_s, t_{s+1}]} dP(t, T_s + t_s) + c_t d\beta(t)
\]

The assumption 5 which states the proportions invested in each type of asset remain constant over time, gives us the following relations:
Risk Measure and Fair Valuation of an Investment Guarantee in Life Insurance.

\[ a_t = \frac{x_a V(t)}{S(t)} \]

\[ b_{t}^{s} = \frac{x_b V(t)}{P(t, T_{s} + t_{s})} \]

\[ c_t = \frac{x_c V(t)}{\beta(t)} \]

Using these relations into Equation (2.6), we find the following differential equation:

\[ dV(t) = \frac{x_a V(t)}{S(t)} dS(t) + \sum_{s=0}^{n-1} \frac{x_b V(t)}{P(t, T_{s} + t_{s})} I_{t \in [t_{s}, t_{s+1}]} dP(t, T_{s} + t_{s}) + \frac{x_c V(t)}{\beta(t)} d\beta(t) \]

Or equivalently, we can rewrite this formula as:

\[ dV(t) \]

\[ V(t) = x_a \frac{dS(t)}{S(t)} + x_b \frac{dP(t, T_{s} + t_{s})}{P(t, T_{s} + t_{s})} + x_c \frac{d\beta(t)}{\beta(t)} \]

for \( t \in [t_{s}, t_{s+1}] \). Without loss of generality, we can assume the initial value \( V(0) \) of the portfolio is equal to 1.

2.2.2. The portfolio value at maturity. We are interested in the value of the portfolio at maturity, \( V(T) \). The self financing assumption allows us to write the return of the portfolio as the product of the returns over the \( n \) different sub periods:

\[ V(T) = \frac{V(T)}{V(0)} = \prod_{s=0}^{n-1} \frac{V(t_{s+1})}{V(t_{s})} \]

Taking the logarithm, we get:

\[ \ln V(T) - \ln V(0) = \sum_{s=0}^{n-1} [\ln V(t_{s+1}) - \ln V(t_{s})] \]

In order to determine the distribution of \( V(T) \), we start by determining the distribution of each term of this sum conditionally on the information known at time 0, \( \mathcal{F}_0 \), (the distribution of \( \ln V(t_{s+1}) - \ln V(t_{s}) | \mathcal{F}_0 \)). In the following, let us fix a time \( t \in [t_{s}, t_{s+1}] \). In this interval, we know the value of portfolio follows the differential equation (2.7):

\[ \frac{dV(t)}{V(t)} = x_a \frac{dS(t)}{S(t)} + x_b \frac{dP(t, T_{s} + t_{s})}{P(t, T_{s} + t_{s})} + x_c \frac{d\beta(t)}{\beta(t)} \]

Inserting Equations (2.1), (2.3) and (2.4), we get:
\[
\frac{dV(t)}{V(t)} = x_a \left( \mu dt + \sigma_S \rho dW_1(t) + \sigma_S \sqrt{1 - \rho^2} dW_2(t) \right) + x_b \left[ (r(t) - \lambda \sigma_r B(T_s + t_s - t)) dt - \sigma_r B(T_s + t_s - t) dW_1(t) \right] + x_c r(t) dt
\]
\[
= (x_a \mu - x_b \lambda \sigma_r B(T_s + t_s - t) + (x_b + x_c) r(t)) dt + (x_a \sigma_S \rho - x_b \sigma_r B(T_s + t_s - t)) dW_1(t) + \left( x_a \sigma_S \sqrt{1 - \rho^2} \right) dW_2(t)
\]

So that, we have

\[
\ln V(T_s + 1) - \ln V(t_s) = \int_{t_s}^{t_s+1} d\ln V(u)
\]
\[
= \int_{t_s}^{t_s+1} x_a \mu - x_b \lambda \sigma_r B(T_s + t_s - u) du + \int_{t_s}^{t_s+1} (x_b + x_c) r(u) du
\]
\[
- \frac{1}{2} \int_{t_s}^{t_s+1} x_a^2 \sigma_S^2 + x_b^2 \sigma_r^2 B(T_s + t_s - u)^2 - 2x_a x_b \sigma_S \sigma_r \rho B(T_s + t_s - u) du
\]
\[
+ \int_{t_s}^{t_s+1} x_a \sigma_S \rho - x_b \sigma_r B(T_s + t_s - u) dW_1(u) + \int_{t_s}^{t_s+1} x_a \sigma_S \sqrt{1 - \rho^2} dW_2(u)
\]

The difference between the logarithms of the value of the portfolio is then equal to the sum of four terms, the first one being deterministic and the three others random. Furthermore, it is easy to see they are all normally distributed. So the difference \( \ln V(T_s + 1) - \ln V(t_s) \) also follows a normal random variable. Since this is equally true for each \( s = 0, 1, \ldots, n - 1 \), we have the following proposition:

**Proposition 2.1.** The logarithm of the value of the portfolio at maturity

\[
\ln V(T) = \sum_{s=0}^{n-1} [\ln V(t_{s+1}) - \ln V(t_s)]
\]

follows a normal random variable with expectation given by:

\[
\mu_P(T) = E[\ln V(T) | \mathcal{F}_0]
\]
\[
= \sum_{s=0}^{n-1} E[\ln V(t_{s+1}) - \ln V(t_s) | \mathcal{F}_0]
\]

and variance given by:
\[ \sigma_P (T) = \text{Var} [\ln V(T) | \mathcal{F}_0] \]
\[ = \sum_{s=0}^{n-1} \text{Var} [\ln V(t_{s+1}) - \ln V(t_s) | \mathcal{F}_0] \]
\[ + 2 \sum_{s=0}^{n-2} \sum_{j=s+1}^{n-1} \text{Covar} [\ln V(t_{s+1}) - \ln V(t_s); \ln V(t_{j+1}) - \ln V(t_j) | \mathcal{F}_0] \]

So knowing the expectations, the variances and the covariances of the differences of the logarithms of the portfolio value over the different sub periods, we can easily find the distribution of the portfolio value at the term of the guarantee. These expectations, variances and covariances are given by the following formulas. (Detailed calculations can be found in Appendix 1).

Expectation of \( \ln V(t_{s+1}) - \ln V(t_s) \):

\[ E [\ln V(t_{s+1}) - \ln V(t_s) | \mathcal{F}_0] = K(T_s, t_{s+1} - t_s) \]
\[ + (x_b + x_c) \left[ b (t_{s+1} - t_s) + \frac{1}{a} (b - r(0)) e^{-at_s} (e^{-a(t_{s+1} - t_s)} - 1) \right] \]

where

\[ K(T_s, t_{s+1} - t_s) = \int_{t_s}^{t_{s+1}} x_a \mu - x_b \lambda \sigma_r B(T_s + t_s - u) du \]
\[ - \frac{1}{2} \int_{t_s}^{t_{s+1}} x_a^2 \sigma_S^2 + x_b^2 \sigma_r^2 B(T_s + t_s - u)^2 - 2x_a x_b \sigma_S \sigma_r B(T_s + t_s - u) du \]
\[ = \left( x_a \mu - \frac{1}{2} x_a^2 \sigma_S^2 \right) (t_{s+1} - t_s) - (x_b \lambda \sigma_r - x_a x_b \sigma_S \sigma_r \rho) IB_1(T_s, t_s) - \frac{1}{2} x_b^2 \sigma_r^2 IB_2(T_s, t_s) \]

\[ IB_1(T_s, t_s) = \int_{t_s}^{t_{s+1}} B(T_s + t_s - u) du \]
\[ = \frac{1}{a} \left[ (t_{s+1} - t_s) + \frac{e^{-aT_s}}{a} \left( 1 - e^{a(t_{s+1} - t_s)} \right) \right] \]
Var \[ IB_2(T_s, t_s) \] = \( \int_{t_s}^{t_{s+1}} B(T_s + t_s - u)^2 du \)
\[
= \frac{1}{a^2} \left[ (t_{s+1} - t_s) - \frac{e^{-2aT_s}}{2a} \left( 1 - e^{-2a(t_{s+1} - t_s)} \right) + \frac{2e^{-aT_s}}{a} \left( 1 - e^{a(t_{s+1} - t_s)} \right) \right]
\]

Variance of \( R_{t_s} \) = \( \ln V(t_{s+1}) - \ln V(t_s) \):

\[
\text{Var} \left[ R_{t_s} | F_0 \right] = (x_b + x_c)^2 \frac{\sigma_r^2}{a^2} \left( t_{s+1} - t_s + \frac{1}{2a} (e^{-a(t_{s+1} - t_s)} - 1)^2 (1 - e^{-2a t_s}) \right)
+ (x_b + x_c)^2 \frac{\sigma_r^2}{a^2} \left( \frac{1}{2a} (1 - e^{-2a(t_{s+1} - t_s)}) - \frac{2}{a} (1 - e^{-a(t_{s+1} - t_s)}) \right)
+ (x_b \sigma_r)^2 IB_2(T_s, t_s) - 2x_a x_b \sigma_S \sigma_r \rho IB_1(T_s, t_s) + (x_a \sigma_S)^2 (t_{s+1} - t_s)
+ 2 (x_b + x_c) \left( \frac{x_a \sigma_S \sigma_r}{a} \left[ (t_{s+1} - t_s) - \frac{1}{a} (1 - e^{-a(t_{s+1} - t_s)}) \right] - \frac{x_a^2 \sigma_r^2}{2a} IB_1(T_s, t_s) \right)
+ 2 (x_b + x_c) \frac{x_b \sigma_r^2}{a^2} \left( \frac{1}{a} (1 - e^{-a(t_{s+1} - t_s)}) - \frac{e^{-aT_s}}{2a} \left( e^{a(t_{s+1} - t_s)} - e^{-a(t_{s+1} - t_s)} \right) \right)
\]

Covariance between \( R_{t_i} = \ln V(t_{i+1}) - \ln V(t_i) \) and \( R_{t_j} = \ln V(t_{j+1}) - \ln V(t_j) \):

\[
\text{Covar} \left[ R_{t_i} ; R_{t_j} | F_0 \right] = (x_b + x_c)^2 \frac{\sigma_r^2}{2a^3} \left( 1 - e^{-a(t_{i+1} - t_i)} \right) \left( 1 - e^{-a(t_{j+1} - t_j)} \right) \left( e^{-a(t_j - t_i)} - e^{-a(t_{i+1} - t_j)} \right)
+ (x_b + x_c) \left( \frac{\sigma_r^2}{a^3} \left( 1 - e^{-a(t_{j+1} - t_j)} \right) \left( e^{-a(t_j - t_i)} - e^{-a(t_{i+1} - t_j)} \right) \right)
+ (x_b + x_c) \frac{\sigma_r^2}{a^2} \left( 1 - e^{-a(t_{j+1} - t_j)} \right) \left( \frac{x_a \sigma_S \sigma_r}{a} \left( e^{-a(t_j - t_i)} - e^{-a(t_{j+1} - t_j)} \right) \right)
- (x_b + x_c) \frac{\sigma_r^2}{a} \left( 1 - e^{-a(t_{j+1} - t_j)} \right) \left( \frac{x_b \sigma_r}{a^2} \left( e^{-a(t_j - t_i)} - e^{-a(t_{j+1} - t_j)} \right) \right)
+ (x_b + x_c) \frac{\sigma_r}{a} \left( 1 - e^{-a(t_{j+1} - t_j)} \right) \frac{e^{-aT_i}}{2a} \left( e^{a(t_{i+1} - t_i)} - e^{-a(t_{i+1} - t_j)} \right)
\]

for \( t_j \geq t_{i+1} \).

2.2.3. The Investment Strategies. The bond strategy described in the preceding section, is too general to be really useful. We will restrict our attention to a few special cases. In this section, we present the three bond strategies we choose to analyze.

- **Strategy 1:** It consists in investing in zero coupon bonds with a time-to-maturity equal to the term of guarantee. We suppose we keep them up to this term. This strategy corresponds to the special case where \( n \) is equal to one and where \( T_1 \) is equal to \( T \).
- **Strategy 2:** It consists in investing initially in zero-coupon bonds with a time-to-maturity inferior to the term of the guarantee. We assume we keep these bonds up to their maturity. Then we reinvest in zero coupon bonds with the same time-to-maturity than the first ones. Again, we keep them up to their maturity and so on to the term
of the guarantee. This strategy corresponds to the special case where $T_s$ is equal to $T_1$ for all $s = 0, 1, \ldots, n - 1$ and $T_1 = T/n$.

- **Strategy 3:** It consists in investing initially in zero-coupon bonds with a time-to-maturity inferior or superior to the term of the guarantee. After a period of time $\Delta t$, we sell these bonds and reinvest in other zero coupon bonds with the same time-to-maturity as the first. Again, we keep them during a period of time $\Delta t$, and at this time, we sell them and reinvest in other zero coupon bonds, and so on up to the term of the guarantee. This strategy corresponds to the special case where $T_s$ is equal to $T_1$ for all $s = 0, 1, \ldots, n - 1$ and $\Delta t = T/n < T_1$.

### 3. Investment Guarantee and Risk Measures.

For the insurer, the risk comes from the possibility of having a financial portfolio value at time $T$ lower than the liability value $L(T)$ ("downside risk"). In order to measure this kind of risk, a number of risk measures have been developed in the literature. In our setting, the value of a risk measure will depend on the guaranteed rate and the financial strategy. Notice the participation rate here does not induce any additional risk. The whole risk comes only from the level of the guaranteed rate and the financial strategy. So given these elements, we can find the value of the risk measure. But the reverse is also true: given a risk measure, a maximal accepted value for this risk measure and a financial strategy, we can find the maximal interest rate we can guarantee on a single premium.

In the next sections, we analyze in more detail these relations for two well-established risk measures, the value at risk and the expected shortfall.

#### 3.1. Investment Guarantee.

As already said, we assume a single premium is paid at date 0. The insurer guarantees a minimum annual return $i$ on this single premium at time $T$. Without loss of generality, we can assume the premium value is 1. Accordingly, the liability value at date $T$ is equal to $L(T) = (1 + i)^T$.

#### 3.2. Value at Risk.

This is historically the first risk measure described in the literature. It is defined as the supremum discounted loss we can experience with a probability not less than a given threshold $\epsilon$. Mathematically, for given $V(T)$, $L(T)$ and $\epsilon$, the value at risk is defined as the value VaR such that

$$\text{VaR} = -\inf \left\{ x \left| \Pr \left[ V(T) - L(T) \leq \frac{x}{P(0, T)} \right] \geq \epsilon \right\} \right.$$

In our setting, we can easily find an explicit solution for the VaR:
Pr \left[ \ln V(T) \leq \ln \left( \frac{-VaR}{P(0,T)} + L(T) \right) \right] = \epsilon \\
Pr \left[ Z \leq \frac{\ln \left( \frac{-VaR}{P(0,T)} + L(T) \right) - \mu_P(T)}{\sigma_P(T)} \right] = \epsilon \\
\ln \left( \frac{-VaR}{P(0,T)} + L(T) \right) = \mu_P(T) + \sigma_P(T)\phi^{-1}(\epsilon) \\
VaR = P(0,T) \left[ L(T) - \exp \left[ \mu_P(T) + \sigma_P(T)\phi^{-1}(\epsilon) \right] \right]

where \( Z \) is a standard normal random variable and \( \phi \) is the cdf of \( Z \); hence \( \phi^{-1}(\epsilon) \) is the \( \epsilon \)-percentile of the standard normal distribution. Finally, we get:

\( (3.1) \quad VaR = P(0,T) \left[ (1 + i)^T - \exp \left[ \mu_P(T) + \sigma_P(T)\phi^{-1}(\epsilon) \right] \right] \)

For a given value at risk, a given strategy and a given guaranteed rate, we can also determine the probability \( \epsilon \). The two most natural values for the VaR are 0 and the amount of capital set aside for this contract, assuming we invest this capital in a risk free asset. In the former case, \( \epsilon \) reduces to the well-known probability of ruin. In our setting, this probability \( \epsilon \) is given by:

\( (3.2) \quad \epsilon = \phi \left( \frac{\ln \left( \frac{-VaR}{P(0,T)} + L(T) \right) - \mu_P(T)}{\sigma_P(T)} \right) \)

The risk measures have been initially developed for risk management purposes. But here, we could extend their uses to a pricing purpose. Indeed, for a given VaR (or equivalently a given probability \( \epsilon \)) and a given financial strategy (on which \( \mu_P(T) \) and \( \sigma_P(T) \) depend), we can determine the corresponding value of the guaranteed rate \( i \). In our setting, this guaranteed rate is given by the formula:

\( (3.3) \quad \overline{i}_\epsilon = \left\{ \frac{VaR}{P(0,T)} + \exp \left[ \mu_P(T) + \sigma_P(T)\phi^{-1}(\epsilon) \right] \right\}^{\frac{1}{T}} - 1 \)

This has the advantage the guaranteed rate is consistent with the risk management strategy.

3.3. Expected Shortfall or CoValue at Risk. Other well-known risk measures are the expected shortfall (ES) and the conditional value at risk (CVaR). When the distributions are continuous, these measures coincide. The CVaR is defined as the discounted expected loss if the expected loss is higher than a given threshold \( R \). Mathematically, given \( V(T) \), \( L(T) \) and \( R \), the conditional value at risk is defined as:

\[ CVaR = E \left[ -P(0,T)(V(T) - L(T)) \ \mid \ V(T) - L(T) \leq -\frac{R}{P(0,T)} \right] \]
In the literature, $R$ is actually the Value at Risk hence the name conditional value at risk. The most natural value for $R$ is probably 0. In this case, the CVaR is simply the discounted expected loss if we are in loss.

In our setting, we can find the following explicit formula for the CVaR:

\[
CVaR = P(0, T) \frac{(1 + i)^T \phi(D_P) - e^{(\mu_P(T) + 0.5\sigma_P^2(T))}\phi(D_P - \sigma_P(T))}{\phi(D)}
\]

where

\[
D_P = \frac{\ln\left(-\frac{R}{P(0, T)} + (1 + i)^T\right) - \mu_P(T)}{\sigma_P(T)}
\]

Even if this measure has also been initially developed for risk management purposes, we can use it for a pricing purpose. Given $R$, the expected shortfall and a financial strategy, we can find a guaranteed rate which is consistent with the risk management strategy defined by the insurer. There is no explicit formula for this guaranteed rate but we can easily invert numerically the last one.

4. Fair Valuation and Participation Rate.

4.1. Participation rate: Classical life insurance products are based on the one hand on a guaranteed technical rate and on the other hand on the distribution of a part of the surplus if any (participation process). If we consider for instance the following assumptions:

- contract with single premium $P = 1$ paid at time $t = 0$
- no mortality assumption
- liability to pay at maturity $t = T$
- guaranteed technical rate $i$
- participation rate $\eta \geq 0$, on the final surplus (terminal bonus)

then the fair value of the contract at maturity can be written as:

\[
FV(T) = L(T) + \eta \max(V(T) - L(T); 0)
\]

where $L(T) = (1 + i)^T$ is the guarantee and $V(T)$ is the final value of the underlying portfolio.

Two parameters have to be fixed by the insurer: the guaranteed rate and the participation rate. As seen before a way to define the guaranteed rate is based on risk management principles, for instance by using the VaR measure.

How to fix then the participation rate? A possible approach is to rely on fair valuation principles. Consider for this the market value of the contract at time $t = 0$ denoted by $FV(0)$. It seems fair to impose that this value coincides with the premium paid at time $t = 0$ (classical actuarial equivalence principle). In order to compute this fair value we can try to use the classical risk neutral approach; in our setting markets are complete and arbitrage free, hence we can express the initial pricing of the product as:
(4.2) \[ FV(0) = (1 + i)^T P(0, T) + \eta P(0, T) E_Q \{ (V(T) - (1 + i)^T)^+ \} \]

where \( Q_T \) is the \( T \)-forward risk neutral measure.

So if \( i_\epsilon \) is the guaranteed rate determined following the principles of Section 3 and for a level \( \epsilon \), then the corresponding equilibrium value of the participation rate is given by:

\[
\bar{\eta} = \frac{1 - (1 + i_\epsilon)^T P(0, T)}{P(0, T) E_Q \{ (V(T) - (1 + i_\epsilon)^T)^+ \}} = \frac{1 - (1 + i_\epsilon)^T P(0, T)}{C(0, (1 + i_\epsilon)^T)}
\]

where \( C(0, (1 + i_\epsilon)^T) \) is the price at time 0 of a European call option on the underlying portfolio with maturity \( T \) and strike \( (1 + i_\epsilon)^T \). Since the price of a call option is always positive, in order to have the participation rate positive, we should have

\[(1 + i_\epsilon)^T P(0, T) \leq 1 \iff i_\epsilon \leq R(0, T)\]

where \( R(0, T) \) is the annually compounded risk free interest rate with term \( T \). Notice in the previous section, we did not have this constraint and a guaranteed rate higher than the risk free rate was acceptable from a risk management perspective. So the guaranteed rate is eventually given by:

\[i_\epsilon = \min(R(0, T), \bar{i}_\epsilon)\]

and the participation rate:

(4.3) \[ \eta = \frac{1 - (1 + i_\epsilon)^T P(0, T)}{C(0, (1 + i_\epsilon)^T)} \]

**4.2. Equilibrium value in the CBS model.** We can explicitly compute the value of the equilibrium participation rate in the framework of the CBS model presented in Section 2. In order to keep computations as easy as possible we will develop them using only ALM strategy 1 as defined in Section 2.2.3 (zero coupon bonds with maturity corresponding to the duration of the guarantee). The explicit formula in the general case is then given in Section 4.2.2.

**4.2.1. Strategy 1.** We first give the processes followed by the different assets under the risk neutral measure; then we move to the \( T \)-forward martingale measure. Under the risk neutral measure the three assets have the following form:

- **Cash:**
  \[ d\beta(t) = r(t)\beta(t)dt \]

- **Bonds:**
  \[ dP(t, T) = P(t, T)(r(t)dt - \sigma_r B(T - t)dW_1(t)) \]

- **Stocks:**
  \[ dS(t) = S(t)(r(t)dt + \sigma_S \rho dW_1(t) + \sigma_S \sqrt{1 - \rho^2} dW_2(t)) \]
where the new Brownian motions are defined by:

\[ dW_1(t) = dW_1(t) + \lambda dt \]

\[ dW_2(t) = dW_2(t) + (\mu - r(t) - \lambda \rho \sigma_S)/(\sigma_S \sqrt{1 - \rho^2}) dt \]

\[ = dW_2(t) + \theta(t) dt \]

The equation of portfolio \( V \) of Section 2.2.1 becomes in the risk neutral world:

\[ \frac{dV(t)}{V(t)} = r(t) dt + (x_a \sigma_p - x_b \sigma_r B(T - t)) d\overline{W}_1(t) + x_a \sigma_S \sqrt{1 - \rho^2} d\overline{W}_2(t) \]

Finally the risk free rate follows a modified Ornstein-Uhlenbeck process:

\[ dr(t) = a(b - \frac{\sigma_r \lambda}{a} - r(t)) dt + \sigma_r d\overline{W}_1(t) \]

The \( T \)-forward martingale measure is then generated by the change:

\[ d\overline{W}_1(t) = d\overline{W}_1(t) + \sigma_r B(T - t) dt \]

\[ d\overline{W}_2(t) = d\overline{W}_2(t) \]

Under this \( T \)-forward martingale measure and recalling \( V(0) = 1 \) the portfolio value can be written as:

\[ V(t) = \exp\left\{ \int_0^t (r(s) - \overline{\sigma}_1(s) \sigma(s, T)) ds + \int_0^t \overline{\sigma}_1(s) d\overline{W}_1(s) + \int_0^t \overline{\sigma}_2(s) d\overline{W}_2(s) - \frac{1}{2} \int_0^t (\overline{\sigma}_1^2(s) + \overline{\sigma}_2^2(s)) ds \right\} \]

In this model, we have then to price a call option:

\[ C(0, K) = P(0, T) E_{Q_T}((V(T) - K)^+) \]

\[ = P(0, T) E_{Q_T}(V(T)1_{V(T) > K}) - KP(0, T) P_{Q_T}(V(T) > K) \]

\[ = I - II \]

Using standard tools for computing derivatives prices in a stochastic term structure we get successively:

\[ II = KP(0, T) \Phi\left( \frac{\ln(1/K) - \ln P(0, T) - v^2(T)/2}{v(T)} \right) \]
with

\[ v^2(T) = \int_0^T (\sigma_1(s) + \sigma(s,T))^2 ds + \int_0^T \sigma_2^2 ds \]

\[ = x_\alpha^2 \sigma_3^2 T + \sigma_r^2 (1 - x_b)^2 B_2(T) + 2 x_a (1 - x_b) \sigma_{S \sigma_r} \rho B_1(T) \]

\[ B_1(T) = \int_0^T B(T - u) du \]

\[ = \frac{T}{a} - \frac{1}{a^2} (1 - e^{-at}) \]

\[ B_2(T) = \int_0^T B^2(T - u) du \]

\[ = \frac{T}{a^2} + \frac{1}{2a^3} (1 - e^{-2at}) - \frac{2}{a^3} (1 - e^{-at}) \]

and

\[ I = \Phi(\frac{\ln(1/K) - \ln P(0,T) + v^2(T)/2}{v(T)}) \]

Finally putting

\[ K = (1 + i_e)^T \]

we get for the price of the call:

\[ C(0, (1 + i_e)^T) = \Phi\left( -\frac{T \ln(1 + i_e) - \ln P(0,T) + v^2(T)/2}{v(T)} \right) \]

\[ - (1 + i_e)^T P(0,T) \Phi\left( -\frac{T \ln(1 + i_e) - \ln P(0,T) - v^2(T)}{v(T)} \right) \]

Or

\[ (4.4) \quad C(0, (1 + i_e)^T) = \Phi(D_+(i_e)) - (1 + i_e)^T P(0,T) \Phi(D_-(i_e)) \]

Putting the Equation (4.4) in Equation (4.3) we obtain the equilibrium value of the participation rate:

\[ (4.5) \quad \eta = \frac{1 - (1 + i_e)^T P(0,T)}{\Phi(D_+(i_e)) - (1 + i_e)^T P(0,T) \Phi(D_-(i_e))} \]
4.2.2. The general case. Similar calculations lead to an explicit formula for the participation rate in the general case. The detailed calculations can be found in Appendix 2. The participation rate is still given by:

$$\eta = \frac{1 - (1 + i_c)^T P(0, T)}{C(0, (1 + i_c)^T)}$$

The price of the call is still a Black-Scholes like formula:

$$C(0, (1 + i_c)^T) = \Phi(D_+(i_c)) - (1 + i_c)^T P(0, T) \Phi(D_-(i_c))$$

with

$$D_+(i_c) = \frac{-T \ln (1 + i_c) - \ln P(0, T) + \frac{1}{2} v^2}{v}$$

$$D_-(i_c) = \frac{-T \ln (1 + i_c) - \ln P(0, T) - \frac{1}{2} v^2}{v}$$

But in the general case the parameter $v^2$ is given by:

$$v^2 = \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} (v_V (u, t_i + T_i) + v_P (u, T))^2 \, du$$

where

$$v_V (u, t_i + T_i) = x_a v_S - x_b v_P (u, t_i + T_i)$$

$$v_S = \begin{bmatrix} \sigma_S \\ \sigma_S \sqrt{1 - \rho^2} \end{bmatrix}$$

$$v_P (u, t_i + T_i) = \begin{bmatrix} \sigma_r B (t_i + T_i - u) \\ 0 \end{bmatrix}$$

In the first expression, we define $a^2 = a' \cdot a$ where $a$ is a $n \times 1$ matrix and $a'$ is its transpose.

5. Numerical Illustrations

We estimated the parameters of our market model on Belgian data’s. The stock parameters are estimated from January 1991 to March 2004, on the Euronext Belgian All Share Index. The interest rate parameters are estimated from January 2000 to March 2004 on the 1 to 10 years swap rates. The maximum likelihood estimators of these parameters are given in table 5.1 and 5.2.

<table>
<thead>
<tr>
<th>Table 5.1. Interest rate</th>
<th>$a$</th>
<th>$b$</th>
<th>$\sigma_r$</th>
<th>$\lambda$</th>
<th>$r(0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1577</td>
<td>5.49%</td>
<td>0.89%</td>
<td>-0.2304</td>
<td>1.23%</td>
<td></td>
</tr>
</tbody>
</table>
Table 5.2. **Stock model**

<table>
<thead>
<tr>
<th></th>
<th>$\mu$</th>
<th>$\sigma$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>9.41%</td>
<td>14.71%</td>
</tr>
</tbody>
</table>

In the next subsections, we illustrate the impact of the financial strategy on different variables, namely the Value at Risk, the probability of ruin, the guaranteed rate, the participation rate based on the VaR and the conditional value at risk and the guaranteed rate, the participation rate based on the CVaR. In each case, 4 graphs are given, representing different types of asset allocation. On the x-axis, we have the number $n$ of times we reinvest in new zero-coupon bonds and on the y-axis, we have the time-to-maturity $T_1$ of the bonds we invest in. In all these illustrations, the term of the guarantee $T$ is set to 15 years and $\rho$ to 0.15.

As already said, the strategy 1 corresponds to the case where $n$ is equal to 1 and $T_1 = T = 15$. On all these graphs, when it does make sense, this strategy is the first point on the y-axis. The second strategy corresponds to the case where $nT_1 = T$. This strategy is represented by the points on the left edge of the curve. The strategy 3 corresponds to all the other points of the curve.

### 5.1. Value at Risk

Figure 5.1 illustrates the effect of the financial strategy on the Value at Risk for a probability $\epsilon$ equal to 1%. In this figure, the guaranteed rate is equal to 3.25%. In table 5.3, we give some numerical examples.

<table>
<thead>
<tr>
<th>$x_a=15%$, $x_b=80%$, $x_c=5%$</th>
<th>Value at risk</th>
</tr>
</thead>
<tbody>
<tr>
<td>Strategy 1:</td>
<td>-0.1865</td>
</tr>
<tr>
<td>Strategy 2:</td>
<td></td>
</tr>
<tr>
<td>$n = 3$ and $T_1 = 5$</td>
<td>0.0374</td>
</tr>
<tr>
<td>$n = 5$ and $T_1 = 3$</td>
<td>0.1039</td>
</tr>
<tr>
<td>Strategy 3:</td>
<td></td>
</tr>
<tr>
<td>$n = 3$ and $T_1 = 8$</td>
<td>-0.1048</td>
</tr>
<tr>
<td>$n = 5$ and $T_1 = 5$</td>
<td>-0.0282</td>
</tr>
</tbody>
</table>

The first important point to notice is that the value at risk is not necessarily increasing with the part invested in the risky assets ($x_a$): the second graph exhibits globally lower values at risk than the first one. As the third and fourth graph show, this result is reversed the more we invest in risky assets. With respect to the part invested in the risky asset, the value at risk is thus at first decreasing then increasing. This result comes from the higher risk premium of the stocks compared to the bonds: the increase in expected return can offset in a certain extent, the increase of risk taking into account the necessity of the guarantee. The second important point is that, in the first graph, we see the value at risk exhibits a minimum with respect to the term of the bonds. The value at risk is lower for bonds with medium time to maturity than for short or long time to maturity.
5.1.1. Probability of ruin: As already said, the probability $\epsilon$ corresponds to the probability of ruin if the VaR is set to 0. Figure 5.2 illustrates the effect of the financial strategy on the probability of ruin for a guaranteed rate equal to 3.25%.

<table>
<thead>
<tr>
<th>Strategy</th>
<th>$n_a=15%$, $x_b=80%$, $x_c=5%$</th>
<th>Pr of ruin</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td>0.02%</td>
</tr>
<tr>
<td>2</td>
<td>$n = 3$ and $T_1 = 5$</td>
<td>1.62%</td>
</tr>
<tr>
<td></td>
<td>$n = 5$ and $T_1 = 3$</td>
<td>3.29%</td>
</tr>
<tr>
<td>3</td>
<td>$n = 3$ and $T_1 = 8$</td>
<td>0.17%</td>
</tr>
<tr>
<td></td>
<td>$n = 5$ and $T_1 = 5$</td>
<td>0.66%</td>
</tr>
</tbody>
</table>
In Figure 5.2, we see the probability of ruin is at first decreasing with the part invested in the risky asset then globally increasing, as it is with the value at risk. Investing a small part of the financial portfolio in risky assets can be profitable in terms of probability of ruin.

**Figure 5.2. Guaranteed rate = 3.25%**

5.1.2. **Guaranteed rate and participation rate:** Here, we also set the VaR equal to 0. Figures 5.3 and 5.4 illustrate the effect of the financial strategy on the guaranteed rate and the participation rate for a probability of ruin equal to 1%.

<table>
<thead>
<tr>
<th>$x_a=15%$, $x_b=80%$, $x_c=5%$</th>
<th>Guaranteed rate</th>
<th>Participation Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>Strategy 1:</td>
<td>4.00%</td>
<td>96.56%</td>
</tr>
<tr>
<td>Strategy 2:</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$n = 3$ and $T_1 = 5$</td>
<td>3.09%</td>
<td>99.35%</td>
</tr>
<tr>
<td>$n = 5$ and $T_1 = 3$</td>
<td>2.79%</td>
<td>99.55%</td>
</tr>
<tr>
<td>Strategy 3:</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$n = 3$ and $T_1 = 8$</td>
<td>3.68%</td>
<td>98.23%</td>
</tr>
<tr>
<td>$n = 5$ and $T_1 = 5$</td>
<td>3.37%</td>
<td>98.98%</td>
</tr>
</tbody>
</table>
Firstly, in Figure 5.3, the guaranteed rate is increasing at first with the part invested in the risky asset then decreasing. This result is obviously consistent with what we found for the value at risk. The fourth graph illustrates the fact that when we invest massively in the risky asset, we can be in a situation where we cannot find a positive guaranteed rate consistent with our risk policy. Secondly, we also see in Graph 1 that the guaranteed rate exhibits a maximum with respect to the time to maturity of the zero coupon bonds. Finally, notice the guaranteed rate is very sensitive to the financial strategy chosen.

In Figure 5.4, we show the participation rates given the financial strategies and the guaranteed rates found above. The participation rate depends on both the guaranteed rate and the risk of the financial strategy in terms of standard deviation. Globally, the participation rate should be decreasing with the guaranteed rate and decreasing with the standard deviation of the strategy. Both effects can act in the same direction. For example, we find lower participation rates in Graph 2 than in Graph 1 due to both a higher guaranteed rate and a higher standard deviation. They can also act on opposite directions. Although we have higher standard deviations in Graph 3 than in Graph 4, we find higher participation rates in Graph 4 due to offsetting lower guaranteed rates.
5.2. Conditional Value at Risk. Figure 5.5 illustrates the effect of the financial strategy on the conditional value at Risk for a guaranteed rate equal to 3.25%. Here, $R$ is set to 0.

<table>
<thead>
<tr>
<th>$x_a=15%$, $x_b=80%$, $x_c=5%$</th>
<th>CvaR</th>
</tr>
</thead>
<tbody>
<tr>
<td>Strategy 1:</td>
<td>1.66%</td>
</tr>
<tr>
<td>Strategy 2:</td>
<td></td>
</tr>
<tr>
<td>$n = 3$ and $T_1 = 5$</td>
<td>3.41%</td>
</tr>
<tr>
<td>$n = 5$ and $T_1 = 3$</td>
<td>4.06%</td>
</tr>
<tr>
<td>Strategy 3:</td>
<td></td>
</tr>
<tr>
<td>$n = 3$ and $T_1 = 8$</td>
<td>2.32%</td>
</tr>
<tr>
<td>$n = 5$ and $T_1 = 5$</td>
<td>2.85%</td>
</tr>
</tbody>
</table>

Firstly, the conditional value at risk is increasing with the part invested in the risky assets and does not seem to exhibit a first decreasing movement, in contrast with the value at risk.
Secondly, in the first graph, as the value at risk, the conditional value at risk exhibits a minimum with respect to the term of the bonds.

5.2.1. Guaranteed rate and Participation rate: Figures 5.6 and 5.7 illustrate the effect of the financial strategy on respectively the guaranteed rate and the participation rate for a conditional value at risk equal to 4%.

Firstly, in Figure 5.6, we see the guaranteed rate is globally decreasing with the part invested in the risky asset. The fourth graph illustrates the situation where we cannot find a positive guaranteed rate consistent with our risk policy. Secondly, here also, the guaranteed rate exhibits a maximum with respect to the time to maturity of the zero coupon bonds. In graph one, the guaranteed rates are actually capped for a number of strategies because they exceed the risk free rate. We can find these strategies in Figure 5.7: these are those exhibiting a participation rate equal to 0.
Figure 5.6. Conditional Value at Risk = 4%

Table 5.7

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Guaranteed rate</th>
<th>Participation Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>Strategy 1:</td>
<td>4.69%</td>
<td>45.94%</td>
</tr>
<tr>
<td>Strategy 2:</td>
<td></td>
<td></td>
</tr>
<tr>
<td>n = 3 and $T_1 = 5$</td>
<td>3.57%</td>
<td>97.03%</td>
</tr>
<tr>
<td>n = 5 and $T_1 = 3$</td>
<td>3.22%</td>
<td>98.42%</td>
</tr>
<tr>
<td>Strategy 3:</td>
<td></td>
<td></td>
</tr>
<tr>
<td>n = 3 and $T_1 = 8$</td>
<td>4.26%</td>
<td>85.87%</td>
</tr>
<tr>
<td>n = 5 and $T_1 = 5$</td>
<td>3.91%</td>
<td>93.98%</td>
</tr>
</tbody>
</table>

As far as the participation rate is concerned, we can make the same comments than with the Value-at-Risk since it does not depend on the chosen risk measure (apart through the guaranteed rate). Globally, it is decreasing with the standard deviation of the financial strategy and the guaranteed rate. Sometimes, these effects act in the same direction, sometimes in opposite directions.
6. Conclusions and Extensions.

In this chapter, we studied the interrelationship between the risk management and the pricing of a classical single premium life insurance contract with profit. The pricing is considered in terms of a guaranteed rate on the premium and a participation rate on the financial surplus. We argued it is a utopia to think we can perfectly hedge the liabilities of long term insurance contracts. There are two main reasons to this. First of all, derivative instruments with terms of 10, 15 or 20 years are not traded on the markets. Secondly, self-hedging would be too expensive. So, at least for these very long term liabilities, financial risk is unavoidable. By using only the risk neutral valuation principle to determine the guaranteed rate and the participation rate, we cannot properly, take into account this financial risk. Moreover, using this pricing principle alone, we would have to pick up arbitrarily, a pair of guaranteed and participation rates among the set of pairs consistent with the no arbitrage assumption. Instead, we argued these parameters should be determined by taking into account explicitly the risk the insurer accepts to bear. Moreover, this acceptable level of risk should be set consistently with the risk management policy of the insurance company, in terms of a risk measure such as the value-at-risk or the conditional value-at-risk. In order to answer these points, we suggested to divide our...
Appendix 1.

In this appendix, we look for the portfolio Value distribution. First, we need a few preliminary results:

\[ r(t) = e^{-at}r(0) + (1 - e^{-at})b + \int_0^t \sigma_r e^{-a(t-u)} dW_1(u) \]
\[
\int_{t_s}^{t_{s+1}} r(u) du = \int_{t_s}^{t_{s+1}} e^{-au} r(0) + b \left(1 - e^{-au}\right) du + \int_{t_s}^{t_{s+1}} \int_{0}^{u} \sigma_\tau e^{-a(u-v)} dW_1(v) du
\]

\[
= b(t_{s+1} - t_s) + \left(\frac{b}{a} - \frac{r(0)}{a}\right) \left(e^{-a(t_{s+1})} - e^{-a(t_s)}\right)
\]

\[
+ \int_{t_s}^{t_{s+1}} \int_{t_s}^{u} \sigma_\tau e^{-a(u-v)} dW_1(v) + \int_{t_s}^{t_{s+1}} \left[\int_{v}^{t_{s+1}} \sigma_\tau e^{-a(u-v)} du\right] dW_1(v)
\]

\[
= b(t_{s+1} - t_s) + \frac{1}{a} (b - r(0)) e^{-ats_s} \left(e^{-a(t_{s+1} - t_s)} - 1\right)
\]

\[
- \int_{0}^{t_s} \int_{t_s}^{u} \sigma_\tau e^{-a(t_s - v)} \left(e^{-a(t_{s+1} - t_s)} - 1\right) dW_1(v) + \int_{t_s}^{t_{s+1}} \frac{\sigma_\tau}{a} \left(1 - e^{-a(t_{s+1} - v)}\right) dW_1(v)
\]

Expectation of \(R_{ts}\) = \(\ln V(t_{s+1}) - \ln V(t_s)\):

\[
E[R_{ts} | F_0] = E \left[\int_{t_s}^{t_{s+1}} x_\mu - x_b \lambda \sigma_\tau B(T_s + t_s - u) \; du \mid F_0\right]
\]

\[- \frac{1}{2} E \left[\int_{t_s}^{t_{s+1}} x_\sigma^2 \sigma_S^2 + x_b^2 \sigma^2_\tau B(T_s + t_s - u)^2 - 2x_\sigma x_b \sigma S \sigma_\tau \rho B(T_s + t_s - u) \; du \mid F_0\right]
\]

\[+ E \left[\int_{t_s}^{t_{s+1}} (x_b + x_c) r(u) du \mid F_0\right]
\]

\[+ E \left[\int_{t_s}^{t_{s+1}} [x_\sigma \sigma \rho - x_b \sigma_\tau B(T_s + t_s - u)] dW_1(u) + \int_{t_s}^{t_{s+1}} x_\sigma \sigma S \sqrt{1 - \rho^2} dW_2(u) \mid F_0\right]
\]

Since the two first integrals are deterministic, we have

\[
E[R_{ts} | F_0] = \int_{t_s}^{t_{s+1}} x_\mu - x_b \lambda \sigma_\tau B(T_s + t_s - u) \; du
\]

\[- \frac{1}{2} \int_{t_s}^{t_{s+1}} x_\sigma^2 \sigma_S^2 + x_b^2 \sigma^2_\tau B(T_s + t_s - u)^2 - 2x_\sigma x_b \sigma S \sigma_\tau \rho B(T_s + t_s - u) du
\]

\[+ E \left[\int_{t_s}^{t_{s+1}} (x_b + x_c) r(u) du \mid F_0\right]
\]

\[= K(T_s, t_{s+1} - t_s) + (x_b + x_c) E \left[\int_{t_s}^{t_{s+1}} r(u) du \mid F_0\right]
\]
Hence

\[
E[R_{t_s} | \mathcal{F}_0] = K(T_s, t_{s+1} - t_s) + (x_b + x_c) \left[ b(t_{s+1} - t_s) + \frac{1}{a} (b - r(0)) e^{-a(t_{s+1} - t_s)} \right] e^{a(t_{s+1} - t_s) - 1}
\]

Variance of \( R_{t_s} = \ln V(t_{s+1}) - \ln V(t_s) \):

\[
Var[R_{t_s} | \mathcal{F}_0] = Var \left[ \int_{t_s}^{t_{s+1}} (x_b + x_c) r(u) du \right] | \mathcal{F}_0
\]

\[
= Var \left[ (x_b + x_c) \int_{t_s}^{t_{s+1}} r(u) du | \mathcal{F}_0 \right] \tag{A}
\]

\[
+ Var \left[ \int_{t_s}^{t_{s+1}} x_a \sigma S \rho - x_b \sigma r B(T_s + t_s - u) dW_1(u) | \mathcal{F}_0 \right] \tag{B}
\]

\[
+ Var \left[ \int_{t_s}^{t_{s+1}} x_a \sigma S \sqrt{1 - \rho^2} dW_2(u) | \mathcal{F}_0 \right] \tag{C}
\]

\[
+ 2Covar \left[ (x_b + x_c) \int_{t_s}^{t_{s+1}} r(u) du ; \int_{t_s}^{t_{s+1}} x_a \sigma S \rho - x_b \sigma r B(T_s + t_s - u) dW_1(u) | \mathcal{F}_0 \right] \tag{D}
\]
We have that

\[ (A) = (x_b + x_c)^2 \left( \frac{\sigma_r}{a} \right)^2 \left( e^{-a(t_{s+1}-t_s)} - 1 \right)^2 \int_0^{t_s} e^{-2a(t_s-u)} du \]

\[ + \int_0^{t_{s+1}} \left( 1 - e^{-a(t_{s+1}-u)} \right)^2 du \]

\[ = (x_b + x_c)^2 \left( \frac{\sigma_r}{a} \right)^2 \left( e^{-a(t_{s+1}-t_s)} - 1 \right)^2 \frac{1}{2a} \left( 1 - e^{-2at_s} \right) \]

\[ + x_b + x_c)^2 \left( \frac{\sigma_r}{a} \right)^2 \left[ \left( t_{s+1} - t_s \right) + \frac{1}{a} \left( 1 - e^{-2a(t_{s+1}-t_s)} \right) - \frac{2}{a} \left( 1 - e^{-a(t_{s+1}-t_s)} \right) \right] \]

\[ (B) = \int_0^{t_{s+1}} \left[ x_a \sigma_s \rho - x_b \sigma_r B(T_s + t_s - u) \right]^2 d(u) \]

\[ = (x_a \sigma_s \rho)^2 (t_{s+1} - t_s) + (x_b \sigma_r)^2 IB_2(T_s, t_s) - 2x_a x_b \sigma_s \sigma_r \rho IB_1(T_s, t_s) \]

\[ (C) = \int_0^{t_{s+1}} \left[ x_a \sigma_s \sqrt{1 - \rho^2} \right]^2 du \]

\[ = (x_a \sigma_s)^2 (1 - \rho^2) (t_{s+1} - t_s) \]

\[ (D) = 2(x_b + x_c) \int_0^{t_{s+1}} \left( \sigma_r \right)^2 \left( 1 - e^{-a(t_{s+1}-u)} \right) \left( x_a \sigma_s \rho - x_b \sigma_r B(T_s + t_s - u) \right) du \]

\[ = 2(x_b + x_c) \left[ x_a \sigma_s \sigma_r \rho \left[ \left( t_{s+1} - t_s \right) - \frac{1}{a} \left( 1 - e^{-a(t_{s+1}-t_s)} \right) \right] - \frac{x_a \sigma_r^2}{a} IB_1(T_s, t_s) \right] \]

\[ + 2(x_b + x_c) \left[ \frac{x_b \sigma_r^2}{a^2} \left( \frac{1}{a} \left( 1 - e^{-a(t_{s+1}-t_s)} \right) - \frac{e^{-aT_s}}{2a} \left( e^{a(t_{s+1}-t_s)} - e^{-a(t_{s+1}-t_s)} \right) \right) \right] \]

Eventually, we get

\[ Var [R_{t_s} | \mathcal{F}_0] = (x_b + x_c)^2 \frac{\sigma_r^2}{a^2} \left[ \left( t_{s+1} - t_s \right) + \frac{1}{2a} \left( e^{-a(t_{s+1}-t_s)} - 1 \right)^2 \left( 1 - e^{-2at_s} \right) \right] \]

\[ + (x_b + x_c)^2 \frac{\sigma_r^2}{a^2} \left[ \frac{1}{2a} \left( 1 - e^{-2a(t_{s+1}-t_s)} \right) - \frac{2}{a} \left( 1 - e^{-a(t_{s+1}-t_s)} \right) \right] \]

\[ + (x_b \sigma_r)^2 IB_2(T_s, t_s) - 2x_a x_b \sigma_s \sigma_r \rho IB_1(T_s, t_s) + (x_a \sigma_s)^2 (t_{s+1} - t_s) \]

\[ + 2(x_b + x_c) \left[ x_a \sigma_s \sigma_r \rho \left[ \left( t_{s+1} - t_s \right) - \frac{1}{a} \left( 1 - e^{-a(t_{s+1}-t_s)} \right) \right] - \frac{x_a \sigma_r^2}{a} IB_1(T_s, t_s) \right] \]

\[ + 2(x_b + x_c) \left[ \frac{x_b \sigma_r^2}{a^2} \left( \frac{1}{a} \left( 1 - e^{-a(t_{s+1}-t_s)} \right) - \frac{e^{-aT_s}}{2a} \left( e^{a(t_{s+1}-t_s)} - e^{-a(t_{s+1}-t_s)} \right) \right) \right] \]
Covariances between \( R_{ti} = \ln V(t_{i+1}) - \ln V(t_i) \) and \( R_{tj} = \ln V(t_{j+1}) - \ln V(t_j) \) with \( t_j \geq t_{i+1} \):

\[
\text{Covar} \left[ R_{tt_i}; R_{jj} \mid \mathcal{F}_0 \right] = \text{Covar} \left[ \int_{t_i}^{t_{i+1}} r(u)du + \int_{t_i}^{t_{j+1}} (x_a \sigma_S \rho - x_b \sigma_r B(T_i + t_i - u)) dW_1(u) \mid \mathcal{F}_0 \right]
\]

\[
= \text{Covar} \left[ \int_{t_i}^{t_{j+1}} (x_a \sigma_S \rho - x_b \sigma_r B(T_i + t_i - u)) dW_1(u) \mid \mathcal{F}_0 \right]
\]

It leads to

\[
\text{Covar} \left[ R_{tt_i}; R_{jj} \mid \mathcal{F}_0 \right] = (x_b + x_c)^2 \int_{t_i}^{t_{j+1}} \frac{\sigma_r^2}{a^2} (1 - e^{-a(t_{i+1} - t_i)}) (1 - e^{-a(t_{j+1} - t_j)}) e^{-a(t_i-u)} e^{-a(t_j-u)} du
\]

\[
+ (x_b + x_c) \int_{t_i}^{t_{j+1}} \left( \frac{\sigma_r}{a} \right)^2 (1 - e^{-a(t_{i+1} - t_i)}) (1 - e^{-a(t_{j+1} - t_j)}) e^{-a(t_j-u)} du
\]

\[
+ (x_b + x_c) \int_{t_i}^{t_{j+1}} \frac{\sigma_r}{a} (x_a \sigma_S \rho - x_b \sigma_r B(T_i + t_i - u)) (1 - e^{-a(t_{j+1} - t_j)}) e^{-a(t_j-u)} du
\]
Using Ito’s formula, we get:

\[
\begin{align*}
&= (x_b + x_c)^2 \frac{\sigma_r^2}{2a^3} \left( 1 - e^{-a(t_{i+1} - t_i)} \right) \left( 1 - e^{-a(t_{j+1} - t_j)} \right) \left( e^{-a(t_j - t_i)} - e^{-a(t_i + t_j)} \right) \\
&+ (x_b + x_c)^2 \frac{\sigma_r^2}{a^3} \left( 1 - e^{-a(t_{j+1} - t_j)} \right) \left( e^{-a(t_j - t_i)} - e^{-a(t_i + t_j)} \right) \\
&- \frac{1}{2} (x_b + x_c)^2 \frac{\sigma_r^2}{a^3} \left( 1 - e^{-a(t_{j+1} - t_j)} \right) \left( e^{-a(t_j - t_i)} - e^{-a(t_i + t_j)} e^{-a(t_j - t_i)} \right) \\
&+ (x_b + x_c)^2 \frac{\sigma_r}{a} \left( 1 - e^{-a(t_{j+1} - t_j)} \right) \left[ \frac{x_a \sigma_S \rho}{a} \left( e^{-a(t_j - t_i)} - e^{-a(t_j - t_i)} \right) \right] \\
&- (x_b + x_c) x_b \frac{\sigma_r^2 e^{-aT_i}}{2a^3} \left( 1 - e^{-a(t_{j+1} - t_j)} \right) \left( e^{a(t_{i+1} - t_i)} e^{-a(t_j - t_i)} - e^{-a(t_j - t_i)} \right)
\end{align*}
\]

**Appendix 2.**

In this appendix, we look for the value of \( C(0, (1 + i_e)T) \) in the general case. Under the martingale measure \( Q \), we know by using standard results that:

\[
\begin{align*}
    dP(t, T) &= P(t, T) \left[ r(t) dt - v'_P(t, T) dW(t) \right] \\
    dS(t) &= S(t) \left[ r(t) dt + v'_S(t, T) dW(t) \right] \\
    d\beta(t) &= r(t) \beta(t) dt
\end{align*}
\]

where

\[
v_p(t, T) = \begin{bmatrix} \sigma_r B(T-t) \\ 0 \end{bmatrix} , \quad v_S = \begin{bmatrix} \sigma_S \rho \\ \sigma_S \sqrt{1 - \rho^2} \end{bmatrix} \quad \text{and} \quad W(t) = \begin{bmatrix} W_1(t) \\ \overline{W_2}(t) \end{bmatrix}
\]

The portfolio value for is then given by \( t \in [t_s, t_{s+1}] \):

\[
\frac{dV(t)}{V(t)} = r(t) dt + [x_a v'_S - x_b v'_P(t, t_s + T_s)] d\overline{W}(t)
\]

Put

\[
v'_V(t, t_s + T_s) = x_a v'_S - x_b v'_P(t, t_s + T_s)
\]

Using Ito’s formula, we get:

\[
d \ln V(t) = \left[ r(t) - \frac{1}{2} v'_V(t, t_s + T_s) . v_V(t, t_s + T_s) \right] dt + v'_V(t, t_s + T_s) d\overline{W}(t)
\]

\[
V(t_{s+1}) = V(t_s) \exp \left\{ \int_{t_s}^{t_{s+1}} \left[ r(u) - \frac{1}{2} v'_V(u, t_s + T_s) . v_V(u, t_s + T_s) \right] du + \int_{t_s}^{t_{s+1}} v'_V(u, t_s + T_s) d\overline{W}(u) \right\}
\]
Under the $T$-forward measure $Q^T$, we know that $\overline{W}(t)$ defined by

$$d\overline{W}(t) = d\overline{W}(t) + v_p(t, T)dt$$

is a 2-dimensional Brownian motion. Under this measure, the prices of the different assets are then given by:

$$dP(t, t_s + T_s) = \left[ r(t)dt + v'_{p}(t, t_s + T_s), v_{p}(t, T)dt - v'_{p}(t, t_s + T_s)d\overline{W}(t) \right]$$

$$dS(t) = \left[ r(t)dt - v'_{S}, v_{p}(t, T)dt + v'_{S}d\overline{W}(t) \right]$$

$$d\beta(t) = \left[ r(t)\beta(t)dt \right]$$

The portfolio value for $t \in [t_s, t_{s+1}]$ is then given by:

$$\frac{dV(t)}{V(t)} = \left[ r(t) - (x_a v'_{S} - x_b v'_{p}, t_s + T_s) , v_{p}(t, T) \right] dt + \left[ x_a v'_{S} - x_b v'_{p}, t_s + T_s \right] d\overline{W}(t)$$

$$d\ln V(t) = \left[ r(t) - v'_{V}(t, t_s + T_s) , v_{p}(t, T) - \frac{1}{2} v''_{V}(t, t_s + T_s) \right] dt + v'_{V}(t, t_s + T_s) d\overline{W}(t)$$

$$V(t_{s+1}) = V(t_s) \exp \left\{ \int_{t_s}^{t_{s+1}} \left[ r(u) - v'_{V}(u, t_s + T_s) , v_{p}(u, T) - \frac{1}{2} v''_{V}(u, t_s + T_s) \right] du \right\}$$

$$+ \int_{t_s}^{t_{s+1}} v'_{V}(u, t_s + T_s) d\overline{W}(u)$$

$$V(T) = V(0) \exp \left\{ \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \left[ r(u) - v'_{V}(u, t_i + T_i) , v_{p}(u, T) - \frac{1}{2} v''_{V}(u, t_i + T_i) \right] du \right\}$$

$$+ \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} v'_{V}(u, t_i + T_i) d\overline{W}(u)$$

We know that under this $T$-forward measure, the price of the call is given by:

$$C = P(0, T)E^{Q^T} \left[ \max \left( F_V(T, T) - K; 0 \right) \right]$$

$$= P(0, T)E^{Q^T} \left[ F_V(T, T)1_{F_V(T, T) > K} \right] - P(0, T)E^{Q^T} \left[ K1_{F_V(T, T) > K} \right]$$

$$= P(0, T) \left[ I - II \right]$$

Where $F_V(t, T)$ is the forward price of the portfolio at time $t$ for a maturity $T$:

$$F_V(t, T) = \frac{V(T)}{P(t, T)}$$
The value $P(T, T)$ can be expressed as

$$P(T, T) = P(0, T) \exp \left\{ \int_0^T r(u)du + \frac{1}{2} \int_0^T v'_P(u, T)v_P(u, T)du - \int_0^T v'_P(u, T)d\overline{W}(u) \right\}$$

Using the formulas for $V(T)$ and $P(T, T)$, $F_V(T, T)$ can be expressed as:

$$F_V(T, T) = \frac{V(0)}{P(0, T)} \exp \left\{ -\sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \left[ v'_V(u, t_i + T_i) . v_P(u, T) + \frac{1}{2} v'_P(u, T) v_P(u, T) \right] du ight\}$$

$$= F_V(0, T) \exp \left\{ -\frac{1}{2} \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} (v'_V(u, t_i + T_i) + v'_P(u, T))^2 du ight\}$$

We can now compute the value of $I$ and $II$.

**Calculation of $II$:**

$$II = KQ_T (F_V(T, T) > K)$$

$$= KQ_T (\ln F_V(T, T) > \ln K)$$

$$= K\Phi(D_{-}(i_\epsilon))$$

where $\Phi(x)$ is the cumulative probability distribution of a standard normal random variable, and

$$D_{-}(i_\epsilon) = -\frac{T \ln (1 + i_\epsilon) - \ln P(0, T) - \frac{1}{2} v'^2}{v}$$

and

$$v'^2 = \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} (v'_V(u, t_i + T_i) + v'_P(u, T))^2 du$$

In the last expression, we define $a^2 = a' \cdot a$ where $a$ is $n \times 1$ matrix and $a'$ is its transpose.

**Calculation of $I$:**

$$I = E^{Q_T} [F_V(T, T) 1_{F_V(T, T) > K}]$$

In order to find this expectation, we can once again change our measure. Let us consider a measure $\tilde{Q}_T$ such that $\tilde{W}$ defined by

$$\tilde{W}(t) = \overline{W}(t) - \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \left[ v'_V(u, t_i + T_i) + v'_P(u, T) \right] 1_{u \leq t} du$$
is a Brownian motion. Then
\[ \overline{dW}(t) = d\tilde{W}(t) + \sum_{i=0}^{n-1} \left[ v'_V(t, t_i + T_i) + v'_p(t, T) \right] 1_{\{t \in [t_i, t_{i+1}]\}} dt \]

Under this new measure, I is given by:
\[
I = F_V(0, T) E^{\tilde{Q}} \left[ 1_{F_V(T, T) > K} \right] \\
= F_V(0, T) \tilde{Q} \left[ F_V(T, T) > K \right]
\]

and \( F_V(T, T) \) is now expressed as:
\[
F_V(T, T) = F_V(0, T) \exp \left\{ +\frac{1}{2} \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} (v'_V(u, t_i + T_i) + v'_p(u, T))^2 du \right. \\
+ \left. \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} (v'_V(u, t_i + T_i) + v'_p(u, T)) d\tilde{W}(u) \right\}
\]

Hence:
\[
I = F_V(0, T) \tilde{Q} \left[ F_V(T, T) > K \right] \\
= F_V(0, T) \tilde{Q} \left[ \ln F_V(T, T) > \ln K \right] \\
= F_V(0, T) \Phi \left[ D_+ (i_\epsilon) \right] \\
= \frac{V(0)}{P(0, T)} \Phi \left[ D_+ (i_\epsilon) \right]
\]

where \( D_+ (i_\epsilon) = D_- (i_\epsilon) + v \).

So eventually,
\[
C(0, (1 + i_\epsilon)^T) = \Phi(D_+(i_\epsilon)) - (1 + i_\epsilon)^T P(0, T) \Phi(D_- (i_\epsilon))
\]

with \( D_+(i_\epsilon) \) and \( D_- (i_\epsilon) \) as given above.
Part 2

Life Insurance Contracts and the Surrender Risk.
CHAPTER 2

Risk-Neutral Valuation of Life Insurance Contracts with a 
Surrender Option.

1. Introduction.

In this chapter, we study the valuation of life insurance contracts with a surrender option. Two broad approaches are usually distinguished in the literature. In the first, the surrender time is defined exogenously whereas in the second one, it is defined endogenously. The first approach consists, roughly, in using a priori fixed probabilities for the surrender time without trying to find a structural explanation for the surrender decision. These probabilities can be defined like survival probabilities are, i.e. through a (time-varying) deterministic instantaneous mortality (here, surrender) rate. In probability theory, these instantaneous rates are more commonly known as intensities. A very general form of this kind of models can be found in the literature on disability insurance. See Linnemann \[62\] for a recent application of this approach to the valuation of life insurance with a surrender option. However, this approach has never really been favored by academics. There are mainly two related reasons. The first one is that the deterministic intensities used in these models imply the independence between the surrender time and the evolution of economic factors. This is not, obviously, a realistic assumption. The second reason is that the surrender is a decision taken by the policyholders and not a purely random event. In order to answer to these drawbacks, the academic literature on fair valuation of insurance contracts has modelled the surrender time as an optimal stopping time with respect to the filtration generated by the prices of the financial assets (see Bacinello \[14\], \[15\], \[16\], \[17\], Grosen and Jorgensen \[47\]), Simon and Van Wouwen \[73\], Tak Kuen Siu \[80\] or Weixi Shen and Huiping Xu \[81\]. In the following, we will called this kind of optimal stopping time model the “traditional model” because of its widespread acceptance in the academic literature. Albizzati and Geman \[6\] described also a kind of ad-hoc model where both approaches are mixed together.

In this chapter, we argue that, if we are to model a realistic surrender time, we should avoid using a stopping time with respect to the filtration generated by the financial prices, mainly because it implicitly assumes that the insurance company and the policyholder have the same set of information and assumes that the surrender decision is accordingly based on this set only. On the contrary, we argue that a policyholder takes his surrender decision on a larger set of information, which is in part, not available to the insurer. Introducing (implicitly or explicitly) this asymmetry of information allows us to model the surrender time as a random time that admits a so-called hazard process. This class of random times is extremely large since essentially any random time not adapted to the filtration generated by the prices of financial
assets, belongs to this class. These random times have been introduced in the default risk literature by Jeanblanc and Rutkowski [58]. See also the book of Bielecki and Rutkowski [23]. In the financial literature, such models are known as the reduced-form models. We show this class of random times includes exogenously as well as endogenously defined surrender times, so that we can reconcile the two main approaches under the asymmetric information assumption. In particular, this model includes the deterministic intensity models as well as the stochastic intensity models which have been recently introduced in the fair valuation literature to model the stochastic mortality (see Dahl [37]).

For the sake of simplicity, we assume there is no mortality risk.

This chapter is organized as follows. In Section 2, we present the theoretical framework. In particular, in Subsection 2.1, we present the financial market and, in Subsection 2.2, the life insurance contract we are going to study. In Section 3, we describe how is modelled the information shared by the policyholder and the insurer and study the impact of introducing a random time. In particular, in Subsection 3.4, we show the equivalence between the absence of arbitrage and the so-called \( (H) \)-hypothesis when the financial market is complete. In Section 4, we present how the surrender time is usually modelled in the literature on fair valuation and give its strengths and weaknesses. Then we introduce alternative random time models based on a hazard process. In Section 5, we show how the general risk neutral valuation formulas can be simplified in this setting. In Section 6, we study these random time in a Brownian filtration. In Section 7, we apply this approach to the valuation of a single premium unit-linked contract in a Gaussian framework.

2. the Theoretical Framework.

2.1. The Financial market. Let \((\Omega, \mathcal{F}, P)\) be a complete probability space. A perfect frictionless financial market is defined on this space. We assume there is one locally risk free asset denoted by \(S_0^0 = S^0(t, \omega)\) and \(s\) risky assets \(S_i^i = S^i(t, \omega), i = 1 \ldots s\) following real cádlág stochastic processes. The price of the locally risk free asset is assumed to follow a strictly positive, continuous path of finite variation process so that we can always find a continuous, finite variation process \(D\) such that \(S_0^0 = e^{Dt}\). The discounted values of these assets are denoted by

\[
X_i^t = \frac{S_i^t}{S_0^0}, \quad i = 0, \ldots s
\]

Even though this assumption is not necessary, it is often assumed there exists a positive predictable stochastic process \(r_t = r(t, \omega)\) called the instantaneous risk free rate such that the value of the locally risk free asset evolves according to the following differential equation

\[
dS_0^0 = r_t S_0^0 dt
\]

with \(S_0^0 = 1\). In this case, we have \(D_t = \int_0^t r_u du\).

Let \(\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}\) be the filtration generated by the stochastic processes \(S_i^i\) for \(i = 0, \ldots s\). We assume this filtration respects the usual hypotheses. Intuitively, it models the information one has about the financial market as it evolves over time. In our context, we will assume that this filtration models the information shared by the insurers and the policyholders.
We assume furthermore that the no-arbitrage hypothesis holds in this financial market with regard to the information set $\mathbb{F}$. This assumption is equivalent to the existence of at least one probability measure $Q$, equivalent to $P$, for which the discounted prices $X^i_t$, $i = 0 \ldots s$ are $(\mathbb{F}, Q)$-local martingales. Such a measure $Q$ is called a local martingale measure.

2.2. The Life Insurance Contract. In life insurance, the traditional premium principle rests on the equivalence between the present value of the insurer’s payments and the present value of the policyholder’s payments. In this section, we describe the way we model these payments. Ultimately, the actual payoffs the insurer gets or pays, depend on the surrender decision. The date at which the surrender occurs is denoted $\tau$. It is a strictly positive random variable ($\mathbb{F}$-measurable) on $\Omega$. In other words, $\tau = \tau(\omega)$ is a random time. We do not assume $\tau$ is an $\mathbb{F}$-stopping time. In the following, we will use the following notation: $t_0$ will denote the initial date of the insurance contract, $T$ the term of the contract and $t$ an arbitrary date between $t_0$ and $T$. Without loss of generality, we will assume $t_0 = 0$. For the sake of simplicity, we do not take the mortality into account.

2.2.1. Payments of the insurer. We mainly need three building blocks. The first one is the payoff the insurer has to pay at the term $T$ of the contract if the policyholder has not surrendered. This payoff is assumed to be a $\mathbb{F}_T$-measurable random variable and is denoted by $g(T, \omega)$. At the term of the contract, the insurer has to pay

$$g(T, \omega)1_{\{\tau > T\}}$$

For a traditional participating life insurance contract, this could be the guaranteed capital. For an annuity contract, this could be the present value (at time $T$) of the annuities. For a unit-linked contract, this could be, for example, the value of the units bought, with or without a guarantee.

The second building block is the amount the insurer has to pay when the policyholder surrenders before the term $T$. This amount is denoted by $1_{\{0 < \tau \leq T\}} R(\tau, \omega)$ where $(R(t, \omega))_{t \geq 0}$ is assumed to be an $\mathbb{F}$-predictable stochastic process. If we define $H_t = H_t(\omega) = 1_{\{\tau(\omega) \leq t\}}$, we can write

$$1_{\{0 < \tau \leq T\}} R(\tau, \omega) = \int_0^T R(u, \omega) dH_u$$

For a traditional participating contract, the policyholder could be allowed, for example, to get back the premiums he has already paid, with a time-dependent penalty or not, and with or without a guaranteed interest rate on these premiums. For a unit-linked contract, he could for example, get back the value of the units he bought with or without penalty, or could be allowed to get back the premiums he paid to buy these units.

The third building block is the payoffs the insurer has to pay as long as the policyholder has not surrendered. We model these payoffs through their cumulative value up to time $t$, $C(t, \omega)$, assumed to be a right-continuous increasing $\mathbb{F}$-adapted process. The cumulative payoff up to the surrender time is given by

$$C(T, \omega)1_{\{\tau > T\}} + C(\tau-, \omega)1_{\{0 < \tau \leq T\}} = \int_0^T (1 - H_u) dC(u, \omega)$$
where we assume $C(0, \omega) = 0$ and $C(T, \omega) = C(T_, \omega)$. This could be the payments of a constant or a time-dependent annuity. Notice, this last payoff can be written as a combination of the two first ones.

2.2.2. Payments of the policyholder. The policyholder is committed to pay premiums periodically as long as he has not surrendered. This is similar to the second building block of the previous subsection. We model these premiums through their cumulative value up to time $t$, $CP(t, \omega)$, assumed to be an $\mathbb{F}$-adapted right-continuous increasing process. The cumulative payoff up to the surrender time is given by

$$CP(T, \omega)1\{\tau > T\} + CP(\tau_, \omega)1\{t_0 < \tau \leq T\} = CP(t_0, \omega) + \int_{t_0}^{T} (1 - H_u) dCP(u, \omega)$$

We assume $CP(T, \omega) = CP(T_, \omega)$. For the sake of simplicity, we will usually consider these premiums are paid at $N$ fixed discrete dates $t_i$ with $i = 0, \ldots, N - 1$. In this case, $CP(\cdot, \omega)$ is a step-wise function. The premium paid at time $t_i$ is denoted by $P(t_i, \omega)$ and is equal to $P(t_i, \omega) = CP(t_i, \omega) - CP(t_{i-1}, \omega)$. We assume $CP(t_0, \omega) = 0$. $P(t_i, \omega)$ is then a random variable $\mathcal{F}_{t_i}$-measurable. The payments stream of the policyholder becomes then

$$\sum_{i=0}^{N-1} P(t_i, \omega)1\{\tau > t_i\}$$

This way of describing a life insurance contract is fairly general. As we have seen, it can accommodate unit-linked products as well as traditional participating contracts.


In order to compute the present values of these payoffs, we rely on the risk neutral valuation principle. In this setting, the value at date $t$ of the liabilities of the insurer or of the policyholder, is given by the expected discounted payoffs under a risk neutral measure $Q$, conditionally on the information known at date $t$. Since our aim is to determine the fair value of an insurance contract from the point of view of an insurer, this expectation should be taken conditionally on the insurer’s information. In our context, this information has a central place. Before studying the valuation formulas, we first precisely describe how this information is modelled.

3.1. The Information. We already defined the process $H_t = 1\{\tau \leq t\}$. Let us introduce a few other definitions.

**Definition 3.1.** Let $\mathbb{H} = (\mathcal{H}_t)_{t \geq 0}$ be the filtration generated by the process $H_t$ i.e $\mathcal{H}_t = \sigma(H_s, 0 \leq t \leq s)$.

The filtration $\mathbb{H}$ models the information an insurance company has over the fact that the policyholder has already surrendered or not.

**Definition 3.2.** Let $\mathbb{G} = (\mathcal{G}_t)_{t \geq 0}$ be the filtration defined as $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t, \forall t \geq 0$.

The filtration $\mathbb{G}$ models altogether the information from the financial market and if the policyholder has surrendered or not. $\mathbb{F}$ is thus a subfiltration of $\mathbb{G}$, i.e., $\mathbb{F} \subseteq \mathbb{G}$. We assume $\mathbb{G}$ is the complete information of the insurer. In particular, this means the insurer has no
private information about its policyholders apart from whether they have already surrendered or not. According to this assumption, \( G \) is thus the filtration under which we should take our expectations in our valuation formulas.

Notice that if we assume \( \tau \) is an \( F \)-stopping time then \( H \subset F \). This means that it is sufficient to observe the financial market to know if one has surrendered or not. In this case, the filtration \( H \) does not offer any additional information and the filtrations \( F \) and \( G \) coincide.

Finally, in general, \( \tau \) is a \( G \)-stopping time and \( H_t \) is \( G \)-adapted.

3.2. Present value of the insurer payment streams. By applying the risk neutral valuation principle, we have the following proposition for the present value of the total liabilities.

**Proposition 3.3.** The value at date \( t \), \( L^C_t \), of the total liabilities of the insurer is given by

\[
L^C_t = E^Q \left[ e^{-(D_T-D_t)} g(T, \omega) 1_{\{\tau>T\}} | G_t \right] \\
+ E^Q \left[ \int_t^T e^{-(D_u-D_t)} R(u, \omega) dH_u | G_t \right] \\
+ E^Q \left[ \int_t^T e^{-(D_u-D_t)} (1 - H_u) dC_u | G_t \right]
\]

(3.1)

**Proof.** This is a straightforward application of the risk neutral valuation principle to the three building blocks described in the previous section. \( \square \)

3.3. Present value of the premium payment streams. As far as the present value \( A_t \) of the liabilities of the policyholder is concerned, we have according to the risk neutral valuation principle, the following proposition.

**Proposition 3.4.** The value at time \( t \), \( A_t \), of the liabilities of the policyholder is given by

\[
A_t = E^Q \left[ \int_t^T e^{-(D_u-D_t)} (1 - H_u) dC_P(u, \omega) | G_t \right]
\]

If we assume the premiums are paid at \( N \) fixed discrete dates, we have

\[
A_t = E^Q \left[ \sum_{i=\lceil t \rceil}^{N-1} e^{-(D_{t_i}-D_t)} 1_{\{\tau>t_i\}} P(t_i, \omega) | F_t \right]
\]

(3.2)

where \( \lceil t \rceil = \inf \{i | t_i > t \} \).

**Proof.** Straightforward. \( \square \)

In order to have a fairly valued contract at the initial date \( t_0 \) of the contract, the present values of the liabilities of both parties must be equal: \( L_{t_0} = A_{t_0} \). At a later date \( t \geq t_0 \), the difference \( V_t = A_t - L_t \) gives the fair value of the contract for the insurer.

It is worth noticing that we can imagine alternative ways of modelling the payoff when one surrenders. In the model we described above, the payment is made at the exact time of surrender. But we can also consider that the payment is made at discrete times. For example, it is probably more realistic to consider that the policyholder will get the amount \( R \) at the end
of the week or the month during which he surrenders. Let us denote $t_j$ the different times of payment with $j = 1 \ldots K$ and $t_K = T$. In this case, the payoff at time $t_j$ is given by:

$$R(t_j, \omega)1_{\{t_{j-1} < \tau \leq t_j\}}$$

The present value of the commitments of the insurer in the discrete case are then

$$L^D_t = E^Q[e^{-(D_T - D_t)}g(T, \omega)1_{\tau > T} + \int_t^T e^{-(D_u - D_t)}(1 - H_u)dC(u, \omega) + \sum_{j=\lfloor t \rfloor}^k e^{-(D_j - D_t)}R(t_j, \omega)1_{\{t_{j-1} < \tau \leq t_j\}}|G_t]$$

This model can be a useful alternative to the one described above. It should also be a good approximation of $L^C_t$ as the intervals $|t_{j-1}, t_j]$ get shorter. We could also have discretized the process $C(\cdot, \omega)$ as we did for $CP(\cdot, \omega)$.

### 3.4. The Absence of Arbitrage and the Enlargement of Filtration.

In Section 2.1, we assumed that there was no-arbitrage in the financial market with respect to the information set $\mathcal{F}$. This is a standard assumption in the financial literature. But in the last section, we actually made a stronger assumption: by using the risk neutral valuation principle, we assumed the existence of an equivalent measure under which the discounted prices $X^i$ are $(\mathcal{G}, Q)$-local martingales. Accordingly, we implicitly assumed there was no-arbitrage, not only, under the filtration $\mathcal{F}$ but also under the new filtration $\mathcal{G}$. Is this a reasonable economic assumption? Does the observation of the surrender time affect the no-arbitrage hypothesis in the financial market? We think the answer to this question is obvious: we cannot expect the observation of the surrender time will give rise to arbitrage opportunities in the financial market if there is not initially. Accordingly, the discounted prices $X^i$ should also be $(\mathcal{G}, Q)$-local martingales.

In general, going from an initial filtration $\mathcal{F}$ to an enlarged one $\mathcal{G}$, $(\mathcal{F}, Q)$-local martingales are not necessarily $(\mathcal{G}, Q)$-local martingales. Accordingly, the no-arbitrage hypothesis under the filtration $\mathcal{F}$ does not necessarily imply the no-arbitrage hypothesis under the filtration $\mathcal{G}$. In the following, we derive necessary and sufficient condition for this assumption to hold under $\mathcal{G}$. As we will see, it implies a constraint on the $Q$-distribution of the random time $\tau$.

Let us first recall a definition and a couple of results.

**Definition 3.5.** Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let $\mathcal{G}$ be a filtration and $\mathcal{F}$ be an arbitrary sub-filtration of $\mathcal{G}$ i.e., for every $t$, $\mathcal{F}_t \subseteq \mathcal{G}_t$. We say the $(H)$-hypothesis holds under $Q$ between the filtrations $\mathcal{F}$ and $\mathcal{G}$ if and only if: for every $t$, $\mathcal{F}_\infty$ and $\mathcal{G}_t$ are conditionally independent with respect to $\mathcal{F}_t$.

The $(H)$-hypothesis is equivalent to the invariance of (local) martingales for $\mathcal{F}$ and $\mathcal{G}$. More precisely, according to Brémaud and Yor [32], we have

**Lemma 3.6.** The following assertions are equivalent.

1. The $(H)$-Hypothesis holds.
2. Every $(\mathcal{F}, Q)$-local martingale is a $(\mathcal{G}, Q)$-local martingale.

In particular, in our setting, we have the following lemma.
Lemma 3.7. The \((H)\)-hypothesis is equivalent to \(Q(\tau \leq t|\mathcal{F}_\infty) = Q(\tau \leq t|\mathcal{F}_t), \forall t \geq 0\).

Proof. See Jeanblanc and Rutkowski [58]. □

We can now prove the relations between the \((H)\)-hypothesis and the no-arbitrage hypothesis under \(\mathcal{G}\).

Proposition 3.8. If there is no-arbitrage in our financial market with respect to \(\mathcal{F}\) and if the \((H)\)-hypothesis holds, then there is no-arbitrage with respect to \(\mathcal{G}\).

Proof. Straightforward. Since there is no-arbitrage opportunity with respect to \(\mathcal{F}\), there exists at least one equivalent (to \(P\)) measure \(Q\) such that the discounted prices \(X^i\) are \((\mathcal{F},Q)\)-local martingales. By Lemma 3.6, if the \((H)\)-hypothesis holds, these discounted prices \(X^i\) are also \((\mathcal{G},Q)\)-local martingales so that there is no-arbitrage opportunity in the financial market considering the filtration \(\mathcal{G}\). □

The following proposition gives almost the converse.

Proposition 3.9. Assume our financial market is arbitrage free and complete under \(\mathcal{F}\). If there is no-arbitrage in the financial market under \(\mathcal{G}\) then the \((H)\)-hypothesis holds between \(\mathcal{F}\) and \(\mathcal{G}\).

Proof. See Blanchet-Scalliet and Jeanblanc [30]. □

If we assume market completeness under \(\mathcal{F}\), the \((H)\)-hypothesis is equivalent to the no-arbitrage hypothesis thanks to Propositions 3.8 and 3.9. Notice though that Proposition 3.8 does not require market completeness; the \((H)\)-hypothesis is thus a sufficient condition for the absence of arbitrage even under market incompleteness.

Actually, in general, this \((H)\)-hypothesis is not necessarily invariant if we go from the measure \(Q\) to an equivalent one, like the physical measure \(P\) for example, see [58] for a counter example due to Kusuoka. If we nevertheless assume this condition is also true under the physical measure \(P\), we have \(P(\tau \leq t|\mathcal{F}_\infty) = P(\tau \leq t|\mathcal{F}_t), \forall t\), then in our setting, this condition is intuitively very clear: it tells us the information over the future evolution of the financial market (after \(t\)) does not give us more information on the surrender before \(t\) than does the information on the financial market up to time \(t\). As a consequence, it also tells us that the fact that one surrenders will not affect the evolution of the financial market. This implication is very natural. Accordingly, even though, strictly speaking, the \((H)\)-hypothesis is only crucial under \(Q\), it seems to be harmless to equally impose this \((H)\)-hypothesis under \(P\).

Notice also this setting covers two important special cases:

1. when \(\tau\) is an \(\mathcal{F}\)-stopping time, \(\mathcal{F}\) and \(\mathcal{G}\) coincide so that there is no enlargement of filtration. The risk neutral valuation framework can then be use safely. Indeed, in this case, the condition \(Q(\tau \leq t|\mathcal{F}_\infty) = Q(\tau \leq t|\mathcal{F}_t)\) is trivially satisfied.

2. when \(\tau\) is \(Q\)-independent of \(\mathcal{F}\), this \((H)\)-hypothesis directly holds since in this case, \(Q(\tau \leq t|\mathcal{F}_\infty) = Q(\tau \leq t) = Q(\tau \leq t|\mathcal{F}_t), \forall t\). We have this assumption when the surrender time is exogenously defined as described in the introduction. This assumption also appears in the stochastic mortality literature. The arguments of this section
show, as expected, the mortality cannot induce any arbitrage opportunity, under this independence assumption.

4. The Surrender Time.

Up to now, we have not made any assumption yet on the date $\tau$ at which the surrender occurs. In the first subsection, we describe how $\tau$ is traditionally modelled in the literature on fair valuation of insurance contracts (see Bacinello [14] and [15] or Grosen and Jorgensen [47]). We give the pros and cons. In the second subsection, we present an alternative model based on an $F$-hazard characterization of $\tau$ and explain how it could answer to the drawbacks of the traditional model.

4.1. $\tau$ as an optimal $F$-stopping time. The literature on fair valuation of insurance contracts has mainly relied on the following definition of $\tau$:

$$\tau = \inf \{ t | R(t, \omega) \geq -V(t, \omega) \}$$

where $V(t, \cdot) = A_t - L_t$ is the fair value of the insurance contract at time $t$ if one has not surrendered before $t$ and does not surrender at time $t$. This definition comes from the optimal stopping time literature, in general, and from the finance literature on the optimal exercise time of American style options, in particular. This definition tells us that it is optimal for the policyholder to surrender when the fair value of his contract is lower or equal to the amount that he can get by immediately surrendering this contract.

At first, this definition seems very appealing from a theoretical point of view. By defining the surrender behavior in this way, we endogenously determine the time of surrender. Indeed, this surrender time actually arises from the very characteristics of the insurance contract and in the meantime, depends intrinsically on the evolution of the financial market. Unfortunately, it appears that this definition has also some drawbacks.

Firstly, since $\tau$ is defined as an optimal stopping time, the policyholder is assumed to be able to process all the information, to make all the required complex calculations and is able to act in a perfectly rational manner. Obviously, this is not the way people behave, probably not even the best actuaries.

Secondly, policyholders are not only assumed to act rationally but are also assumed to take their decisions on exactly the same set of information $F$. As a consequence, it means, for a given generation, all the policyholders would behave as one: we would observe a period without any surrender and suddenly all would surrender at the same time. Even though the surrender decisions among different policyholders are indeed correlated, a perfect correlation is not realistic. This also means no room is left for idiosyncratic information that could trigger the decision of surrender. Notice that this criticism has nothing to do with the fact $\tau$ is defined as "optimal" but does come from the fact that $\tau$ is an $F$-stopping time. Notice also, the information $F$ is assumed to be known by the insurer so that this definition of $\tau$ does not allowed for any asymmetry of information between the policyholders and the insurer.

Thirdly, since to our knowledge, all the continuous-time literature assumes $F$ to be a Brownian filtration, $\tau$ is then even an $F$-predictable stopping time. This means intuitively that, at each time $t$, the insurance company is able to predict if its policyholders are going to surrender
or not during the next small interval of time $dt$. There is never an unexpected surrender. Ob-
viously, in the real world, an insurance company cannot perfectly predict if someone is going to
surrender or not, because the decision is taken, in part, on information that are not accessible
to the insurance company (even though the evolution of the financial market plays a role in
the decision and can give the insurer a valuable information on the probability of surrender).
One could argue we could overcome this "predictability" problem simply by using a model of
financial prices with an unpredictable component, such as a Lévy process with jumps for exam-
ple. In this case, we would have indeed unpredictable surrenders while keeping $\tau$ an $\mathbb{F}$-stopping
time. The trouble with this approach is the "unpredictability" of the surrender would come
from the characteristics of the financial market and not from the surrender decision itself.

Fourthly, when $\tau$ is an $\mathbb{F}$-stopping time, if the financial market is complete, so is the insur-
ance market. It gives the false impression that the surrender risk can be, at least theoretically,
perfectly hedged away and therefore, that the surrender is not really risky for an insurance
company. As we said, it misses the point that surrender is an unpredictable event and that
for this reason, surrender is risky for an insurance company. From a risk management point
of view, it is clear we cannot rely on and base decisions on this kind of model. For pricing
purposes, one could still accept this definition, arguing it gives us an upper bound of the fair
value of a life insurance contract. The trouble is that using different models for pricing and
risk management, that are by nature, not consistent with each other, is probably not very
appropriate to manage an insurance company.

Finally, the value $V(t, \omega)$ that we compare to the surrender value $R(t, \omega)$, should be the
value of the contract from the policyholder perspective. Unfortunately, usually, the value
$V(t, \omega)$ used in the literature is the value of the contract from the insurer perspective (see also
Bacinello [17] for a discussion of this point). The value of the contract from the policyholder
perspective depends actually on his own information and its own risk aversion. It is thus a
subjective value. By definition, a subjective value is different from one policyholder to another
and is unknown for the insurer. Accordingly, this argument implies once again the existence of
an asymmetry of information.

In conclusion, contrary to the traditional model of $\tau$, we argue that the surrender decision
should be random with respect to the shared information $\mathbb{F}$, even though it should not be
independent of it. This information should actually influence the randomness of the surrender
decision. This leads to the conclusion that the $\mathbb{F}$-measurability of the surrender decision is
really the crucial point. In other words, we argue that we should avoid the hypothesis that $\tau$
is an $\mathbb{F}$-stopping time.

4.2. $\tau$ as a random time characterized by an $\mathbb{F}$-hazard process. In the previous
subsection, we saw that using optimal stopping times to model the surrender decisions, does not
allow us to model realistically neither the behavior of the policyholder nor the unpredictable
component of the surrender decision for the insurance company (at least when $\mathbb{F}$ is the Brownian
filtration). We also saw that it was crucial to avoid defining $\tau$ as an $\mathbb{F}$-stopping time, but that
it was also crucial to allow the probability of surrender to depend on the information $\mathbb{F}$. In this
subsubsection, we argue that we can introduce an asymmetry of information by modelling $\tau$ as a
random time that admits an $\mathcal{F}$-hazard process. Let us first introduce our definition of $\tau$ and the definition of the associated hazard process.

**Definition 4.1.** Let $\tau$ be a non negative random variable defined on $(\Omega, \mathcal{F}, P)$. Assume $P(\tau = 0) = 0$ and $P(\tau > t) > 0$, $\forall \ t \geq 0$. We define the $(\mathcal{F}, P)$-hazard process of $\tau$, denoted $\Gamma$ by

$$F_t = 1 - e^{-\Gamma_t}$$

$\forall \ t \geq 0$.

In order to have $\Gamma_t$ well-defined for all $t$, we have to impose the condition $F_t < 1$ because $\Gamma_t = -\ln(1 - F_t)$. This excludes $\tau$ of being an $\mathcal{F}$-stopping time. In other words, an $\mathcal{F}$-stopping time cannot admit an $\mathcal{F}$-hazard process. Notice that $F_t$ is an $\mathcal{F}$-adapted stochastic process and admits a right continuous modification. Accordingly, $\Gamma_t$ is also an $\mathcal{F}$-adapted càdlàg stochastic processes with $\Gamma_t < \infty \ \forall \ t$ and $\Gamma_0 = 0$.

As already explained, when $\tau$ is an $\mathcal{F}$-stopping time, we can tell from the observation of the financial market if a policyholder has already surrendered or not. By modelling $\tau$ as defined above, we exclude this situation. Here, the surrender decision depends necessarily on elements outside the information $\mathcal{F}$. These elements are idiosyncratic information can be different from a policyholder to another (unless the surrender decision would still be perfectly correlated). Accordingly, in defining $\tau$ as above, we implicitly model an asymmetry of information between the policyholders and the insurer. For the same reason, the surrender decision is indeed unpredictable for the insurer, even when $\mathcal{F}$ is a Brownian filtration. This unpredictability here does not come from the characteristics of the financial market but from the asymmetry of information itself. Notice that even though the surrender decision is unpredictable, the probability of surrender is stochastic and depends on the evolution of the financial market (through $\Gamma_t$). Let us give some examples of such random times.

**Example.**

1. Probably the most simple random time having a hazard characterization is an exponential random time $\tau$ with constant intensity $\lambda > 0$. Indeed we have:

$$F_t = 1 - P(\tau > t | \mathcal{F}_t)$$

$$= 1 - P(\tau > t)$$

$$= 1 - e^{-\lambda t}$$

The hazard process is here an increasing positive deterministic function, equals to $\Gamma_t = \lambda t$.

2. The time of the first jump of inhomogeneous Poisson process independent of $\mathcal{F}$, with hazard function $\Lambda(t)$ (a deterministic càdlàg function), is also such a random time.

$$F_t = 1 - P(N_t = 0)$$

$$= 1 - e^{-\Lambda(t)}$$

Again, the hazard process $\Gamma$ is here an increasing positive deterministic function $\Gamma_t = \Lambda(t)$. More formally, this function $\Lambda(t)$ is actually a measure defined on the
measurable space \((\mathbb{R}, \mathcal{B})\) where \(\mathcal{B}\) is the Borel \(\sigma\)-algebra on \(\mathbb{R}\). If this measure is absolutely continuous with regard to the Lebesgue measure, we can find the intensity \(\lambda(t)\) of the Poisson process, which is a positive function, given by

\[
\Lambda(t) = \int_0^t \lambda(u) du
\]

The definition of \(\tau\) given above also includes the time of the first jump of a Cox process. This process is a counting process that generalizes the Poisson process to the case where the measure \(\Lambda\) is a random measure on \((\mathbb{R}, \mathcal{B})\). Such a counting process \(N_t\) is also known as a doubly stochastic Poisson process. We can define a Cox process in the following manner. \(N_t\) follows a Cox process if and only if conditionally on \(\mathcal{F}_t\), \(N_t\) follows an inhomogeneous Poisson process for all \(t\).

The hazard function of the inhomogeneous process can be given, conditionally on \(\mathcal{F}_t\), by the realization of an increasing positive \(\mathcal{F}\)-adapted stochastic process \(\Lambda_t\). Since \(\Lambda_t\) is assumed \(\mathcal{F}\)-adapted, \(\Lambda_t\) is indeed, conditionally on \(\mathcal{F}_t\), a deterministic function. Furthermore, since each path is increasing, cádlág and positive, \(\Lambda_t\) induced, conditionally on \(\mathcal{F}_t\), a deterministic measure on \((\mathbb{R}, \mathcal{B})\) and unconditionally, a random measure on the same space. See Grandell [45] for a more formal description.

According to this definition, conditionally on \(\mathcal{F}_t\), the distribution of the time of the first jump of a Cox process is given by

\[
F_t = 1 - P(N_t = 0|\mathcal{F}_t) = 1 - e^{-\Lambda_t}
\]

Once again, this random time admits an increasing, positive \(\mathcal{F}\)-hazard process \(\Gamma_t\) given by \(\Gamma_t = \Lambda_t\). Unconditionally, we also have

\[
P(\tau \leq t) = E^P[1 - e^{-\Lambda_t}] = 1 - E^P[e^{-\Lambda_t}]
\]

If \(\Lambda_t\) is absolutely continuous with respect to the Lebesgue measure, it exists a unique positive \(\mathcal{F}\)-adapted predictable process \(\lambda_t\) called the \((\mathcal{F},\mathcal{P})\)-intensity process of \(N\), such that

\[
\Lambda_t = \int_0^t \lambda_u du
\]

A few elements are worth underlining. Firstly, in the two first examples, the hazard process is a deterministic function and \(\tau\) cannot depend on the financial market. In the case of the Cox process, the hazard process is a stochastic process which allows the probability of the surrender time to depend on the evolution of the financial market.

Secondly, the examples of random times given here are, all three, the time of the first arrival of a counting process. But we would like to highlight that these are just special cases of our definition and do not represent all the random times that admit a hazard process. Actually, as long as we construct a random time with \(P(\tau \leq t|\mathcal{F}_t) \neq 1\), we can find a process \(\Gamma_t\).

Thirdly, in these three examples, the hazard functions or the hazard process are increasing since each three induces a (deterministic or random) measure on the Borel sets. Our definition
of \( \tau \) does not depend on this assumption. As we said, these examples are just special cases of our definition and in general we can find random times admitting positive non necessarily increasing hazard process. So, \( \Gamma_t \) increasing is not a necessary assumption. However, the \((H)\)-hypothesis, we have seen in Section 3.4, implies the process \( \hat{F}_t \) defined as \( \hat{F}_t = Q(\tau \leq t|\mathcal{F}_t) \), is an increasing process. The \((\mathcal{F}, Q)\)-hazard process \( \hat{\Gamma}_t \) defined as \( \hat{\Gamma}_t = -\ln(1 - \hat{F}_t) \) similarly to the Definition 4.1, is then also an increasing process and if \( \hat{\lambda}_t \) defined as \( \hat{\Gamma}_t = \int_0^t \hat{\lambda}_u du \), exists, then it is a positive process. So if we assume the \((H)\)-hypothesis holds then \( \hat{\Gamma}_t \) increasing is a necessary condition for constructing \( \tau \) but \( \Gamma_t \) increasing, is not since as we said, the \((H)\)-hypothesis is not necessarily true under \( P \) and does not have to be\(^1\). This fact justifies that we can focus, under \( Q \), on Cox processes to model the surrender time.

4.3. Construction of \( \tau \). The hypothesis made on \( \tau \) in the previous subsection, are very weak and allows us to construct \( \tau \) in a great variety of ways. Actually, the two broad approaches described in the introduction (the exogenous and endogenous one) are compatible with our definition of \( \tau \).

The first one consists in describing explicitly the surrender behavior and the asymmetry of information between the policyholder and the insurer. In other words, it consists in describing the surrender from the policyholder's point of view. For example, we can generalize the traditional modelling of the surrender time. We saw that the decision of surrender was taken on a larger set of information that includes idiosyncratic information. Let us denote this policyholder's information set by a filtration \( \mathcal{J} \) with \( \mathcal{F} \subset \mathcal{J} \). We could consider \( \tau \) to be a \( \mathcal{J} \)-optimal stopping time instead of an \( \mathcal{F} \)-optimal stopping time. For example, we could model a stochastic mortality which would depend on information related to the health of the policyholder and only known by himself and not the insurer. In this case, we could still defined \( \tau \) as \( \tau = \inf \{ t | R(t, \omega) \geq -V(t, \omega) \} \) but where \( -V(t, \omega) \) is the value of the insurance contract for the policyholder but, this time, given his own information \( \mathcal{J} \). In this case, \( \tau \) would be endogenously defined but still not \( \mathcal{F} \)-measurable. Accordingly, we could then derive a probability of surrender \( \hat{F}_t \neq 1 \forall t \) such that \( \hat{\Gamma}_t = -\ln(1 - \hat{F}_t) \) makes sense. Even if this model would probably be too complex to deal with, it is interesting to realize that this approach does not exclude structural models of the surrender time. Another simpler example of this approach is the following. Let us assume that one surrenders when the value of a given financial asset \( S_t \) reaches a certain level, \( E \). This level at which one surrenders, is specific to each policyholder and is unknown by the insurer. The insurance contract portfolio is then heterogeneous with respect to this level. Finally, let assume this level follows a unit exponential distribution under \( Q \) and that \( E \) is independent of \( \mathcal{F} \). In this case, we have \( \hat{F}_t = 1 - e^{-\sup_{0 < s \leq t} S_s} \) so that the \((\mathcal{F}, Q)\)-hazard process is given by \( \hat{\Gamma}_t = \sup_{0 < s \leq t} S_s \) which is indeed \( \mathcal{F} \)-adapted, cádlág, positive and increasing. Our point in giving these examples is to show our definition allows a structural model of \( \tau \) or in other words, allows an endogenous definition of the surrender time.

The second approach considers the insurer point of view and is closer to an exogenous specification of \( \tau \). It simply consists in specifying directly the process followed by \( \hat{\Gamma} \) without

\(^1\)In Section 3.4, we argued it is probably a reasonable and harmless assumption to assume the \((H)\)-hypothesis holds under \( P \) too. In this case, \( \Gamma_t \) is obviously also an increasing process.
necessarily trying to describe explicitly the actual surrender behavior nor the asymmetry of information. One way to construct the random time \( \tau \) given a hazard process, is the following. Let \( E \) be a random variable (\( F \)-measurable) defined on \( \Omega \), but which is independent of the filtration \( (F_t)_{0 \leq t \leq T} \). \( E \) is assumed to have an unit exponential distribution under \( Q \). Let \( \hat{\Lambda} \) be an \( F \)-adapted increasing positive process with \( \hat{\Lambda}_0 = 0 \) and \( \hat{\Lambda}_t < \infty \). If \( \tau \) is defined as
\[
\tau = \inf \{ t | \hat{\Lambda}_t \geq E \}
\]
then \( \tau \) is a random time with an \( F \)-hazard process under \( Q \) given by \( \hat{\Gamma} = \hat{\Lambda} \). Indeed, we have:
\[
Q(\tau \leq t | F_t) = Q(\hat{\Lambda}_t \geq E | F_t)
= 1 - e^{-\hat{\lambda}_t}
\]
Alternatively, we can also start with a positive \( F \)-adapted predictable stochastic process \( \hat{\lambda} \) and define \( \tau \) as
\[
\tau = \inf \{ t | \int_0^t \hat{\lambda}_u du \geq E \}
\]
In this case, \( \tau \) is a random time with an \((F, Q)\)-hazard process given by \( \hat{\Gamma}_t = \int_0^t \hat{\lambda}_u du \) and with an \((F, Q)\)-intensity process equals to \( \hat{\lambda} \). Indeed, in both cases, the random times constructed can be seen as the time of the first jump of Cox processes. Both of these constructions also offer an easy way to simulate the random time \( \tau \). We simply have to simulate the hazard process and an independent exponential random variable. The two methods described above are standard constructions which have the advantage of being very convenient but we can go further at the cost of constructing random times less tractable. For example, even if \( \hat{\Lambda} \) is not an increasing process, we can still define \( \tau \) in the same way. Let \( \hat{\Lambda} \) be an \( F \)-adapted positive process with \( \hat{\Lambda}_0 = 0 \). We can define \( \tau = \inf \{ t | \hat{\Lambda}_t \geq E \} \). In this case, we have
\[
Q(\tau \leq t | F_t) = Q( \sup_{0 \leq s \leq t} \hat{\Lambda}_s \geq E | F_t)
= 1 - e^{-\sup_{0 \leq s \leq t} \hat{\Lambda}_s}
\]
The random time \( \tau \) admits an \((F, Q)\)-hazard process characterization with the hazard process \( \hat{\Gamma}_t = \sup_{0 \leq s \leq t} \hat{\Lambda}_s \). This random time can still be seen as the time of the first jump of a Cox process since \( \sup_{0 \leq s \leq t} \hat{\Lambda}_s \) is an increasing càdlàg positive \( F \)-adapted process. Obviously, this form is less tractable than the other two, but can be sometime justified. Notice the second example of the first approach can be seen as an application of this last construction. Accordingly, the 2 approaches we gave, are not necessarily antagonist.

In practice, one would probably prefer to specify directly \( \hat{\lambda} \) defined as \( \hat{\Gamma}_t = \int_0^t \hat{\lambda}_u du \) instead of \( \hat{\Gamma} \) since the first one has an intuitive meaning the second lacks. It is well known that the intensity parameter \( \lambda \) or the intensity function \( \lambda(t) \) of a Poisson process corresponds to what is known in life insurance as the instantaneous mortality rate. When it exists, we have the well-known following expression:
\[
\lambda(t) = - \frac{d \ln (1 - P(\tau \leq t))}{dt} = \lim_{\Delta t \downarrow 0} \frac{P(t < \tau \leq t + \Delta t | \tau > t)}{\Delta t}
\]
The Risk Neutral Valuation Formulas.

So informally, $\lambda(t)dt$ is equal to the probability one surrenders during the next interval $dt$ if one has not yet surrendered before time $t$. This conditional probability is here deterministic for all $t$. When it exists, we have a similar result for the $(\mathbb{F}, Q)$-intensity process

$$\hat{\lambda}_t = \lim_{\Delta t \downarrow 0} \frac{Q(t < \tau \leq t + \Delta t | \tau > t \cup \mathcal{F}_t)}{\Delta t}$$

So informally, $\hat{\lambda}_t dt$ is again equal to the probability one surrenders, under $Q$, during the next interval $dt$ if one has not surrendered before time $t$ and knowing the information $\mathcal{F}_t$. Obviously, this probability is stochastic and for a given time $t$, its value depends on the financial market. To model this instantaneous probability, we can make an explicit reference to the financial asset prices, by specifying $\hat{\lambda}$ as a function of these prices. In most applications, in order to get tractable formulas, we will restrict ourselves to Markov intensity processes. Let $\hat{\lambda}(\cdot, \cdot)$ be a function from $\mathbb{R} \times \mathbb{R}^{s+1} \to \mathbb{R}_+$. In this case, we will define the intensity process of $\tau$ as:

$$\hat{\lambda}_t(\omega) = \hat{\lambda}(t, S_t(\omega))$$

Notice this stochastic process is indeed $\mathbb{F}$-adapted.

To conclude, the definition of $\tau$ as a random time characterized by an $(\mathbb{F}, Q)$-hazard process allows us to specify the surrender time in a great variety of ways and seem to be a valuable alternative to the traditional modelling. Notice that we constructed the distribution of $\tau$ directly under $Q$ through the specification of $\hat{\Gamma}$. To find an explicit functional between $\hat{\Gamma}$ and $\Gamma$ is not easy task in general. In Section 6, we give a simple functional under certain simplifying assumptions.

5. The Risk Neutral Valuation Formulas.

In this section, we show how the risk neutral valuation formulas of Sections 3.2 and 3.3 can be modified when we assumed $\tau$ is characterized by an $\mathbb{F}$-hazard process or an $\mathbb{F}$-intensity process. In the first subsection, we introduce a number of useful results that allows us to get rid of the indicator function by taking conditional expectations with respect to $\mathcal{F}_t$ instead of $\mathcal{G}_t$. In the second and third subsections, we apply these formulas to our valuation problem.

5.1. Conditional expectations with respect to $\mathcal{F}_t$. In this subsection, we assume the random time $\tau$ admits an $\mathbb{F}$-hazard process $\Gamma$ under the probability measure considered.

**Lemma 5.1.** Let $X$ be a $\mathcal{F}_s$-measurable random variable with $s \geq t$. We have

$$E \left[ 1_{\{\tau>s\}} X \mid \mathcal{F}_t \right] = 1_{\{\tau>t\}} E \left[ e^{-\int_s^\tau \Gamma_u du} X \mid \mathcal{F}_t \right]$$

If moreover, $\Gamma$ is absolutely continuous, it exists an $\mathbb{F}$-predictable measurable process, referred to as the $\mathbb{F}$-intensity process of $\tau$, such that $\Gamma_t = \int_0^t \lambda_u du$. The last formula becomes

$$E \left[ 1_{\{\tau>s\}} X \mid \mathcal{G}_t \right] = 1_{\{\tau>t\}} E \left[ e^{-\int_t^\tau \lambda_u du} X \mid \mathcal{F}_t \right]$$

**Proof.** See Jeanblanc and Rutkowski [58].

In Section 7 when we study unit-linked contracts, we will actually only use this first lemma.
LEMMA 5.2. Let \( Z \) be an \( \mathbb{F} \)-predictable process. Then for any \( t < s \leq \infty \), we have
\[
E\left[ \int_t^s Z_u dH_u | \mathcal{G}_t \right] = 1_{\{\tau > t\}} e^{\Gamma_t} E\left[ \int_t^s Z_u dF_u | \mathcal{F}_t \right]
\]
If \( F_t \) is a continuous increasing process, then
\[
E\left[ \int_t^s Z_u dH_u | \mathcal{G}_t \right] = 1_{\{\tau > t\}} e^{\Gamma_t} E\left[ \int_t^s Z_u e^{-\Gamma_u} d\Gamma_u | \mathcal{F}_t \right]
\]
If moreover, \( \tau \) admits an \( \mathbb{F} \)-intensity process such that \( \Gamma_t = \int_0^t \lambda_u du \), we have
\[
E\left[ \int_t^s Z_u dH_u | \mathcal{G}_t \right] = 1_{\{\tau > t\}} e^{\Gamma_t} E\left[ \int_t^s \lambda_u Z_u e^{-\int_t^u \lambda_v d\Gamma_v} du | \mathcal{F}_t \right]
\]
**Proof.** See Jeanblanc and Rutkowski [58]. □

LEMMA 5.3. Let \( C \) be a right continuous \( \mathbb{F} \)-adapted process. If \( F_t \) follows a process of finite variation then for every \( t \leq s \)
\[
E\left[ \int_t^s (1 - H_u) dC_u | \mathcal{G}_t \right] = 1_{\{\tau > t\}} e^{\Gamma_t} E\left[ \int_t^s e^{-\Gamma_u} dC_u | \mathcal{F}_t \right]
\]
If moreover, \( \tau \) admits an \( \mathbb{F} \)-intensity process such that \( \Gamma_t = \int_0^t \lambda_u du \), we have
\[
E\left[ \int_t^s (1 - H_u) dC_u | \mathcal{G}_t \right] = 1_{\{\tau > t\}} e^{\Gamma_t} E\left[ \int_t^s e^{-\int_t^u \lambda_v d\Gamma_v} dC_u | \mathcal{F}_t \right]
\]
**Proof.** A proof can be found in Jeanblanc and Rutkowski for an \( \mathbb{F} \)-predictable process \( C \) but this assumption is actually not used in their proof. □

If the \((H)\)-hypothesis holds under the probability measure considered, it implies \( F_t \) is an increasing process, and \( F_t \) is then indeed of finite variation.

5.2. Present value of the insurer’s payments. In order to simplify Equation (3.1), we can directly apply the 3 lemmas of the last subsection. We have the following proposition.

**Proposition 5.4.** Let us assume \( \hat{F}_t \) is an increasing process\(^2\), the present value of the insurer payments is given by
\[
L^C_t = 1_{\{\tau > t\}} E^Q \left[ e^{-\left(D_T - D_t\right)} e^{-\left(\hat{\Gamma}_t - \Gamma_t\right)} g(T, \omega) | \mathcal{F}_t \right] + 1_{\{\tau > t\}} E^Q \left[ \int_t^T e^{-\left(D_u - D_t\right)} e^{\hat{\Gamma}_u} R(u, \omega) d\hat{F}_u | \mathcal{F}_t \right]
\]
\[
+ 1_{\{\tau > t\}} E^Q \left[ \int_t^T e^{-\left(D_u - D_t\right)} e^{-\left(\hat{\Gamma}_u - \Gamma_t\right)} dC_u | \mathcal{F}_t \right]
\]
\[(5.1)\]
\(^2\)This assumption is necessary if we want the \((H)\)-hypothesis to hold.
If $\hat{F}_t$ is an increasing continuous process and $\tau$ admits an $(\mathbb{F}, Q)$-intensity process $\hat{\lambda}$ and if an instantaneous risk-free rate $r_t$ exists, then the present value of the insurer payments is given by

$$L^C_t = 1_{\{\tau > t\}}E^Q \left[ e^{-\int_t^T (r_u + \hat{\lambda}_u)du} g(T, \omega) | \mathcal{F}_t \right]$$

$$+ 1_{\{\tau > t\}}E^Q \left[ \int_t^T \hat{\lambda}_u e^{-\int_u^T (r_v + \hat{\lambda}_v)dv} R(u, \omega) du | \mathcal{F}_t \right]$$

$$+ 1_{\{\tau > t\}}E^Q \left[ \int_t^T e^{-\int_u^T (r_v + \hat{\lambda}_v)dv} dC_u | \mathcal{F}_t \right].$$

(5.2)

5.3. Present value of the policyholder’s payments. Again, we can apply Lemma 5.1 to Equation (3.2). We then have the following proposition.

**Proposition 5.5.** Let us assume the premiums are paid at fixed discrete dates $t_i$ with $i = 0, \ldots, N - 1$, then the present value of the policyholder’s payments is given by

$$A_t = 1_{\{\tau > t\}}E^Q \left[ \sum_{i=\lceil t \rceil}^{N-1} e^{-\int_{t_i}^{t_{i+1}} \hat{\Gamma}_u - \hat{\Gamma}_t} P(t_i, \omega) | \mathcal{F}_t \right]$$

If $\tau$ admits an $(\mathbb{F}, Q)$-intensity process $\hat{\lambda}$ and if an instantaneous risk-free rate $r_t$ exists, we have

$$A_t = 1_{\{\tau > t\}}E^Q \left[ \sum_{i=\lceil t \rceil}^{N-1} e^{-\int_{t_i}^{t_{i+1}} r_u + \hat{\lambda}_u du} P(t_i, \omega) | \mathcal{F}_t \right]$$

where $\lceil t \rceil = \inf \{i | t_i > t \}.$

These expressions generalize well known results when $\tau$ is stochastic but independent of $\mathbb{F}$ like in the stochastic mortality models. We see $\hat{\Gamma}_t$ plays the same role as $D_t$ or when they exist, the intensity process $\hat{\lambda}_t$ is similar to the short term rate $r_t$. This means the value of a life insurance contract with a surrender risk can be considered, except for the second term in Equations (5.1) and (5.2), to be equal to the value of the same contract without surrender risk but under a modified stochastic term structure $D_t + \hat{\Gamma}_t$. This also means all the results developed in the extensive literature on option pricing models with stochastic interest rates can be applied, mutatis mutandis, to the valuation of life insurance contracts with stochastic surrender.

5.4. Change of measure. Change of measure techniques have been very useful to derive closed form solution of option prices with stochastic term structure, especially in Gaussian models. We can use a similar change of measure in models with stochastic surrender characterized by a hazard or an intensity process when we have expectations of the form

$$1_{\tau > t}E^Q \left[ e^{-(D_t - D_i) - \hat{\Gamma}_t} \hat{\Gamma}_t \right]$$

where $X$ is a $\mathcal{F}_T$-measurable random variable. We can rely on the following proposition to simplify this kind of expression.
Proposition 5.6. Let $Q^T$ be the measure equivalent to $Q$ on $(\Omega, \mathcal{F}_T)$ whose Radon-Nikodym derivative is given by

$$\frac{dQ^T}{dQ} |_{\mathcal{F}_T} = e^{-D_T} \frac{E^Q [e^{-D_T} | \mathcal{F}_0]}{E^Q[e^{-D_T} | \mathcal{F}_0]}$$

$Q$-a.s. Let $Q^{S(T)}$ be the measure equivalent to $Q^T$ on $(\Omega, \mathcal{F}_T)$ whose Radon-Nikodym derivative is given by

$$\frac{dQ^{S(T)}}{dQ^T} |_{\mathcal{F}_T} = e^{-\hat{\Gamma}_T} \frac{E^{Q^T}[e^{-\hat{\Gamma}_T} | \mathcal{F}_T]}{E^{Q^T}[e^{-\hat{\Gamma}_T} | \mathcal{F}_T]}$$

$Q^T$-a.s.

If $X$ is $Q^{S(T)}$-integrable, then we have

$$1_{\tau>t}E^Q \left[ e^{-(D_T-D_t)}e^{-(\hat{\Gamma}_T-\hat{\Gamma}_t)}X | \mathcal{F}_t \right] = 1_{\tau>t}P(t,T)E^{Q^T} \left[ e^{-(\hat{\Gamma}_T-\hat{\Gamma}_t)}X | \mathcal{F}_t \right] = 1_{\tau>t}P(t,T)E^{Q^T} \left[ e^{-(\hat{\Gamma}_T-\hat{\Gamma}_t)} | \mathcal{F}_t \right] E^{Q^{S(T)}} \left[ X | \mathcal{F}_t \right] = P(t,T)Q^T(\tau > T | \mathcal{G}_t) E^{Q^{S(T)}} \left[ X | \mathcal{F}_t \right]$$

Proof. See Appendix 1. □

In the financial literature, the measure $Q^T$ is called the $T$-forward martingale measure. By analogy, we called the measure $Q^{S(T)}$ the $T$-forward surrender measure. In the last proposition, $P(t,T) = E^Q \left[ e^{-(D_T-D_t)} | \mathcal{F}_t \right]$ is obviously the price of a zero-coupon bond at time $t$ with maturity $T$, and $Q^T(\tau > T | \mathcal{G}_t)$ is the probability under the $T$-forward martingale measure of no surrender before $T$ conditionally on $\mathcal{G}_t$.

6. The Brownian Filtration.

Up to now, we have made weak assumptions about the distribution of the different processes. In this section we’re going to make more specific assumptions, namely we assume

1. the filtration $\mathbb{F}$ is the Brownian filtration
2. $\tau$ admits a continuous increasing $(\mathbb{F}, P)$-hazard process $\Gamma_t$
3. the $(H)$-hypothesis holds under $P$ so that the Brownian motion is also a Brownian motion with respect to $\mathcal{G}$ under $P$.

In this setting, we give a change of measure formula that allows us to go from the real measure $P$ to an equivalent one. It also provides a simple functional between $\Gamma$ and $\hat{\Gamma}$.

6.1. Change of measure and choice of the local martingale measure. The risk neutral valuation principle rests on the choice of a local martingale measure $Q$ equivalent to $P$. The following lemma gives the set of measures $Q$ equivalent to $P$, that includes the equivalent local martingale measure that we should use in our valuation formulas.
Lemma 6.1. Any probability measure $Q$ equivalent to $P$, $\eta_t$, with the following representation

$$\eta_t = 1 + \int_0^t \xi_u dW_u + \int_0^t \zeta_u dM_u$$

where $W_t$ is a $\mathbb{G}$-Brownian motion under $P$, $M_t = H_t - \Gamma_{t \wedge \tau}$ is a $\mathbb{G}$-martingale under $P$ and $\xi_t$ and $\zeta_t$ are $\mathbb{G}$-predictable processes.

Since $\eta_t$ is strictly positive, it can also always be written as

$$\eta_t = 1 - \int_0^t \eta_u - \beta_u dW_u + \int_0^t \eta_u - \kappa_u dM_u$$

where $\beta_t$ and $\kappa_t > -1$ are $\mathbb{G}$-predictable processes. It is well-known that such a stochastic differential equation has a unique solution given by the following product of Doleans exponentials

$$\eta_t = \mathbb{E}_t(\int_0^\tau \kappa_u dM_u) \mathbb{E}_t(\int_0^t \beta_u dW_u) = (1 + \kappa_t 1_{\{\tau \leq t\}}) \exp\left(-\int_0^{t \wedge \tau} \kappa_u d\Gamma_u\right) \exp\left(-\int_0^t \beta_u dW_u - \frac{1}{2} \int_0^t |\beta_u|^2 du\right)$$

Remark 6.2. For any $\mathbb{G}$-predictable process $\kappa_u$, we can find an $\mathbb{F}$-predictable process $\kappa'_u$ such that $\kappa'_u = \kappa_u$ on $\{\tau > t\}$. Since in this section, we only use the value of $\kappa_t$ on $\tau > t$, we can, without loss of generality, assume $\kappa$ is $\mathbb{F}$-predictable.

A Girsanov-like theorem can be formulated in this setting.

Lemma 6.3. Any probability measure $Q$ equivalent to $P$ has a Radon-Nikodym derivative given by

$$\frac{dQ}{dP}_{|\mathcal{G}_t} = \eta_t = \varepsilon^1_t \varepsilon^2_t$$

with

$$\varepsilon^1_t = (1 + \kappa_t 1_{\{\tau \leq t\}}) \exp\left(-\int_0^{t \wedge \tau} \kappa_u d\Gamma_u\right)$$

$$\varepsilon^2_t = \exp\left(-\int_0^t \beta_u dW_u - \frac{1}{2} \int_0^t |\beta_u|^2 du\right)$$

where $\kappa$ and $\beta$ are $\mathbb{G}$-predictable processes. Furthermore, the process

$$\hat{W}_t = W_t + \int_0^t \beta_u du$$

follows a $(\mathbb{G}, Q)$-Brownian motion, and the process $\hat{M}_t$ is given by

$$\hat{M}_t = H_t - \int_0^{t \wedge \tau} (1 + \kappa_u) d\Gamma_u$$
is a \((\mathbb{G}, Q)\)-martingale.

**Proof.** The proofs of Lemmas 6.1 and 6.3 can be found in Jeanblanc and Rutkowski [58]. □

However, this set of equivalent measure is too large for our purpose. As we said, it is crucial that the \((H)\)-hypothesis holds under \(Q\) and we know that this property is not necessarily invariant from an equivalent measure to another. The set of equivalent measures, such that the \((H)\)-hypothesis holds, forms only a subset of all the equivalent measures. A sufficient condition for the equivalent measure \(Q\) to respect the \((H)\)-hypothesis, is that \(\beta\) should not be \(\mathbb{G}\)-predictable but \(\mathbb{F}\)-predictable processes. We have the following proposition.

**Proposition 6.4.** Let \(Q\) be an equivalent measure to \(P\). If the process \(\beta\) given in Lemma 6.3 is \(\mathbb{F}\)-predictable then the \((H)\)-hypothesis holds under \(Q\).

**Proof.** We have to prove

\[
Q(\tau \leq t \mid \mathcal{F}_\infty) = Q(\tau \leq t \mid \mathcal{F}_t)
\]

Let us first consider the left-hand term of this equality. We have

\[
Q(\tau \leq t \mid \mathcal{F}_\infty) = \frac{E^P[\varepsilon_1^\infty \varepsilon_2^\infty 1_{\{\tau \leq t\}} \mid \mathcal{F}_\infty]}{E^P[\varepsilon_2^\infty \mid \mathcal{F}_\infty]}
= \frac{E^P[\varepsilon_1^\infty 1_{\{\tau \leq t\}} \mid \mathcal{F}_\infty]}{E^P[\varepsilon_2^\infty \mid \mathcal{F}_\infty]}
= \frac{E^P[\varepsilon_1^\infty 1_{\{\tau \leq t\}} \mid \mathcal{F}_\infty]}{E^P[\varepsilon_1^\infty \mid \mathcal{F}_\infty]}
= \frac{E^P[(1 + \kappa_\tau 1_{\{\tau \leq t\}}) e^{-\int_0^{\tau \wedge \infty} \kappa_u d\Gamma_u} 1_{\{\tau \leq t\}} \mid \mathcal{F}_\infty]}{E^P[\varepsilon_1^\infty \mid \mathcal{F}_\infty]}
= \frac{E^P[(1 + \kappa_\tau 1_{\{\tau \leq t\}}) e^{-\int_0^{\tau \wedge t} \kappa_u d\Gamma_u} 1_{\{\tau \leq t\}} \mid \mathcal{F}_t]}{E^P[\varepsilon_1^\infty \mid \mathcal{F}_\infty]}
= \frac{E^P[(1 + \kappa_\tau 1_{\{\tau \leq t\}}) e^{-\int_0^{\tau \wedge t} \kappa_u d\Gamma_u} 1_{\{\tau \leq t\}} \mid \mathcal{F}_t]}{E^P[\varepsilon_1^\infty \mid \mathcal{F}_\infty]}
\]

The second equality comes from the assumption that \(\varepsilon_2^\infty\) is \(\mathcal{F}_\infty\)-measurable. The last one comes from the fact that the \((H)\)-hypothesis holds under \(P\) and that this hypothesis implies for any random variable \(A\), \(\mathcal{G}_t\)-measurable, we have \(E^P[A \mid \mathcal{F}_\infty] = E^P[A \mid \mathcal{F}_t]\). If we now consider the
denominator of this last equation, we have

\[
E^P \left[ \varepsilon_1^\infty \middle| \mathcal{F}_\infty \right] = \int_0^\infty (1 + \kappa_v) e^{-\int_0^v \kappa_u d\Gamma_u} dP (\tau \leq v \middle| \mathcal{F}_\infty) \\
= \int_0^\infty (1 + \kappa_v) e^{-\int_0^v \kappa_u d\Gamma_u} dP (\tau \leq v \middle| \mathcal{F}_v) \\
= \int_0^\infty (1 + \kappa_v) e^{-\int_0^v \kappa_u d\Gamma_u} d\left(1 - e^{-\Gamma_v}\right) \\
= \int_0^\infty d\left(1 - e^{-\int_0^v \kappa_u d\Gamma_u}\right) \\
= 1
\]

The first equality comes from the assumption \( \kappa \) is \( \mathbb{F} \)-adapted. The second equality comes from the fact the \((H)\)-hypothesis holds under \( P \). The fourth and fifth equalities come from the fact \( \Gamma_u \) is a finite variation continuous process. Eventually, we get for the left hand term of Equation (5.3)

\[
Q (\tau \leq t \middle| \mathcal{F}_\infty) = E^P \left[ \varepsilon_1^1 1_{\{\tau \leq t\}} \middle| \mathcal{F}_t \right]
\]

As far as the right-hand side of Equation (5.3) is concerned, we have

\[
Q (\tau \leq t \middle| \mathcal{F}_t) = \frac{E^P \left[ \varepsilon_1^\infty \varepsilon_2^\infty 1_{\{\tau \leq t\}} \middle| \mathcal{F}_t \right]}{E^P \left[ \varepsilon_1^\infty \varepsilon_2^\infty \middle| \mathcal{F}_t \right]} \\
= \frac{E^P \left[ \varepsilon_1^1 \varepsilon_2^1 1_{\{\tau \leq t\}} \middle| \mathcal{F}_t \right]}{E^P \left[ \varepsilon_1^1 \varepsilon_2^1 \middle| \mathcal{F}_t \right]} \\
= \frac{E^P \left[ \varepsilon_1^1 1_{\{\tau \leq t\}} \middle| \mathcal{F}_t \right]}{E^P \left[ \varepsilon_1^1 \middle| \mathcal{F}_t \right]} 
\]

since \( \varepsilon_2^1 \) is \( \mathcal{F}_t \)-measurable. Since we know \( E^P \left[ \varepsilon_1^\infty \middle| \mathcal{F}_\infty \right] = 1 \), by the law of iterated expectation, we can easily show \( E^P \left[ \varepsilon_1^1 \middle| \mathcal{F}_t \right] = 1 \).

We already said that it is not in general an easy task to find an explicit functional between \( \Gamma \) and \( \hat{\Gamma} \). In the case of the Brownian filtration and when \( \beta \) (and \( \kappa \)) is \( \mathbb{F} \)-predictable, we can find a simple relation between \( \Gamma \) and \( \hat{\Gamma} \). We have the following proposition.

**Proposition 6.5.** Let \( Q \) be an equivalent measure to \( P \). If the processes \( \beta \) and \( \kappa \) of Lemma 6.3 are \( \mathbb{F} \)-predictable then the process \( \hat{\Gamma}_t \) defined by

\[
\hat{\Gamma}_t = \int_0^t (1 + \kappa_u) d\Gamma_u
\]

is the \( \mathbb{F} \)-hazard process of \( \tau \) under \( Q \). If \( \tau \) admits an \( \mathbb{F} \)-intensity process \( \lambda \) under \( P \), then the process \( \hat{\lambda}_t \) defined by

\[
\hat{\lambda}_t = (1 + \kappa_u) \lambda_u
\]
is the $\mathbb{F}$-intensity of $\tau$ under $Q$.

**Proof.** We have

\[
e^{-\hat{\Gamma}_t} = \frac{Q(\tau > t | \mathcal{F}_t)}{E^P[\varepsilon_t 1_{\{\tau > t\}} | \mathcal{F}_t]} = \frac{E^P[\varepsilon_t 1_{\{\tau > t\}} | \mathcal{F}_t]}{E^P[\varepsilon_t | \mathcal{F}_t]},
\]

since $\varepsilon_t^2$ is $\mathcal{F}_t$-measurable. From the previous proposition, we know that $E^P[\varepsilon_t | \mathcal{F}_t] = 1$. Then let us first consider the numerator. We have

\[
E^P[\varepsilon_t 1_{\{\tau > t\}} | \mathcal{F}_t] = E^P\left[\left(1 + \kappa_t 1_{\{\tau \leq t\}}\right)e^{-\int_0^t \kappa_t \, d\Gamma_u} 1_{\{\tau > t\}} | \mathcal{F}_t\right]
\]

\[
= E^P\left[e^{-\int_0^t \kappa_t \, d\Gamma_u} 1_{\{\tau > t\}} | \mathcal{F}_t\right]
\]

\[
= e^{-\int_0^t \kappa_t \, d\Gamma_u} E^P\left[1_{\{\tau > t\}} | \mathcal{F}_t\right]
\]

\[
= e^{-\int_0^t (1+\kappa_t) \, d\Gamma_u}
\]

The third equality comes from the assumption that $\kappa$ is $\mathbb{F}$-adapted. $\Box$

As we said, $\eta$ is the Radon-Nikodym derivative of $Q$ with respect to $P$. The problem is now to find the processes $\beta_t$ and $\kappa_t$ that we should use in our valuation formulas.

Let us start with $\beta_t$ first. If we had no surrender risk, $\eta$ would reduce to $\varepsilon_2$ and we would get the standard case extensively studied in financial economics. In this case, we should choose $\beta_t$ such that the discounted financial prices are local martingales under the associated equivalent measure. $\beta_t$ would then correspond to the price of market risk. There is no difference in our setting; we should simply choose $\beta_t$ as if there was no surrender risk. Let us finally notice that when the financial market is complete, the local martingale measure of the financial market is unique and so is $\beta_t$.

The choice of $\kappa_t$ is more problematic. $\kappa_t$ is the price of the pure surrender risk. It determines how the hazard or the intensity process should be adjusted for the pure risk of surrender. Unfortunately, we do not have any market on which assets affected by surrender risk are traded. So, even if the financial market is complete, the "life insurance market" is not with respect to this surrender risk. The equivalent local martingale measure related to this risk is therefore not unique. We can neither extract the market price of this risk from the values of assets affected by surrender since as we just said there is no such market. However, this pure surrender risk correspond to the idiosyncratic risk component of the surrender decision. If we assume the idiosyncratic component is independent from a policyholder to another, we can argue that this pure surrender risk can be diversified away. Accordingly, in a competitive market, no remuneration for this risk should be required and its price could be considered equal to 0. Another argument pleading for a pure surrender risk equal to 0, is to recognize that the associated local martingale measure corresponds to the minimal martingale measure of Schweizer. See Chapter 4 for more details.

If we set $\kappa_t = 0 \forall t$, we see that $\tau$ admits the hazard process $\hat{\Gamma}_t = \Gamma_t$ or the intensity process $\hat{\lambda}_t = \lambda_t$ under $Q$. This last equality simply means the processes $\Gamma$ or $\lambda$ are not modified if we
set the price of surrender risk equal to 0, but it does not mean the probabilities of surrender are equal under the local martingale measure and the physical measure. Indeed, these processes depend on the Brownian motion and are accordingly adjusted under the local martingale measure, by the price of market risk. In conclusion, by setting $\kappa = 0$, the idiosyncratic component of the surrender decision is not adjusted for the risk, but the common component of the surrender risk, which is linked to the financial market and is not diversifiable, is indeed adjusted for the risk through the market price of risk.

7. Application to Unit-Linked Contracts.

In this section, we apply the framework described in the previous sections to the valuation of unit-linked contracts. We study single premium contracts and periodic premium contracts. We do not make any specific assumptions on the distribution followed by our stochastic processes other than that the discounted prices of the financial assets are, under $Q$, not only $\mathbb{F}$-martingales but also $\mathbb{G}$-martingales. This is a slightly stronger assumption than the $(H)$-hypothesis. In this section, we derive general formulas for the valuation of unit-linked contracts. You will notice we only need Lemma 5.1 of Section 5.1 to derive these valuation formulas so that $\tau$ does not need to admit an $(\mathbb{F}, Q)$-intensity process nor does $\hat{\Gamma}$ need to be continuous.

7.1. Single premium contract. We assume a single premium $P$ is paid at time $t$. With this premium, the policyholder acquires $n$ units of a stock index with value $S(t)$ at time $t$, as well as a guarantee $g$ at the term $T$ of the contract. This guarantee could be for example a guaranteed rate on the initial value of the units, a guaranteed rate on the premium paid, etc.

When the policyholder surrenders, we assume he receives a proportion $(1 - \alpha)$ of the $n$ units. So, $\alpha$ is here the penalty when one surrenders. We assume the payment is made at the precise time of surrender $\tau$. Accordingly, using our notation, we have:

$$R(\tau) = (1 - \alpha)nS(\tau)$$

$$g(T, \omega) = \max(nS(T), g)$$

The value at date $t$ of the liabilities is given by:

$$L_t = E^Q \left[ e^{-(D_T - D_t)}g(T, \omega)1_{\tau > T} + e^{-(D_\tau - D_\tau)}R(\tau)1_{\{t < \tau \leq T\}} \mid \mathcal{G}_t \right]$$
We can simplify this equation as followed:

\[
L_t = E^Q \left[ e^{-(D_T-D_t)} 1_{\{\tau>T\}} \max(nS(T), g) | \mathcal{G}_t \right] \\
+ E^Q \left[ e^{-(D_T-D_t)} (1 - \alpha) nS(\tau) (1_{\{\tau>t\}} - 1_{\{\tau>T\}}) | \mathcal{G}_t \right] \\
= E^Q \left[ e^{-(D_T-D_t)} 1_{\{\tau>T\}} \max(nS(T), g) | \mathcal{G}_t \right] \\
+ \sum_{\{\tau>t\}} E^Q \left[ (1 - \alpha) nS(\tau) e^{-(D_T-D_t)} 1_{\{\tau>t\}} | \mathcal{G}_t \right] \\
- \sum_{\{\tau>T\}} E^Q \left[ (1 - \alpha) nS(\tau) e^{-(D_T-D_t)} 1_{\{\tau>T\}} | \mathcal{G}_t \right]
\]

The terms I, II and III can be simplified as follows:

\[
I = E^Q \left[ e^{-(D_T-D_t)} 1_{\{\tau>T\}} \max(nS(T), g) | \mathcal{G}_t \right] \\
= 1_{\{\tau>t\}} E^Q \left[ e^{-(D_T-D_t)} e^{-\hat{\Gamma}_T - \hat{\Gamma}_t} \max(nS(T), g) | \mathcal{F}_t \right]
\]

\[
II = E^Q \left[ (1 - \alpha) nS(\tau) e^{-(D_T-D_t)} 1_{\{\tau>t\}} | \mathcal{G}_t \right] \\
= 1_{\{\tau>t\}} (1 - \alpha) nE^Q \left[ e^{-(D_T-D_t)} S(\tau) | \mathcal{G}_t \right] \\
= 1_{\{\tau>t\}} (1 - \alpha) nS(t)
\]

In the last equality, we use the hypothesis that all \( \mathcal{F} \)-martingales are also \( \mathcal{G} \)-martingales.

\[
III = (1 - \alpha) nE^Q \left[ e^{-(D_T-D_t)} S(\tau) 1_{\{\tau>T\}} | \mathcal{G}_t \right] \\
= (1 - \alpha) nE^Q \left[ e^{-(D_T-D_t)} E^Q \left[ S(\tau) e^{-(D_T-D_T)} 1_{\{\tau>T\}} | \mathcal{G}_T \right] | \mathcal{G}_t \right] \\
= (1 - \alpha) nE^Q \left[ e^{-(D_T-D_t)} 1_{\{\tau>T\}} E^Q \left[ S(\tau) e^{-(D_T-D_T)} | \mathcal{G}_T \right] | \mathcal{G}_t \right] \\
= (1 - \alpha) nE^Q \left[ e^{-(D_T-D_t)} 1_{\{\tau>T\}} S(T) | \mathcal{G}_t \right] \\
= 1_{\{\tau>t\}} (1 - \alpha) nE^Q \left[ e^{-(D_T-D_t)} e^{-\hat{\Gamma}_T - \hat{\Gamma}_t} S(T) | \mathcal{F}_t \right]
\]

In the last but one equality we use the hypothesis that the discounted price of the financial asset is, under \( Q \), a \( \mathcal{G} \)-martingale. Eventually, we have

\[
L_t = 1_{\{\tau>t\}} e^{(D_t + \hat{\Gamma}_t)} E^Q \left[ e^{-(D_T+\hat{\Gamma}_T)} \max(nS(T), g) | \mathcal{F}_t \right] \\
+ 1_{\{\tau>t\}} (1 - \alpha) n \left\{ S(t) - e^{(D_t + \hat{\Gamma}_t)} E^Q \left[ e^{-(D_T+\hat{\Gamma}_T)} S(T) | \mathcal{F}_t \right] \right\}
\]
When an intensity process and an instantaneous risk free rate exist, this last expression becomes

\[ L_t = 1_{\{\tau>t\}} E^Q \left[ e^{-\int_t^T (r_u + \hat{\lambda}_u) du} \max \left( nS(T), g \right) |\mathcal{F}_t \right] \]

\[ + 1_{\{\tau>t\}} (1 - \alpha) n \left\{ S(t) - E^Q \left[ e^{-\int_t^T (r_u + \hat{\lambda}_u) du} S(T) |\mathcal{F}_t \right] \right\} \]

These last two expressions are very simple ones that involve the calculation of the price of a call options with a modified term structure model \( D_t^\hat{\Gamma} = D_t + \hat{\Gamma}_t \) or in the second case, with a modified instantaneous risk free interest rate \( r_t^\hat{\lambda} = r_t + \hat{\lambda}_t \). It can be solved, in a number of cases, in closed-form or in quasi-closed form by Fourier transform for example. Notice, we have not made any assumption about the form of the processes \( S(t), D_t \) or \( \hat{\Gamma}_t \), so that Equation (7.1) is thus a general valuation formula for single premium unit linked contracts with stochastic interest rate and stochastic surrender.

### 7.2. Periodic Premium contract

Here, we assume premiums are paid at time \( t_i \) with \( i = 0 \ldots N - 1 \) if the policyholder has not surrendered. We assume the value of these premiums is constant and equal to \( P \). As we’ve already seen, the payoff is equal to:

\[ \sum_{i=0}^{N-1} P 1_{\tau>t_i} \]

The value of this payoff at date \( t \), is given by

\[ A_t = E^Q \left[ \sum_{i=\lceil t \rceil}^{N-1} e^{-(D_{t_i} - D_t)} P 1_{\tau>t_i} |\mathcal{G}_t \right] \]

Using the results of the previous sections, we get

\[ A_t = 1_{\{\tau>t\}} P \sum_{i=\lceil t \rceil}^{N-1} E^Q \left[ e^{-(D_{t_i} - D_t)} e^{-(\hat{\Gamma}_{t_i} - \hat{\Gamma}_t)} |\mathcal{F}_t \right] \]

When an intensity process and an instantaneous risk free rate exist, this last expression becomes

\[ A_t = 1_{\{\tau>t\}} P \sum_{i=\lceil t \rceil}^{N-1} E^Q \left[ e^{-\int_t^{t_i} (r_u + \hat{\lambda}_u) du} |\mathcal{F}_t \right] \]  

(7.2)

As far as the value of the liabilities is concerned, we distinguish two cases. In the first one, the number of units that the policyholder receives, depends on the value of the units at the time of payment. In the second case, we assume that the payment of a premium gives the right to the policyholder, to \( n \) units whatever the value of these units at the time of payment. We call the first kind of contract, a periodic premium contract of type I and the second one, a periodic premium contract of type II.
7.2.1. Periodic Premiums contract of type I. In this case, we assume a proportion $(1 - \rho)$ of the premium $P$ paid at time $t_i$ is used to buy $n_{t_i}$ units. This number of unit $n_{t_i}$ is given by

$$n_{t_i} = \frac{(1 - \rho)P}{S(t_i)}$$

The present value of the liabilities is given by

$$L_t = E^Q \left[ e^{-(D_T - D_t)\max\left(n_{t_i}, g\right)} \sum_{i=0}^{N-1} n_{t_i} 1_{\tau > T} |\mathcal{G}_t \right]$$

$$+ E^Q \left[ e^{-(D_T - D_t)(1-\alpha)S(\tau)} \sum_{i=0}^{\lfloor \tau \rfloor} n_{t_i} 1_{\{t < \tau \leq T\}} |\mathcal{G}_t \right]$$

We show in Appendix 2 that we can rewrite this equation as:

$$L_t = 1_{\{\tau > t\}} e^{(D_t + \tilde{\tau})} E^Q \left[ e^{-(D_T + \tilde{\tau})\max\left(n_{t_i}, g\right)} (1 - \rho)P \sum_{i=0}^{N-1} S(T) \sum_{i=0}^{\lfloor \tau \rfloor} 1_{\{t < \tau \leq T\}} |\mathcal{F}_t \right]$$

$$+ 1_{\{\tau > t\}} (1 - \alpha)(1 - \rho)P \sum_{i=0}^{\lfloor \tau \rfloor} \frac{1}{S(t_i)} \left\{ S(t) - e^{(D_t + \tilde{\tau})} E^Q \left[ S(T) e^{-(D_T + \tilde{\tau})} |\mathcal{F}_t \right] \right\}$$

$$(7.3) + 1_{\{\tau > t\}} (1 - \alpha)(1 - \rho)P \sum_{i=0}^{\lfloor \tau \rfloor} \left\{ e^{(D_t + \tilde{\tau})} E^Q \left[ \left(1 - \frac{S(T)}{S(t_i)}\right) e^{-(D_t + \tilde{\tau})} |\mathcal{F}_t \right] \right\}$$

When an intensity process and an instantaneous risk free rate exist, we get

$$L_t = 1_{\{\tau > t\}} E^Q \left[ e^{-\int_t^T (r_u + \lambda_u) du} \max\left(n_{t_i}, g\right)} (1 - \rho)P \sum_{i=0}^{N-1} S(T) \sum_{i=0}^{\lfloor \tau \rfloor} 1_{\{t < \tau \leq T\}} |\mathcal{F}_t \right]$$

$$+ 1_{\{\tau > t\}} (1 - \alpha)(1 - \rho)P \sum_{i=0}^{\lfloor \tau \rfloor} \frac{1}{S(t_i)} \left\{ S(t) - E^Q \left[ S(T) e^{-\int_t^T (r_u + \lambda_u) du} |\mathcal{F}_t \right] \right\}$$

$$+ 1_{\{\tau > t\}} (1 - \alpha)(1 - \rho)P \sum_{i=0}^{\lfloor \tau \rfloor} \left\{ E^Q \left[ \left(1 - \frac{S(T)}{S(t_i)}\right) e^{-\int_t^T (r_u + \lambda_u) du} |\mathcal{F}_t \right] \right\}$$

Notice if we take $N = 1$ and pose $(1 - \rho)P = nS(t)$, we get the same result as for the single premium contract. The last two expectations can often be found in closed form or in quasiclosed form. Unfortunately, the first expectation is much harder to compute. No closed form solution is available for this one, even in the simplest case. However, we can still rely on the extensive literature on the numerical techniques used for the pricing of Asian options in finance.

7.2.2. Periodic Premiums contract of Type II. We assume the payment of a premium gives the right to the policyholder, to $n$ units whatever the value of these units at the time of payment.
We assume there are \( N \) dates of payment. In this case, we have
\[
R(\tau) = (1 - \alpha) \sum_{i=0}^{[\tau]} nS(\tau)
\]
\[
= (1 - \alpha) [\tau] nS(\tau)
\]
\[
g(T, \omega) = \max \left( \sum_{i=0}^{N-1} nS(T), g \right)
\]
\[
= \max (N nS(T), g)
\]

The present value of the liabilities is given by
\[
L_t = E^Q \left[ e^{-(D_T - D_t)} \max (N nS(T), g) 1_{\tau > T} | \mathcal{G}_t \right] + E^Q \left[ e^{-(D_T - D_t)} (1 - \alpha) [\tau] nS(\tau) 1_{t < \tau \leq T} | \mathcal{G}_t \right]
\]

We show in Appendix 3 that we can rewrite this equation as
\[
L_t = 1_{\{\tau > t\}} E^Q \left[ e^{-(D_T + \tilde{\Gamma}_t)} \max (N nS(T), g) | \mathcal{F}_t \right]
\]
\[
+ 1_{\{\tau > t\}} (1 - \alpha) n \sum_{i=0}^{[t]} S(t_i)
\]
\[
+ 1_{\{\tau > t\}} (1 - \alpha) n e^{-(D_T + \tilde{\Gamma}_t)} \left\{ \sum_{i=[t]}^{N-1} E^Q \left[ e^{-(D_{t_i} + \tilde{\Gamma}_{t_i})} S(t_i) | \mathcal{F}_t \right] - NE^Q \left[ e^{-(D_T + \tilde{\Gamma}_T)} S(T) | \mathcal{F}_t \right] \right\}
\]

When an intensity process and an instantaneous risk-free rate exist, we have
\[
L_t = 1_{\{\tau > t\}} E^Q \left[ e^{-\int_t^T (r_u + \tilde{\lambda}_u) du} \max (N nS(T), g) | \mathcal{F}_t \right]
\]
\[
+ 1_{\{\tau > t\}} (1 - \alpha) n \sum_{i=0}^{[t]} S(t_i)
\]
\[
(7.4) + 1_{\{\tau > t\}} (1 - \alpha) n \left\{ \sum_{i=[t]}^{N-1} E^Q \left[ e^{-\int_{t_i}^{t} (r_u + \tilde{\lambda}_u) du} S(t_i) | \mathcal{F}_t \right] - NE^Q \left[ e^{-\int_t^T (r_u + \tilde{\lambda}_u) du} S(T) | \mathcal{F}_t \right] \right\}
\]

7.3. Gaussian Model. In the past subsections, we made very weak assumptions on the processes followed by \( S_t, D_t \) and \( \tilde{\Gamma}_t \). The results derived above are very general and do not depend on the specific assumptions that one chooses to model these variables. For example, the value of the stock index could be driven by a jump-diffusion process, a Lévy process, etc. . .

In this section, we now consider a particular model: a Gaussian model. Firstly, we assume \( \tau \) admits an \( \mathbb{F} \)-intensity process \( \lambda(t) \) and there exists an instantaneous risk free rate \( r(t) \). Secondly, we assume they, both, as well as the logarithm of financial asset prices, follow Gaussian processes.
As far as the surrender decision is concerned, we assume the policyholder does not react in a perfectly rational manner. Instead, we assume he follows a simple heuristic that consists of comparing the return of the fund $I$, he has invested in, to the return of another fund $R$, he takes as a reference, so that the higher the return of the reference fund with respect to the return of the contract fund, the higher the instantaneous surrender rate.

7.3.1. *The financial market.* We assume the term structure is modelled by a multi-factors Gaussian affine model. The class of affine models of the term structure has been studied in details by Dai and Singleton [38] and extended by Duffee [42]. We follow the classification and notation of Dai and Singleton. We assume the term structure is described by the maximally affine model denoted by $AM_0(3)$ which corresponds to a three correlated factors Gaussian term structure model. Under $P$, the canonical representation of this model is given by

$$r(t) = \delta_0 + \delta'_1 X(t)$$

where $\delta_0 \in \mathbb{R}$, $\delta_1 \in \mathbb{R}^3$. The factors $X(t)$ are solutions of the following system of stochastic differential equations.

$$dX(t) = -KX(t)dt + \Sigma^X dW(t)$$

where $W(t)$ is a 3-dimensional standard Brownian motion.

$$K = \begin{pmatrix} \kappa_{11} & 0 & 0 \\ \kappa_{21} & \kappa_{22} & 0 \\ \kappa_{31} & \kappa_{32} & \kappa_{33} \end{pmatrix}, \Sigma^X = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

In order to take into account the existence of our two funds, we rewrite this model in the following manner. The instantaneously risk free rate is still given by

$$r(t) = \delta_0 + \delta'_1 X(t)$$

with

$$dX(t) = -KX(t)dt + \Sigma dW(t)$$

but $W(t)$ is this time a 5-dimensional standard Brownian motion and

$$\Sigma = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

The matrix $K$ remains unchanged.

As we said, we have to model two different funds. The first one, $S_I$, is the one the policyholder has invested in. The second one $S_R$, is the reference fund. We assume the values of these funds evolve as geometric Brownian motions. Under the real measure $P$, the values of these funds are then the solutions of the following differential equations:

$$dln(S_I(t)) = \left[ \mu_I(t) - \frac{1}{2}\sigma'_I\sigma_I \right] dt + \sigma'_I dW(t)$$

$$dln(S_R(t)) = \left[ \mu_R(t) - \frac{1}{2}\sigma'_R\sigma_R \right] dt + \sigma'_R dW(t)$$
where $\sigma_R$ and $\sigma_I \in \mathbb{R}^5$. We do not specify any further the drift coefficients $\mu_i$ and $\mu_R$ since they will be removed under the local martingale measure and do not play any role in the pricing formula.

7.3.2. The time of surrender. Here, we use the second of the two approaches that we described to define a surrender time i.e. we do not explicitly construct our surrender time but instead directly specify its intensity process. The simple specification that we choose, is not based on any empirical work and we do not want to suggest this model corresponds to any actual surrender behavior. It only aims at illustrating our methodology.

Let $h$ be a positive real function $h(\cdot, \cdot) : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^+$, increasing and differentiable in its second argument. We define the intensity process by

$$
\lambda_t = h \left( t, \left( \frac{S_R(t)}{S_R(0)}, \frac{S_I(t)}{S_I(0)} \right) \right)
$$

The intensity process depends on the time and on the ratio of the funds returns. The higher the return of the reference fund compare to the return of the fund I, the higher the instantaneous surrender rate. We approximate this function by the first term of its Taylor expansion in the second argument and at the point $x_2 = 1$.

$$
\lambda_t \approx h(t, 1) + \left[ \ln \left( \frac{S_R(t)}{S_R(0)} \right) - \ln \left( \frac{S_I(t)}{S_I(0)} \right) \right] h_{x_2}(t, 1)
$$

where $h_{x_2}(\cdot, \cdot)$ is the derivative function of $h(\cdot, \cdot)$ in its second argument. We obtain a new intensity process which is linear with respect to difference of the logarithmic returns of the funds. Since these logarithmic returns follow Gaussian processes, this approximate intensity process follows also a Gaussian process. Strictly speaking, this process is not a positive process as it should be. The same kind of problem occurs in Gaussian term structure model where the instantaneous risk free rate can take negative value whereas it theoretically should be positive. Gaussian term structure model are although widely used by practitioners and academics as well. Their use is justified by the argument that, for most realistic specification, the probability of having a path with negative values should be very low. We make the same implicit assumption for $\lambda_t$.

We can, for example, assume $h(\cdot, \cdot)$ is a linear function in its second argument, that is

$$
h(t, x) = \alpha(t) + \beta(t)x
$$

with $\alpha(t) > 0 \ \forall \ t$ and $\beta(t) > 0 \ \forall \ t$. $\beta(t)$ is then the sensibility of the instantaneous surrender rate with respect to the difference of the returns of the funds. $\alpha(t)$ represent the instantaneous surrender rate when a policyholder is not sensitive to the evolution of the financial market. The time dependence of $\alpha(t)$ and $\beta(t)$ can express for example, a tax effect on the surrender probability.
The approximate intensity process is then given by

\[ \lambda_t \approx \alpha(t) + \beta(t) \left[ 1 + \ln \left( \frac{S_R(t)}{S_R(0)} \right) - \ln \left( \frac{S_I(t)}{S_I(0)} \right) \right] \]

7.3.3. The equivalent martingale measure \( Q \). Lemma 6.3 allows us to separate the specification of the price of market risk and the price of the surrender risk. We start with the price of market risk. Duffee introduced a price of risk \( \Lambda(t) \) with the following form

\[ \Lambda(t) = \gamma_1 + \gamma_2 X(t) \]

where \( \gamma_1 \in \mathbb{R}^5 \) and \( \gamma_2 \in \mathbb{R}^{5 \times 3} \). If we define by \( Q \), the \( P \)-equivalent measure that has a Radon-Nikodym derivatives with respect to \( P \), given by:

\[ \frac{dQ}{dP} |_{\mathcal{F}} = \mathcal{E} \left( - \int_0^t \Lambda(u) du \right) \]

where \( \mathcal{E}(\cdot) \) is the Dooleans exponential, then

\[ W^Q(t) = W(t) + \int_0^t \Lambda(u) du \]

is a Brownian motion under the measure \( Q \). Under this equivalent measure, we have this time

\[ dX(t) = \tilde{K} \left[ \tilde{\Theta} - X(t) \right] dt + \Sigma dW^Q(t) \]

where

\[ \tilde{K} = K + \Sigma \gamma_2 \]

\[ \tilde{\Theta} = (K + \Sigma \gamma_2)^{-1} (\Sigma \gamma_1) \]

In order to \( Q \) to be a martingale measure, \( \gamma_1 \) and \( \gamma_2 \) are to be chosen such that under \( Q \), we have:

\[ d\ln(S_I(t)) = \left[ r(t) - \frac{1}{2} \sigma_I^2 \right] dt + \sigma_I' dW^Q(t) \]

\[ d\ln(S_R(t)) = \left[ r(t) - \frac{1}{2} \sigma_R^2 \right] dt + \sigma_R' dW^Q(t) \]

As far as the price of surrender risk is concerned, we assume the pure surrender risk can be diversified away in a large portfolio so that the price of surrender risk \( K \) is set to 0. We then have:

\[ \hat{\lambda}_t = \lambda_t \]
7.3.4. The valuation formula. In the following we illustrate the change of measure technique we explained in Proposition 5.6, by deriving closed form solution in the model described above for a periodic premiums contract of type II.

By applying the results of Proposition 5.6 to Equations (7.4) and (7.2), we get

\[ L_t = 1_{\{\tau > t\}} P(t,T) E^{Q^T} \left[ e^{-\int_t^T \lambda(u) du} | F_t \right] E^{Q^{S(T)}} \left[ \max \left( \sum_{i=0}^{\left\lfloor t \right\rfloor} S(t_i), g \right) | F_t \right] \]
\[ + 1_{\{\tau > t\}} (1 - \alpha) \sum_{i=0}^{\left\lfloor t \right\rfloor} S(t_i) \]
\[ \quad + 1_{\{\tau > t\}} (1 - \alpha) \sum_{i=\left\lceil t \right\rceil}^{N-1} P(t,t_i) E^{Q^{S(t_i)}} \left[ e^{-\int_t^{t_i} \lambda(u) du} | F_t \right] E^{Q^{S(t_i)}} \left[ S(t_i) | F_t \right] \]
\[ \quad - 1_{\{\tau > t\}} (1 - \alpha) n N P(t,T) E^{Q^T} \left[ e^{-\int_t^T \lambda(u) du} | F_t \right] E^{Q^{S(T)}} \left[ S(T) | F_t \right] \]

and

\[ A_t = 1_{\{\tau > t\}} P \sum_{i=\left\lfloor t \right\rfloor}^{N-1} P(t,t_i) E^{Q^{S(t_i)}} \left[ e^{-\int_t^{t_i} \lambda(u) du} | F_t \right] \]

We derive closed-form solutions for the expressions (A), (B), (C) and (D). These solutions can then be applied mutatis mutandis for their associated expressions in \( t_i \).

Let us begin by solving (A). The price \( P(t,T) \) at time \( t \) of a zero coupon bond with maturity \( T \) is given in our 3 factors Gaussian affine model by:

\[ P(t,T) = e^{-A(t,T) - B(t,T)'X(t)} \]

where

\[ A(t,T) = \delta_0 (T-t) + \left( (T-t) \delta_1 I - B(t,T)' \right) \tilde{\Theta} - \frac{1}{2} \int_{u=t}^T B(u,T)' B(u,T) du \]
\[ B(t,T)' = \delta_1' \left[ N \frac{1}{K_d} (I - e^{-K_d(T-t)})N^{-1} \right] \]

See Appendix 4 for a proof. In order to value (B), we need to know the distribution of \( \lambda \) under \( Q^T \). Here, the Radon-Nikodym derivative of the \( T \)-forward martingale measure \( Q^T \) defined in Proposition 5.6, is given by

\[ \frac{dQ^T}{dQ} | F_t = e^{-\frac{1}{2} \int_0^T B(u,T)' \Sigma \Sigma' B(u,T) du - \int_0^T B(u,T)' \Sigma dW_Q \left( u \right)} \]
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See Appendix 5 for a proof. Let us define the process $\varepsilon_t$ by:

$$\varepsilon_t = E^Q \left[ \frac{dQ^T}{dQ} \bigg| \mathcal{F}_t \right]$$

This last process is the solution of the following differential equation

$$d\varepsilon_t = -\varepsilon_t B(t,T)'\Sigma dW^Q(t)$$

Thanks to Girsanov’s theorem, we know the process $W^Q_T$ defined by

$$W^Q_T(t) = W^Q(t) + \int_0^t \Sigma' B(u,T)du, \quad \forall t \in [0,T]$$

is a standard Brownian motion under $Q^T$.

$\lambda$ is given by

$$\lambda(u) = \alpha(u) + \beta(u) \left[ 1 + \ln \left( \frac{S_R(u)}{S_R(0)} \right) - \ln \left( \frac{S_I(u)}{S_I(0)} \right) \right]$$

Under $Q^T$, the values of the funds are then solutions of the following stochastic differential equations:

$$\ln S_I(u) = \ln S_I(0) + \int_0^u \left[ r(s) - \frac{1}{2}\sigma'^I\sigma_I - \sigma'^I\Sigma' B(s,T) \right] ds + \int_0^u \sigma'_I dW^Q^T(s)$$

$$\ln S_R(u) = \ln S_R(0) + \int_0^u \left[ r(s) - \frac{1}{2}\sigma'^R\sigma_R - \sigma'^R\Sigma' B(s,T) \right] ds + \int_0^u \sigma'_R dW^Q^T(s)$$

We then have

$$\lambda(u) = \alpha(u) + \beta(u) \left[ 1 + \int_t^u \frac{1}{2}(\sigma'^R \sigma_R - \sigma'^I \sigma_I) + (\sigma'^R - \sigma'^I) \Sigma' B(s,T) ds + \int_t^u (\sigma'_I - \sigma'_R) dW^Q^T(s) \right]$$

In Appendix 6, we show the distribution of $\int_t^T \lambda(u)\,du$ under $Q^T$ is given by

$$\int_t^T \lambda(u)\,du = \mu^Q^T_T(t,T) + \int_t^T \sigma^Q^T(u,T)\,dW^Q^T(u)$$

where

$$\mu^Q^T_T(t,T) = \int_t^T \alpha(u)\,du + \int_t^T \beta(u)\,du + \frac{1}{2}(\sigma'^R \sigma_R - \sigma'^I \sigma_I) \int_t^T \beta(u)(u-t)\,du$$

$$+ \int_t^T (\sigma'^R - \sigma'^I) \Sigma' B(u,T) \left( \int_u^T \beta(s)\,ds \right) du$$

$$\sigma^Q^T(t,T)' = (\sigma'^I - \sigma'^R) \int_t^T \beta(s)\,ds$$

Accordingly, we have for (B)

$$E^Q \left[ e^{-\int_t^T \lambda(u)\,du} \bigg| \mathcal{F}_t \right] = e^{-\mu^Q^T_T(t,T) + \frac{1}{2} \int_t^T \sigma^Q^T(u,T)\sigma^Q^T(u,T)'\,du}$$
We can now find (C). We have the following useful equality.

\[ E^{Q^{(T)}}[S_t(T) | \mathcal{F}_t] = E^{Q^{(T)}}\left[ \frac{S_t(T)}{P(t,T)} | \mathcal{F}_t \right] \]

\[ = E^{Q^{(T)}}[F_{S_t}(T,T) | \mathcal{F}_t] \]

where \( F_{S_t}(t,T) \) is the forward price at time \( t \) of the fund \( S_t \) for a maturity \( T \). Let us find the distribution of this forward price under \( Q^{(T)} \). The Radon-Nikodym derivative of the \( T \)-forward surrender measure \( Q^{(T)} \) defined in Proposition 5.6, is given, in our model, by

\[ \frac{dQ^{(T)}}{dQ^T} = e^{-\frac{1}{2} \int_0^T \left( (\sigma'_I - \sigma'_R)(f^T_u \beta(s)ds) \right) \left( (\sigma'_I - \sigma'_R)(f^T_u \beta(s)ds) \right) ds} dt - \int_0^T \left( (\sigma'_I - \sigma'_R)(f^T_u \beta(s)ds) \right) dW^{Q^T}(u) \]

See the Appendix 7 for a proof. Let us define the process \( S^\epsilon_t \) by:

\[ S^\epsilon_t = E^{Q^T}\left[ \frac{dQ^{(T)}}{dQ^T} | \mathcal{F}_t \right] \]

This last process is the solution of the following differential equation

\[ dS^\epsilon_t = -S^\epsilon_t (\sigma'_I - \sigma'_R) \left( \int_t^T \beta(s)ds \right) dW^{Q^T}(t) \]

Thanks to Girsanov’s theorem, we know the process \( W^{Q^{(T)}} \) defined by

\[ W^{Q^{(T)}}(t) = W^{Q^T}(t) + \int_0^T \left( \int_u^T \beta(s)ds \right) [(\sigma'_I - \sigma'_R)]' du, \forall t \in [0, T] \]

is a standard Brownian motion under \( Q^{(T)} \).

Under \( Q^T \), the forward price follows a martingale and is the solution of the following differential equations:

\[ d(F_{S_t}(t,T)) = F_{S_t}(t,T) (\sigma'_I + B(t,T)'\Sigma) dW^{Q^T}(t) \]

See Appendix 8 for a proof. Under \( Q^{(T)} \), this forward price is then the solution of

\[ d(F_{S_t}(t,T)) = F_{S_t}(t,T) \left[ - \int_t^T \beta(s)ds(\sigma'_I + B(t,T)'\Sigma)(\sigma_I - \sigma_R)dt + (\sigma'_I + B(t,T)'\Sigma)dW^{Q^T}(t) \right] \]

The solution of this equation is given by

\[ \ln F_{S_t}(T,T) = \ln F_{S_t}(t,T) - \int_t^T \int_u^T \beta(s)ds(\sigma'_I + B(u,T)'\Sigma)(\sigma_I - \sigma_R)du \]

\[ - \frac{1}{2} \int_t^T (\sigma'_I + B(u,T)'\Sigma)(\sigma'_I + B(u,T)'\Sigma)' du + \int_t^T (\sigma'_I + B(u,T)'\Sigma)dW^{Q^T}(u) \]

or

\[ \ln F_{S_t}(T,T) = \ln F_{S_t}(t,T) - \mu^{Q^T} F_{S_t}(t,T) - \frac{1}{2} \int_t^T \sigma_{F_S}(u,T)\sigma_{F_S}(u,T)' du + \int_t^T \sigma_{F_S}(u,T)'dW^{Q^T}(u) \]
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where

$$\mu_{FS}^{Q_T}(t, T) = \int_t^T \int_u^T \beta(s) ds (\sigma'_u + B(u, T)'\Sigma)(\sigma_l - \sigma_R) du$$

$$\sigma_{FS}(u, T)' = \sigma'_u + B(u, T)'\Sigma$$

so that

$$E^{Q_S(T)}[S_l(T) | \mathcal{F}_t] = E^{Q_S(T)}[F_{S_1}(T, T) | \mathcal{F}_t]$$

$$= \frac{S_l(t)}{P(t, T)} e^{-\int_t^T \int_u^T \beta(s) ds (\sigma'_u + B(u, T)'\Sigma)(\sigma_l - \sigma_R) du}$$

Finally, we still have to find the value of

$$E^{Q_S(T)}[\max(NnS_l(T), g) | \mathcal{F}_t] = NnE^{Q_S(T)}\left[\max(S_l(T), \frac{g}{Nn}) | \mathcal{F}_t\right]$$

$$= NnE^{Q_S(T)}\left[\max(F_{S_1}(T, T), \frac{g}{Nn}) | \mathcal{F}_t\right]$$

$$= Nn \frac{E^{Q_S(T)}[F_{S_1}(T, T) 1_{\{F_{S_1}(T, T) > \frac{g}{Nn}\}} | \mathcal{F}_t]}{f_T}$$

$$+ g \frac{E^{Q_S(T)}[1_{\{F_{S_1}(T, T) \leq \frac{g}{Nn}\}} | \mathcal{F}_t]}{f_T}$$

Let us start by (II). We have:

$$E^{Q_S(T)}\left[1_{\{F_{S_1}(T, T) \leq \frac{g}{Nn}\}} | \mathcal{F}_t\right] = Q^{S(T)}\left(F_{S_1}(T, T) \leq \frac{g}{Nn} | \mathcal{F}_t\right)$$

$$= Q^{S(T)}\left(\ln F_{S_1}(T, T) \leq \ln \frac{g}{Nn} | \mathcal{F}_t\right)$$

$$= N(-d_2)$$

Where N is the cumulative distribution function of a standard normal random variable, and

$$d_2 = -\ln \frac{g}{Nn} + \ln F_{S_1}(t, T) - \mu_{FS}^{Q_T}(t, T) - \frac{1}{2} \int_t^T \sigma_{FS}(u, T)'\sigma_{FS}(u, T) du$$

$$\left[\int_{t-T}^T \sigma_{FS}(u, T)'\sigma_{FS}(u, T) du\right]^{1/2}$$

For (I), we can again change our measure. Let the new measure $Q^{S(T)*}$ be defined by the following Radon Nikodym derivative with respect to $Q^{S(T)}$:

$$\frac{dQ^{S(T)*}}{dQ^{S(T)}} = \frac{F_{S_1}(T, T)}{E^{Q_S(T)}[F_{S_1}(T, T) | \mathcal{F}_t]}$$

We have the following equality

$$E^{Q_S(T)}\left[F_{S_1}(T, T) 1_{\{F_{S_1}(T, T) > \frac{g}{Nn}\}} | \mathcal{F}_t\right] = E^{Q_S(T)}[F_{S_1}(T, T) | \mathcal{F}_t] E^{Q^{S(T)*}}\left[1_{\{F_{S_1}(T, T) > \frac{g}{Nn}\}} | \mathcal{F}_t\right]$$
Let us define the process $\varepsilon^*_t$ by:

$$
\varepsilon^*_t = E^{Q^{(T)}} \left[ \frac{dQ^{(T)*}}{dQ^{(T)}} | \mathcal{F}_t \right]
$$

This process is the solution of the following differential equation

$$
d\varepsilon^*_t = \varepsilon^*_t (\sigma'_1 + B(t, T)' \Sigma) \, dW^{Q^{(T)*}}(t)
$$

Thanks to Girsanov’s theorem, we know the process $W^{Q^{(T)*}}$ defined by

$$
W^{Q^{(T)*}}(t) = W^{Q^{(T)}}(t) - \int_0^t (\sigma'_1 + B(u, T)' \Sigma)' \, du, \quad \forall t \in [0, T]
$$

is a standard Brownian motion under $Q^{(T)*}$. Under this measure the forward price $F_{S_t}(t, T)$ is given by

$$
\ln F_{S_t}(T, T) = \ln F_{S_t}(t, T) - \mu^{Q^T}_{F_S}(t, T) + \frac{1}{2} \int_t^T \sigma_{F_S}(u, T)' \sigma_{F_S}(u, T) \, du + \int_t^T \sigma_{F_S}(u, T)' dW^Q(u)
$$

We can now find

$$
E^{Q^{(T)*}} \left[ 1_{\{F_{S_t}(T, T) > \frac{g}{Nn}\}} \mid \mathcal{F}_t \right] = Q^{(T)*} \left( F_{S_t}(T, T) > \frac{g}{Nn} \mid \mathcal{F}_t \right)
$$

$$
= \text{Prob}(Z > -d_1)
$$

$$
= \text{Prob}(Z \leq d_1)
$$

$$
= N(d_1)
$$

with

$$
d_1 = \frac{-\ln \frac{g}{Nn} + \ln F_{S_t}(t, T) - \mu^{Q^T}_{F_S}(t, T) + \frac{1}{2} \int_t^T \sigma_{F_S}(u, T)' \sigma_{F_S}(u, T) \, du}{\left[ \int_t^T \sigma_{F_S}(u, T)' \sigma_{F_S}(u, T) \, du \right]^{\frac{1}{2}}}
$$

We then find

$$
E^{Q^{(T)}} \left[ \max \left( NnS_t(T), g \right) \mid \mathcal{F}_t \right] = NnE^{Q^{(T)}} \left[ F_{S_t}(T, T) \mid \mathcal{F}_t \right] N(d_1) + gN(-d_2)
$$

with

$$
E^{Q^{(T)}} \left[ F_{S_t}(T, T) \mid \mathcal{F}_t \right] = \frac{S_t(t)}{P(t, T)} e^{-\int_t^T (\int_s^T \beta(s) \, ds) \sigma'_1 + B(u, T)' \Sigma (\sigma_1 - \sigma_2) \, du}
$$

$$
d_1 = \frac{-\ln \frac{g}{Nn} + \ln F_{S_t}(t, T) - \mu^{Q^T}_{F_S}(t, T) + \frac{1}{2} \int_t^T \sigma_{F_S}(u, T)' \sigma_{F_S}(u, T) \, du}{\left[ \int_t^T \sigma_{F_S}(u, T)' \sigma_{F_S}(u, T) \, du \right]^{\frac{1}{2}}}
$$

$$
d_2 = \frac{-\ln \frac{g}{Nn} + \ln F_{S_t}(t, T) - \mu^{Q^T}_{F_S}(t, T) - \frac{1}{2} \int_t^T \sigma_{F_S}(u, T)' \sigma_{F_S}(u, T) \, du}{\left[ \int_t^T \sigma_{F_S}(u, T)' \sigma_{F_S}(u, T) \, du \right]^{\frac{1}{2}}}
$$
8. Conclusion.

In this chapter, we presented an alternative model of the surrender time in life insurance. Technicalities aside, the main difference with the traditional model of the surrender time, is the way we model the insurer’s and policyholder’s information. In the traditional model, the insurer and the policyholder have exactly the same set of information $\mathbb{F}$ and all their decisions, in particular the surrender decision, are based on this set. Accordingly, the surrender time has to be an $\mathbb{F}$-stopping time. We argue this is not realistic. On the contrary, we assume that the policyholder takes his surrender decision on a larger set of information. In this situation, the surrender time can be characterized by its so-called $\mathbb{F}$-hazard process.

We also studied the impact of this framework on the fair valuation of life insurance contract. In particular, we gave fair valuation formulas for single premium and periodic premiums unit linked contracts with stochastic term structure and stochastic surrender. We also illustrate these formulas in a Gaussian framework.

Appendix 1.

**Proposition.** Let $Q^T$ be the measure equivalent to $Q$ on $(\Omega, \mathcal{F}_T)$ whose Radon-Nikodym derivative is given by

$$
\frac{dQ^T}{dQ} |_{\mathcal{F}_T} = \frac{e^{-D_T}}{E_Q[e^{-D_T} | \mathcal{F}_0]}
$$

$Q$-a.s. Let $Q^{S(T)}$ be the measure equivalent to $Q^T$ on $(\Omega, \mathcal{F}_T)$ whose Radon-Nikodym derivative is given by

$$
\frac{dQ^{S(T)}}{dQ^T} |_{\mathcal{F}_T} = \frac{e^{-\tilde{\Gamma}_T}}{E_{Q^T}[e^{-\tilde{\Gamma}_T} | \mathcal{F}_0]}
$$

$Q^T$-a.s.

If $X$ is $Q^{S(T)}$-integrable, then we have

$$
1_{\tau > t} E_Q \left[ e^{-(D_T-D_t)} e^{-(\tilde{\Gamma}_T-\tilde{\Gamma}_t)} X | \mathcal{F}_t \right] = 1_{\tau > t} P(t, T) E_{Q^T} \left[ e^{-(\tilde{\Gamma}_T-\tilde{\Gamma}_t)} X | \mathcal{F}_t \right]
$$

$$
= 1_{\tau > t} P(t, T) E_{Q^T} \left[ e^{-(\tilde{\Gamma}_T-\tilde{\Gamma}_t)} | \mathcal{F}_t \right] E_{Q^{S(T)}} \left[ X | \mathcal{F}_t \right]
$$

$$
= P(t, T) Q^T (\tau > T | \mathcal{G}_t) E_{Q^{S(T)}} \left[ X | \mathcal{F}_t \right]
$$
PROOF. We have

\[
E^Q e^{-(\hat{\Gamma}_T - \hat{\Gamma}_t) X | F_t} = \frac{E^Q e^{-D_T} e^{-(\hat{\Gamma}_T - \hat{\Gamma}_t) X | F_t}}{E^Q e^{-D_T} e^{-(\hat{\Gamma}_T - \hat{\Gamma}_t) X | F_t}}
\]

\[
= \frac{e^{-D_t} E^Q e^{-(D_T - D_t) e^{-(\hat{\Gamma}_T - \hat{\Gamma}_t) X | F_t}}}{e^{-D_T} E^Q e^{-(D_T - D_t) X | F_t}}
\]

\[
= \frac{E^Q e^{-(D_T - D_t) e^{-(\hat{\Gamma}_T - \hat{\Gamma}_t) X | F_t}}}{E^Q e^{-(D_T - D_t) X | F_t}}
\]

So that we have the first equality of our proposition.

\[
1_{\tau > t} E^Q e^{-(D_T - D_t) e^{-(\hat{\Gamma}_T - \hat{\Gamma}_t) X | F_t}} = 1_{\tau > t} P(t, T) E^Q e^{-(\hat{\Gamma}_T - \hat{\Gamma}_t) X | F_t}
\]

(8.1)

Similarly, we have

\[
E^{Q^{S(T)}} X | F_t = \frac{E^{Q^{T}} e^{-\hat{\Gamma}_T} X | F_t}{E^{Q^{T}} e^{-\hat{\Gamma}_T} | F_t}
\]

\[
= \frac{e^{-\hat{\Gamma}_T} E^{Q^{T}} e^{-\hat{\Gamma}_T} X | F_t}{e^{-\hat{\Gamma}_T} E^{Q^{T}} e^{-\hat{\Gamma}_T} | F_t}
\]

\[
= \frac{E^{Q^{T}} e^{-\hat{\Gamma}_T} X | F_t}{E^{Q^{T}} e^{-\hat{\Gamma}_T} | F_t}
\]

So that

\[
E^{Q^{T}} e^{-(\hat{\Gamma}_T - \hat{\Gamma}_t) X | F_t} = E^{Q^{T}} e^{-(\hat{\Gamma}_T - \hat{\Gamma}_t) X | F_t} E^{Q^{S(T)}} X | F_t
\]

Eventually, using this last expression in (8.1), we have the second equality of our proposition.

\[
1_{\tau > t} E^Q e^{-(D_T - D_t) e^{-(\hat{\Gamma}_T - \hat{\Gamma}_t) X | F_t}} = 1_{\tau > t} P(t, T) E^{Q^{T}} e^{-(\hat{\Gamma}_T - \hat{\Gamma}_t) X | F_t} E^{Q^{S(T)}} X | F_t
\]
Appendix 2.

Proposition. The value of the liabilities of the periodic premium unit-linked contract of type I is given by

\[ L_t = 1_{\{\tau > t\}} e^{(D_t + \tilde{\Gamma}_t)} E^Q \left[ e^{-(D_T + \tilde{\Gamma}_T) \max \left( 1 - \rho P \sum_{i=0}^{N-1} S(T) / S(t_i), g \right)} | F_t \right] \]

\[ + 1_{\{\tau > t\}} (1 - \alpha) (1 - \rho) P \sum_{i=0}^{N-1} \left\{ \frac{S(t)}{S(t_i)} - e^{(D_t + \tilde{\Gamma}_t)} E^Q \left[ \frac{S(T)}{S(t_i)} e^{-(D_T + \tilde{\Gamma}_T)} | F_t \right] \right\} \]

\[ + 1_{\{\tau > t\}} (1 - \alpha) (1 - \rho) P \sum_{i=0}^{N-1} \left\{ e^{(D_t + \tilde{\Gamma}_t)} E^Q \left[ \left( 1 - \frac{S(T)}{S(t_i)} \right) e^{-(D_t + \tilde{\Gamma}_t)} | F_t \right] \right\} \]

Proof. The value of these liabilities are given by

\[ L_t = E^Q \left[ e^{-(D_T - D_t)} \max \left( S(T) \sum_{i=0}^{N-1} n_{t_i}, g \right) 1_{\{\tau > T\}} | G_t \right] \]

\[ + E^Q \left[ e^{-(D_T - D_t)} (1 - \alpha) S(\tau) \left( \sum_{i=0}^{N-1} n_{t_i} \right) 1_{\{\tau \leq T\}} | G_t \right] \]

where \([\tau] = \sup \{i | t_i \leq \tau\}\). We can rewrite this equation as

\[ L_t = \underbrace{E^Q \left[ e^{-(D_T - D_t)} \max \left( S(T) \sum_{i=0}^{N-1} n_{t_i}, g \right) 1_{\{\tau > T\}} | G_t \right]}_{I} \]

\[ + \underbrace{E^Q \left[ e^{-(D_T - D_t)} (1 - \alpha) S(\tau) \left( \sum_{i=0}^{N-1} n_{t_i} \right) 1_{\{\tau \leq T\}} | G_t \right]}_{II} \]

\[ - \underbrace{E^Q \left[ e^{-(D_T - D_t)} (1 - \alpha) S(\tau) \left( \sum_{i=0}^{N-1} n_{t_i} \right) 1_{\{\tau \leq T\}} | G_t \right]}_{III} \]

We can study each of these terms separately.
$$I = EQ \left[ e^{-(D_T - D_t)_{\text{max}}} \left( S(T) \sum_{i=0}^{N-1} n_{t_i} g \right) 1_{\tau>T} | G_t \right]$$

$$= 1_{\tau>t} EQ \left[ e^{-(D_T - D_t)_{\text{max}}} \left( S(T) \sum_{i=0}^{N-1} n_{t_i} g \right) 1_{\tau>T} | F_t \right]$$

$$= 1_{\tau>t} EQ \left[ e^{-(D_T - D_t)_{\text{max}}} \left( (1 - \rho) \sum_{i=0}^{N-1} \frac{S(T)}{S(t_i)} g \right) 1_{\tau>T} | F_t \right]$$

$$II = EQ \left[ e^{-(D_T - D_t)} (1 - \alpha) S(\tau) \left( \sum_{i=0}^{\lfloor \tau \rfloor} n_{t_i} \right) 1_{\tau>| G_t} \right]$$

$$= 1_{\tau>t} (1 - \alpha) EQ \left[ e^{-(D_T - D_t)} S(\tau) \left( \sum_{i=0}^{\lfloor \tau \rfloor} \frac{(1 - \rho) P}{S(t_i)} \right) | G_t \right]$$

$$= 1_{\tau>t} (1 - \alpha) (1 - \rho) PEQ \left[ e^{-(D_T - D_t)} S(\tau) \left( \sum_{i=0}^{N-1} \frac{1}{S(t_i)} 1_{\tau>t_i} \right) | G_t \right]$$

$$= 1_{\tau>t} (1 - \alpha) (1 - \rho) P \sum_{i=0}^{\lfloor t \rfloor} S(t_i) \left( \sum_{i=0}^{\lfloor \tau \rfloor} \frac{1}{S(t_i)} 1_{\tau>t_i} \right) | G_t \right]$$

$$+ 1_{\tau>t} (1 - \alpha) (1 - \rho) P \sum_{i=0}^{N-1} E^Q \left[ e^{-(D_T - D_t)} S(\tau) \frac{S(t)}{S(t_i)} 1_{\tau>t_i} | G_t \right]$$

$$+ 1_{\tau>t} (1 - \alpha) (1 - \rho) P \sum_{i=0}^{N-1} E^Q \left[ e^{-(D_T - D_t)} S(\tau) \frac{S(t)}{S(t_i)} 1_{\tau>t_i} | G_t \right]$$

$$= 1_{\tau>t} (1 - \alpha) (1 - \rho) P \sum_{i=0}^{\lfloor t \rfloor} S(t_i)$$

$$+ 1_{\tau>t} (1 - \alpha) (1 - \rho) P \sum_{i=0}^{N-1} E^Q \left[ e^{-(D_T - D_t)} S(\tau) \frac{S(t)}{S(t_i)} 1_{\tau>t_i} | G_t \right]$$

$$= 1_{\tau>t} (1 - \alpha) (1 - \rho) P \sum_{i=0}^{\lfloor t \rfloor} S(t_i)$$

$$+ 1_{\tau>t} (1 - \alpha) (1 - \rho) P \sum_{i=0}^{N-1} E^Q \left[ \{ \tau > t_i \} e^{-(D_{ti} - D_t)} \frac{1}{S(t_i)} E^Q \left[ e^{-(D_T - D_t)} S(\tau) | G_{t_i} \right] | G_t \right]$$
Finally, we have for (II):

\[
II = 1_{\tau>t}(1-\alpha)(1-\rho)P \sum_{i=0}^{\lfloor \tau \rfloor} \frac{S(t)}{S(t_i)} + 1_{\tau>t}(1-\alpha)(1-\rho)P \sum_{i=0}^{N-1} E^Q \left[ \{ \tau > t_i \} e^{-(D_{t_i} - D_t)} \frac{S(t_i)}{S(t)} | \mathcal{G}_t \right]
\]

\[
III = E^Q \left[ e^{-(D_{\tau-D_t})} (1-\alpha)S(\tau) \left( \sum_{i=0}^{\lfloor \tau \rfloor} n_{t_i} \right) 1_{\tau>T} | \mathcal{G}_t \right]
\]

In the above derivation, we used the fact the discounted price of $S$ is a $(Q, G)$-martingale. □

**Appendix 3.**

**Proposition.** The present value of the liabilities of unit-linked contract of type II is given by

\[
L_t = 1_{\{\tau>t\}} e^{(D_{t}+\tilde{\Gamma}_t)} E^Q \left[ e^{-(D_{T}+\tilde{\Gamma}_T)} \max(NnS(T), g) | \mathcal{F}_t \right]
\]

\[
+ 1_{\{\tau>t\}} (1-\alpha)n \sum_{i=0}^{\lfloor \tau \rfloor} S(t_i)
\]

\[
+ 1_{\{\tau>t\}} (1-\alpha)ne^{-(D_{\tau}+\tilde{\Gamma}_t)} \left\{ \sum_{i=0}^{N-1} E^Q \left[ e^{-(D_{t_i}+\tilde{\Gamma}_t)} S(t_i) | \mathcal{F}_t \right] - NE^Q \left[ e^{-(D_{T}+\tilde{\Gamma}_T)} S(T) | \mathcal{F}_t \right] \right\}
\]

**Proof.** The present value of the liabilities is given by

\[
L_t = E^Q \left[ e^{-(D_{\tau-D_t})} \max(NnS(T), g) 1_{\tau>T} | \mathcal{G}_t \right]
\]

\[
+ E^Q \left[ e^{-(D_{\tau-D_t})} (1-\alpha) | \tau| nS(\tau) 1_{t<\tau \leq T} | \mathcal{G}_t \right]
\]
We can rewrite this equation as

\[
L_t = \frac{E^Q}{I} \left[ e^{-(D_T-D_t)max(NnS(T), g)1_{\tau>T} |G_t}} \right] + \frac{E^Q}{II} \left[ e^{-(D_T-D_t)(1-\alpha)|\tau|nS(\tau)1_{\tau>t} |G_t}} \right] - \frac{E^Q}{III} \left[ e^{-(D_T-D_t)(1-\alpha)|\tau|nS(\tau)1_{\tau>T} |G_t}} \right]
\]

We can study each of these terms separately.

\[
I = E^Q \left[ e^{-(D_T-D_t)max(NnS(T), g)1_{\tau>T} |G_t}} \right] = 1_{\{\tau>t\}} e^{(D_T+\hat{\Gamma}t)} E^Q \left[ e^{-(D_T+\hat{\Gamma})max(NnS(T), g) |F_t}} \right]
\]

\[
II = 1_{\{\tau>t\}} (1-\alpha) n E^Q \left[ e^{-(D_T-D_t)} \sum_{i=0}^{N-1} 1_{\{\tau>t_i\}} S(\tau) |G_t}} \right]
\]

\[
= 1_{\{\tau>t\}} (1-\alpha) n \sum_{i=0}^{\lfloor t \rfloor} E^Q \left[ e^{-(D_T-D_t)} 1_{\{\tau>t_i\}} S(\tau) |G_t}} \right] + 1_{\{\tau>t\}} (1-\alpha) n \sum_{i=\lfloor t \rfloor}^{N-1} E^Q \left[ e^{-(D_T-D_t)} 1_{\{\tau>t_i\}} S(\tau) |G_t}} \right]
\]

\[
= 1_{\{\tau>t\}} (1-\alpha) n \sum_{i=0}^{\lfloor t \rfloor} 1_{\{\tau>t_i\}} E^Q \left[ e^{-(D_T-D_t)} S(\tau) |G_t}} \right] + 1_{\{\tau>t\}} (1-\alpha) n \sum_{i=\lfloor t \rfloor}^{N-1} E^Q \left[ 1_{\{\tau>t_i\}} e^{-(D_T-D_t)} E^Q \left[ e^{-(D_T-D_t)} S(\tau) |G_t}} \right] |G_t}} \right]
\]

\[
= 1_{\{\tau>t\}} (1-\alpha) n \left\{ \sum_{i=0}^{\lfloor t \rfloor} S(t_i) + \sum_{i=\lfloor t \rfloor}^{N-1} e^{-(D_T+\hat{\Gamma}t)} E^Q \left[ e^{-(D_T+\hat{\Gamma})} S(t_i) |F_t}} \right] \right\}
\]
By Itô's lemma, we can find the solution of this stochastic differential equation. We have for

\[ III = (1 - \alpha)nE^Q \left[ e^{-(D_T - D_t)} \sum_{i=0}^{N-1} 1_{\{\tau>T\}} |G_t \right] \]

\[ = (1 - \alpha)nNE^Q \left[ e^{-(D_T - D_t)} S(\tau) 1_{\{\tau>T\}} |G_t \right] \]

\[ = (1 - \alpha)nNE^Q \left[ e^{-(D_T - D_t)} E^Q \left[ e^{-(D_T - D_t)} S(\tau) |G_T \right] 1_{\{\tau>T\}} |G_t \right] \]

\[ = (1 - \alpha)nNE^Q \left[ e^{-(D_T - D_t)} S(T) 1_{\{\tau>T\}} |G_t \right] \]

\[ = 1_{\{\tau>T\}}(1 - \alpha)nNe^{-(D_T + \tilde{\alpha}t)} E^Q \left[ e^{-(D_T + \tilde{\alpha}t)} S(T) |F_t \right] \]

In the above derivation, we used the fact the discounted price of \( S \) is a \((Q, G)\)-martingale. \( \square \)

**Appendix 4.**

**Proposition.** The price at time \( t \) of a zero coupon bond with maturity \( T \) is given in our 3 factors Gaussian affine model by:

\[ P(t, T) = e^{-A(t, T) - B(t, T)'X(t)} \]

where

\[ A(t, T) = \delta_0(T-t) + [(T-t)\delta_1 I - B(t, T)'] \tilde{\Theta} - \frac{1}{2} \left[ \int_t^T B(u, T)'B(u, T)du \right] \]

\[ B(t, T)' = \delta_1 \left[ N^{-1} \frac{1}{\tilde{K}_d} (I - e^{-K_d(T-t)}) N^{-1} \right] \]

**Proof.** Assume that \( \tilde{K} \) can be decomposed as \( \tilde{K} = N\tilde{K}_d N^{-1} \) where \( \tilde{K}_d \) is a \((3 \times 3)\) diagonal matrix. Let \( X^*(t) = N^{-1}X(t) \). We have

\[ dX^*(t) = N^{-1}dX(t) \]

\[ = N^{-1}[\tilde{K}(\tilde{\Theta} - X(t)) + \Sigma dW^Q(t)] \]

\[ = \tilde{K}_d N^{-1} (\tilde{\Theta} - X(t)) dt + N^{-1} \Sigma dW^Q(t) \]

\[ = \tilde{K}_d (N^{-1} \tilde{\Theta} - X^*(t)) dt + N^{-1} \Sigma dW^Q(t) \]

By Itô’s lemma, we can find the solution of this stochastic differential equation. We have for each \( i \).

\[ d(e^{(\tilde{K}_d)_i u} X^*_i(u)) = (\tilde{K}_d)_i e^{(\tilde{K}_d)_i u} X^*_i(u) du + e^{(\tilde{K}_d)_i u} dX^*_i(u) \]

\[ = (\tilde{K}_d)_i e^{(\tilde{K}_d)_i u} X^*_i(u) du \]

\[ = e^{(\tilde{K}_d)_i u} (\tilde{K}_d)_i (N^{-1} \tilde{\Theta})_i du + e^{(\tilde{K}_d)_i u} (N^{-1} \Sigma)_{i,i} dW^Q(u) \]
If we integrate this expression, we have

\[
e^{(\tilde{K}_d)_{it} X_i^*(T)} - e^{(\tilde{K}_d)_{it} X_i^*(t)} = \int_t^T e^{(\tilde{K}_d)_{it} u(N^{-1}\tilde{\Theta})_i} du + \int_t^T e^{(\tilde{K}_d)_{it} (N^{-1}\Sigma)_i} dW^Q(u)
\]

\[
e^{(\tilde{K}_d)_{it} X_i^*(T)} = e^{(\tilde{K}_d)_{it} X_i^*(t)} + [e^{(\tilde{K}_d)_{it} T} - e^{(\tilde{K}_d)_{it} t}] (N^{-1}\tilde{\Theta})_i
\] + \int_t^T e^{(\tilde{K}_d)_{it} u(N^{-1}\Sigma)_i} dW^Q(u)

Finally,

\[
X_i^*(T) = e^{-(\tilde{K}_d)_{it} (T-t)} X_i^*(t) + [1 - e^{-(\tilde{K}_d)_{it} (T-t)}] (N^{-1}\tilde{\Theta})_i
\] + \int_t^T e^{-(\tilde{K}_d)_{it} (T-u)} (N^{-1}\Sigma)_i dW^Q(u)

In matrix form,

\[
X^*(T) = e^{-(\tilde{K}_d)(T-t)} X^*(t) + [I - e^{-(\tilde{K}_d)(T-t)}] N^{-1}\tilde{\Theta}
\] + \int_t^T e^{-(\tilde{K}_d)(T-u)} N^{-1}\Sigma dW^Q(u)

To find \(X(T)\), we can multiply this last expression by \(N\).

\[
X(T) = Ne^{-(\tilde{K}_d)(T-t)} X^*(t) + N[I - e^{-(\tilde{K}_d)(T-t)}] N^{-1}\tilde{\Theta}
\] + \int_t^T Ne^{-(\tilde{K}_d)(T-u)} N^{-1}\Sigma dW^Q(u)

= \[Ne^{-(\tilde{K}_d)(T-t)} N^{-1}] X(t) + [I - [Ne^{-(\tilde{K}_d)(T-t)} N^{-1}]] \tilde{\Theta}
\] + \int_t^T [Ne^{-(\tilde{K}_d)(T-u)} N^{-1}\Sigma] dW^Q(u)

To find the distribution of \(\int_t^T r(u) du\), we need to know

\[
\int_t^T r(u) du = \int_t^T \delta_0 + \delta'_1 X(u) du
\] = \int_t^T \delta_0 + \delta'_1 N X^*(u) du

= \delta_0 (T - t) + \delta'_1 N \int_t^T X^*(u) du
For each \( i \), we have

\[
\int_t^T X_i^*(u) du = \frac{1}{(K_d)_i} (1 - e^{-(\tilde{K}_d)_{(T-t)}}) X_i^*(t) \\
+ \left[ \left( T - t \right) - \frac{1}{(K_d)_i} (1 - e^{-(\tilde{K}_d)_{(T-t)}}) \right] (N^{-1}\tilde{\Theta})_i \\
+ \int_t^T \frac{1}{(K_d)_i} (1 - e^{-(\tilde{K}_d)_{(T-u)}}) (N^{-1}\Sigma)_i, dW_Q(u)
\]

In matrix form

\[
\int_t^T X^*(u) du = \frac{1}{(K_d)} (1 - e^{-(\tilde{K}_d)_{(T-t)}}) X^*(t) \\
+ \left[ \left( T - t \right) I - \frac{1}{(K_d)} (I - e^{-(\tilde{K}_d)_{(T-t)}}) \right] (N^{-1}\tilde{\Theta}) \\
+ \int_t^T \frac{1}{(K_d)} (1 - e^{-(\tilde{K}_d)_{(T-u)}}) (N^{-1}\Sigma)dW_Q(u)
\]

To find \( \int_t^T X(u) du \), we can multiply the last expression by \( N \)

\[
\int_t^T X(u) du = N \frac{1}{(K_d)} (1 - e^{-(\tilde{K}_d)_{(T-t)}}) X^*(t) \\
+ \left[ \left( T - t \right) N I - N \frac{1}{(K_d)} (I - e^{-(\tilde{K}_d)_{(T-t)}}) \right] (N^{-1}\tilde{\Theta}) \\
+ \int_t^T N \frac{1}{(K_d)} (I - e^{-(\tilde{K}_d)_{(T-u)}}) (N^{-1}\Sigma)dW_Q(u) \\
= [N \frac{1}{(K_d)} (1 - e^{-(\tilde{K}_d)_{(T-t)}}) N^{-1}] X(t) \\
+ \left[ \left( T - t \right) I - [N \frac{1}{(K_d)} (I - e^{-(\tilde{K}_d)_{(T-t)}}) N^{-1}] \right] \tilde{\Theta} \\
+ \int_t^T [N \frac{1}{(K_d)} (I - e^{-(\tilde{K}_d)_{(T-u)}}) N^{-1}] \Sigma dW_Q(u)
\]
Finally, we obtain
\[
\int_t^T r(u) du = \delta_0(T - t) + \delta_1'(N \frac{1}{K_d}(1 - e^{-(K_d)(T-t)})N^1)]X(t)
+ \delta_1'[[(T-t)I - [N \frac{1}{K_d}(I - e^{-(K_d)(T-t)})N^{-1}]]]\tilde{\Theta}
+ \delta_1' \int_t^T [N \frac{1}{K_d}(I - e^{-(K_d)(T-u)})N^{-1}]\Sigma dW^Q(u)
= \delta_0(T - t) + B(t, T)'X(t) + [(T-t)\delta_1'I - B(t, T)']\tilde{\Theta}
+ \int_t^T B(u, T)'\Sigma dW^Q(u)
\]
where
\[
B(t, T)' = \delta_1'[N \frac{1}{K_d}(I - e^{-K_d(T-t)})N^{-1}]
\]
Thus, \(\int_{u=t}^T r(u) du\) is normally distributed with mean
\[
E^Q \left[ \int_t^T r(u) du | \mathcal{F}_t \right] = \delta_0(T - t) + B(t, T)'X(t) + [(T-t)\delta_1'I - B(t, T)']\tilde{\Theta}
\]
and variance
\[
VAR^Q \left[ \int_t^T r(u) du | \mathcal{F}_t \right] = \int_t^T B(u, T)'\Sigma\Sigma' B(u, T) du
= \int_t^T B(u, T)'B(u, T) du
\]
Finally since \(\int_t^T r(u) du\) is normally distributed, we have
\[
P(t, T) = E^Q \left[ e^{-\int_t^T r(u) du} | \mathcal{F}_t \right] = e^{-\int_t^T r(u) du} + \frac{1}{2} A R^Q \left[ \int_t^T r(u) du | \mathcal{F}_t \right] = e^{-\delta_0(T-t) - B(t, T)'X(t) - [(T-t)\delta_1'I - B(t, T)']\tilde{\Theta} + \frac{1}{2} \int_t^T B(u, T)'B(u, T) du}
\]
The result follows. \(\square\)

\section*{Appendix 5.}

\textbf{Proposition.} \textit{In our model, the Radon-Nikodym derivative of the equivalent T-forward martingale measure \(Q_T\) with respect to \(Q\) is given by}
\[
\frac{dQ_T}{dQ} = e^{-\frac{1}{2} \int_0^T B(u, T)'\Sigma\Sigma' B(u, T) du - \int_0^T B(u, T)'\Sigma dW^Q(u)}
\]
Proof. The $T$-forward measure is defined by the following Radon-Nikodym derivative
\[
\frac{dQ^T}{dQ} = \frac{e^{-\int_0^T r(u)du}}{E_Q \left[ e^{-\int_0^T r(u)du} | \mathcal{F}_t \right] }, \text{ Q.a.s.}
\]
Using the results of Appendix 4, we have
\[
\frac{dQ^T}{dQ} = \frac{e^{-\delta_0(T-0)-B(0,T)X(0)-[(T-0)\delta_B^I I-B(0,T)\hat{\Theta}] - \int_0^T B(u,T) \Sigma dW^Q(u)}
\]
\[
eq \frac{e^{-\delta_0(T-0)-B(0,T)X(0)-[(T-0)\delta_B^I I-B(0,T)\hat{\Theta}] - \int_0^T B(u,T) \Sigma dW^Q(u)}}{e^{-A(0,T)-B(0,T)X(0)}}
\]
\[
= e^{-\int_0^T \beta(u)du - \int_0^T \beta(u)(u-t)du + \int_0^T (\sigma_R^I - \sigma_I^I) \Sigma B(u,T) \left( \int_u^T \beta(s)ds \right) du}
\]
\[
\lambda(u) = \alpha(u) + \beta(u)\left[1 + \int_u^T r(s) - \frac{1}{2} \sigma_R^I \sigma_I - \sigma_I^I \Sigma B(s,T)ds + \int_u^T \sigma_I^I dW^Q(s) - \int_u^T r(s) - \frac{1}{2} \sigma_R^I \sigma_R - \sigma_I^I \Sigma B(s,T)ds - \int_u^T \sigma_I^I dW^Q(s) \right]
\]
\[
= \alpha(u) + \beta(u)\left[1 + \int_u^T \frac{1}{2} (\sigma_R^I \sigma_R - \sigma_I^I \sigma_I) + (\sigma_R^R - \sigma_R^I) \Sigma B(s,T)ds + \int_u^T (\sigma_I^I - \sigma_R^I) dW^Q(s) \right]
\]
\[
\lambda(u) = \alpha(u) + \beta(u)\left[1 + \frac{1}{2} (\sigma_R^I \sigma_R - \sigma_I^I \sigma_I)(u-t) + \int_t^u (\sigma_R^R - \sigma_I^R) \Sigma B(s,T)ds + \int_t^u (\sigma_I^R - \sigma_R^R) dW^Q(s) \right]
\]
\[
\lambda(u) = \alpha(u) + \beta(u)\left[1 + \frac{1}{2} (\sigma_R^I \sigma_R - \sigma_I^I \sigma_I) + (\sigma_R^R - \sigma_I^R) \Sigma B(s,T)ds + \int_t^u (\sigma_I^R - \sigma_R^R) dW^Q(s) \right]
\]
If we integrate this expression, we have

\[
\int_t^T \lambda(u)du = \int_t^T \alpha(u)du + \int_t^T \beta(u)du + \frac{1}{2} \int_t^T \beta(u)(\sigma'_R \sigma_R - \sigma'_I \sigma_I)(u-t)du \\
+ \int_t^T \beta(u) \int_t^u (\sigma'_R - \sigma'_I) \Sigma'(s,T)ds du + \int_t^T \beta(u) \int_t^u (\sigma'_I - \sigma'_R) dW^Q(s)du
\]

Using Fubini’s theorem, we have

\[
\int_t^T \beta(u) \int_t^u (\sigma'_I - \sigma'_R) dW^Q(s)du = \int_t^T \int_s^T \beta(u)(\sigma'_I - \sigma'_R)dudW^Q(s) \\
= \int_t^T (\sigma'_I - \sigma'_R) \int_s^T \beta(u)dudW^Q(s)
\]

and

\[
\int_t^T \beta(u) \int_t^u (\sigma'_R - \sigma'_I) \Sigma'(s,T)ds du = \int_t^T \int_s^T \beta(u)(\sigma'_R - \sigma'_I) \Sigma'(s,T)duds \\
= \int_t^T (\sigma'_R - \sigma'_I) \Sigma'(s,T) \int_s^T \beta(u) duds
\]

So that we can write

\[
\int_t^T \lambda(u)du = \int_t^T \alpha(u)du + \int_t^T \beta(u)du + \frac{1}{2} \int_t^T \beta(u)(\sigma'_R \sigma_R - \sigma'_I \sigma_I)(u-t)du \\
+ \int_t^T (\sigma'_R - \sigma'_I) \Sigma'(s,T)ds du + \int_t^T (\sigma'_I - \sigma'_R) \int_t^u (\sigma'_I - \sigma'_R) dW^Q(s)du
\]

To have more concise expressions, we introduce the following notation

\[
\int_t^T \lambda(u)du = \mu Q^T(t,T) + \int_t^T \sigma^Q(u,T)dW^Q(u)
\]

where

\[
\mu Q^T(t,T) = \int_t^T \alpha(u)du + \int_t^T \beta(u)du + \frac{1}{2} \int_t^T \beta(u)(\sigma'_R \sigma_R - \sigma'_I \sigma_I)(u-t)du \\
+ \int_t^T (\sigma'_R - \sigma'_I) \Sigma'(s,T) \left( \int_u^T \beta(s)ds \right) du
\]

\[
\sigma^Q(t,T) = (\sigma'_I - \sigma'_R) \int_t^T \beta(s)ds
\]
Appendix 7.

Proposition. In our model, the Radon-Nikodym derivative of the $T$-forward surrender measure $Q^{S(T)}$ with respect to $Q^T$ is given by

$$
\frac{dQ^{S(T)}}{dQ^T} = e^{-\frac{1}{2} \int_0^T \left[(\sigma'_I - \sigma'_R) \int_u^T \beta(s) ds\right] du - \int_0^T \left(\sigma'_I - \sigma'_R\right) \int_u^T \beta(s) ds dW^{Q^T}(u)}
$$

Proof. The $T$-forward surrender measure is defined by the following Radon-Nikodym derivative

$$
\frac{dQ^{S(T)}}{dQ^T} = e^{-\gamma_T} \frac{E_Q^{T} \left[e^{-\gamma_T} | F_t \right]}{E_Q^{T} \left[e^{-\gamma_T} | F_t \right]}, Q^T\text{-a.s.}
$$

Using the results of the Appendix 6, we have

$$
\frac{dQ^{S(T)}}{dQ^T} = \frac{e^{-\gamma_T(0,T) - \int_0^T \sigma^V(u,T) \gamma(u) du}}{e^{-\gamma_T(0,T) + \frac{1}{2} \int_0^T \sigma^V(u,T) \sigma^V(u,T) du}}
$$

$$
= e^{-\frac{1}{2} \int_0^T \sigma^V(u,T) \gamma(u) du - \int_0^T \sigma^V(u,T) \gamma(u) du} = e^{-\frac{1}{2} \int_0^T \left[(\sigma'_I - \sigma'_R) \int_u^T \beta(s) ds\right] du - \int_0^T \left(\sigma'_I - \sigma'_R\right) \int_u^T \beta(s) ds dW^{Q^T}(u)}
$$

Appendix 8.

Proposition. Under $Q^T$, the forward price $F_{S_I}(t,T)$ follows a martingale and is solution of

$$
\frac{dF_{S_I}(t,T)}{dF_{S_I}(t,T)} = F_{S_I}(t,T)(\sigma'_I + B(t,T)\Sigma) dW^{Q^T}(t)
$$

Proof. Under $Q^T$, we saw the price of the fund $S_I$ is given by:

$$
\ln S_I(t) = \ln S_I(0) + \int_0^t r(u) du - \frac{1}{2} \sigma'_I \sigma_I - \frac{1}{2} \sigma'_I \Sigma B(u,T) du + \int_0^t \sigma'_I dW^{Q^T}(u)
$$

As far as the price of the zero coupon bond is concerned, we have under $Q$

$$
\frac{dP(t,T)}{dP(t,T)} = P(t,T) [r(t) - B'(t,T) \Sigma dW^Q(t)]
$$

The solution of this differential equation is given by

$$
\ln P(t,T) = \ln P(0,T) + \int_0^t r(u) du - \frac{1}{2} B'(u,T) \Sigma \Sigma B'(u,T) du - \int_0^t B'(u,T) \Sigma dW^Q(u)
$$

Under $Q^T$, the price of a zero coupon is then given by

$$
\ln P(t,T) = \ln P(0,T) + \int_0^t r(u) du + \frac{1}{2} B'(u,T) \Sigma \Sigma B'(u,T) du - \int_0^t B'(u,T) \Sigma dW^Q(u)
$$
Taking the difference between $\ln S_I(t)$ and $\ln P(t, T)$,

$$
\ln \left( \frac{S_I(t)}{P(t, T)} \right) = \ln \left( \frac{S_I(0)}{P(0, T)} \right) - \frac{1}{2} \int_0^t B'(u, T) \Sigma \Sigma' B'(u, T) + \sigma_I' \sigma_I + 2 \sigma_I' \Sigma B(u, T) du \\
+ \int_0^t \sigma_I' + B'(u, T) \Sigma dW^{Q^T}(u)
$$

Then we get

$$F_{S_I}(t, T) = F_{S_I}(0, T) e^{-\frac{1}{2} \int_0^t (\sigma_I' + B'(u, T) \Sigma)^2 du + \int_0^t \sigma_I' + B'(u, T) \Sigma dW^{Q^T}(u)}$$

Which is the solution of the differential equation

$$d(F_{S_I}(t, T)) = F_{S_I}(t, T)(\sigma_I' + B(t, T) \Sigma) dW^{Q^T}(t)$$
CHAPTER 3

Risk-Minimizing Strategies of Life Insurance Contracts with Surrender Option.

1. Introduction.

This chapter aims to study hedging strategies of insurance contracts with surrender option. The academic literature on the valuation of surrender options generally considers the surrender time as an optimal stopping time with respect to the filtration generated by the financial assets prices. See [14, 15, 47, 81, 80] for examples. A life insurance contract with surrender option can then be seen as an American option. Even though there are some good arguments in favor of these models for a valuation purpose, they are not appropriate for a risk management purpose. Indeed, in these models, the surrender option does not induce any additional source of risk to the purely financial one. In particular, if the financial market is complete, the insurance market remains complete and accordingly, any insurance contract with a surrender option can be perfectly hedged (if we neglect the mortality risk). This is not a realistic assumption in a risk management perspective. In order to overcome this drawback, we assume the surrender times are not stopping times with respect to the filtration generated by the financial assets prices. We refer the reader to the previous chapter for more details.

In this framework, even if the financial market is initially complete, an insurance contract with a surrender option cannot be perfectly hedged. It is then worthwhile to study the hedging strategies an insurer should follow to minimize its exposure to this surrender risk. Different approaches have been developed to tackle the problem of contingent claims hedging in an incomplete market. One of the most popular approaches are those based on quadratic hedging strategies as the risk-minimizing, local risk-minimizing and mean-variance hedging strategies. See Schweizer [78] for a review. In this chapter, we focus on the risk-minimizing hedging strategies developed by Föllmer and Sondermann in [44]. This approach has been introduced with success in the insurance literature by Møller [68, 69] and Dahl and Møller [39]. In particular, Møller in [68] extended the risk-minimization theory to payment processes and applied this theory to the hedging of a portfolio of unit-linked insurance contracts subject to non hedgeable mortality risk. In his papers, Møller assumed a complete financial market of the Black and Scholes type and assumed the policyholders’s dates of death are random times independent of each other and independent of the financial market. In this setting, Møller derived the risk-minimizing hedging strategies for portfolios of $n$ policies and showed that, by following these strategies, the remaining (unhedgeable) risk goes down to zero in relative terms with the size $n$ of the portfolio. In other words, he showed we could combine diversification and risk-minimizing hedging to reduce as much as we want the relative risk of such portfolios.
Riesner [72] extended this result by introducing an incomplete financial market driven by a Lévy process. He showed the relative risk can only be reduced down to a limit related to the incompleteness of the financial market. In this chapter, we generalize these results in two ways. Firstly, we do not assume the independence of the random times of payment with the financial market. It is an imperative assumption if we want to model realistic surrender times. Indeed it is a well known empirical fact, the surrenders depend on the evolution of the financial market. This dependence obviously implies the random times are neither independent of each other. Secondly, we consider a general incomplete financial market (not necessarily driven by a Lévy process). We only assume the discounted prices of the financial assets are local martingales. In this setting, we firstly derive the general form of the risk-minimizing hedging strategies for a single policy. Under an additional assumption, namely the $F$-independence of the random times, we extend this result to a portfolio of $n$ policies. Under this assumption, we then generalize Møller’s conclusion: if the financial market is complete, we show we can reduce as much as we want the relative risk of a portfolio by combining diversification and continuous hedging. If, as in Riesner, the financial market is incomplete, by definition, we can only reduce the relative risk down to a limit related to the degree of incompleteness of the financial market.

In a recent paper, Dahl and Møller [39] generalized these results in a still different way by introducing a systematic mortality risk. The random times of payment are thus dependent to each other but independent of the financial market. Our problem is in a sense similar to theirs but not identical. In both cases, the times of payment are subject to a systematic risk but in Dahl and Møller, this systematic risk cannot be hedged whereas in our, it can be hedged perfectly or at least in part whether the financial market is complete or not.

This chapter is organized as follows. In Section 2, we describe the financial market and the insurance payment processes. In Section 3, we briefly recall the main theoretical results of Föllmer and Sondermann and Møller on the risk-minimization theory. In Section 4, we apply these general results to a single insurance policy subject to surrender risk and derive the forms of the risk-minimizing hedging strategies. In Section 5, we extend the results of the previous section to a portfolio with $n$ policies. In Section 6, we study the relative risk process and shows how it evolves with the number $n$ of policies.

2. The Theoretical Framework.

2.1. The financial market. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. A perfect frictionless financial market is defined in this space. We assume there is one locally risk free asset denoted by $B_t = B(t, \omega)$ and $s$ risky assets $S^i_t = S^i(t, \omega), i = 1, \ldots , s$ following real càdlàg stochastic processes. The price of the locally risk free asset is assumed to follow a strictly positive, continuous process of finite variation. The discounted values of these assets are denoted by

$$X^i_t = \frac{S^i_t}{B_t}, i = 1, \ldots , s$$

Let $\mathcal{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$ be the filtration generated by the stochastic processes $S^i$ for $i = 1, \ldots , s$ and $B_t$. We assume this filtration respects the usual hypothesis.
We assume furthermore there is no arbitrage opportunity in this financial market with regard to the filtration $\mathcal{F}$. This assumption is equivalent to the existence of at least one probability measure $Q$, equivalent to $P$, such that the discounted prices $X$ are $(\mathcal{F}, Q)$-local martingales i.e. for each $i = 1, \ldots, s$, $X^i \in \mathcal{M}^{loc}(Q, \mathcal{F})$, the space of local martingales with respect to the filtration $\mathcal{F}$ and the measure $Q$.

### 2.2. The life insurance contract.

The surrender time $\tau$ is assumed to be a random time, i.e. $\tau = \tau(\omega)$ is a strictly positive random variable defined on $\Omega$ and $\mathcal{F}$-measurable. We also defined the process $H_t$ in the following way: $H_t = 1_{\{\tau \leq t\}}$. The crucial assumption here is we assume $\tau$ is not an $\mathcal{F}$-stopping time. We refer the reader to the previous chapter for a discussion of this assumption.

#### 2.2.1. The insurer’s payments.

A large variety of life insurance contracts can be modelled as a combination of the following three building blocks. The first one is the payoff the insurer has to pay at the term of the contract $T$. This payoff is assumed to be a $\mathcal{F}_T$-measurable random variable and is denoted by $g(T, \omega)$. At the term of the contract, the insurer has to pay:

$$g(T, \omega)1_{\{\tau > T\}}$$

The second building block is the amount the insurer has to pay when the policyholder surrenders before the term $T$. This amount is denoted by $1_{\{0 < \tau \leq T\}}R(\tau, \omega)$ where $(R(t, \omega))_{t \geq 0}$ is assumed to be an $\mathcal{F}$-predictable stochastic process. We can write

$$1_{\{0 < \tau \leq T\}}R(\tau, \omega) = \int_0^T R(u, \omega)dH_u$$

The third building block is the payoffs the insurer has to pay as long as the policyholder has not surrendered. We model these payoffs through their cumulative value up to time $t$, denoted by $C(t, \omega)$. This process is assumed to be a right continuous increasing $\mathcal{F}$-adapted process. The cumulative payoff up to surrender is then given by:

$$C(T, \omega)1_{\{\tau > T\}} + C(\tau, \omega)1_{\{0 < \tau \leq T\}} = \int_0^T (1 - H_u) dC(u, \omega)$$

where we assume $C(0, \omega) = 0$ and $C(T, \omega) = C(T, \omega)$.

#### 2.2.2. The policyholder’s payments.

We assume a policyholder pays premiums periodically at $N$ fixed dates $t_i$ with $i = 0, \ldots, N - 1$, as long as he has not surrendered. We denote by $P(t_i, \omega)$ the value of the premium paid at time $t_i$ and assume $P(t_i, \omega)$ is $\mathcal{F}_{t_i}$-measurable for each $i = 0, \ldots, N - 1$. The policyholder’s payments is then given by:

$$\sum_{i=0}^{N-1} P(t_i, \omega)1_{\{\tau > t_i\}}$$
We will here recall the main results of the theory of risk minimization. These results are due to Föllmer and Sondermann [44] for a single payoff and were extended by Møller [68] to payment processes. We mainly follow Schweizer [78] for the presentation.

The financial market is essentially defined as in Section (2.1) but we will here consider an arbitrary filtration $G$ larger than the filtration $F$ generated by the financial assets prices. We assume a specific martingale measure $Q$ has been chosen so that $X$ is a $(Q, G)$-local martingale.

**Definition 3.1.** A trading strategy $\rho$ is a pair of processes $(\epsilon, \eta)$ where $\eta$ is a 1-dimensional real-valued $G$-adapted process and $\epsilon$ is an $s$-dimensional real-valued $G$-predictable process.

The process $\epsilon_t$ represents the amount of risky assets held at time $t$ and the process $\eta_t$ is the discounted amount invested in the risk free asset.

**Definition 3.2.** The value process $V_t(\rho)$ of a trading strategy $\rho$, is defined by $V_t(\rho) = \epsilon_t X_t + \eta_t$ for $0 \leq t \leq T$.

This value process represents the discounted value of the insurer’s financial portfolio following the trading strategy $\rho$. This portfolio is not necessarily self-financed. We define by $L^2(X, Q)$ the following space.

**Definition 3.3.** $L^2(X, Q, G)$ is the space of $\mathbb{R}^s$- and $G$-predictable processes $\epsilon$ such that $\|\epsilon\|_{L^2(X, Q, G)} = \left( E^Q \left[ \int_0^T \epsilon_u d[X, X]_u \right] \right)^{1/2} < \infty$.

We also have the following important lemma due to Schweizer:

**Lemma 3.4.** Suppose $X$ is a $(Q, G)$-local martingale. For any $\epsilon \in L^2(X, Q, G)$, the process $\int_0^T \epsilon_u dX_u \in \mathcal{M}_0^2(Q, G)$. Moreover, the space $\mathcal{I}^2(X, Q, G) = \{ \int \epsilon dX \mid \epsilon \in L^2(X, Q, G) \}$ is a stable subspace of $\mathcal{M}_0^2(Q, G)$.

**Proof.** See Schweizer [78].

**Definition 3.5.** An RM-strategy $\rho$ is any pair $(\epsilon, \eta)$ where $\epsilon \in L^2(X, Q, G)$ and $\eta$ is a càdlàg adapted process such that the value process $V_t(\rho)$ is right continuous and square integrable.

The liabilities of the insurer towards a policyholder are modelled as a process $A_t$. The process $A_t$ is assumed to be càdlàg, $G$-adapted and square integrable. The process $A_t$ represents the discounted value of the cumulative payments up to time $t$.

**Definition 3.6.** The cumulative cost process $C_t(\rho)$ of a RM-strategy $\rho$, is defined by $C_t(\rho) = V_t(\rho) - \int_0^t \epsilon_u dX_u + A_t$.

The initial cumulative cost process $C_0(\rho)$ is given by $C_0(\rho) = V_0(\rho) + A_0$. The first term $V_0(\rho)$ represents the initial amount the insurer has to invest to create his financial portfolio. The second term is the initial payment the insurer has to pay to the policyholder. Usually, $A_0$ will be negative and instead of a payment will represent the initial premium paid by the policyholder.
Definition 3.7. A strategy is said to be mean self financing if its cumulative cost process is a martingale.

The risk process is defined as follows:

Definition 3.8. The risk process of an RM-strategy \( \rho \) is defined by
\[
R_t(\rho) = E_Q\left[ (C_T(\rho) - C_t(\rho))^2 | \mathcal{G}_t \right].
\]

Following Möller [68], we restrict our attention to strategies which are 0-admissible in the following sense:

Definition 3.9. A strategy \( \rho \) is said to be 0-admissible if and only if:
\[
V_T(\rho) = 0, \quad Q\text{-a.s.}
\]

The theory of risk minimization aims at finding an 0-admissible RM-strategy that minimizes the risk process \( R_t(\rho) \) in the following sense.

Definition 3.10. An RM-strategy \( \rho \) is called risk-minimizing if and only if
\[
R_t(\rho) \leq R_t(\tilde{\rho}) \quad Q\text{-a.s. for every } t \in [0, T] \quad \text{and for any RM-strategy } \tilde{\rho} \text{ which is an admissible continuation of } \rho \text{ from } t \text{ on, in the sense that } V_T(\rho) = V_T(\tilde{\rho}) \quad Q\text{-a.s. , } \tilde{\epsilon}_u = \epsilon_u \text{ for } u \leq t \text{ and } \tilde{\eta}_u = \eta_u \text{ for } u < t.
\]

To solve this problem, we need the following result:

Lemma 3.11. Any risk-minimizing RM-strategy is also mean self financing.

Proof. See Möller [68] for a proof. □

This lemma leads directly to the following one:

Lemma 3.12. The discounted value of the risk-minimizing RM-strategy \( \rho^* \) is given by:
\[
V_t(\rho^*) = E_Q [A_T - A_t | \mathcal{G}_t]
\]

Proof. Indeed since by Lemma 3.11, the cost process is a martingale, we have
\[
C_t(\rho^*) = E_Q [C_T(\rho^*) | \mathcal{G}_t]
\]
\[
V_t(\rho^*) - \int_0^t \epsilon_u dX_u + A_t = E_Q \left[ 0 - \int_0^T \epsilon_u dX_u + A_T | \mathcal{G}_t \right]
\]
Since the stochastic integral \( \int \epsilon_u dX_u \) is a martingale by Lemma 3.4, we have the result. □

The risk-minimizing discounted amount invested in the risk free asset is thus given by:
\[
\eta^*_t = E_Q [A_T - A_t | \mathcal{G}_t] - \epsilon^*_t X_t.
\]

Schweizer shows the solution of this problem is closely related to the so-called Galtchouk-Kunita-Watanabe decomposition of \( A_T \). Since \( \mathcal{I}^2(X, Q, \mathcal{G}) \) is a stable subspace of \( \mathcal{M}^2_0(Q, \mathcal{G}) \), any \( A_T \) square integrable random variable can be uniquely written as:
\[
A_T = E_Q [A_T | \mathcal{G}_0] + \int_0^T \epsilon^A u dX_u + L^A_T
\]
Q-a.s., where \( \epsilon^A u \in L^2(X, Q, \mathcal{G}) \) and \( L^A_T \in \mathcal{M}^2_0(Q, \mathcal{G}) \), is \( Q \)-strongly orthogonal to \( \mathcal{I}^2(X, Q, \mathcal{G}) \).

With these notation, we can give the solution of our risk-minimizing problem in the following lemma:
Lemma 3.13. The unique risk-minimizing RM-strategy \( \rho^* = (\varepsilon^*, \eta^*) \) of \( A \) is given by 
\[
\varepsilon^* = \varepsilon^A \quad \text{and} \quad \eta^* t = E_Q [A_T - A_t | G_t] - \varepsilon^*_t X_t.
\]
The cumulative cost process of \( \rho^* \) is given by 
\[
C_t(\rho^*) = E_Q [A_T | G_0] + L^A_t = C_0(\rho^*) + L^A_t
\]
and the value process of \( \rho^* \) is given by 
\[
V_t(\rho^*) = E_Q [A_T - A_t | G_t].
\]

Proof. See Schweizer [78] for the single payoff case or Møller [68] for the extension to payment processes.

As Schweizer noticed in [78], we do not have to assume \( X \) is (locally) square integrable for these results to hold.


In this section, we apply the risk minimization theory to a single insurance contract. In Section 5, we apply this theory to a portfolio with \( n \) insurance contracts.

Henceforth, we will assume a specific equivalent martingale measure \( Q \) has been chosen and we will assume we are in the theoretical framework described in Section 2.

4.1. Preliminary assumptions and results.

4.1.1. The insurer’s information. We assume the insurer observes the financial market and whether the policyholder has surrendered or not. We define \( \mathcal{H} = (\mathcal{H}_t)_{0 \leq t \leq T} \) with \( \mathcal{H}_t = \sigma(H_s, 0 \leq s \leq t) \), the filtration generated by the process \( H_t \). We defined \( \mathcal{G} = (\mathcal{G}_t)_{0 \leq t \leq T} \) where \( \mathcal{G}_t = \mathcal{F}_t \lor \mathcal{H}_t \), the filtration corresponding to the progressive enlargement of the filtration \( \mathcal{F} \) by \( \mathcal{H} \). Intuitively, the filtration \( \mathcal{G} \) corresponds to the insurer’s information. Since we assume \( \tau \) is not an \( \mathcal{F} \)-stopping time, the filtration \( \mathcal{G} \) is strictly larger than \( \mathcal{F} \).

4.1.2. The hazard process. In this subsection, we introduce the \((\mathcal{F}, Q)\)-hazard process of a random time \( \tau \).

Definition 4.1. Let \( \tau \) be a non negative random variable defined on \((\Omega, \mathcal{F})\). Assume \( Q(\tau = 0) = 0 \) and \( Q(\tau > t) > 0 \), \( \forall t \in \mathbb{R}_+ \). We define the \((\mathcal{F}, Q)\)-hazard process \( \hat{\Gamma} \) of \( \tau \) by 
\[
\hat{F}_t = 1 - e^{-\hat{\Gamma}_t},
\]
where \( \hat{F}_t = Q(\tau \leq t | \mathcal{F}_t), \forall t \in \mathbb{R}_+ \). Notice this hazard process is well defined since our surrender time \( \tau \) is not an \( \mathcal{F} \)-stopping time.

Before studying the risk-minimizing strategies, we first need a few preliminary results. The first results can be found in Jeanblanc and Rutkowski [58] or in Bielecki and Rutkowski [23] but for completeness we give them here.

Theorem 4.2. Let the process \( L_t \) be defined by 
\[
L_t = 1_{\{\tau > t\}} e^{\hat{\Gamma}_t};
\]
then \( L_t \) is a \((Q, \mathcal{G})\)-martingale.

\(^1\) We assume \( \mathcal{G} \) is complete and satisfies the usual hypothesis.
PROOF. Let us prove that $E^Q_L_s|G_t| = L_t$, $\forall s > t$. We have:

$$
E^Q_L_{1_{\tau>s} e^{\hat{\Gamma}_s}}|G_t| = 1_{\tau>t} e^{\hat{\Gamma}_t} E^Q_e^{-\hat{\Gamma}_s} e^{\hat{\Gamma}_s} |F_t|
$$

$$
= 1_{\tau>t} e^{\hat{\Gamma}_t}

= L_t
$$

\[\square\]

**Theorem 4.3.** If the stochastic process $\hat{\Gamma}_t$ is continuous and of finite variation then we have $L_t = 1 - \int_0^t e^{\hat{\Gamma}_u} dM_u$ where $M_u$ is given by: $H_u - \hat{\Gamma}_{u\wedge \tau}$.

**Proof.** By the integration by parts, we have:

$$
L_t = \int_0^t 1_{\tau>u} e^{\hat{\Gamma}_u} + \int_0^t e^{\hat{\Gamma}_u} - d1_{\tau>u} + [e^{\hat{\Gamma}_u}, 1_{\tau>u}]_u
$$

The square bracket is equal to

$$
[e^{\hat{\Gamma}_u}, 1_{\tau>u}]_u = \sum_{0 \leq s \leq u} \Delta e^{\hat{\Gamma}_s} \Delta 1_{\tau>s}
$$

$$
= \Delta e^{\hat{\Gamma}_0} \Delta 1_{\tau>0}

= 1
$$

The second equality comes from the hypothesis that $\hat{\Gamma}_u$ is continuous. The last one comes from the assumption $e^{\hat{\Gamma}_0} = 1_{\tau>0} = 0$ and $e^{\hat{\Gamma}_0} = 1_{\tau>0} = 1$. Since $\hat{\Gamma}_u$ is continuous and of finite variation, we also have

$$
\int_0^t 1_{\tau>u} - e^{\hat{\Gamma}_u} d\hat{\Gamma}_u = \int_0^t 1_{\tau>u} - e^{\hat{\Gamma}_u} d\hat{\Gamma}_u
$$

$$
= \int_0^t e^{\hat{\Gamma}_u} d\hat{\Gamma}_{u\wedge \tau}
$$

and

$$
\int_0^t e^{\hat{\Gamma}_u} d1_{\tau>u} = \int_0^t e^{\hat{\Gamma}_u} d1_{\tau>u}
$$

$$
= -\int_0^t e^{\hat{\Gamma}_u} dH_u
$$

Finally, we get the result:

$$
L_t = 1 + \int_0^t e^{\hat{\Gamma}_u} d\hat{\Gamma}_{u\wedge \tau} - \int_0^t e^{\hat{\Gamma}_u} dH_u
$$

$$
= 1 - \int_0^t e^{\hat{\Gamma}_u} dM_u
$$

\[\square\]
The corollary of the two previous theorems is that when $\hat{\Gamma}_t$ is continuous and of finite variation, $\hat{\Gamma}_{t\wedge \tau}$ corresponds to the $(Q,G)$-compensator of the counting process $H_t$. It is not true when $\hat{\Gamma}_t$ is not continuous. From now on, we assume Condition 2.3 holds.

**Condition 4.4.** The $(Q,F)$-hazard process $\Gamma_t$ of $\tau$ is continuous.

4.1.3. The enlargement of filtration and the $(H)$-hypothesis. We assumed in Section 2.1, there is no arbitrage in the financial market with respect to the filtration $F$. Unfortunately, the financial market is not necessarily arbitrage free with respect to the enlarged filtration $G$. A sufficient condition for the financial market to be arbitrage free with respect to $G$, is that the $(H)$-hypothesis defined below, holds under the martingale measure $Q$.

Definition 4.5. Let $(\Omega,F,Q)$ be a probability space. Let $G$ be a filtration and $F$ an arbitrary sub-filtration of $G$ i.e. for every $t$, $F_t \subseteq G_t$. We say the $(H)$-hypothesis holds under $Q$ between the filtrations $F$ and $G$ if and only if: for every $t$, $F_\infty$ and $G_t$ are conditionally independent with respect to $F_t$.

From now on, we assume this $(H)$-hypothesis holds.

**Condition 4.6.** The $(H)$-hypothesis holds under $Q$ between $F$ and $G$.

The following lemma shows this hypothesis is equivalent to the invariance of local martingales from $F$ to $G$.

Lemma 4.7. The following assertions are equivalent

1. The $(H)$-hypothesis holds under $Q$.
2. Every $(F,Q)$-local martingale is a $(G,Q)$-local martingale.
3. $\forall t \geq 0, \forall H \in F_\infty$, $E^Q [H | G_t] = E^Q [H | F_t]$

Proof. See Brémaud and Yor [33]. □

As we already said, this condition is sufficient for the absence of arbitrage to hold under $G$. If furthermore the financial market is complete, it is even a necessary condition. See Blanchet-Scalliet and Jeanblanc [58] for a proof in a slightly different framework. In our setting, we also have the following equivalence:

Lemma 4.8. The $(H)$-hypothesis under $Q$ is equivalent to

$Q(\tau \leq t | F_\infty) = Q(\tau \leq t | F_t)$

Proof. See Jeanblanc and Rutkowski [58]. □

The corollary of this lemma is that under the $(H)$-hypothesis, $\hat{F}_t$ and thus $\hat{\Gamma}_t$ are increasing and accordingly, are of finite variation. The following proposition is essential.

Proposition 4.9. If the $(H)$-hypothesis holds for $Q$ and if the $(F,Q)$-hazard process $\hat{\Gamma}_t$ is continuous then $\Delta U_\tau = 0$ a.s. for any $F$-adapted càdlàg process $U_t$.
Proof. First, we have
\[\begin{align*}
E^Q [1_{\{\tau < t\}} | F_t] &= E^Q \left[ \lim_{\epsilon \to 0} 1_{\{\tau \leq t - \epsilon\}} | F_t \right] \\
&= \lim_{\epsilon \to 0} E^Q \left[ 1_{\{\tau \leq t - \epsilon\}} | F_{t-\epsilon} \right] \\
&= \lim_{\epsilon \to 0} \left( 1 - e^{-\Gamma_{t-\epsilon}} \right) \\
&= 1 - e^{-\Gamma_t}
\end{align*}\]
where \(\epsilon > 0\). In the third equality, we use the fact that when the \((H)\)-hypothesis holds, we have for all \(s > t\),
\[\begin{align*}
E^Q [1_{\{\tau \leq t\}} | F_s] &= E^Q \left[ E^Q \left[ 1_{\{\tau \leq t\}} | F_\infty \right] | F_s \right] \\
&= E^Q \left[ E^Q \left[ 1_{\{\tau \leq t\}} | F_t \right] | F_s \right] \\
&= E^Q [1_{\{\tau \leq t\}} | F_t]
\end{align*}\]
where in the second equality we use Lemma 4.2. Since \(U_t\) is càdlàg, there is only a countable number of jumps on the bounded interval \([0, T]\). Let us denote \(S_n\) the times of jump of \(U_t\) in this interval. We have:
\[\begin{align*}
E^Q \left[ \sum_n 1_{\{\tau = S_n\}} \right] &= \sum_n E^Q \left[ E^Q \left[ 1_{\{\tau = S_n\}} | F_{S_n} \right] \right] \\
&= \sum_n E^Q \left[ E^Q \left[ (1_{\{\tau \leq S_n\}} - 1_{\{\tau < S_n\}}) | F_{S_n} \right] \right] \\
&= \sum_n E^Q \left[ (1 - e^{-\Gamma_{S_n}}) - E^Q \left[ 1_{\{\tau < S_n\}} | F_{S_n} \right] \right] \\
&= \sum_n E^Q \left[ (1 - e^{-\Gamma_{S_n}}) - (1 - e^{-\Gamma_{S_n}^-}) \right] \\
&= \sum_n E^Q \left[ e^{-\Gamma_{S_n}^-} \{1 - e^{-\Delta \Gamma_{S_n}}\} \right] \\
&= 0
\end{align*}\]
In the first equality, we use the fact the number of jumps is countable. In the last equality, we use the fact \(\Gamma\) is continuous. Since \(\sum_n 1_{\{\tau = S_n\}}\) is a positive random variable, if its expectation is equal to zero, it means \(\sum_n 1_{\{\tau = S_n\}} = 0\) a.s. \(\square\)

We now apply the theory of risk minimization to the different building blocks we introduced earlier.

4.2. Payment at the term \(T\) of the contract. We want to find the risk-minimizing strategy of the payoff \(1_{\tau > T} \frac{g_T}{B_T}\) where we assume \(g_T\) is square integrable with respect to \(Q\). We have the following proposition.

Proposition 4.10. Assume conditions 2.3 and 4.1 hold. Let us denote the \((Q, F)\)-Galtchouk-Kunita-Watanabe decomposition of the purely financial contingent claim \(U^g_t = E^Q \left[ \frac{g_T}{B_T} e^{-\Gamma_T} | F_t \right] \)
by
\[ U_t^g = U_0^g + \int_0^t \phi_u dX_u + L_t^g \]

Then the discounted value \( V_t^g(\rho^*) \) of the risk-minimizing strategy for a claim paying at time \( T \), \( 1_{T>gT} \) where \( gT \) is a \( \mathcal{F}_T \)-measurable random variable, is given by
\[ V_t^g(\rho^*) = 1_{T>t} e^{\hat{\Gamma}_T} \mathbb{E}^Q \left[ \frac{gT}{B_T} e^{-\hat{\Gamma}_T} \bigg| \mathcal{F}_t \right] \]

Its risk-minimizing strategy \( \rho^* \) is given by
\[ \rho^* = (1_{T-t} e^{\hat{\Gamma}_t} \phi_t, V_t^g - 1_{T-t} e^{\hat{\Gamma}_t} \phi_t X_t) \]
where \( \phi_t \) is thus the amount of risky assets of the strategy risk-minimizing the purely financial contingent claim \( U_t^g \). The cost process \( C_t^g(\rho^*) \) is given by:
\[ C_t^g(\rho^*) = C_0^g(\rho^*) + L_t^g \]
where
\[ C_0^g(\rho^*) = \mathbb{E}^Q \left[ \frac{gT}{B_T} e^{-\hat{\Gamma}_T} \bigg| \mathcal{F}_0 \right] \]

and
\[ L_t^g = \int_0^t L_u dL_u^g - \int_0^t U_u^g e^{\hat{\Gamma}_u} dM_u \]

**Proof.** The cumulative discounted payment process is given by
\[ A_t = 1_{T>T} \frac{gT}{B_T} 1_{\{t=T\}} \].
According to Lemma 3.12, the value process \( V_t^g(\rho^*) \) of the risk-minimizing strategy \( \rho^* \) for the claim \( A_t \) is given by:
\[ V_t^g(\rho^*) = \mathbb{E}^Q \left[ \frac{gT}{B_T} e^{-\hat{\Gamma}_T} \bigg| \mathcal{G}_t \right] = 1_{T>t} e^{\hat{\Gamma}_t} \mathbb{E}^Q \left[ \frac{gT}{B_T} e^{-\hat{\Gamma}_T} \bigg| \mathcal{F}_t \right] \]
since \( \tau \) admits an \((Q,\mathbb{F})\)-hazard process \( \hat{\Gamma}_t \).

Finding the amount \( \varepsilon^* \) of risky assets comes down to finding the Galtchouk-Kunita-Watanabe of the square integrable \((Q,\mathbb{G})\)-martingale \( J_t^g \) defined as \( J_t^g = \mathbb{E}^Q \left[ 1_{T>T} \frac{gT}{B_T} \bigg| \mathcal{G}_t \right] \]. Let us denote \( U_t^g = \mathbb{E}^Q \left[ \frac{gT}{B_T} e^{-\hat{\Gamma}_T} \bigg| \mathcal{F}_t \right] \). It is a \((Q,\mathbb{F})\)-square integrable martingale and thanks to the

\[ \text{Here, we obviously have } J_t^g = V_t^g(\rho^*). \text{ But this will not be true for the two other payment processes. This is why we distinguish between the processes.} \]
(H)-hypothesis, it is even a \((Q, \mathbb{G})\)-square integrable martingale. Applying the integration by parts formula to \(J_t^g\), we can write:

\[
J_t^g = L_t U_t^g = L_0 U_0^g + \int_0^t L_u dU_u^g + \int_0^t U_u^g dL_u + \int_0^t d[L_u, U_u^g]_u
\]

\[
= L_0 U_0^g + \int_0^t L_u dU_u^g - \int_0^t U_u^g e^{\hat{\Gamma}_u} dM_u - \int_0^t e^{\hat{\Gamma}_u} d[M_u, U_u^g]_u
\]

In the third equality, we use the Theorem 4.8 since \(\hat{\Gamma}_t\) is, by assumption, continuous and increasing thanks to the (H)-hypothesis. Furthermore thanks to Proposition 4.10, the continuity of \(\hat{\Gamma}_t\) implies \(U_t^g\) does not jump at time \(\tau\). We can write the square bracket as

\[
[M_u, U_u^g]_t = \sum_{0 \leq s \leq t} \Delta M_s \Delta U_s^g
\]

\[
= M_0 U_0^g
\]

We then have:

\[
(4.1) \quad J_t^g = L_0 U_0^g + \int_0^t L_u dU_u^g - \int_0^t U_u^g e^{\hat{\Gamma}_u} dM_u
\]

The \((Q, \mathbb{G})\)-local martingales \(\int_0^t L_u dU_u^g\) and \(\int_0^t U_u^g e^{\hat{\Gamma}_u} dM_u\) are even \((Q, \mathbb{G})\)-square integrable martingales. Indeed, we have

\[
[J_t^g, J_t^g]_t = \int_0^t L_u^2 d[U_u^g, U_u^g]_u + \int_0^t \left(\frac{U_u^g e^{\hat{\Gamma}_u}}{U_u^g} \right)^2 d[M_u, M_u]_u - 2 \int_0^t L_u U_u^g e^{\hat{\Gamma}_u} d[U_u^g, M_u]_u
\]

\[
= \int_0^t L_u^2 d[U_u^g, U_u^g]_u + \int_0^t \left(\frac{U_u^g e^{\hat{\Gamma}_u}}{U_u^g} \right)^2 d[M_u, M_u]_u
\]

Since \(E[[J_t^g, J_t^g]_t] < \infty\), we have \(E \left[ \int_0^t L_u^2 d[U_u^g, U_u^g]_u \right] < \infty\) and \(E \left[ \int_0^t \left(\frac{U_u^g e^{\hat{\Gamma}_u}}{U_u^g} \right)^2 d[M_u, M_u]_u \right] < \infty\).

We now apply the Galtchouk-Kunita-Watanabe decomposition to the \((Q, \mathbb{F})\)-square integrable martingale \(U_t^g\), we get

\[
(4.2) \quad U_t^g = U_0^g + \int_0^t \phi_u dX_u + L_t^U
\]

where \(\phi\) is an \(\mathbb{F}\)-predictable stochastic process in \(L^2(X, Q, \mathbb{F})\) and \(L_t^U\) is in \(\mathcal{M}_0^2(\mathbb{Q}, \mathbb{F})\) and strongly \((Q, \mathbb{F})\)-orthogonal to \(\mathcal{T}^2(X, Q, \mathbb{F})\). Replacing Equation (4.2) in (4.1), we then obtain the following decomposition

\[
(4.3) \quad J_t^g = L_0 U_0^g + \int_0^t L_u \phi_u dX_u + L_t^g
\]
where $L^g_t = \int_0^t L_u - dL^g_t - \int_0^t U^g_u e^{\hat{\Gamma}_u} dM_u$.

Let us prove the Equation (4.3) is indeed the required Galtchouk-Kunita-Watanabe decomposition. Firstly, $\int_0^t L_u - \phi_u dX_u$ and $\int_0^t L_u - dL^g_t$ are obviously in $\mathcal{M}_0^2(Q, G)$. Secondly, we have to prove $L^g_t$ is strongly orthogonal to $\mathcal{T}^2(X, Q, G)$. It is equivalent to showing the following products $\left(\int_0^t \psi_u dX_u\right) \left(\int_0^t L_u - dL^g_t\right)$ and $\left(\int_0^t \psi_u dX_u\right) \left(\int_0^t U^g_u e^{\hat{\Gamma}_u} dM_u\right)$ are $(Q, G)$-martingales for any $G$-predictable process $\psi \in L^2(X, Q, G)$. On one hand, we have $\left[\int_0^t \psi_u dX_u, \int_0^t U^g_u e^{\hat{\Gamma}_u} dM_u\right]_t = \int_0^t \psi_u U^g_u e^{\hat{\Gamma}_u} d[X, M]_u = 0$ $\forall t \in [0, T]$ $Q$-a.s. since $M_t$ is a finite variation process and there is no common jumps between $X_t$ and $M_t$. On the other hand, knowing that on $\{\tau > t\}$, any $G$-predictable process $\psi \in L^2(X, Q)$ can be written as an $\mathbb{F}$-predictable process $\hat{\psi} \in L^2(X, Q)$, we have

$$\left\langle \int_0^t \psi_u dX_u, \int_0^t L_u - dL^g_t \right\rangle^G_t = \int_0^t 1_{\{\tau > u\}} e^{\hat{\Gamma}_u} d\left\langle \int_0^t \hat{\psi}_u dX_u, L^g_t \right\rangle_u^F$$

$$= \int_0^t 1_{\{\tau > u\}} e^{\hat{\Gamma}_u} d\left\langle \int_0^t \hat{\psi}_u dX_u, L^g_t \right\rangle_u^G$$

$$= 0$$

In the second equality, we use the fact under the ($H$)-hypothesis, the sharp brackets under $G$ and under $\mathbb{F}$ are indistinguishable. The third equality, we use the fact $L^g_t$ is strongly by $(Q, \mathbb{F})$-orthogonal to $\mathcal{T}^2(X, Q, \mathbb{F})$.

As far as the cost process is concerned, we immediately have by definition:

$$C^g_t(\rho^*) = C^g_0(\rho^*) + L^g_t$$

$$= C^g_0(\rho^*) + \int_0^t L_u - dL^g_t - \int_0^t U^g_u e^{\hat{\Gamma}_u} dM_u$$

where $M_t = H_u - \hat{\Gamma}_{\{\tau \wedge u\}}$. $\square$

Notice the cost process is the sum of two terms which are strongly $(Q, G)$-orthogonal to each other. If the market is complete $L^g_T = 0$ for any $t \in [0, T]$ and the cost process becomes $C^g_t(\rho^*) = C^g_0(\rho^*) - \int_0^T U^g_u e^{\hat{\Gamma}_u} dM_u$. The first component is thus related to the incompleteness of the market and the second one is related to the unpredictability of the surrender time. Notice also, if the market is complete, $\phi$ is the self financing replicating portfolio of the modified contingent claim $U^g_t$. The same proposition holds, mutatis mutandis, for the premium payments $P(t_i, \omega) 1_{\{\tau > t_i\}}$.

### 4.3. Payments at the surrender time.

We now want to find the risk-minimizing strategy of the discounted payment process $A_t = \int_0^t \frac{R_u}{B_u} dH_u$ which is assumed to be square integrable with respect to $Q$. We have the following proposition.

**Proposition 4.11.** Assume conditions 2.3 and 4.1 hold. Let us denote the $(Q, \mathbb{F})$-Galtchouk-Kunita-Watanabe decomposition of the purely financial contingent claim $U^R_t = E^Q \left[ \int_0^T \frac{R_u}{B_u} d\hat{F}_u \mid \mathcal{F}_t \right]$. 


by

\[ U_t^R = U_0^R + \int_0^t \psi_u dX_u + L_t^R \]

Then the discounted value \( V_t^R(\rho^*) \) of the risk-minimizing portfolio for the payment \( R_\tau \) at the surrender time \( \tau \) is given by:

\[ V_t^R(\rho^*) = 1_{\tau > t} e^{\hat{\Gamma}_t} E^Q \left[ \int_t^T \frac{R_u}{B_u} d\hat{F}_u \bigg| \mathcal{F}_t \right] \]

Its risk-minimizing strategy \( \rho^* \) is given by

\[ \rho^* = \left( 1_{\tau > t} e^{\hat{\Gamma}_t} \psi_t, 1_{\tau > t} e^{\hat{\Gamma}_t} E^Q \left[ \int_t^T \frac{R_u}{B_u} d\hat{F}_u \bigg| \mathcal{F}_t \right] - 1_{\tau > t} e^{\hat{\Gamma}_t} \psi_t, B_t \right) \]

where \( \psi_t \) is the amount of risky assets of the strategy risk-minimizing the purely financial contingent claim \( U_t^R \). The cost process \( C_t^R(\rho^*) \) of this risk-minimizing strategy is given by

\[ C_t^R(\rho^*) = C_0^R(\rho^*) + L_t^R \]

where

\[ C_0^R(\rho^*) = E^Q \left[ \int_0^T \frac{R_s}{B_s} d\hat{F}_s \bigg| \mathcal{F}_0 \right] \]

and

\[ L_t^R = \int_0^t L_u dL_t^U + \int_0^t \left( \frac{R_u}{B_u} - E^Q \left[ \int_u^T \frac{R_s}{B_s} e^{-(\hat{\Gamma}_s - \hat{\Gamma}_u)} d\hat{F}_s \bigg| \mathcal{F}_u \right] \right) dM_u \]

**Proof.** We essentially follow the same steps as in Proposition 4.10. According to Lemma 3.12, the discounted value of the risk-minimizing portfolio is given by

\[ V_t^R(\rho^*) = E^Q \left[ \int_0^T \frac{R_u}{B_u} dH_u - \int_0^t \frac{R_u}{B_u} dH_u \bigg| \mathcal{G}_t \right] \]

\[ = E^Q \left[ \int_t^T \frac{R_u}{B_u} dH_u \bigg| \mathcal{G}_t \right] \]

\[ = 1_{\tau > t} e^{\hat{\Gamma}_t} E^Q \left[ \int_t^T \frac{R_u}{B_u} d\hat{F}_u \bigg| \mathcal{F}_t \right] \]

since \( \tau \) admits a hazard process \( \hat{\Gamma}_t \).

The amounts of risky assets to hold is given by the Galtchouk-Kunita-Watanabe of the square integrable \((Q, \mathcal{G})\)-martingale \( J_t^R = E^Q \left[ \int_0^T \frac{R_u}{B_u} dH_u \bigg| \mathcal{G}_t \right] \). Using the integration by parts
in the fourth equality and denoting \( U_t^R = EQ \left[ \int_0^T \frac{R_u}{B_u} d\hat{F}_u \right] \), we have

\[
J_t^R = \int_0^t \frac{R_u}{B_u} dH_u + EQ \left[ \int_0^T \frac{R_u}{B_u} dH_u \right] G_t \\
= \int_0^t \frac{R_u}{B_u} dH_u + 1_{t > \tau} e^{\hat{\Gamma}_t} EQ \left[ \int_0^T \frac{R_u}{B_u} d\hat{F}_u \right] F_t \\
= \int_0^t \frac{R_u}{B_u} dH_u - 1_{t > \tau} e^{\hat{\Gamma}_t} \int_0^t \frac{R_u}{B_u} d\hat{F}_u + 1_{t > \tau} e^{\hat{\Gamma}_t} EQ \left[ \int_0^T \frac{R_u}{B_u} d\hat{F}_u \right] F_t \\
= \int_0^t \frac{R_u}{B_u} dH_u - 1_{t > \tau} e^{\hat{\Gamma}_t} \int_0^t \frac{R_u}{B_u} d\hat{F}_u + L_0 U_t^R + \int_0^t L_u - dU_t^R - \int_0^t U_t^R dF_u - \int_0^t e^{\hat{\Gamma}_t} [M, U_t^R]_u
\]

Since \( \hat{\Gamma}_t \) is continuous, thanks to Proposition 4.10, there is no discontinuity in \( U_t^R \) at time \( \tau \), the square bracket is thus equal to 0. Furthermore, we have

\[
1_{t > \tau} e^{\hat{\Gamma}_t} \int_0^t \frac{R_u}{B_u} d\hat{F}_u = \int_0^t 1_{t > u} e^{\hat{\Gamma}_u} \frac{R_u}{B_u} d\hat{F}_u - \int_0^t \left( \int_0^u \frac{R_s}{B_s} d\hat{F}_s \right) e^{-\hat{\Gamma}_u} dM_u + \left[ L, \int_0^t \frac{R_u}{B_u} d\hat{F}_u \right]_t
\]

\[
= \int_0^t 1_{t > u} e^{\hat{\Gamma}_u} \frac{R_u}{B_u} d\hat{\Gamma}_u - \int_0^t \left( \int_0^u \frac{R_s}{B_s} d\hat{F}_s \right) e^{-\hat{\Gamma}_u} dM_u
\]

\[
= \int_0^t \frac{R_u}{B_u} d\hat{\Gamma}_u - \int_0^t \left( \int_0^u \frac{R_s}{B_s} d\hat{F}_s \right) e^{-\hat{\Gamma}_u} dM_u
\]

In the second equality, we use the fact the square bracket is null since \( \hat{F} \) is continuous. Finally, since \( \hat{\Gamma}_{t \wedge \tau} \) is the \((Q, G)\)-compensator of \( H_t \), we obtain

\[
(4.4) \quad J_t^R = L_0 U_t^R + \int_0^t L_u - dU_t^R + \int_0^t \left( \frac{R_u}{B_u} - EQ \left[ \int_u^T \frac{R_s}{B_s} e^{-\hat{\Gamma}_s} d\hat{\Gamma}_s \right] F_u \right) dM_u
\]

Thanks to the \((H)\)-hypothesis, the square integrable \((Q, F)\)-martingale \( U_t^R \) is even a square integrable \((Q, G)\)-martingale. Using the same argument as in Proposition 4.10, we can easily show

\[
\int_0^t L_u - dU_t^R + \int_0^t \left( \frac{R_u}{B_u} - EQ \left[ \int_u^T \frac{R_s}{B_s} e^{-\hat{\Gamma}_s} d\hat{\Gamma}_s \right] F_u \right) dM_u
\]

are even square integrable \((Q, G)\)-martingales.

Since \( U_t^R \) is a square integrable \((Q, F)\)-martingale, we can find its Galtchouk-Kunita-Watanabe decomposition. We have

\[
(4.5) \quad U_t^R = U_0 + \int_0^t \psi_u dX_u + L_t^U_R
\]

where \( \psi \) is an \(F\)-predictable stochastic process in \( L^2(X, Q, F) \) and \( L_t^U_R \) is in \( M_2^2(Q, F) \) and strongly \((Q, F)\)-orthogonal to \( T^2(X, Q, F) \). Replacing Equation (4.5) in Equation (4.4), we
obtain the following decomposition:

\[ J_t^R = E^Q \left[ \int_0^T \frac{R_s}{B_s} d\hat{F}_s \right] F_0 + \int_0^t L_u \psi_u dX_u + L_t^R \]

where \( L_t^R = \int_0^t L_u dL_t^{UR} + \int_0^t \left( \frac{R_u}{B_u} - E^Q \left[ \int_u^T \frac{R_s}{B_s} e^{-(\hat{\Gamma}_s-\hat{\Gamma}_u)} d\hat{\Gamma}_s \right] \right) dM_u \). Using the same argument as in Proposition 4.10, we can easily prove this decomposition is indeed the Galtchouk-Kunita-Watanabe decomposition.

As far as the cost process is concerned, we immediately have:

\[ C_t^R(\rho^*) = C_0^R(\rho^*) + L_t^R \]

Again, the cost process can be decomposed in two strongly orthogonal components: the first one is related to the incompleteness of the market and the second one again related to the unpredictability of the surrender time.

### 4.4. Payments up to surrender

We now study the risk-minimizing strategy of the discounted payment process \( A_t = \int_0^t 1_{\tau > u} \frac{1}{B_u} dC_u \) where \( C_t \) is assumed to be a square integrable process with respect to \( Q \).

**Proposition 4.12.** Assume conditions 2.3 and 4.1 hold. Let us denote the \((Q, \mathbb{F})\)-Galtchouk-Kunita-Watanabe decomposition of the purely financial claims \( U_t^C = E^Q \left[ \int_0^T 1_{\tau > u} \frac{1}{B_u} e^{-\hat{\Gamma}_u} dC_u \right] \mathcal{F}_t \) by

\[ U_t^C = U_0^C + \int_0^t \nu_u dX_u + L_t^{UC} \]

Then the discounted value \( V_t^C(\rho^*) \) of the risk-minimizing portfolio for the payment process \( \int_0^T 1_{\tau > u} dC_u \) is given by:

\[ V_t^C(\rho^*) = 1_{\{\tau > t\}} e^{\hat{\Gamma}_t} E^Q \left[ \int_t^T 1_{\tau > u} \frac{1}{B_u} e^{-\hat{\Gamma}_u} dC_u \right] \mathcal{F}_t \]

The risk-minimizing strategy \( \rho^* \) is given by

\[ \rho^* = \left( 1_{\tau > t} e^{\hat{\Gamma}_t} \nu_t, 1_{\tau > t} e^{\hat{\Gamma}_t} \frac{1}{B_u} e^{-(\hat{\Gamma}_s-\hat{\Gamma}_u)} d\hat{\Gamma}_s \right) - 1_{\tau > t} e^{\hat{\Gamma}_t} \nu_t X_t \]

where \( \nu_t \) is the amount of risky assets of the strategy risk-minimizing the contingent claim \( U_t^C \). The cost process \( C_t^C(\rho^*) \) of the risk-minimizing strategy is given by:

\[ C_t^C(\rho^*) = C_0^C(\rho^*) + L_t^C \]
where

\[ C_0^C(\rho^*) = E^Q \left[ \int_0^T \frac{1}{B_u} e^{-\hat{\Gamma}_u} dC_u \bigg| \mathcal{F}_0 \right] \]

and

\[ L_t^C = \int_0^t L_{u_0} - dL_t^C - \int_t^T E^Q \left[ \int_0^{T_{t\tau >u}} e^{-\hat{\Gamma}_u} \frac{1}{B_u} dC_u \bigg| \mathcal{F}_u \right] dM_u \]

**Proof.** The discounted value of the risk-minimizing portfolio is given by:

\[ V_t^C(\rho^*) = E^Q \left[ \int_0^T 1_{T>t\tau >u_1} \frac{1}{B_u} dC_u - \int_0^t 1_{T>t\tau >u_1} \frac{1}{B_u} dC_u \bigg| \mathcal{G}_t \right] \]

\[ = E^Q \left[ \int_t^T 1_{T>t\tau >u_1} \frac{1}{B_u} dC_u \bigg| \mathcal{G}_t \right] \]

When \( \tau \) admits a hazard process \( \hat{\Gamma}_t \) of finite variation, the last equation can be written as:

\[ V_t^C(\rho^*) = 1_{\{\tau>t\}} e^{\hat{\Gamma}_t} E^Q \left[ \int_t^T \frac{1}{B_u} e^{-\hat{\Gamma}_u} dC_u \bigg| \mathcal{F}_t \right] \]

As already explained the optimal amounts of risky assets \( \nu \) to hold is given by the Galtchouk-Kunita-Watanabe of the square integrable \((Q, \mathcal{G})\)-martingale \( J_t^C = E^Q \left[ \int_0^T 1_{T>t\tau >u_1} \frac{1}{B_u} dC_u \bigg| \mathcal{G}_t \right] \).

We have:

\[ J_t^C = E^Q \left[ \int_0^T 1_{T>t\tau >u_1} \frac{1}{B_u} dC_u \bigg| \mathcal{G}_t \right] \]

\[ = \int_0^t 1_{T>t\tau >u_1} \frac{1}{B_u} dC_u + 1_{T>t\tau >u_1} e^{\hat{\Gamma}_t} E^Q \left[ \int_t^T \frac{1}{B_u} e^{-\hat{\Gamma}_u} dC_u \bigg| \mathcal{F}_t \right] \]

\[ = \int_0^t 1_{T>t\tau >u_1} \frac{1}{B_u} dC_u - 1_{T>t\tau >u_1} e^{\hat{\Gamma}_t} \int_0^t e^{-\hat{\Gamma}_u} \frac{1}{B_u} dC_u + 1_{T>t\tau >u_1} e^{\hat{\Gamma}_t} E^Q \left[ \int_0^T \frac{1}{B_u} dC_u \bigg| \mathcal{F}_t \right] \]

Let us denote by \( U_t^C \) the following square integrable \((Q, \mathcal{F})\)-martingale \( U_t^C = E^Q \left[ \int_0^T \frac{1}{B_u} e^{\hat{\Gamma}_u} dC_u \bigg| \mathcal{F}_t \right] \).

By integration by parts, we have:

\[ J_t^C = \int_0^t 1_{T>t\tau >u_1} \frac{1}{B_u} dC_u - 1_{T>t\tau >u_1} e^{\hat{\Gamma}_t} \int_0^t e^{-\hat{\Gamma}_u} \frac{1}{B_u} dC_u + \int_0^t L_{u_1} - dL_t^C - \int_0^t U_t^C e^{\hat{\Gamma}_u} dM_u - \int_0^t e^{\hat{\Gamma}_u} \left[ M_u, U_t^C \right]_t \]
Thanks to Proposition 4.10, $U^C_t$ has no discontinuity at time $\tau$, the square bracket is then equal to $M_0U^C_0 = 0$. Furthermore, we have

$$1_{\tau > t} e^{\hat{\Gamma}_t} \int_0^t \frac{e^{-\hat{\Gamma}_u}}{B_u} dC_u = \int_0^t 1_{\tau > u} e^{\hat{\Gamma}_u} \int_0^u \frac{e^{-\hat{\Gamma}_s}}{B_s} dC_s - \int_0^t \left( \int_0^u \frac{e^{-\hat{\Gamma}_s}}{B_s} dC_s \right) e^{\hat{\Gamma}_u} dM_u$$

$$+ \left[ 1_{\tau > t} e^{\hat{\Gamma}_t} \int_0^t \frac{e^{-\hat{\Gamma}_u}}{B_u} dC_u \right]_t$$

$$= \int_0^t 1_{\tau > u} e^{\hat{\Gamma}_u} \frac{e^{-\hat{\Gamma}_u}}{B_u} dC_u - \int_0^t \left( \int_0^u \frac{e^{-\hat{\Gamma}_s}}{B_s} dC_s \right) e^{\hat{\Gamma}_u} dM_u$$

$$= \int_0^t 1_{\tau > u} \frac{1}{B_u} dC_u - \int_0^t \left( \int_0^u \frac{e^{-\hat{\Gamma}_s}}{B_s} dC_s \right) e^{\hat{\Gamma}_u} dM_u$$

We have thus

\[(4.6) \quad J^C_t = L_0U^C_0 + \int_0^t L_u dU^C_u - \int_0^t E^Q \left[ \int_u^T e^{-(\hat{\Gamma}_s - \hat{\Gamma}_u)} \frac{S_s}{B_s} dC_s \Bigg| \mathcal{F}_u \right] dM_u \]

From now on, we can easily prove the proposition using the same arguments as in Propositions 4.10 and 4.11. This is left to the reader. □

Once again, the cost process can be decomposed in two parts, one related to the incompleteness of the financial market $\int_0^t L_u dU^C_t$ and the second one related to the unpredictability of the surrender time $-\int_0^t E^Q \left[ \int_u^T e^{-(\hat{\Gamma}_s - \hat{\Gamma}_u)} \frac{S_s}{B_s} dC_s \Bigg| \mathcal{F}_u \right] dM_u$.


We now assume our portfolio consists of $n$ policyholders. We denote by $\tau^i$ the surrender time for each policyholder and define $H^i_t = 1_{\{\tau^i \leq t\}}$ with $i = 1, \ldots, n$. Möller in [68] studied a similar situation but he assumed the random times $\tau^i$ were independent of each other and he assumed the financial market was the (complete) Black-Scholes model. In this setting, he showed that by following the risk-minimizing strategy, the risk of such a portfolio can be in part, hedged away and showed that the unhedgeable part of the risk (the risk process) tends to zero in relative terms with the size $n$ of the portfolio. In other words, he showed that by taking a sufficiently large portfolio and by following the risk-minimizing strategy, we can reduce as much as we want the relative risk of this portfolio. Riesner [72] extended this result by introducing an incomplete market driven by a Lévy process.

In our framework, the random times $\tau^i$ are obviously not independent of each other since they all depend on the financial market. It is not obvious whether or not we can obtain Möller’s conclusion in our setting. Our aim here is to study how far we can go and keep Möller’s result.

In the first subsection, we will give some preliminary assumptions and results. Then in the next subsections, we will study the risk-minimizing strategies for the different building blocks.
when there are initially $n$ policyholders. In Section 6, we will study the risk processes of these strategies.

5.1. Preliminary assumptions and results.

5.1.1. The insurer’s information. We assume here the insurer observes the financial market and whether the different policyholders have already surrendered or not. This information can be modelled through the following filtration $G_p = (G^p_t)_{0 \leq t \leq T}$ where $G^p_t = F_t \lor \mathcal{H}^1_t \lor \mathcal{H}^2_t \lor \cdots \lor \mathcal{H}^n_t$ and $\mathcal{H}^i_t = \sigma(\mathcal{H}^i_t)$ for each $i = 1, \ldots, n$. We also introduce the following notation.

Notation 5.1. We denote the filtration $G^i = (G^i_t)_{0 \leq t \leq T}$ where $G^i_t = F_t \lor \mathcal{H}^i_t$ and the filtration $G^{p-i} = (G^{p-i}_t)_{0 \leq t \leq T}$ where $G^{p-i}_t = F_t \lor \mathcal{H}^1_t \lor \cdots \lor \mathcal{H}^{i-1}_t \lor \mathcal{H}^{i+1}_t \lor \cdots \lor \mathcal{H}^n_t$.

5.1.2. The enlargement of filtration and the $(H)$-hypothesis. As in Section 4.1.3, we have to introduce an assumption that avoids the existence of arbitrage opportunities under the enlarged filtration $G^p$. As we already know the $(H)$-hypothesis is a sufficient condition for the no arbitrage hypothesis to hold.

Condition 5.2. The $(H)$-hypothesis between the filtration $F$ and $G^p$ holds under $Q$.

This assumption implies the $(H)$-hypothesis holds for any subfiltration. Indeed, we have the following proposition.

Proposition 5.3. If the $(H)$-hypothesis holds under $Q$ between the filtration $F$ and $G^p$ then it holds between $F$ and any filtration $\mathcal{I}$ such that $F \subseteq \mathcal{I} \subseteq G^p$.

Proof. We can easily prove this proposition thanks to the third condition of Lemma 4.7. Let $H \in F_\infty$ and $t \geq 0$, we have

\[
E^Q[H | I_t] = E^Q[E^Q[H | G^p_t] | I_t] = E^Q[E^Q[H | F_t] | I_t] = E^Q[H | F_t]
\]

In the second equality, we use the fact the $(H)$-hypothesis holds under $Q$ between the filtration $F$ and $G^p$. \hfill $\square$

In particular, the $(H)$-hypothesis holds for the filtrations $G^i = (F_t \lor \mathcal{H}^i_t)_{0 \leq t \leq T}$ for any $i = 1, \ldots, n$.

5.1.3. The $F$-independence. A well-know property of the risk-minimization theory is its linearity: the risk-minimizing strategy of the sum of two payoffs is simply the sum of the risk-minimizing strategies of each separate payoff. It would then be tempting to directly use this property to conclude the risk-minimizing strategy of a portfolio of $n$ insurance contracts is simply given by the sum of the risk-minimizing strategies given in Section 4. But this solution is not true in general. It comes from the fact that when we have $n$ contracts the insurer’s information set is different from the one of Section 4. Indeed, if we have initially $n$ policyholders, the filtration we should consider is now the filtration $G^p$. The linearity property of the risk minimization indeed holds under the filtration $G^p$: the risk-minimizing strategy for a portfolio
of $n$ contracts is thus the sum of the risk-minimizing strategies of the individual contracts but with respect to the filtration $\mathcal{G}^p$. Unfortunately, the risk-minimizing strategy with respect to $\mathcal{G}^p$, for a single contract is not necessarily the same as the risk-minimizing strategy with respect to the filtration $\mathcal{G}$ (which we derived in the preceding section), for the same single contract. Intuitively, since in general the different surrender times are not independent of each other, the fact someone has surrendered or not, does give a valuable information on the probability someone else has surrendered or not. In establishing our risk-minimizing strategies, we should accordingly take into account this information. This is not possible if we only observe $\mathcal{G}$.

Deriving these risk-minimizing strategies with respect to $\mathcal{G}^p$ seems to be a difficult task because we could have in general complex dependencies between the different surrender times. So, we would like to find conditions under which we could use the results we derived in the previous sections and which are in the meantime, sufficiently realistic. An obvious assumption is to assume the surrender times are independent of each other. In this case, intuitively, the fact someone has surrendered should not give any information on the surrender of another. The individual risk-minimizing strategies with respect to $\mathcal{G}^p$ would then be the same as the risk-minimizing strategies with respect to $\mathcal{G}$. This is the assumption made in Møller’s papers and Riesner’s and is indeed realistic if we were to model the dates of decease of different individuals, but is unfortunately too strong if we are to model surrender times. Here we would like to find an intermediary assumption weaker than the pure independence. Let us introduce the following condition:

**Condition 5.4.** For each $i, j = 1, \ldots, n$ with $i \neq j$, $\tau^i$ and $\tau^j$ are $(Q, F)$-independent.

This $F$-conditional independence assumption has a direct intuitive meaning. As we already argued, the fact the surrender times are not $F$-stopping times, means the policyholders take their surrender decisions partly on financial variables which are observable by the insurer, and partly on other variables unknown to the insurer. The fact the different surrender times are $F$-independent means these unobservable variables which influence the surrender decisions, are purely idiosyncratic variables. They are not “macro” variables which could have a systematic influence on the other policyholders decisions. In other word, the only “macro” variables which have a systematic influence on the surrender decisions comes from the financial market and are also observable by the insurer. From a mathematical point of view, this assumption has a number of consequences. In particular, this implies the following equality:

\[
E^Q [1_{\tau^i > T} 1_{\tau^j > T} | \mathcal{F}_T] = E^Q [1_{\tau^i > T} | \mathcal{F}_T] E^Q [1_{\tau^j > T} | \mathcal{F}_T]
\]

We also have the following propositions:

**Proposition 5.5.** If Condition 5.4 holds then, for $t < T$ and $H$ a $\mathcal{G}^p_t$-measurable random variable, we have

\[
E^Q [H | \mathcal{G}^p_t] = E^Q [H | \mathcal{F}_t \vee \mathcal{H}_t]
\]

**Proof.** Straightforward. \qed

5.1.4. The hazard process. Based on Definition 4.1, we can here define a hazard process with respect to any sub-filtration of $\mathcal{G}^p$. However, we have the following proposition.
PROPOSITION 5.6. If Condition 5.4 holds then \( Q \left( \tau^i \leq t \mid G^{i \leftarrow \tau}_t \right) = Q \left( \tau^i \leq t \mid \mathcal{F}_t \right) \) for each \( i = 1, \ldots, n \).

PROOF. Straightforward. \qed

In other words, the \((Q, \mathcal{G}^{i \leftarrow \tau})\)-hazard process of \( \tau^i \) is equal to its \((Q, \mathcal{F})\)-hazard process. Intuitively, the fact the insurer can observe the financial market and the surrender of the other policyholders does not give us more information on the surrender of a policyholder than the observation of the financial market alone. Intuitively, this property should allow us to say the risk-minimizing strategy for \( n \) contracts is given by the sum of the individual risk-minimizing strategies as derived in Section 4. This is what we will prove formally in the next subsections.

In the rest of the chapter, we will denote by \( \hat{\Gamma}^i_t \) the \((Q, \mathcal{F})\)-hazard process of \( \tau^i \). For the sake of simplicity, we will assume the policyholders have homogeneous behaviors with respect to the evolution of the financial market. This assumption translates into \( \hat{\Gamma}^i_t = \hat{\Gamma}^j_t \) \( \forall t \in [0, T] \), \( \forall i, j = 1, \ldots, n \). In this case, we will simply write \( \hat{\Gamma}_t = \hat{\Gamma}^i_t \) \( \forall i = 1, \ldots, n \). As in Section 4, we also have the following propositions

PROPOSITION 5.7. Let the process \( L^i_t \) be defined by \( L^i_t = 1_{\{t > \tau^i\}} e^{\hat{\Gamma}^i_t} \) then if Condition 5.4 holds, \( L^i_t \) is a \((Q, \mathcal{G}^p)\)-martingale.

PROOF. Let us prove that \( E^Q \left[ L^i_s \mid G^{p \leftarrow \tau}_t \right] = L^i_t \), \( \forall s > t \). We have:

\[
E^Q \left[ 1_{s > \tau^i} e^{\hat{\Gamma}^i_t} \right] = E^Q \left[ 1_{s > \tau^i} e^{\hat{\Gamma}^i_t} \mid G^{p \leftarrow \tau}_t \right] = E^Q \left[ 1_{s > \tau^i} e^{\hat{\Gamma}^i_t} E^Q \left[ e^{-\hat{\Gamma}^i_t} \mid G^{p \leftarrow \tau}_t \right] \right] = L^i_t
\]

where in the second and third equalities, we use Proposition 5.6. \qed

PROPOSITION 5.8. If the stochastic process \( \hat{\Gamma}^i_t \) is continuous and of finite variation then we have \( L^i_t = 1 - \int_0^t e^{\hat{\Gamma}^i_u} dM^i_u \) where \( M^i_u \) is given by: \( H^i_u - \hat{\Gamma}^i_{\{u \wedge \tau^i\}} \).

PROOF. Same proof than Theorem 4.8. \qed

The corollary of the two previous theorems is that when Condition 5.4 holds and \( \hat{\Gamma}^i_t \) is continuous and of finite variation, \( \hat{\Gamma}^i_{\{u \wedge \tau^i\}} \) is the \((Q, \mathcal{G}^p)\)-compensator of \( H^i_t \). If we denote \( N_t = \sum_{i=1}^n H_i \) then the \((Q, \mathcal{G}^p)\)-compensator of \( N_t \) is given by \( \sum_{i=1}^n \hat{\Gamma}^i_{\{u \wedge \tau^i\}} \) or by \( n - N_{u-} \hat{\Gamma}^i_u \) if the policyholders are homogeneous.

As in Section 4, we assume from now on the following condition holds:

CONDITION 5.9. For each \( i = 1, \ldots, n \), the \((Q, \mathcal{F})\)-hazard process \( \hat{\Gamma}^i_t \) is continuous.

Notice since the \((H)\)-hypothesis holds between \( \mathcal{F} \) and \( \mathcal{G}^i \) for each \( i = 1, \ldots, n \), then \( \hat{\Gamma}^i_t \) is an increasing process for each \( i = 1, \ldots, n \) thanks to Lemma 4.2, and thus a finite variation process.
As in Proposition 4.10, we can prove there is no jump in the $\mathcal{F}$-adapted càdlàg processes at any time $\tau^i$.

**Proposition 5.10.** For any $i = 1, \ldots, n$, if the (H)-hypothesis holds between $\mathcal{F}$ and $\mathcal{G}^i$ under $Q$ and if the $(\mathcal{F}, Q)$-hazard process $\hat{\Gamma}^i_t$ of $\tau^i$ is continuous then $\Delta U_{\tau^i} = 0$ $Q$-a.s. for any $\mathcal{F}$-adapted càdlàg process $U_t$.

**Proof.** See Proposition 4.10. $\square$

In the three next subsections, we study the risk-minimizing strategies for each of our building blocks.

**5.2. Payment at the term.** For a portfolio of $n$ homogeneous policies paying a payoff $g_T$ if the policyholder has not surrender before the term $T$, we have the following proposition.

**Proposition 5.11.** If the Conditions 5.2, 5.6 and 5.9 hold, then the discounted value $V^g,p_t(\rho^*)$ of the risk-minimizing $n$-policyholders portfolio is given by

$$V^g,p_t(\rho^*) = (n - N_t) e^{\hat{\Gamma}^i_t} E^Q \left[ \frac{g_T}{B_T} e^{-\hat{\Gamma}^i_T} \bigg| \mathcal{F}_t \right]$$

where $N_t = \sum_{i=0}^{n} 1_{r^i \leq t}$.

The risk-minimizing strategy $\rho^*$ is given by

$$\rho^*_t = \left( (n - N_{t-}) e^{\hat{\Gamma}^i_t} \phi_t, (n - N_t) e^{\hat{\Gamma}^i_t} E^Q \left[ \frac{g_T}{B_T} e^{-\hat{\Gamma}^i_T} \bigg| \mathcal{F}_t \right] - (n - N_{t-}) e^{\hat{\Gamma}^i_t} \phi_t X_t \right)$$

where $\phi$ is the amount of risky assets of the strategy risk-minimizing the purely financial contingent claim $U^g_t = E^Q \left[ \frac{g_T}{B_T} e^{-\hat{\Gamma}^i_T} \bigg| \mathcal{F}_t \right]$. The cost process is given by:

$$C^g,p_t(\rho^*) = C^g,p_0(\rho^*) + L^g,p_t$$

where

$$C^g,p_0(\rho^*) = n E^Q \left[ \frac{g_T}{B_T} e^{-\hat{\Gamma}^i_T} \bigg| \mathcal{F}_0 \right]$$

$$L^g,p_t = \int_0^t (n - N_{u-}) e^{\hat{\Gamma}^i_u} dU^g_u + \int_0^t U^g_u e^{\hat{\Gamma}^i_u} dM^p_u$$

and $M^p_t = N_t - (n - N_{t-}) \hat{\Gamma}_t$. 


PROOF. The discounted value of the risk-minimizing $n$-policyholders portfolio $V^{g,p}(\rho^*)$ is given by:

$$V_t^{g,p}(\rho^*) = E^Q \left[ \sum_{i=1}^{n} \frac{g_T}{B_T} T_{i}^{p} \bigg| \mathcal{F}_t \right]$$

$$= \sum_{i=1}^{n} E^Q \left[ 1_{T_i > T} \frac{g_T}{B_T} \bigg| \mathcal{F}_t \mathcal{G}_i \mathcal{F}_t \right]$$

$$= \sum_{i=1}^{n} 1_{T_i > T} e^{\Gamma_i} E^Q \left[ \frac{g_T}{B_T} e^{-\Gamma_i} \bigg| \mathcal{F}_t \right]$$

where in the second equality, we use Proposition 5.5. For each $i = 1, \ldots, n$, we can use the decomposition given in Proposition 4.10. If for each $i = 1, \ldots, n$, we write the $(Q, F)$-Galtchouk-Kunita-Watanabe decomposition of the $(Q, F)$-martingale $U_t^{g,i} = E^Q \left[ \frac{g_T}{B_T} e^{-\Gamma_i} \bigg| \mathcal{F}_t \right]$ as:

$$E^Q \left[ \frac{g_T}{B_T} e^{-\Gamma_i} \bigg| \mathcal{F}_t \right] = E^Q \left[ \frac{g_T}{B_T} e^{-\Gamma_i} \bigg| \mathcal{F}_0 \right] + \int_0^t \phi_u dX_u + L_t^{g,i}$$

then, we get:

$$V_t^{g,p}(\rho^*) = \sum_{i=1}^{n} E^Q \left[ \frac{g_T}{B_T} e^{-\Gamma_i} \bigg| \mathcal{F}_0 \right] + \int_0^t \left( \sum_{i=1}^{n} 1_{\{T_i > T\}} e^{\Gamma_i} \phi_u \right) dX_u$$

$$+ \sum_{i=1}^{n} \int_0^t 1_{\{T_i > T\}} e^{\Gamma_i} dL_t^{U^{g,i}} + \sum_{i=1}^{n} \int_0^t U_t^{g,i} e^{\Gamma_i} dM_u$$

This is indeed the $(Q, G^p)$-Galtchouk-Kunita-Watanabe decomposition of $V_t^{g,p}(\rho^*)$ since thanks to Condition 5.2, $\int_0^t \left( \sum_{i=1}^{n} 1_{\{T_i > T\}} e^{\Gamma_i} \phi_u \right) dX_u$ and $\int_0^t 1_{\{T_i > T\}} e^{\Gamma_i} dL_t^{U^{g,i}}$ are $(Q, G^p)$-square integrable martingales strongly $(Q, G^p)$-orthogonal for any $j = 1, \ldots, n$ and since $M^j$ is also a $(Q, G^p)$-square integrable martingale strongly $(Q, G^p)$-orthogonal to $\int_0^t \left( \sum_{i=1}^{n} 1_{\{T_i > T\}} e^{\Gamma_i} \phi_u \right) dX_u$ for any $j = 1, \ldots, n$ thanks to Proposition 5.10. Accordingly, we can say the $(Q, G^p)$-Galtchouk-Kunita-Watanabe decomposition of the payoff of the portfolio is given by the sum of the $(Q, G^q)$-Galtchouk-Kunita-Watanabe decomposition of the payoff of each contract.

If furthermore, we assume the policyholders are homogeneous, we have

$$V_t^{g,p}(\rho^*) = (n - N_t) e^{\Gamma_T} E^Q \left[ \frac{g_T}{B_T} e^{-\Gamma_T} \bigg| \mathcal{F}_t \right]$$

and the decomposition:

$$V_t^{g,p}(\rho^*) = n E^Q \left[ \frac{g_T}{B_T} e^{-\Gamma_T} \bigg| \mathcal{F}_0 \right] + \int_0^t (n - N_u) e^{\Gamma_u} \phi_u dX_u$$

$$+ \int_0^t (n - N_u) e^{\Gamma_u} dL^{U^{g}} + \int_0^t U_t^{g} e^{\Gamma_u} dM_u$$
where $M^p_u = \sum_{i=1}^{n} M^i_u = N_u - (n - N_u-)\hat{\Gamma}_u$.

As an illustration, we can easily show this proposition give the same result than Møller’s. He made two additional assumptions. He assumed the $\tau^i$ were independents of the filtration $\mathcal{F}_{\cdot}$. In this case, the process $\hat{\Gamma}_t$ is a continuous deterministic function. Furthermore, he assumed the market was complete. In this case, we have $\mathbb{E}_{\mathbb{P}}[\hat{\Gamma}_T] = \hat{\Gamma}_T$. Eventually, we get

$$V_t^{g,p}(\rho^*) = n\mathbb{E}_Q\left[\frac{g_T}{B_T} e^{-\hat{\Gamma}_T} \bigg| \mathcal{F}_t\right] + \int_0^t (n - N_u-) e^{\hat{\Gamma}_u} \phi_u dX_u + \int_0^t u^g \mathbb{E}_{\mathbb{P}}_u e^{\hat{\Gamma}_u} dM^p_u$$

$$= n\mathbb{Q}(\tau > T)\mathbb{E}_Q\left[\frac{g_T}{B_T} \bigg| \mathcal{F}_0\right] + \int_0^t (n - N_u-) e^{-(\hat{\Gamma}_T-\hat{\Gamma}_u)} \gamma u dX_u$$

$$+ \int_0^t \mathbb{E}_Q\left[\frac{g_T}{B_T} \bigg| \mathcal{F}_u\right] e^{-(\hat{\Gamma}_T-\hat{\Gamma}_u)} dM^p_u$$

Notice $\gamma_u$ is $\mathcal{F}$-predictable and is the self financing replication strategy of the discounted payoff $g_T$. Indeed, we have:

$$U^g_t = \mathbb{E}_Q\left[\frac{g_T}{B_T} e^{-\hat{\Gamma}_T} \bigg| \mathcal{F}_t\right]$$

$$= e^{-\hat{\Gamma}_T} \mathbb{E}_Q\left[\frac{g_T}{B_T} \bigg| \mathcal{F}_0\right] + \int_0^t \gamma u dX_u$$

For a portfolio of $n$ policyholders, we then have the following strategy

$$(n - N_u-) e^{-(\hat{\Gamma}_T-\hat{\Gamma}_u)} \gamma t, (n - N_u) e^{-(\hat{\Gamma}_T-\hat{\Gamma}_u)} \mathbb{E}_Q\left[\frac{g_T}{B_T} \bigg| \mathcal{F}_t\right] - (n - N_u-) e^{-(\hat{\Gamma}_T-\hat{\Gamma}_u)} \gamma t X_t$$

This is exactly Møller’s result.

5.3. Payment at the surrender time. If we have a portfolio with initially $n$ policyholders with homogeneous behavior, we have the following proposition.

**Proposition 5.12.** If the Conditions 5.2, 5.6 and 5.9 hold then the discounted value $V^{R,p}_t(\rho^*)$ of the risk-minimizing $n$-policyholders portfolio is given by:

$$V^{R,p}_t(\rho^*) = (n - N_t) e^{\hat{\Gamma}_t} \mathbb{E}_Q\left[\int_t^T \frac{R_u}{B_u} d\hat{F}_u \bigg| \mathcal{F}_t\right]$$

where $N_t = \sum_{i=0}^{n} 1_{\tau^i \leq t}$.

The risk-minimizing strategy $\rho^*$ is given by

$$\rho^*_t = (n - N_t) e^{\hat{\Gamma}_t} \psi_t, (n - N_t) e^{\hat{\Gamma}_t} \mathbb{E}_Q\left[\int_t^T \frac{R_u}{B_u} d\hat{F}_u \bigg| \mathcal{F}_t\right] - (n - N_t-) e^{\hat{\Gamma}_t} \psi t X_t$$
where $\psi$ is the amount of risky assets of the strategy risk-minimizing the purely financial contingent claim $U_t^R = E^Q \left[ \int_0^T \frac{R_u}{B_u} d\hat{F}_u \big| \mathcal{F}_t \right]$. The cost process is given by

$$C_t^{R,p}(\rho^*) = C_0^{R,p}(\rho^*) + L_t^{R,p}$$

where

$$C_0^{R,p}(\rho^*) = nE^Q \left[ \int_0^T \frac{R_u}{B_u} d\hat{F}_u \big| \mathcal{F}_0 \right]$$

$$L_t^{R,p} = \int_0^t (n - N_{u-}) e^{\tilde{r}_u} dL_t^{U^R} \int_0^t \left( \frac{R_u}{B_u} - E^Q \left[ \int_u^T \frac{R_s}{B_s} e^{-(\tilde{r}_s - \tilde{r}_u)} d\hat{r}_s \big| \mathcal{F}_u \right] \right) dM^p_u$$

and $M^p_t = N_t - (n - N_{t-}) \tilde{\Gamma}_t$.

**Proof.** The discounted value of the risk-minimizing portfolio is given by

$$V_t^{R,p}(\rho^*) = E^Q \left[ \sum_{i=1}^n \int_t^T \frac{R_u}{B_u} dH^i_{u} \bigg| \mathcal{G}^p_t \right]$$

$$= \sum_{i=1}^n E^Q \left[ \int_t^T \frac{R_u}{B_u} dH^i_{u} \bigg| \mathcal{F}_t \cup \mathcal{H}^i_t \right]$$

$$= \sum_{i=1}^n 1_{t^i > t} e^{\tilde{r}^i_t} E^Q \left[ \int_t^T \frac{R_u}{B_u} d\hat{F}^i_{u} \bigg| \mathcal{F}_t \right]$$

where in the second equality, we use Proposition 5.5. To find the risk-minimizing strategy, we have to find the $(Q, \mathbb{G}^p)$-Galtchouk-Kunita-Watanabe decomposition of

$$J_t^{R,p} = E^Q \left[ \sum_{i=1}^n \int_0^T \frac{R_u}{B_u} dH^i_{u} \bigg| \mathcal{G}^p_t \right]$$

$$= \sum_{i=1}^n E^Q \left[ \int_0^T \frac{R_u}{B_u} dH^i_{u} \bigg| \mathcal{F}_t \cup \mathcal{H}^i_t \right]$$

As in the previous proposition, we can easily see the $(Q, \mathbb{G}^p)$-Galtchouk-Kunita-Watanabe decomposition for the portfolio is given by the sum of the $(Q, \mathbb{G}^i)$-Galtchouk-Kunita-Watanabe decomposition for each contract. If for each $i = 1, \ldots, n$, we write the $(Q, \mathbb{F})$-Galtchouk-Kunita-Watanabe decomposition of the square integrable $(Q, \mathbb{F})$-martingale $U_t^{R,i} = E^Q \left[ \int_0^T \frac{R_s}{B_s} e^{-\tilde{r}^i_s} d\tilde{\Gamma}^i_s \big| \mathcal{F}_t \right]$ as:

$$E^Q \left[ \int_0^T \frac{R_s}{B_s} e^{-\tilde{r}^i_s} d\tilde{\Gamma}^i_s \bigg| \mathcal{F}_t \right] = E^Q \left[ \int_0^T \frac{R_s}{B_s} e^{-\tilde{r}^i_s} d\tilde{\Gamma}^i_s \bigg| \mathcal{F}_0 \right] + \int_0^t \psi^i_u dX_u + L_t^{R,i}$$
Then using the results of Proposition 4.11, we have

\[ J_t^{R, p} = \sum_{i=1}^{n} E^Q \left[ \int_0^T \frac{R_s}{B_s} e^{-\hat{\Gamma}_s} d\hat{\Gamma}_s \bigg| \mathcal{F}_t \right] + \int_0^t \left( \sum_{i=1}^{n} 1_{\{\tau_t^i > u\}} e^{\hat{\Gamma}_s^i} \nu_i^u \right) dX_u \]

\[ + \sum_{i=1}^{n} \int_0^T 1_{\{\tau_t^i > u\}} e^{\hat{\Gamma}_s^i} dL_t^{R,i} + \sum_{i=1}^{n} \int_0^t \frac{R_u}{B_u} - E^Q \left[ \int_u^T \frac{R_s}{B_s} e^{-\hat{\Gamma}_s} d\hat{\Gamma}_s \bigg| \mathcal{F}_u \right] \] \[ dM_u \]

If the policyholders are homogeneous, we have:

\[ V_t^{R, p}(\rho^*) = (n - N_t) e^{\hat{\Gamma}_t} E^Q \left[ \int_t^T \frac{R_u}{B_u} d\hat{\Gamma}_u \bigg| \mathcal{F}_t \right] \]

and

\[ J_t^{R, p} = n E^Q \left[ \int_0^T \frac{R_s}{B_s} e^{-\hat{\Gamma}_s} d\hat{\Gamma}_s \bigg| \mathcal{F}_t \right] + \int_0^t (n - N_u^t) e^{\hat{\Gamma}_s} \nu_u dX_u \]

\[ + \int_0^t (n - N_u^t) e^{\hat{\Gamma}_s} dL_t^{R} + \int_0^t \frac{R_u}{B_u} - E^Q \left[ \int_u^T \frac{R_s}{B_s} e^{-(\hat{\Gamma}_s - \hat{\Gamma}_u)} d\hat{\Gamma}_s \bigg| \mathcal{F}_u \right] dM_u \]

Where \( N_t = \sum_{i=0}^{k} H_i = \sum_{i=0}^{n} 1_{\{\tau_t^i \leq u\}} \) and \( M_u^p = N_u - (n - N_u^t) \hat{\Gamma}_u \).

**5.4. Payment up to the surrender time.** If we have a portfolio with initially \( n \) policyholders with homogeneous behavior, we obtain this time the following proposition:

**Proposition 5.13.** If the Conditions 5.2, 5.6 and 5.9 hold, then the discounted value \( V_t^{C, p}(\rho^*) \) of the risk-minimizing \( n \)-policyholders portfolio is given by

\[ V_t^{C, p}(\rho^*) = (n - N_t) e^{\hat{\Gamma}_t} E^Q \left[ \int_t^T \frac{1}{B_u} e^{-\hat{\Gamma}_u} dC_u \bigg| \mathcal{F}_t \right] \]

where \( N_t = \sum_{i=0}^{n} 1_{\{\tau_t^i \leq t\}} \).

The risk-minimizing strategy \( \rho^* \) is given by

\[ \rho_t^* = \left( (n - N_t^t) e^{\hat{\Gamma}_t} \nu_t, (n - N_t) e^{\hat{\Gamma}_t} E^Q \left[ \int_t^T \frac{1}{B_u} e^{-\hat{\Gamma}_u} dC_u \bigg| \mathcal{F}_t \right] - (n - N_t^t) e^{\hat{\Gamma}_t} \nu_t X_t \right) \]

where \( \nu \) is the amount of risky assets of the strategy risk-minimizing the purely financial contingent claim \( U_t^C = E^Q \left[ \int_0^T \frac{e^{-\hat{\Gamma}_u}}{B_u} dC_u \bigg| \mathcal{F}_t \right] \). The cost process is given by

\[ C_t^{C, p}(\rho^*) = C_0^{C, p}(\rho^*) + L_t^{C, p} \]

where

\[ C_0^{C, p}(\rho^*) = E^Q \left[ \int_0^T \frac{1}{B_u} e^{-\hat{\Gamma}_u} dC_u \bigg| \mathcal{F}_0 \right] \]
\[ L^C_t = \int_0^t (n - N_u) e^{\hat{\tau}_u} dL^C_t - \int_0^t E^Q \left[ \int_u^T \frac{e^{-(\hat{\tau}_s - \hat{\tau}_u)}}{B_s} dC_s \mid \mathcal{F}_u \right] dM^p_u \]

and \( M^p_u = \Delta n - (n - N_u - \hat{\Gamma}_u) \)

**Proof.** Same proof than the previous proposition.


In this section, we study the risk process of a portfolio. In particular, we want to study the behavior of the relative risk process when the number of contract \( n \) increases. Before studying the risk processes in more details, we need the following result.

**Proposition 6.1.** If Condition 5.4 holds then \( L^i_t L^j_t \) is a \((Q, \mathcal{G}^p)\)-martingale for each \( i, j = 1, \ldots n \) with \( i \neq j \).

**Proof.** For \( t \geq s \), we have:

\[
E^Q \left[ L^i_t L^j_t \mid \mathcal{G}^p_s \right] = E^Q \left[ e^{(\hat{\tau}^i_t + \hat{\tau}^j_t)} 1_{\tau^i_t > t, \tau^j_t > t} \mid \mathcal{G}^p_s \right]
\]

\[
= E^Q \left[ e^{(\hat{\tau}^i_t + \hat{\tau}^j_t)} 1_{\tau^i_t > s, \tau^j_t > s} \mid \mathcal{G}^p_s \right]
\]

\[
= E^Q \left[ e^{(\hat{\tau}^i_t + \hat{\tau}^j_t)} 1_{\tau^i_t > s} \mid \mathcal{G}^p_s \right]
\]

\[
= L^i_s L^j_s E^Q \left[ e^{(\hat{\tau}^i_t + \hat{\tau}^j_t)} 1_{\tau^i_t > t} \mid \mathcal{G}^p_s \right]
\]

\[
= L^i_s L^j_s E^Q \left[ e^{(\hat{\tau}^i_t + \hat{\tau}^j_t)} 1_{\tau^i_t > t} \mid \mathcal{F}_t \right] E^Q \left[ 1_{\tau^j_t > t} \mid \mathcal{F}_t \right]
\]

\[
= L^i_s L^j_s E^Q \left[ e^{(\hat{\tau}^i_t + \hat{\tau}^j_t)} 1_{\tau^i_t > t} \mid \mathcal{F}_t \right] E^Q \left[ 1_{\tau^j_t > t} \mid \mathcal{F}_t \right] \mathcal{G}^p_s
\]

In the third equality, we use Proposition 5.6. In the fifth equality, we use the independence of \( \tau^i \) and \( \tau^j \) with respect to \( \mathbb{H}^k \) for \( k \neq i, j \) conditionally on \( \mathbb{F} \). In the sixth equality, we use Equation (5.1).

\( L^i_t \) and \( L^j_t \) are thus strongly \((Q, \mathcal{G}^p)\)-orthogonal. This leads to the following corollary.

**Corollary 6.2.** If Condition 5.4 holds and if the stochastic processes \( \hat{\Gamma}^i_t \) with \( i = 1, \ldots n \), are continuous and of finite variation then \( M^i_t \) and \( M^j_t \) are \((Q, \mathcal{G}^p)\)-orthogonal for each \( i, j = 1, \ldots n \) with \( i \neq j \).

**Proof.** Straightforward thanks to Propositions 5.8 and 6.1.
Now, we can study the risk process of the whole portfolio. For \( d \in \{g, R, C\} \), the risk process is given by

\[
R_0^{d,p}(\rho^*) = E^Q \left[ \left( C_T^{d,p}(\rho^*) - C_0^{d,p}(\rho^*) \right)^2 \right] 
\]

(6.1)

\[
= E^Q \left[ \left( \sum_{i=1}^n \int_0^T L_u^i dL_u^{d,i} + \sum_{i=1}^n \int_0^T K_{u-}^{d,i} dM_u^i \right)^2 \right] 
\]

where \( K_{u-}^{g,i} = -E^Q \left[ e^{-(\tilde{r}_u^g - \tilde{r}_u^i) B_T} |\mathcal{F}_u\right] \), \( K_{u-}^{R,i} = \frac{R_u}{B_u} - E^Q \left[ \int_u^T \frac{R_u}{B_u} e^{-(\tilde{r}_u^g - \tilde{r}_u^i) \tilde{r}_u^i} d\tilde{r}_u^i |\mathcal{F}_u \right] \) and \( K_{u-}^{C,i} = -E^Q \left[ \int_u^T \frac{e^{-\tilde{r}_u^g - \tilde{r}_u^j}}{B_u} dC_s |\mathcal{F}_u \right] \). We have the following result.

**Proposition 6.3.** If we assume the Conditions 5.2, 5.4 and 5.9 hold then, for each \( d \in \{g, R, C\} \), the risk process \( R_0^{d,p}(\rho^*) \) for a \( n \)-policyholders portfolio is given by

\[
R_0^{d,p}(\rho^*) = E^Q \left[ \left( \sum_{i=1}^n \int_0^T L_u^i dL_u^{d,i} \right)^2 \right] + \sum_{i=1}^n E^Q \left[ \left( \int_0^T K_{u-}^{d,i} dM_u^i \right)^2 \right] 
\]

where \( K_{u-}^{g,i} = -E^Q \left[ e^{-(\tilde{r}_u^g - \tilde{r}_u^i) B_T} |\mathcal{F}_u\right] \), \( K_{u-}^{R,i} = \frac{R_u}{B_u} - E^Q \left[ \int_u^T \frac{R_u}{B_u} e^{-(\tilde{r}_u^g - \tilde{r}_u^i) \tilde{r}_u^i} d\tilde{r}_u^i |\mathcal{F}_u \right] \) and \( K_{u-}^{C,i} = -E^Q \left[ \int_u^T \frac{e^{-\tilde{r}_u^g - \tilde{r}_u^j}}{B_u} dC_s |\mathcal{F}_u \right] \). Furthermore, if the policyholders are homogeneous then, as \( n \) increases, the relative risk process tends to:

\[
\lim_{n \to \infty} \frac{\sqrt{R_0^{d,p}(\rho^*)}}{n} = \sqrt{E^Q \left[ \left( L^{d,i}, L^{d,j} \right)_T \right]} 
\]

**Proof.** According to Equation (6.1), we can write

\[
R_0^{d,p}(\rho^*) = E^Q \left[ \left( \sum_{i=1}^n \int_0^T L_u^i dL_u^{d,i} \right)^2 \right] 
\]

\[
+ 2 \sum_{j=1}^n E^Q \left[ \left( \sum_{i=1}^n \int_0^T L_u^i dL_u^{d,i} \right) \left( \int_0^T K_{u-}^{d,j} dM_u^j \right) \right] 
\]

\[
+ \sum_{i=1}^n E^Q \left[ \left( \int_0^T K_{u-}^{d,i} dM_u^i \right)^2 \right] 
\]

\[
+ 2 \sum_{i=1}^{n-1} \sum_{j=i+1}^n E^Q \left[ \left( \int_0^T K_{u-}^{d,i} dM_u^i \right) \left( \int_0^T K_{u-}^{d,j} dM_u^j \right) \right] 
\]
Thanks to Corollary 6.2, we know the square integrable \((Q, \mathcal{G}_p^p)\)-martingales \(\int_0^T K_{u_i}^{d,i} dM_{u_i}^i\) and \(\int_0^T K_{u_j}^{d,j} dM_{u_j}^j\) are strongly \((Q, \mathcal{G}_p^p)\)-orthogonal for \(i \neq j\). We thus have
\[
E^Q \left[ \left( \int_0^T K_{u_i}^{d,i} dM_{u_i}^i \right) \left( \int_0^T K_{u_j}^{d,j} dM_{u_j}^j \right) \right] = 0 \text{ for } i \neq j.
\]
We also have
\[
E^Q \left[ \sum_{i=1}^n \int_0^T L_{u_i}^{i} dL_{u_i}^{i} \right] = 0
\]
since these integrals are also strongly \((Q, \mathcal{G}_p^p)\)-orthogonal thanks to Proposition 5.10.

If furthermore, we assume the \(n\) policyholders are homogeneous, the risk process becomes:
\[
R_0^{d,p}(\rho^*) = \sum_{i=1}^n E^Q \left[ \left( \int_0^T L_{u_i}^{i} dL_{u_i}^{i} \right)^2 \right] + 2 \sum_{i=1}^{n-1} \sum_{j=i+1}^n E^Q \left[ \left( \int_0^T L_{u_i}^{i} dL_{u_i}^{i} \right) \left( \int_0^T L_{u_j}^{j} dL_{u_j}^{j} \right) \right]
\]
\[
+ \sum_{i=1}^n E^Q \left[ \left( \int_0^T K_{u_i}^{d,i} dM_{u_i}^i \right)^2 \right]
\]
\[
= nE^Q \left[ \left( \int_0^T L_{u_i}^{i} dL_{u_i}^{i} \right)^2 \right] + (n^2 - n) E^Q \left[ \left( \int_0^T L_{u_i}^{i} dL_{u_i}^{i} \right) \left( \int_0^T L_{u_j}^{j} dL_{u_j}^{j} \right) \right]
\]
\[
+ nE^Q \left[ \left( \int_0^T K_{u_i}^{d,i} dM_{u_i}^i \right)^2 \right]
\]
for arbitrary \(i \neq j\). The limit of the relative risk process is then given
\[
\lim_{n \to \infty} \frac{\sqrt{R_0^{d,p}(\rho^*)}}{n} = \sqrt{E^Q \left[ \left( \int_0^T L_{u_i}^{i} dL_{u_i}^{i} \right) \left( \int_0^T L_{u_j}^{j} dL_{u_j}^{j} \right) \right]}
\]
\[
= \sqrt{E^Q \left[ \int_0^T L_{u_i}^{i} dL_{u_i}^{i} \left[ L_{u_i}^{i}, L_{u_j}^{j} \right]_u \right]}
\]
\[
= \sqrt{E^Q \left[ \int_0^T E^Q \left[ L_{u_i}^{i} - L_{u_j}^{j} \mid \mathcal{F}_u^- \right] d \left[ L_{u_i}^{i}, L_{u_j}^{j} \right]_u \right]}
\]
\[
= \sqrt{E^Q \left[ \left[ L_{u_i}^{i}, L_{u_j}^{j} \right]_T \right]}
\]
where in the third equality we use the predictable projection theorem and in the fourth, the fact that
\[
E^Q \left[ L_{u_i}^{i} - L_{u_j}^{j} \mid \mathcal{F}_u^- \right] = E^Q \left[ L_{u_i}^{i} \mid \mathcal{F}_u^- \right] E^Q \left[ L_{u_j}^{j} \mid \mathcal{F}_u^- \right] = 1
\]

As \(n\) increases, the risk related to the surrender, goes to 0 which means this risk can be diversified away, even though the surrender times are not independent. For a sufficiently large portfolio, the only risk remaining is thus the one related to the incompleteness of the financial
market. This risk could never be diversified away since it is systematic risk pertaining the whole economy. However, if we assume the financial market is complete, the term $L_t^{ud} = 0$ $\forall t$ $Q$-a.s. and the relative risk process will go to 0 as $n$ increases. This proposition generalizes Møller’s and Riesner’s results. Notice as $n$ increases, the relative risk process of our portfolio tends to the relative risk process of a portfolio of modified purely financial claims.

7. Conclusion.

In this chapter, we studied the risk-minimizing strategies for a single life insurance contract with a surrender option and for a portfolio of such policies. We showed that if the random times of surrender are $\mathbb{F}$-independent, we can combine diversification and hedging to reduce the risk of such a portfolio.

Notice that, in this chapter, assuming the $(H)$-hypothesis is crucial. Otherwise, the financial prices are no longer martingales and the theory of risk-minimization does not apply anymore. In the next chapter, we remove this assumption when studying the theory of local risk-minimization.
CHAPTER 4

Locally Risk-Minimizing Strategies of Life Insurance Contracts with Surrender Option.

1. Introduction.

The motivation of this chapter is to extend the local risk-minimization theory initially developed by Schweizer [76] for a single payoff, to a payment stream, and to apply this extended theory to insurance contracts with surrender option or equivalently to default sensitive contingent claims. In an independent work [79], Schweizer recently studied the same problem and even extended the local risk-minimization theory to a multi-dimensional setting.

Unlike the traditional actuarial literature on the fair valuation of surrender option (see [14, 15, 47, 81, 80] for examples), we do not assume that the surrender time is a(n) (optimal) stopping time with respect to the filtration generated by the financial assets prices. Instead, we assume that the surrender time is a random time that admits a hazard process. We refer to Chapter 2 for a discussion of this assumption. Such random times have been applied to model the default time in the credit risk literature and they have been thoroughly studied by Jeanblanc and Rutkowski in [58]. See also the book of Bielecki and Rutkowski [23]. In the credit risk literature, such models are known as the reduced-form models.

The assumption that the surrender time admits a hazard process, has an important impact for an insurer. Indeed, in this case, the surrender time is not predictable and it implies that an insurance contract with a surrender option cannot be perfectly hedged even if the financial market is initially complete. In other words, in this case, the surrender option introduces a genuine surrender risk. It is thus interesting to study the hedging strategies an insurer should follow to mitigate its exposure to this surrender risk. In the previous chapter, we already studied the risk-minimizing strategies of such contracts. Here, we focus on the locally risk-minimizing strategies.

Except for our previous work, the actuarial literature on the hedging of the surrender risk is extremely limited. However, in our setting, the problem of hedging such insurance contracts is, technically speaking, essentially the same than the problem of hedging default sensitive contingent claims. It thus makes sense to review the financial literature. Quadratic hedging strategies for default sensitive contingent claims have been studied in a number of papers. Mean-variance hedging strategies have been extensively studied in the context of defaultable markets in [24, 25, 26, 27, 28, 58]. As far as the local risk-minimization theory is concerned, it has already been applied in the credit risk literature in Lotz [63] and in Biagini and Cretarola [21, 22]. Lotz studied the locally risk-minimizing strategies of a defaultable claim when a risk free bond and a defaultable bond are traded. In [21], Biagini and Cretarola studied the locally
risk-minimizing strategies of a defaultable claim with two non defaultable primitive assets, a money market account and a risky asset, assuming a mutual dependence of the risky asset and the default time. In [22], they extended their previous work to deal with random recovery at default time.

This chapter can be seen as an extension of the works of Biagini and Cretarola. Actually, we extend [22] in at least three different directions. Firstly, in [22], Biagini and Cretarola showed how to locally risk-minimize a payment at the time of default but they did not give a local risk-minimization theory for more general payment streams. Here, we show that the local risk-minimization theory can be extended to arbitrary (square integrable) payment processes in the same way as Møller did in [68], for the risk-minimization theory. Secondly, Biagini and Cretarola considered a complete financial market with a single risky asset whose price followed a (generalized\(^1\)) geometric Brownian motion. Here, we assume that the discounted financial assets prices follow a general continuous \(s\)-dimensional semimartingale. Thirdly, unlike these authors, we do not assume that the so-called \((H)\)-hypothesis necessarily holds. In addition to this, we also give a detailed study of the impact of a progressive enlargement of filtration on the so-called minimal martingale measure.

The conclusion of this chapter is that, in each case, we can write the (pseudo) locally risk-minimizing hedging strategies of an insurance contract with surrender option as a function of the (pseudo) locally risk-minimizing hedging strategies of a properly modified purely financial claims.

This chapter is organized as follows. In Section 2, we introduce the theoretical framework; we describe the financial market and the different types of payments of our insurance contract. In Section 3, we introduce the local risk-minimization theory. In Section 3.1, we briefly present the local risk-minimization theory for a single payoff as developed in Schweizer [76] and in Section 3.2, we show how this theory can be extended to payment processes. In Sections 4 and 5, we apply this extended theory to the insurance payments described in Section 2. In Sections 4, we assume that the \((H)\)-hypothesis holds whereas in Section 5, we remove this assumption.

2. The Theoretical Framework.

In this section, we introduce the financial market, the surrender time (resp. the default time) and the payments of the insurance contracts (resp. of the default sensitive contingent claim) we want to study.

2.1. Description of the financial market. Let \((\Omega, \mathcal{F}, P)\) be a complete probability space. A perfect frictionless financial market is defined in this space. We assume there is one locally risk free asset denoted by \(B_t = B(t, \omega)\) and \(s\) risky assets \(S_t^i = S^i(t, \omega), i = 1, \ldots s\) following real càdlàg stochastic processes. The price of the locally risk free asset is assumed to follow a strictly positive, continuous process of finite variation. The discounted values of these assets are denoted by

\[
X_t^i = \frac{S_t^i}{B_t}, i = 1, \ldots s
\]

\(^1\)The drift and the diffusion terms can be stochastic in their model.
Let $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ be a filtration that satisfies the usual hypothesis. We assume the stochastic processes $S^i_t$ for $i = 1, \ldots, s$ and $B_t$ are adapted to this filtration.

We now recall some standard assumptions about $X_t$ that appear in the mathematical finance literature. We assume $X_t$ is a special semimartingale in $S^2_{loc}(P, \mathbb{F})$. We write its $(P, \mathbb{F})$-canonical decomposition as $X_t = X_0 + N_t + A_t$ where $N_t$ is $\mathbb{R}^s$-valued locally square integrable $(P, \mathbb{F})$-local martingale null at 0, i.e. $N_t \in \mathcal{M}^2_{0,loc}(P, \mathbb{F})$, and $A_t$ is an $\mathbb{R}^s$-valued $\mathbb{F}$-predictable process of finite variation null at 0 such that the variation of $A^i$, $\|A^i\|_t$, is assumed to be locally square integrable for $i = 1, \ldots, s$. We assume that, for $i = 1, \ldots, s$, $A^i_t \ll \langle N^i, N^i \rangle_t$ with predictable density $\alpha^i_t$, i.e. we have

$$A^i_t = \int_0^t \alpha^i_u d\langle N^i, N^i \rangle_u$$

$P$-a.s. for all $t$ and all $i = 1, \ldots, s$. Let $\tilde{B}_t$ a fixed increasing predictable càdlàg process null at 0 such that $\langle N^i, N^j \rangle_t \ll \tilde{B}_t$ for each $i = 1, \ldots, s$ (we can choose $\tilde{B}_t = \sum_{i=1}^s \langle N^i, N^i \rangle_t$). The Kunita-Watanabe inequality implies $\langle N^i, N^j \rangle_t \ll \tilde{B}_t$ for all $i, j = 1, \ldots, s$. We denote by $\sigma^{ij}_t$ the predictable density of $\langle N^i, N^j \rangle_t$ with respect to $\tilde{B}_t$ i.e.

$$\langle N^i, N^j \rangle_t = \int_0^t \sigma^{ij}_u d\tilde{B}_u$$

$P$-a.s. for all $t$ and all $i, j = 1, \ldots, s$. We can then write

$$A^i_t = \int_0^t \gamma^i_u d\tilde{B}_u$$

where $\gamma^i_t = \alpha^i_t \sigma^{ii}_t$ $P$-a.s. for all $t$ and all $i = 1, \ldots, s$. Finally, we assume that $X_t$ satisfies the structure condition. This structure condition is defined as follows:

**Definition 2.1.** (The structure condition) We say that $X_t$ satisfies the structure condition if there exists an $\mathbb{R}^s$-valued predictable process $\hat{\lambda}_t \in L^2_{loc}(N)$ such that

$$\sigma_t \hat{\lambda}_t = \gamma_t$$

$P$-a.s. for all $t$.

A càdlàg version of the process $\hat{K}_t := \int_0^t \hat{\lambda}_u \gamma_u d\tilde{B}_u$ is called the mean variance trade-off process of $X$.

### 2.2. The random time $\tau$.

We now introduce a random time $\tau$, i.e. $\tau = \tau(\omega)$ is a strictly positive random variable defined on $\Omega$ and $\mathcal{F}$-measurable. It describes the time of surrender of a policyholder (or the time of default of a company in the credit risk literature). We define the process $H_t := 1_{\{\tau \leq t\}}$. The filtration generated by this process $H_t$ is denoted by $\mathbb{H}$, i.e. $\mathbb{H} = (\mathcal{H}_t)_{t \geq 0}$ where $\mathcal{H}_t = \sigma(H_s, 0 \leq s \leq t)$. We define $\mathbb{G} = (G_t)_{t \geq 0}$ where $G_t = \mathcal{F}_t \vee \mathcal{H}_t$, the filtration corresponding to the progressive enlargement of the filtration $\mathbb{F}$ by $\mathbb{H}$.

We assume that $\tau$ satisfies the following condition.

---

2 We assume $\mathbb{G}$ is complete and satisfies the usual hypothesis.
CONDITION 2.2. \( \tau \) is not an \( \mathcal{F} \)-stopping time.

See Chapter 2 for a discussion of this assumption for the surrender time. As in Jeanblanc and Rutkowski [58], we now introduce the so-called \((\mathcal{F}, P)\)-hazard process \( \Gamma_t \) of \( \tau \) by:

\[
F_t = 1 - e^{-\Gamma_t}
\]

where \( F_t := P(\tau \leq t | \mathcal{F}_t), \forall t \geq 0 \). We assume \( P(\tau > t) > 0 \) for all \( t \geq 0 \). Thanks to Condition 2.2, it implies \( F_t < 1 \) for all \( t \geq 0 \) and \( \Gamma_t = -\ln(1 - F_t) \) is indeed well defined. For simplicity, we assume \( F_0 = 0 \).

\( F_t \) is a \((P, \mathcal{F})\)-submartingale. Accordingly, it admits a unique Doob-Meyer decomposition that we write \( F_t = D_t + Y_t \) where \( D_t \) is an \( \mathcal{F} \)-predictable increasing process and \( Y_t \) a \((P, \mathcal{F})\)-martingale.

In [58] or in [23], we can find that the \((P, \mathbb{G})\)-compensator \( \Lambda_t \) of \( H_t \) is given by

\[
\Lambda_t = \int_0^{\tau \wedge t} \frac{1}{1 - F_u} dD_u \tag{2.1}
\]

We denote the compensated process \( M_t := H_t - \Lambda_t \). Throughout this chapter, we will assume the following condition holds.

CONDITION 2.3. The \((P, \mathcal{F})\)-hazard process \( \Gamma_t \) of \( \tau \) is continuous.

Before describing the payments of our insurance contract, we have to stress a last point. At the beginning of this subsection, we introduce the enlargement of the filtration \( \mathcal{F} \) by the random time \( \tau \). This enlargement of filtration deserves some comments. In general, a (local) martingale with respect to a given filtration is not necessarily a (local) martingale with respect to a larger one. In our setting, it means the \((P, \mathcal{F})\)-local martingales are not necessarily \((P, \mathbb{G})\)-local martingales. Let us introduce the following definition.

DEFINITION 2.4. We say the \((H)\)-hypothesis holds under \( P \) between \( \mathcal{F} \) and \( \mathbb{G} \) if any \((P, \mathcal{F})\)-local martingale is a \((P, \mathbb{G})\)-local martingale.

In this chapter, if this \((H)\)-hypothesis does not hold, then \( N_t \), in particular, is not necessarily a \((P, \mathbb{G})\)-local martingale. It implies that the canonical decomposition of \( X_t \) is not necessarily the same if we consider the filtration \( \mathcal{F} \) or the filtration \( \mathbb{G} \). Actually, we do not even know if \( X_t \) is still a semimartingale in the enlarged filtration. Accordingly, without this \((H)\)-hypothesis, the dynamics of the financial market can be deeply altered by this enlargement of filtration. We should thus be very cautious and distinguish when this \((H)\)-hypothesis holds and when it does not. In this chapter, we study in details these two cases: Section 4 deals with the locally risk-minimizing strategies of life insurance contracts when this \((H)\)-hypothesis is assumed and Section 5 deals with the same problem when this \((H)\)-hypothesis is removed.

2.3. The insurance contract. We now describe the payments we wish to hedge.

\[^3\text{or at least for all } t \in [0, T] \text{ where } T \text{ is the term of the insurance contract.}\]
2.3.1. The insurer’s payments. A large variety of life insurance contracts (or default-sensitive contingent claims) can be modelled as a combination of the following three building blocks.

The first one is the payoff the insurer has to pay at the term of the contract \( T \). This payoff is assumed to be a \( \mathcal{F}_T \)-measurable random variable and is denoted by \( g(T, \omega) \). At the term of the contract, the insurer has to pay:

\[
g(T, \omega)1_{\{\tau > T\}}
\]

The second building block is the amount the insurer has to pay when the policyholder surrenders before the term \( T \). This amount is denoted by \( 1_{\{0 < \tau \leq T\}}R(\tau, \omega) \) where \((R(t, \omega))_{t \geq 0}\) is assumed to be an \( \mathbb{F} \)-predictable stochastic process. We have

\[
1_{\{0 < \tau \leq T\}}R(\tau, \omega) = \int_0^T R(u, \omega) dH_u
\]

The third building block is the payoffs the insurer has to pay as long as the policyholder has not surrendered. We model these payoffs through their cumulative value up to time \( t \), denoted by \( C(t, \omega) \). This process is assumed to be a right continuous increasing \( \mathbb{F} \)-adapted process. The cumulative payoff up to surrender is then given by:

\[
C(T, \omega)1_{\{\tau > T\}} + C(\tau, \omega)1_{\{0 < \tau \leq T\}} = \int_0^T (1 - H_u) dC(u, \omega)
\]

where we assume \( C(0, \omega) = 0 \) and \( C(T, \omega) = C(T_-, \omega) \). Notice this third payoff can be written as the sum of the two first ones with \( g(T, \omega) = C(T, \omega) \) and \( R(u, \omega) = C(u, \omega) \).

2.3.2. The policyholder’s payments. We assume a policyholder pays premiums periodically at \( N \) fixed dates \( t_i \) with \( i = 0, \ldots, N - 1 \), as long as he has not surrendered. We denote by \( P(t_i, \omega) \) the value of the premium paid at time \( t_i \) and assume \( P(t_i, \omega) \) is \( \mathcal{F}_{t_i} \)-measurable for each \( i = 0, \ldots, N - 1 \). The policyholder’s payments is then given by:

\[
\sum_{i=0}^{N-1} P(t_i, \omega)1_{\{\tau > t_i\}}
\]

We will not refer to these payments in the rest of the chapter since they are similar to the insurer’s payment at the term of the contract.

3. The Local Risk-Minimization Theory.

In Section 3.1, we briefly review the theory of local risk-minimization developed by Schweizer in [76], for a single payoff made at a fixed date \( T \). We follow the presentation of Schweizer in [78]. In Section 3.2, we show how to extend this theory to payment processes.
3.1. A brief review of the local risk-minimization theory. The financial market is defined as in Section 2.1. Let us start with a few definitions.

**Definition 3.1.** $\Theta_S$ is the space of all processes $\vartheta \in L(X)$ for which the stochastic integral $\int \vartheta dX$ is in the space $S^2(P)$ of semimartingales.

**Definition 3.2.** An $L^2$-strategy $\rho$ is a pair $\rho = (\vartheta, \eta)$ where $\vartheta \in \Theta_S$ and $\eta$ is a real valued adapted process such that the value process $V_t(\rho) = \vartheta_t X_t + \eta_t$ is right continuous and square integrable.

**Definition 3.3.** The cumulative cost process $C_t(\rho)$ of a strategy $\rho = (\vartheta, \eta)$, is defined by $C_t(\rho) = V_t(\rho) - \int_0^t \vartheta_u dX_u$.

**Definition 3.4.** A strategy $\rho$ is said to be mean-self-financing if its cost process is a $(P, \mathcal{F})$-martingale.

**Definition 3.5.** The risk process $R_t(\rho)$ of a $L^2$-strategy $\rho$ is defined by

$$R_t(\rho) = E^P \left[ (C_T(\rho) - C_t(\rho))^2 | \mathcal{G}_t \right]$$

We now restrict our attention to the case $s = 1$ for which the local risk-minimization theory has been developed.

**Definition 3.6.** A trading strategy $\Delta = (\delta, \varepsilon)$ is called a small perturbation if it satisfies the following conditions:

a) $\delta$ is bounded
b) The variation of $\int \delta dA$ is bounded
c) $\delta_T = \varepsilon_T = 0$

For any subinterval $(s, t]$ of $[0, T]$, if $\Delta = (\delta, \varepsilon)$ is a small perturbation then we can define the small perturbation $\Delta \mid_{(s, t]}$ by

$$\Delta \mid_{(s, t]} := (\delta \mid_{(s, t]}, \varepsilon \mid_{(s, t]})$$

We can now introduce the main definitions:

**Definition 3.7.** For an $L^2$-strategy $\rho$, a small perturbation $\Delta$ and a partition $\pi$ of $[0, T]$, we define the $R$-quotient $r^\pi [\rho, \Delta] (t, \omega)$

$$r^\pi [\rho, \Delta] (t, \omega) := \sum_{t_i \in \pi} \frac{R_{t_i} (\rho + \Delta \mid_{(t_i, t_{i+1}]}) - R_{t_i} (\rho)}{E \left[ (M)_{t_{i+1}} - (M)_{t_i} | \mathcal{F}_{t_i} \right]} (\omega) 1_{(t_i, t_{i+1}[}(t)$$
We can now introduce the definition of a locally risk-minimizing strategy.

**Definition 3.8.** A strategy \( \rho \) is called locally risk-minimizing if

\[
\liminf_{n \to \infty} r_{\pi_n}^{\rho} \geq 0, \quad P_M - a.e.
\]

for every small perturbation \( \Delta \) and every increasing 0-convergent sequence \( \pi_n \) of partitions of \([0, T]\).

Schweizer in [76] established the following theorem:

**Theorem 3.9.** Let \( H \in L^2(\mathcal{F}_T, P) \) be a contingent claim and let \( \rho \) be an \( L^2 \)-strategy with \( V_T(\rho) = H, \ P - a.s. \) If the following assumptions (1-6) hold then \( \rho \) is locally risk-minimizing if and only if \( \rho \) is mean-self-financing and the martingale \( C_t(\rho) \) is strongly \((P, \mathbb{F})\)-orthogonal to \( N_t \).

1. \( s = 1 \).
2. \( N_t \) is a \((P, \mathbb{F})\)-square integrable martingale with \( N_0 = 0 \).
3. \((N_t, N_t)\) is strictly increasing \( P\)-a.s.
4. \( A_t \) is continuous.
5. \( A \ll (N_t, N_t) \) with predictable density \( \alpha_t \) i.e. \( A_t = \int_0^t \alpha_u d(N_u, N_u) \ P\)-a.s. for all \( t \).
6. \( E \left[ \int_0^T |\alpha_u|^2 d(N_u, N_u) \right] < \infty. \)

Strictly speaking, the local risk-minimization theory has only been developed for \( s = 1 \). When \( s \geq 1 \), inspired by the last theorem, Schweizer defines pseudo locally risk-minimizing strategies in the following way:

**Definition 3.10.** Let \( H \in L^2(\mathcal{F}_T, P) \) be a contingent claim. An \( L^2 \)-strategy \( \rho \) with \( V_T(\rho) = H, \ P-a.s. \), is called pseudo-locally risk-minimizing for \( H \) if \( \rho \) is mean-self-financing and the martingale \( C_t(\rho) \) is strongly \((P, \mathbb{F})\)-orthogonal to \( N_t \).

This definition leads to the following decomposition.

**Theorem 3.11.** Let \( H \in L^2(\mathcal{F}_T, P) \) be a contingent claim. \( H \) admits a pseudo-locally risk-minimizing \( L^2 \)-strategy \( \rho \) with \( V_T(\rho) = H \ P-a.s. \), if and only if \( H \) can be written as:

\[
H = H_0 + \int_0^T \varepsilon_u^H dX_u + L_t^H
\]

with \( H_0 \in L^2(\mathcal{F}_0, P), \varepsilon^H \in \Theta_S \) and \( L_t^H \in \mathcal{M}_0^2(P, \mathbb{F}) \) strongly \((P, \mathbb{F})\)-orthogonal to \( N_t \). The pseudo-locally risk-minimizing strategy \( \rho = (\vartheta, \eta) \) is then given by \( \vartheta_t = \varepsilon^H_t \) and \( \eta_t = V_t(\rho) - \varepsilon^H_t X_t \) where \( V_t(\rho) = H_0 + \int_0^t \varepsilon_u^H dX_u + L_t^H, \ \forall t \in [0, T]. \)

**Proof.** See Schweizer [78] or see the proof of Proposition 3.19. \( \square \)

**Remark 3.12.** We can also write the value process as \( V_t(\rho) = E^P \left[ H - \int_0^T \varepsilon_u^H dX_u | \mathcal{F}_t \right]. \)

\(^4\text{However, in a recent independent work [79], Schweizer extended the local risk minimization theory to a multi-dimensional setting.}\)
The decomposition of $H$ described in the previous theorem is called the Föllmer-Schweizer decomposition of $H$.

Unfortunately, it is often difficult to directly find this decomposition. However, under some further assumptions (the continuity of the assets prices for example), the Föllmer-Schweizer decomposition can be found by finding the Galtchouk-Kunita-Watanabe (GKW) decomposition of $H$ but with respect of a particular martingale measure $\hat{P}$ called the minimal martingale measure.

**Theorem 3.13.** Assume the structure condition holds and the canonical decomposition of $X_t$ is given by $X_t = X_0 + N_t + A_t$. Define the process $\hat{Z}_t := \mathbb{E}_t\left( -\int_0^\cdot \hat{\lambda} dN \right)$. If $X$ is continuous then $\hat{Z}_t$ is a strict martingale density for $X_t$.

**Proof.** See Schweizer [77] for examples.

If furthermore $\hat{Z}_T \in L^2(\mathcal{F}_T, P)$ then the measure $\hat{P}$ defined as $d\hat{P} / dP := \hat{Z}_T$ is called the minimal martingale measure. When such a minimal martingale measure exists, it is unique. We then have the following theorem.

**Theorem 3.14.** Assume the contingent claim $H$ admits a Föllmer-Schweizer decomposition. Let $X$ be continuous and assume the minimal martingale measure $\hat{P}$ exists. Then the $\hat{P}$-martingale $V_{t}^{H,\hat{P}}$ defined as $V_{t}^{H,\hat{P}} = E^{\hat{P}}[H | \mathcal{F}_t]$ admits a Galtchouk Kunita Watanabe decomposition under $\hat{P}$ with respect to $X_t$ as

$$V_{t}^{H,\hat{P}} = V_{0}^{H,\hat{P}} + \int_0^t \varepsilon_{u}^{H,\hat{P}} dX_{u} + L_{t}^{H,\hat{P}}$$

For $t = T$, this decomposition is the Föllmer-Schweizer decomposition of $H$ with $\varepsilon_{t}^{H,\hat{P}} = \varepsilon_t^H$. We also have $H_0 = V_{0}^{H,\hat{P}} = E^{\hat{P}}[H | \mathcal{F}_0]$.

**Proof.** See Schweizer [78].

### 3.2. Extension of the local risk-minimization theory for payment processes.

In this section, we extend the local risk-minimization theory described in Section 3.1. Here, we assume the payoffs are not necessarily paid at a fixed date $T$ but can be paid continuously on the time interval $[0, T]$. In other words, the payments are now modelled as a stochastic process $E_t$. More precisely, the process $E_t$ represents the discounted value of the cumulative payments up to time $t$ and is assumed to be càdlàg, $\mathbb{F}$-adapted and square integrable. As already explained, the local risk-minimization has also been independently extended to payment streams in Schweizer [79]. To derive a local risk-minimization theory for payment processes, we can basically follow Møller [68] when he extended the risk-minimization theory of Föllmer and Sondermann to payment processes. As Møller did, we first slightly modify the definition of the cumulative cost process.

**Definition 3.15.** The cumulative cost process $C_t(\rho)$ of a strategy $\rho$, is defined by $C_t(\rho) = V_t(\rho) - \int_0^t \varepsilon_a dX_u + E_t$. 
The initial cumulative cost process $C_0(\rho)$ is given by $C_0(\rho) = V_0(\rho) + E_0$. The second term $E_0$ is the initial payment the insurer has to pay to the policyholder. The first term $V_0(\rho)$ represents the initial amount the insurer has to invest to create his financial portfolio (after the payment of $E_0$). Following Møller [68], we restrict our attention to strategies which are 0-admissible in the following sense:

**Definition 3.16.** A strategy $\rho$ is said to be 0-admissible if and only if: $V_T(\rho) = 0$, $P$-a.s.

All the other definitions introduced in the previous section, remain unchanged.

The following proposition is the counterpart of Theorem 3.9 for payment processes. Here, we assume $s = 1$. It is a straightforward extension of Schweizer’s proof.

**Proposition 3.17.** Let $E_t$ be a càdlàg, $\mathbb{F}$-adapted and square integrable payment process and let $\rho$ be an $L^2$-strategy 0-admissible. If the 6 conditions of Theorem 3.9 hold then $\rho$ is locally risk-minimizing if and only if $\rho$ is mean self financing and the $(P,\mathbb{F})$-martingale $C_t(\rho)$ is strongly $(P,\mathbb{F})$-orthogonal to $N_t$.

**Proof.** We can follow Schweizer’s proof with the new definition of the cost process and for a 0-admissible strategy. Let us decompose this proof in three steps:

1) In Lemma 2.1 in [76], Schweizer shows that if a trading strategy $\rho$ is locally risk-minimizing, it is mean self financing. We can follow the same proof than in [76] if, following the same notation than Schweizer, we construct the trading strategy $\hat{\rho} = (\hat{\varepsilon}, \hat{\eta})$ as:

$$
\hat{\varepsilon} := \varepsilon \\
\hat{\eta}_s := E \left[ V_T(\rho) - \int_0^T \varepsilon_u dX_u + E_T | \mathcal{F}_s \right] + \int_0^s \varepsilon_u dX_u - E_s - \varepsilon_s X_s 
$$

for $s \in [0, T]$. In this case, we thus have:

$$
V_s(\hat{\rho}) = E \left[ V_T(\rho) - \int_0^T \varepsilon_u dX_u + E_T | \mathcal{F}_s \right] + \int_0^s \varepsilon_u dX_u - E_s 
$$

and in particular $V_T(\hat{\rho}) = V_T(\rho) = 0$. Using the modified definition of the cost process, we have, as in Schweizer, that

$$
C_T(\hat{\rho}) = V_T(\hat{\rho}) - \int_0^T \hat{\varepsilon}_u dX_u + E_T \\
= V_T(\rho) - \int_0^T \varepsilon_u dX_u + E_T \\
= C_T(\rho)
$$
and that

\[ E[C_T(\hat{\rho}) | F_s] = E[V_T(\hat{\rho}) - \int_0^T \hat{\epsilon}_u dX_u + E_T | F_s] = E[V_T(\rho) - \int_0^T \epsilon_u dX_u + E_T | F_s] = V_s(\hat{\rho}) - \int_0^s \epsilon_u dX_u + E_s = C_s(\hat{\rho}) \]

With these results, the rest of the proof is identical to Schweizer’s. A locally risk-minimizing strategy \( \rho \) is thus mean self-financing and the modified version of the cost process \( C_t(\rho) \) is a \((P, \mathbb{F})\)-martingale.

2) In Lemma 2.2 in [76], Schweizer shows that basically we can find a locally risk-minimizing trading strategy by varying only the \( \varepsilon \)-component. The proof of this lemma depends on the fact the cost process \( C_t(\rho) \) is a martingale but not on the specific definition of this process. We can thus directly use Schweizer’s proof.

3) Finally, in Lemma 2.3, Schweizer shows that if \( \rho \) is locally risk-minimizing, the martingale \( C_t(\rho) \) is strongly orthogonal to \( N_t \). To prove this lemma, Schweizer uses Theorem 3.2 of [75]. Again, this theorem relies on the fact \( C_t(\rho) \) is a martingale and not on its specific definition. This completes the proof. □

When \( s \geq 1 \), we can define pseudo-locally risk-minimizing strategies for payment processes as in Section 3.1.

**Definition 3.18.** Let \( E_t \) be a càdlàg, \( \mathbb{F} \)-adapted and square integrable payment process. An \( 0 \)-admissible \( L^2 \)-strategy \( \rho \) is called pseudo-locally risk-minimizing for \( E_t \) if \( \rho \) is mean-self-financing and the \((P, \mathbb{F})\)-martingale \( C_t(\rho) \) is strongly \((P, \mathbb{F})\)-orthogonal to \( N_t \).

We now have the counterpart of Theorem 3.11 for payment processes:

**Proposition 3.19.** Let \((E_t)_{0 \leq t \leq T} \in L^2(\Omega, P)\) be a payment process. \((E_t)_{0 \leq t \leq T} \) admits a pseudo-locally risk-minimizing \( L^2 \)-strategy \( \rho \) with \( V_T(\rho) = 0 \) \( P \)-a.s. if and only if \( E_T \) admits a Föllmer-Schweizer decomposition, i.e. if \( E_T \) can be written as:

\[
E_T = J_0 + \int_0^T \varepsilon^E_u dX_u + L^E_T
\]

with \( J_0 \in L^2(\mathcal{F}_0, P), \varepsilon^E \in \Theta_S \) and \( L^E_t \in M^2(\mathcal{P}, \mathbb{F}) \) strongly \((P, \mathbb{F})\)-orthogonal to \( N_t \).

The pseudo-locally risk-minimizing strategy \( \rho = (\vartheta, \eta) \) is then given by \( \vartheta_t = \varepsilon^E_t \) and \( \eta_t = V_t(\rho) - \varepsilon^E_t X_t \) where its value process \( V_t(\rho) \) is given by

\[
V_t(\rho) = J_0 + \int_0^t \varepsilon^E_u dX_u + L^E_t - E_t
\]

\( \forall t \in [0, T] \).
Proof. If \( \rho = (\vartheta, \eta) \) is pseudo-locally risk-minimizing for \( E_t \), we know thanks to the previous proposition, that \( L_t^E := C_t(\rho) - C_0(\rho) \in M_0^2(P, \mathbb{F}) \) and is strongly \((P, \mathbb{F})\)-orthogonal to \( N_t \). Moreover, by the definition of the cost process, we have that for all \( t \):

\[
L_t^E = C_t(\rho) - C_0(\rho) = V_t(\rho) - \int_0^t \vartheta_u dX_u + E_t - C_0(\rho)
\]

so we indeed have \( V_t(\rho) = C_0(\rho) + \int_0^t \vartheta_u dX_u + L_t^E - E_t \). Moreover since by definition \( V_T(\rho) = 0 \), this last equation leads, for \( t = T \), to Equation (3.1).

Notice, we also have the following relationships:

\[
V_t(\rho) = E^P \left[ E_T - E_t - \int_t^T \varepsilon_u^E dX_u | \mathcal{F}_t \right]
\]

and

\[
J_0 = C_0(\rho) = V_0(\rho) + E_0 = E^P \left[ E_T - \int_0^T \varepsilon_u^E dX_u | \mathcal{F}_0 \right]
\]

4. Application to a Life Insurance Contract When the \((H)\)-Hypothesis Holds.

In this section, we apply the (extended) theory of local risk-minimization to the payments described in Section 2.3.1, assuming the \((H)\)-hypothesis holds. More precisely, we study their pseudo locally risk-minimizing strategies.

In Section 5.1, we first give some preliminary results and in Section 2.1, we briefly study the impact of the enlargement of filtration on the financial market. In Section 4.3, we first study the minimal martingale measure and its properties. Then, assuming the continuity of \( X_t \), we derive the forms of the pseudo locally risk-minimizing strategies by finding the GKW decomposition of our payments under this minimal martingale measure. In Section 4.4, we remove the continuity of \( X_t \) and give the forms of the pseudo locally risk-minimizing strategies by directly finding the Föllmer-Schweizer decomposition.

Our main conclusion is that finding the pseudo locally risk-minimizing strategies of any of these payments comes down to finding the pseudo locally risk-minimizing strategies of a properly modified purely financial claims.

Throughout this section, we assume the following condition holds.

**Condition 4.1.** The \((H)\)-hypothesis holds under \( P \) between \( \mathbb{F} \) and \( \mathbb{G} \).

An alternative definition of the \((H)\)-Hypothesis often appears in the literature.

**Definition 4.2.** We say the \((H)\)-hypothesis holds under \( P \) between \( \mathbb{F} \) and \( \mathbb{G} \) if and only if: for every \( t \), \( \mathcal{F}_\infty \) and \( \mathcal{G}_t \) are conditionally independent with respect to \( \mathcal{F}_t \).

See Brémaud and Yor [33] for a proof of the equivalence between this definition and Definition 2.4.
4.1. Preliminary results. Before studying the Föllmer-Schweizer decomposition of our payments or their Galtchouk-Kunita-Watanabe decomposition under the minimal martingale measure, we introduce some preliminary results.

**Proposition 4.3.** Let \( m_t \) and \( n_t \) be two \( \mathbb{F} \)-adapted processes such that \( \langle m_t, n_t \rangle^P_t \) exists. Then, if the \((H)\)-hypothesis holds under \( P \), \( \langle m_t, n_t \rangle^P_t \) and \( \langle m_t, n_t \rangle^G_t \) are \( P \)-indistinguishable.

**Proof.** By definition, \( \langle m_t, n_t \rangle^P_t \) is the \((P,\mathbb{F})\)-compensator of \( [m_t, n_t]_t \), i.e. \( [m_t, n_t]_t - \langle m_t, n_t \rangle^P_t \) is a \((P,\mathbb{F})\)-local martingale. Thanks to the \((H)\)-hypothesis, \( [m_t, n_t]_t - \langle m_t, n_t \rangle^P_t \) is thus also a \((P,\mathbb{G})\)-local martingale. By the uniqueness of the compensator, \( \langle m_t, n_t \rangle^P_t \) is thus also the \((P,\mathbb{G})\)-compensator of \( [m_t, n_t]_t \).

Also, Condition 4.1 ensures all \((P,\mathbb{F})\)-strongly orthogonal martingales will also be \((P,\mathbb{G})\)-strongly orthogonal martingales.

**Corollary 4.4.** Let \( m_t \) and \( n_t \) be two square integrable \((P,\mathbb{F})\)-martingales strongly \((P,\mathbb{F})\)-orthogonal. Then \( m_t \) and \( n_t \) are \((P,\mathbb{G})\)-strongly orthogonal.

**Proof.** \((P,\mathbb{F})\)-strong orthogonality is equivalent to \( \langle m_t, n_t \rangle^P_t = 0 \). Since under the \((H)\)-hypothesis, \( \langle m_t, n_t \rangle^P_t = \langle m_t, n_t \rangle^G_t \), we thus also have the \((P,\mathbb{G})\)-strong orthogonality.

In our setting, i.e. when a filtration \( \mathbb{F} \) is enlarged with the observation of a random time \( \tau \), we have also the following equivalence:

**Lemma 4.5.** The \((H)\)-hypothesis under \( P \) is equivalent to

\[
P(\tau \leq t \mid \mathcal{F}_\infty) = P(\tau \leq t \mid \mathcal{F}_t)
\]

**Proof.** See Jeanblanc and Rutkowski [58].

The corollary of this lemma is that under the \((H)\)-hypothesis, \( F_t \) and thus \( \Gamma_t \) are increasing and accordingly, are of finite variation. Recall that the Doob-Meyer decomposition of the submartingale \( F_t \) is written \( F_t = Y_t + D_t \). As far as \( Y_t \) is concerned, we have the following lemma:

**Lemma 4.6.** If Conditions 4.1 and 2.3 hold then the \((P,\mathbb{F})\)-martingale \( Y_t \) is equal to 0 for all \( t \geq 0 \) \( P \)-a.s.

**Proof.** Conditions (4.1) and (2.3) imply \( F_t \) is increasing and continuous. \( F_t \) is thus a predictable finite variation process. By the uniqueness of the Doob-Meyer decomposition, it leads to \( F_t = D_t \) for all \( t \geq 0 \) \( P \)-a.s.

As far as the \((P,\mathbb{G})\)-compensator \( \Lambda_t \) of \( H_t \) is concerned, we have the following lemma.

**Lemma 4.7.** If \( \Gamma_t \) is continuous and increasing, the \((P,\mathbb{G})\)-compensator \( \Lambda_t \) of \( H_t \) is given by \( \Lambda_t = \Gamma_{t \wedge \tau} \).

**Proof.** This proof can be found in Jeanblanc and Rutkowski [58] or in Bielecki and Rutkowski [23]. We give it here for the sake of completeness. Injecting \( D_t = F_t \) in Equation (2.1), we have:
\[ \Lambda_t = \int_0^{t \wedge \tau} e^{\Gamma_u} d(1 - e^{-\Gamma_u}) = \int_0^{t \wedge \tau} d\Gamma_u \]

Under these assumptions, the compensated process \( M_t \) can be written \( M_t = H_t - \Gamma_t \wedge \tau \). We will also need the following lemma.

**Lemma 4.8.** If the \((P, \mathcal{F})\)-hazard process \( \Gamma_t \) is continuous and of finite variation then the \((P, \mathcal{G})\)-martingale \( L_t := 1_{\{\tau > t\}} e^{\Gamma_t} \) can be written as

\[ L_t = 1 - \int_0^t e^{\Gamma_u} dM_u \]

**Proof.** See Jeanblanc and Rutkowski [58]. □

### 4.2. The financial market.

We now study how the enlargement of filtration affects the financial market. Fortunately, Condition 4.1 ensures this enlargement does not modify the main structure of the financial market.

**Proposition 4.9.** \( X_t \in S^2_{\text{loc}}(P, \mathcal{G}) \) and its \((P, \mathcal{G})\)-canonical decomposition is still given by \( X_t = X_0 + N_t + A_t \).

**Proof.** Straightforward. By definition of the \((H)\)-hypothesis, the \((P, \mathcal{F})\)-local martingale \( N_t \) is a \((P, \mathcal{G})\)-local martingale. We even have \( N_t \in \mathcal{M}^2_{\text{loc}}(P, \mathcal{G}) \) since square integrability does not depend on the filtration. \( A_t \) is still a finite variation process with locally square integrable variation, since these properties do not depend on the filtration. □

The following proposition is essential.

**Proposition 4.10.** If the \((H)\)-hypothesis holds for \( P \) and if the \((\mathcal{F}, P)\)-hazard process \( \Gamma_t \) is continuous then \( \Delta U_{\tau} = 0 \) a.s. for any \( \mathcal{F} \)-adapted càdlàg process \( U_t \).

**Proof.** See the previous chapter. □

### 4.3. The minimal martingale measure approach.

We saw in Section 3 that under some further assumptions (for example when \( X_t \) is continuous) we can find the Föllmer-Schweizer decomposition of a contingent claim by finding its Galtchouk-Kunita-Watanabe decomposition under the minimal martingale measure denoted by \( \hat{P} \). In this section, we use this approach to find the pseudo-locally risk-minimizing strategies. We first give some preliminary results on the minimal martingale measure in Section 4.3.1. Then in the subsequent sections, we study the locally risk-minimizing strategies of the different payments. From now on, we assume the minimal martingale measure \( \hat{P} \) exists for the financial market restricted to the filtration \( \mathcal{F} \).
4.3.1. The minimal martingale measure and the enlargement of the filtration. In the first proposition, we study the impact of the enlargement of the filtration on the minimal martingale measure when the (H)-hypothesis holds under \( P \) between the filtrations \( F \) and \( G \). In general, the minimal martingale measure depends on the choice of the filtration. When they exist, we denote \( \hat{P}^F \) and \( \hat{P}^G \) the minimal martingale measures for, respectively, the filtration \( F \) and \( G \). We denote by \( \hat{\Gamma} \) the minimal martingale measure depends on the choice of the filtration. When they exist, we study the impact of the enlargement of the filtration on the minimal martingale measure. Since by assumption, it is square integrable, \( \hat{\Gamma} \) is even a uniformly square integrable \((P,G)\)-martingale. By definition, we thus have \( \hat{Z}^G_t = \hat{Z}^F_t \).

When the (H)-hypothesis holds, the minimal martingale measures are indistinguishable and we can thus talk about the minimal martingale measure \( \hat{P} \). In the next proposition, we study the hazard process under the measure \( \hat{P} \).

**Proposition 4.12. (Invariance of the hazard process)** Let \( \Gamma_t \) be the \((P,F)\)-hazard process of \( \tau \), and \( \hat{\Gamma}_t \) be the \((\hat{P},F)\)-hazard process of \( \tau \). If the (H)-hypothesis holds under \( P \), we have \( \Gamma_t = \hat{\Gamma}_t \) \( \forall t \) \( P \)-a.s.

**Proof.** By Bayes rule, we have:

\[
E^{\hat{P}} \left[ 1_{\{\tau > t\}} \mid \mathcal{F}_t \right] = \frac{E^P \left[ \hat{Z}^G_{\infty} 1_{\{\tau > t\}} \mid \mathcal{F}_t \right]}{\hat{Z}^G_t} = \frac{E^P \left[ \hat{Z}^G_{\infty} \mid \mathcal{F}_t \right] E^P \left[ 1_{\{\tau > t\}} \mid \mathcal{F}_t \right]}{\hat{Z}^G_t} = \frac{E^P \left[ 1_{\{\tau > t\}} \mid \mathcal{F}_t \right]}{\hat{Z}^G_t}
\]

In the second equality, we use Definition 4.2 of the (H)-hypothesis. By definition of the hazard process, we thus have that \( \hat{\Gamma}_t = \Gamma_t \) \( \forall t \) \( P \)-a.s. \( \square \)

Notice it does not mean the probabilities of surrender are equal under \( P \) and \( \hat{P} \). Indeed, we have \( P(\tau > t) = E^P \left[ e^{-\Gamma_t} \right] \neq E^\hat{P} \left[ e^{-\Gamma_t} \right] = E^\hat{P} \left[ e^{-\Gamma_t} \right] = \hat{P}(\tau > t) \).
Remark 4.13. We know that if the hazard process is continuous then $\Gamma_{t\wedge \tau}$ is equal to the $(P, G)$-compensator $\Lambda_t$ of $\tau$. So here, $\Gamma_{t\wedge \tau}$ is also equal to the $(P, G)$-compensator $\hat{\Lambda}_t$ of $\tau$.

Proposition 4.14. (Invariance of the (H)-hypothesis) If the (H)-hypothesis holds under $P$ between the filtration $\mathcal{F}$ and $\mathcal{G}$ then the (H)-hypothesis holds under the minimal martingale measure $\hat{P}$.

Proof. To show that the (H)-hypothesis holds under $\hat{P}$, we have to show that any $(\hat{P}, \mathcal{F})$-local martingale is also a $(\hat{P}, \mathcal{G})$-local martingale. For any $(\hat{P}, \mathcal{F})$-local martingale $R_t$, we have that $\hat{Z}_t^\mathcal{F} R_t$ is $(\hat{P}, \mathcal{F})$-local martingale. Thanks to the (H)-hypothesis under $P$, $\hat{Z}_t^\mathcal{F} R_t$ is also a $(P, \mathcal{G})$-local martingale. Since according to Proposition 4.11, $\hat{Z}_t^\mathcal{F} = \hat{Z}_t^G$ then $\hat{Z}_t^G R_t$ is also a $(P, \mathcal{G})$-local martingale. Since $\hat{Z}_t^G$ is the $\mathcal{G}$-density of $\hat{P}$, $R_t$ is a $(\hat{P}, \mathcal{G})$-local martingale. □

We now study the pseudo-locally risk-minimizing strategies for the payments described in Section 2.3.1. We assume the minimal martingale measure under the filtration $\mathcal{F}$ exists.

4.3.2. Payment at the term of the contract. We want to find the pseudo-locally risk-minimizing strategy of the payment process $1_{\{T=t\}}1_{\{\tau>T\}} \frac{q_T}{\hat{B}_T}$ where we assume $g_T$ is $\mathcal{F}_T$-adapted and square integrable with respect to $P$. The following proposition gives the Föllmer-Schweizer decomposition of this payment.

Proposition 4.15. Let us assume Conditions 2.3 and 4.1 hold. If the $(\hat{P}, \mathcal{F})$-Galtchouk-Kunita-Watanabe decomposition of the $(\hat{P}, \mathcal{F})$-martingale $\hat{U}_t^g := E^\hat{P} \left[ \frac{q_T}{\hat{B}_T} e^{-\Gamma_T} | \mathcal{F}_t \right]$ is written

$$E^\hat{P} \left[ \frac{q_T}{\hat{B}_T} e^{-\Gamma_T} | \mathcal{F}_t \right] = \hat{U}_0^g + \int_0^t \hat{\phi}_u dX_u + \hat{L}_t^U$$

then the $(P, \mathcal{G})$-Föllmer-Schweizer decomposition of $1_{\{\tau>T\}} \frac{q_T}{\hat{B}_T}$ is given by

$$1_{\{\tau>T\}} \frac{q_T}{\hat{B}_T} = \hat{U}_0^g + \int_0^T 1_{\{\tau>u-\}} e^{\Gamma_u} \hat{\phi}_u dX_u + \hat{L}_T^g$$

where

$$\hat{L}_t^g = \int_0^t 1_{\{\tau>u-\}} e^{\Gamma_u} \hat{M}_t^U - \int_0^t e^{\Gamma_u} \hat{V}_t^U dM_u$$

and

$$\hat{V}_t^U = \hat{U}_t^g$$

Proof. According to Theorem 3.14, we know that when $X_t$ is continuous, the $\mathcal{G}$-Föllmer-Schweizer decomposition of $1_{\{\tau>T\}} \frac{q_T}{\hat{B}_T}$ under the measure $P$, is given by the $\mathcal{G}$-Galtchouk-Kunita-Watanabe decomposition under the minimal martingale measure $\hat{P}$ of $E^\hat{P} \left[ 1_{\{\tau>T\}} \frac{q_T}{\hat{B}_T} | \mathcal{G}_t \right]$.

---

5 Since we only have a single payoff at the date $T$, we do not really need here the extension of the local risk-minimization theory to payment processes described above.
Since the \((\hat{P}, \mathbb{F})\)-hazard process of \(\tau\) is \(\Gamma_t\), we have
\[
E^{\hat{P}} \left[ 1_{\{\tau>T\}} \frac{g_T}{B_T} | G_t \right] = 1_{\{\tau>t\}} e^{\Gamma_t} E^{\hat{P}} \left[ \frac{e^{-\Gamma_T} g_T}{B_T} | \mathcal{F}_t \right]
\]
Let us define \(\hat{U}^g_t := E^{\hat{P}} \left[ e^{-\Gamma_T} \frac{g_T}{B_T} | \mathcal{F}_t \right]\). By hypothesis, its \((\hat{P}, \mathbb{F})\)-Galtchouk-Kunita-Watanabe is given by
\[
\hat{U}^g_t = \hat{U}^g_0 + \int_0^t \hat{\phi}_u dX_u + \hat{L}^g_t
\]
Using the integration by parts formula, we can write
\[
E^{\hat{P}} \left[ 1_{\{\tau>T\}} \frac{g_T}{B_T} | G_t \right] = 1_{\{\tau>t\}} e^{\Gamma_t} \hat{U}^g_t
\]
\[
= \int_0^t 1_{\{\tau>u\}} e^{\Gamma_u} \hat{U}^g_u - \int_0^t \hat{U}^g_u e^{\Gamma_u} dM_u + \left[ \hat{U}^g_0, 1_{\{\tau>\}} e^{\Gamma} \right]_t
\]
where \(M_u = 1_{\{\tau\leq u\}} - \Gamma_u \wedge \tau\). Thanks to the \((H)\)-hypothesis, we know \(\Gamma_t\) is increasing and \(L_t = 1_{\{\tau>\}} e^{\Gamma}\) is thus a finite variation process. It implies
\[
\left[ \hat{U}^g_0, 1_{\{\tau>\}} e^{\Gamma} \right]_t = \hat{U}^g_0 L_0 + \sum_{0<s\leq t} \Delta \hat{U}^g_s \Delta L_s
\]
Since \(\hat{\Gamma}_t\) is continuous, thanks to Proposition 4.10, there is no discontinuity in \(\hat{U}^g_t\) at time \(\tau\), the square bracket is thus equal to \(\hat{U}^g_0 L_0 = \hat{U}^g_0\). We thus have
\[
E^{\hat{P}} \left[ 1_{\{\tau>T\}} \frac{g_T}{B_T} | G_t \right] = \hat{U}^g_0 + \int_0^t 1_{\{\tau>u\}} e^{\Gamma_u} \hat{\phi}_u dX_u + \hat{L}^g_t
\]
where \(\hat{L}^g_t = \int_0^t 1_{\{\tau>u\}} e^{\Gamma_u} \hat{L}^g_u - \int_0^t e^{\Gamma_u} \hat{U}^g_u dM_u\).

Let us now prove this decomposition is indeed the \((\hat{P}, \mathcal{G})\)-GKW decomposition of \(E^{\hat{P}} \left[ 1_{\{\tau>T\}} \frac{g_T}{B_T} | G_t \right]\).

First, notice that the \((P, \mathcal{G})\)-martingale \(M_u\) is a \((\hat{P}, \mathcal{G})\)-martingale thanks to Remark 4.13 and also that the \((\hat{P}, \mathbb{F})\)-martingales \(X_t\) and \(\hat{L}^g_t\) are \((\hat{P}, \mathcal{G})\)-martingales thanks to Proposition 4.14.

We still have to prove \(\langle X, \hat{L}^g \rangle_{P, \mathcal{G}}^{\hat{P}, \mathcal{G}} = 0\). We have
\[
\langle X, \hat{L}^g \rangle_{P, \mathcal{G}}^{\hat{P}, \mathcal{G}} = \int_0^t 1_{\{\tau>u\}} e^{\Gamma_u} d\langle X, \hat{L}^g \rangle_{P, \mathcal{G}}^{\hat{P}, \mathcal{G}} - \int_0^t e^{\Gamma_u} \hat{V}^g u d\langle X, M \rangle_{\hat{P}, \mathcal{G}}
\]
\[
= 0
\]
We use the fact \(\langle X, M \rangle_{P, \mathcal{G}}^{\hat{P}, \mathcal{G}} = 0\) since \(M\) is a finite variation process and \(X_t\) is continuous. Moreover, according to Proposition 4.3 and since the \((H)\)-hypothesis holds under \(\hat{P}\) thanks to Proposition 4.14, we know that \(\langle X, \hat{L}^g \rangle_{P, \mathcal{G}}^{\hat{P}, \mathcal{G}} = \langle X, L^g \rangle_{P, \mathcal{F}} = 0\).

We can now give the form of the pseudo-locally risk-minimizing portfolio of \(1_{\{\tau>T\}} \frac{g_T}{B_T}\).
Corollary 4.16. If the Conditions 2.3 and 4.1 hold, then the discounted value process \( \hat{V}_t^g(\hat{\rho}^*) \) of the locally risk-minimizing portfolio for the claim \( 1_{\{\tau > T\}} \) is given by:

\[
\hat{V}_t^g(\hat{\rho}^*) = 1_{\{\tau > t\}} e^{\Gamma_t} \hat{V}_t^{U^g}(\hat{\rho})
\]

and, on \( \{\tau > t\} \), the locally risk-minimizing strategy \( \hat{\rho}^* = (\hat{\phi}^*, \hat{\eta}^*) \) is given by:

\[
\hat{\rho}^*_t = e^{\Gamma_t} \left( \hat{\phi}_t, \hat{\eta}_t \right)
\]

where \( \hat{V}_t^{U^g}(\hat{\rho}) \) and \( \hat{\rho} = (\hat{\phi}, \hat{\eta}) \) are respectively the discounted value process and the strategy of the locally risk-minimizing portfolio of the purely financial claim \( \frac{q_T}{B_T} e^{-\Gamma t} \).

The cost process \( C_t^g(\hat{\rho}^*) \) is given by

\[
C_t^g(\hat{\rho}^*) = C_0^g(\hat{\rho}^*) + \hat{L}_t^g
\]

where \( C_0^g(\hat{\rho}^*) = \hat{U}_0^g \).

Proof. We simply have

\[
\hat{V}_t^g(\hat{\rho}^*) = E^{\hat{P}} \left[ 1_{\{\tau > T\}} \frac{q_T}{B_T} | \mathcal{G}_t \right]
\]

\[
= 1_{\{\tau > t\}} e^{\Gamma_t} E^{\hat{P}} \left[ e^{-\Gamma_t} \frac{q_T}{B_T} | \mathcal{F}_t \right]
\]

\[
= 1_{\{\tau > t\}} e^{\Gamma_t} \hat{V}_t^{U^g}(\hat{\rho})
\]

where in third equality, we use the fact the restriction of \( \hat{P} \) on \( \mathbb{F} \) and \( \mathbb{G} \) are indistinguishable. According to the previous theorem, on \( \{\tau > t\} \), we obviously have \( \hat{\phi}^* = e^{\Gamma_t} \hat{\phi} \). As far as \( \hat{\eta}^* \) is concerned, on \( \{\tau > t\} \), we have

\[
\hat{\eta}^*_t = \hat{V}_t^g(\hat{\rho}^*) - \hat{\phi}^*_t X_t
\]

\[
= e^{\Gamma_t} \hat{V}_t^{U^g}(\hat{\rho}) - e^{\Gamma_t} \hat{\phi}_t X_t
\]

\[
= e^{\Gamma_t} \hat{\eta}_t
\]

\( \square \)

4.3.3. Payment at the surrender time. We now study the pseudo locally risk-minimizing strategies of the discounted cumulative payment process \( \int_0^T \frac{R_u}{B_u} dH_u \).

Proposition 4.17. Let us assume Conditions 2.3 and 4.1 hold. If the \((\hat{P}, \mathbb{F})\)-Galtchouk-Kunita-Watanabe decomposition of the \((\hat{P}, \mathbb{F})\)-martingale \( \hat{U}_t^R := E^{\hat{P}} \left[ \int_0^T \frac{R_u}{B_u} dF_u | \mathcal{F}_t \right] \) is written

\[
E^{\hat{P}} \left[ \int_0^T \frac{R_u}{B_u} dF_u | \mathcal{F}_t \right] = \hat{U}_0^R + \int_0^t \hat{\psi}_u dX_u + \hat{L}_t^{U^R}
\]
then the $(P, \mathbb{G})$-Föllmer-Schweizer decomposition of $\int_0^T \frac{R_u}{B_u} dH_u$ is given by

\[
\int_0^T \frac{R_u}{B_u} dH_u = \hat{U}_0^R + \int_0^T 1_{\{\tau > u\}} e^{\Gamma_u} \hat{\psi}_u dX_u + \hat{L}_T^R
\]

where

\[
\hat{L}_T^R = \int_0^t 1_{\{\tau > u\}} e^{\Gamma_u} d\hat{L}_t^R + \int_0^t \left( \frac{R_u}{B_u} - e^{\Gamma_u} \hat{V}_u^R \right) dM_u
\]

and

\[
\hat{V}_t^R = E^{\hat{P}} \left[ \int_t^T \frac{R_u}{B_u} dF_u \big| \mathcal{F}_t \right]
\]

**Proof.** According to Theorem 3.14, we know that when $X_t$ is continuous, the $\mathbb{G}$-Föllmer-Schweizer decomposition of $\int_0^T \frac{R_u}{B_u} dH_u$ under the measure $P$, is given by the $\mathbb{G}$-Galtchouk-Kunita-Watanabe decomposition under the minimal martingale measure $\hat{P}$ of $E^{\hat{P}} \left[ \int_0^T \frac{R_u}{B_u} dH_u \big| \mathcal{G}_t \right]$. We have

\[
E^{\hat{P}} \left[ \int_0^T \frac{R_u}{B_u} dH_u \big| \mathcal{G}_t \right] = \int_0^t \frac{R_u}{B_u} dH_u + 1_{\{\tau > t\}} E^{\hat{P}} \left[ \int_t^T \frac{R_u}{B_u} dH_u \big| \mathcal{F}_t \right]

= \int_0^t \frac{R_u}{B_u} dH_u + 1_{\{\tau > t\}} E^{\hat{P}} \left[ \int_t^T \frac{R_u}{B_u} dF_u \big| \mathcal{F}_t \right]

= \int_0^t \frac{R_u}{B_u} dH_u - 1_{\{\tau > t\}} e^{\Gamma_t} \int_0^t \frac{R_u}{B_u} dF_u

+ 1_{\{\tau > t\}} e^{\Gamma_t} E^{\hat{P}} \left[ \int_0^T \frac{R_u}{B_u} dF_u \big| \mathcal{F}_t \right]
\]

In the second equality, we use the fact that according to (4.12), the $(\hat{P}, \mathbb{F})$-hazard process of $\tau$ is equal to $\Gamma_t$. Let us define $\hat{U}_t^R := E^{\hat{P}} \left[ \int_0^T \frac{R_u}{B_u} dF_u \big| \mathcal{F}_t \right]$. By hypothesis, its $(\hat{P}, \mathbb{F})$-Galtchouk-Kunita-Watanabe is given by

\[
\hat{U}_t^R = \hat{U}_0^R + \int_0^t \hat{\psi}_u dX_u + \hat{L}_t^R
\]

Using the integration by parts formula and the continuity of $F_t$, we can show that

\[
1_{\{\tau > t\}} e^{\Gamma_t} \int_0^t \frac{R_u}{B_u} dF_u = \int_0^t \frac{R_u}{B_u} d\Gamma_u \wedge \tau - \int_0^t \left( \int_0^u \frac{R_s}{B_s} dF_s \right) e^{\Gamma_u} dM_u
\]

We also have
1_{\{\tau>t\}}e^{\Gamma_t} \hat{U}_t^R = \int_0^t 1_{\{\tau>t-\}}e^{\Gamma_s}d\hat{U}_u^R - \int_0^t \hat{U}_u^R e^{\Gamma_s}dM_u + [1_{\{\tau>t\}}e^{\Gamma_t}, \hat{U}_t^R]_t

= \hat{U}_0^R + \int_0^t 1_{\{\tau>t-\}}e^{\Gamma_s}d\hat{U}_u^R - \int_0^t \hat{U}_u^R e^{\Gamma_s}dM_u

since \[1_{\{\tau>t\}}e^{\Gamma_t}, \hat{U}_t^R]_t = \hat{U}_0^R + \sum_{0<s\leq t} \Delta \hat{U}_s^R \Delta L_s = \hat{U}_0^R. \] We thus obtain

\[E^P \left[ \int_0^T \frac{R_u}{B_u}dH_u \left| \mathcal{G}_t \right. \right] = \hat{U}_0^R + \int_0^t 1_{\{\tau>u-\}}e^{\Gamma_u} \hat{V}_u^R \psi_u dX_u + \hat{L}_t^R \]

where \[\hat{L}_t^R = \int_0^t 1_{\{\tau>u-\}}e^{\Gamma_u} \hat{L}_u^R + \int_0^t \left( \frac{R_u}{B_u} - e^{\Gamma_u} \hat{V}_u^R \right) dM_u. \]

Following the same arguments than in Proposition 4.15, we can easily prove this decomposition is indeed the \((\hat{P}, \mathcal{G})-\text{GKW}\) decomposition of \[E^P \left[ \int_0^T \frac{R_u}{B_u}dH_u \left| \mathcal{G}_t \right. \right]. \]

We can easily find the form of the pseudo-locally risk-minimizing portfolio of \[\int_0^T \frac{R_u}{B_u}dH_u \] as in Corollary 4.16.

The payment \[\int_0^T 1_{\{\tau>u\}}dC_u \] can be studied in a similar fashion. This is left to the reader or see Proposition 4.20.

4.4. The Föllmer-Schweizer decomposition approach. When the \((H)\)-hypothesis holds, we can actually even remove the assumption that \(X_t\) is continuous, by studying directly the Föllmer-Schweizer decomposition. We only study the second and third payments. This is left to the reader.

4.4.1. Payment at the surrender time. We have the following proposition.

**Proposition 4.18.** If the Conditions 2.3 and 4.1 hold and if, for \(R_u, \mathcal{F}\)-predictable and \(P\)-square integrable, the Föllmer-Schweizer decomposition of \[\int_0^T \frac{R_u}{B_u}dF_u \] exists and is written

\[\int_0^T \frac{R_u}{B_u}dF_u = J_0^{UR} + \int_0^T \psi_u dX_u + L_T^{UR} \]

then the Föllmer-Schweizer decomposition of \(\int_0^T \frac{R_u}{B_u}dH_u \) is given by

\[\int_0^T \frac{R_u}{B_u}dH_u = J_0^{UR} + \int_0^T 1_{\{\tau>u\}}e^{\Gamma_u} \psi_u dX_u + L_T^R \]
where

\[ L_T^R = \int_0^T 1_{\{\tau > u\}} e^{\Gamma u} dL_u^R + \int_0^T \left( \frac{R_u}{B_u} - V_{u-}^R e^{\Gamma u} \right) dM_u \]

and

\[ V_t^{UR} = E^P \left[ \int_0^T \frac{R_u}{B_u} dF_u \bigg| \mathcal{F}_t \right] - E^P \left[ \int_0^T \psi_u dX_u \bigg| \mathcal{F}_t \right] \]

**Proof.** Let us define \( U_t^R := E^P \left[ \int_0^T \frac{R_u}{B_u} dF_u \bigg| \mathcal{F}_t \right] \). We have

\[ U_t^R = J_t^{UR} + \int_0^t \psi_u dX_u + L_t^R \]

where \( J_t^{UR} := J_0^{UR} + \int_0^t \psi_u dX_u + L_t^R \) and \( W_t^{UR} := E^P \left[ \int_0^T \psi_u dX_u \bigg| \mathcal{F}_t \right] \). On the other hand, we can write

\[
E^P \left[ \int_0^T \frac{R_u}{B_u} dH_u \bigg| \mathcal{G}_t \right] = \int_0^t \frac{R_u}{B_u} dH_u + E^P \left[ \int_t^T \frac{R_u}{B_u} dH_u \bigg| \mathcal{G}_t \right] \\
= \int_0^t \frac{R_u}{B_u} dH_u - 1_{\{\tau > t\}} e^{\Gamma t} \int_0^t \frac{R_u}{B_u} dF_u + 1_{\{\tau > t\}} e^{\Gamma t} E^P \left[ \int_0^T \frac{R_u}{B_u} dF_u \bigg| \mathcal{F}_t \right] \\
= \int_0^T \frac{R_u}{B_u} dH_u - 1_{\{\tau > t\}} e^{\Gamma t} \int_0^t \frac{R_u}{B_u} dF_u + 1_{\{\tau > t\}} e^{\Gamma t} J_t^{UR} + 1_{\{\tau > t\}} e^{\Gamma t} W_t^{UR}
\]

Using the same arguments than in Proposition 4.17, we can write that

\[
E^P \left[ \int_0^T \frac{R_u}{B_u} dH_u \bigg| \mathcal{G}_t \right] = J_0^{UR} + \int_0^t 1_{\{\tau > t\}} e^{\Gamma t} dJ_t^{UR} - \int_0^t \frac{R_u}{B_u} dM_u \\
- \int_0^t \left[ J_{u-}^{UR} - \left( \int_0^u \frac{R_s}{B_s} dF_s \right) \right] e^{\Gamma u} dM_u + 1_{\{\tau > t\}} e^{\Gamma t} W_t^{UR}
\]

Since \( J_t^{UR} - \int_0^t \frac{R_u}{B_u} dF_s = V_t^{UR} \) and since \( W_T^{UR} = 0 \) then for \( t = T \), we can write the last equation as

\[
\int_0^T \frac{R_u}{B_u} dH_u = J_0^{UR} + \int_0^T 1_{\{\tau > u\}} e^{\Gamma u} dJ_u^{UR} + \int_0^T \left( \frac{R_u}{B_u} - V_{u-}^R e^{\Gamma u} \right) dM_u
\]

Replacing the expression of \( J_t^{UR} \), we have

\[
\int_0^T \frac{R_u}{B_u} dH_u = J_0^{UR} + \int_0^T 1_{\{\tau > u\}} e^{\Gamma u} \psi_u dX_u + L_T^R
\]

where \( L_T^R := \int_0^T 1_{\{\tau > u\}} e^{\Gamma u} dL_u^{UR} + \int_0^T \left( \frac{R_u}{B_u} - V_{u-}^R e^{\Gamma u} \right) dM_u. \)
We still have to prove that this decomposition is indeed the Föllmer-Schweizer decomposition of \( \int_0^T \frac{R_u}{B_u} dH_u \). Since, thanks to the \((H)\)-hypothesis, the canonical decomposition of \( X_t = X_0 + N_t + A_t \) does not vary with the enlargement of filtration, we just have to show that \( L^g_t \) is strongly \((P, \mathbb{F})\)-orthogonal to \( N_t \) or equivalently that \( \langle L^g, N \rangle_{t}^{P, \mathbb{F}} = 0 \).

\[
\langle L^g, N \rangle_{t}^{P, \mathbb{F}} = \left\langle \int_0^t 1_{\{\tau > u\}} e^{G_u} dL^u - \int_0^t V^u \Gamma_u dM_u, N \right\rangle_t^{P, \mathbb{F}}
\]

On the one hand, we have \( \langle M, N \rangle_t^{P, \mathbb{F}} = 0 \) since we know \([M, N] = 0\). On the other hand, thanks to the definition of the Föllmer-Schweizer decomposition, we know the \((P, \mathbb{F})\)-martingales \( L^u \) and \( N_t \) are strongly \((P, \mathbb{F})\)-orthogonal so that \( \langle L^u, N \rangle_t^{P, \mathbb{F}} = 0 \). Thanks to Proposition 4.3, we know that \( \langle L^u, N \rangle_t^{P, \mathbb{G}} = \langle L^u, N \rangle_t^{P, \mathbb{F}} = 0 \) for all \( t \geq 0 \). We thus have \( \langle L^g, N \rangle_{t}^{P, \mathbb{G}} = 0 \).

We can now give the form of the pseudo-locally risk-minimizing portfolio of \( \int_0^T \frac{R_u}{B_u} dH_u \).

**Corollary 4.19.** If the Conditions 2.3 and 4.1 hold then the discounted value \( V_t^R(\rho^*) \) of the locally risk-minimizing portfolio for the payment \( \int_0^T \frac{R_u}{B_u} dH_u \) are respectively given by

\[
V_t^R(\rho^*) = 1_{\{\tau > t\}} e^{\Gamma_t} V_t^{U^R}(\rho)
\]

and, on \( \{\tau > t\} \), the locally risk-minimizing strategy \( \rho^* \) is given by:

\[
\rho^*_t = e^{\Gamma_t} (\psi_t, \eta_t)
\]

where \( V_t^{U^R}(\rho) \) and \( \rho_t = (\psi_t, \eta_t) \) are respectively the discounted value and the strategy of the locally risk-minimizing portfolio of the purely financial claim \( \int_0^T \frac{R_u}{B_u} dF_u \).

The cost process \( C^R(\rho^*) \) is given by

\[
C^R(\rho^*) = C_0^R(\rho^*) + I^R_t
\]

where \( C_0^R(\rho^*) = J_0^{U_t^R} \).

**Proof.** According to Equation (3.2), we have

\[
V_t^R(\rho^*) = E^P \left[ \int_t^T \frac{R_u}{B_u} dH_u - \int_t^T 1_{\{\tau > u\}} e^{\Gamma_u} \psi_u dX_u \big| \mathcal{G}_t \right]
\]

\[
= 1_{\{\tau > t\}} e^{\Gamma_t} E^P \left[ \int_t^T \frac{R_u}{B_u} dF_u \big| \mathcal{F}_t \right] - 1_{\{\tau > t\}} e^{\Gamma_t} E^P \left[ \int_t^T 1_{\{\tau > u\}} e^{\Gamma_u} \psi_u dA_u \big| \mathcal{F}_t \right]
\]

\[
= 1_{\{\tau > t\}} e^{\Gamma_t} E^P \left[ \int_t^T \frac{R_u}{B_u} dF_u \big| \mathcal{F}_t \right] - 1_{\{\tau > t\}} e^{\Gamma_t} E^P \left[ \int_t^T \psi_u dX_u \big| \mathcal{F}_t \right]
\]

\[
= 1_{\{\tau > t\}} e^{\Gamma_t} V_t^{U^R}
\]
In the third equality, we use the predictable projection theorem and the continuity of \( \Gamma_t \). For the rest of the proof, see Corollary 4.16.

4.4.2. Payments up to the surrender time. We now study the pseudo-locally risk-minimizing strategies of the discounted cumulative payment process \( \int_0^T 1_{\{\tau > u\}} \frac{1}{B_u} dC_u \) where \( C_t \) is assumed to be an \( \mathbb{F} \)-adapted square integrable process with respect to \( P \).

**Proposition 4.20.** If the Conditions 2.3 and 4.1 hold and if, for \( C_t \) \( \mathbb{F} \)-adapted and \( P \)-square integrable, the Föllmer-Schweizer decomposition of \( \int_0^T e^{-\Gamma_u} \frac{1}{B_u} dC_u \) exists and is written

\[
\int_0^T e^{-\Gamma_u} \frac{1}{B_u} dC_u = J_0^{UC} + \int_0^T \nu_u dX_u + L_T^{UC}
\]

then the Föllmer-Schweizer decomposition of \( \int_0^T 1_{\{\tau > u\}} \frac{1}{B_u} dC_u \) is given by

\[
\int_0^T 1_{\{\tau > u\}} \frac{1}{B_u} dC_u = J_0^{UC} + \int_0^T 1_{\{\tau > u\}} e^{\Gamma_u} \nu_u dX_u + L_T^{UC}
\]

where

\[
L_T^{UC} = \int_0^T 1_{\{\tau > u\}} e^{\Gamma_u} V_u^{UC} - \int_0^T V_u^{UC} e^{\Gamma_u} dM_u
\]

and

\[
V_t^{UC} = \mathbb{E}^P \left[ \int_t^T 1_{\{\tau > u\}} e^{\Gamma_u} dL_u^{UC} \right] - \mathbb{E}^P \left[ \int_t^T \nu_t dX_u \right] \mathbb{F}_t
\]

**Proof.** Let us define \( U_t^{UC} := \mathbb{E}^P \left[ \int_t^T \frac{1}{B_u} dC_u \right] \mathbb{F}_t \). We have

\[
U_t^{UC} = J_0^{UC} + \int_t^T \nu_u dX_u + L_t^{UC} + \mathbb{E}^P \left[ \int_t^T \nu_u dX_u \right] \mathbb{F}_t
\]

where \( J_t^{UC} := J_0^{UC} + \int_0^t \nu_u dX_u + L_t^{UC} \) and \( W_t^{UC} := \mathbb{E}^P \left[ \int_t^T \nu_u dX_u \right] \mathbb{F}_t \). On the other hand, we can write:

\[
\mathbb{E}^P \left[ \int_0^T 1_{\{\tau > u\}} \frac{1}{B_u} dC_u \right] \mathbb{F}_t = \int_0^t 1_{\{\tau > u\}} \frac{1}{B_u} dC_u - 1_{\{\tau > t\}} e^{\Gamma_t} \int_0^t e^{-\Gamma_u} \frac{1}{B_u} dC_u + 1_{\{\tau > t\}} e^{\Gamma_t} \int_0^T e^{-\Gamma_u} \frac{1}{B_u} dC_u + 1_{\{\tau > t\}} e^{\Gamma_t} J_t^{UC} + 1_{\{\tau > t\}} e^{\Gamma_t} W_t^{UC}
\]
Using the integration by parts formula, we can show
\[ 1_{\{\tau>t\}}e^{\Gamma_t} \int_0^t \frac{e^{-\Gamma_u}}{B_u} dC_u = \int_0^t 1_{\{\tau>u\}} e^{\Gamma_u} \frac{e^{-\Gamma_u}}{B_u} dC_u - \int_0^t \left( \int_0^u \frac{e^{-\Gamma_s}}{B_s} dC_s \right) e^{\Gamma_u} dM_u \]
\[ + \left[ 1_{\{\tau>t\}} e^{\Gamma_t} \int_0^\tau \frac{e^{-\Gamma_u}}{B_u} dC_u \right] \]
\[ = \int_0^t 1_{\{\tau>u\}} e^{\Gamma_u} \frac{e^{-\Gamma_u}}{B_u} dC_u - \int_0^t \left( \int_0^u \frac{e^{-\Gamma_s}}{B_s} dC_s \right) e^{\Gamma_u} dM_u \]
\[ = \int_0^t 1_{\{\tau>u\}} \frac{1}{B_u} dC_u - \int_0^t \left( \int_0^u \frac{e^{-\Gamma_s}}{B_s} dC_s \right) e^{\Gamma_u} dM_u \]
As in other propositions, we also have
\[ 1_{\{\tau>t\}} e^{\Gamma_t} J_t^{UC} = J_0^{UC} + \int_0^t L_u - dJ_t^{UC} - \int_0^t J_t^{UC} e^{\Gamma_u} dM_u \]
We thus obtain
\[ E^P \left[ \int_0^T 1_{\{\tau>u\}} \frac{1}{B_u} dC_u \mid G_t \right] = J_0^{UC} + \int_0^t 1_{\{\tau>u\}} e^{\Gamma_u} dJ_t^{UC} \]
\[ - \int_0^t \left( J_t^{UC} - \int_0^t \frac{e^{-\Gamma_s}}{B_s} dC_s \right) e^{\Gamma_u} dM_u + 1_{\{\tau>t\}} e^{\Gamma_t} W_t^{UC} \]
Since \( J_t^{UC} - \int_0^t \frac{e^{-\Gamma_s}}{B_s} dC_s = V_t^{UC} \) and since \( W_T = 0 \) then for \( t = T \), we can write
\[ \int_0^T 1_{\{\tau>u\}} \frac{1}{B_u} dC_u = J_0^{UC} + \int_0^T 1_{\{\tau>u\}} e^{\Gamma_u} dJ_t^{UC} - \int_0^T V_t^{UC} e^{\Gamma_u} dM_u \]
Replacing the expression of \( J_t^{UC} \), we obtain
\[ \int_0^T 1_{\{\tau>u\}} \frac{1}{B_u} dC_u = J_0^{UC} + \int_0^T 1_{\{\tau>u\}} e^{\Gamma_u} \nu_u dX_u + L_T^{UC} \]
where \( L_T^{UC} = \int_0^T 1_{\{\tau>u\}} e^{\Gamma_u} dL_t^{UC} - \int_0^T V_t^{UC} e^{\Gamma_u} dM_u \). Following the same arguments than in Proposition 4.18, we can easily prove this equation corresponds to the Föllmer-Schweizer decomposition of \( J_0^{T} 1_{\{\tau>u\}} dC_u \).

We can easily find the form of the pseudo-locally risk-minimizing portfolio of \( \int_0^T 1_{\{\tau>u\}} dC_u \) as in Corollary 4.19. This is left to the reader.

Notice that, when \( X_t \) is continuous, the two methods obviously leads to the same decomposition. To see this, compare Propositions 4.17 and 4.18, for examples. Indeed, the integrand \( \psi \) and the martingale \( L^{UR} \) in the Föllmer Schweizer decomposition under \( P \) of the \( \mathbb{F} \)-adapted process \( \int_0^T \frac{R_u}{B_u} dF_u \), are respectively, the same than the integrand \( \dot{\psi} \) and the martingale \( \dot{L}^{UR} \) of the GKW decomposition under \( \tilde{P} \) since according to Theorem 4.11, the restriction of the minimal martingale \( \tilde{P} \) (defined initially for the filtration \( \mathcal{G} \)) corresponds to the minimal martingale measure defined for the filtration \( \mathbb{F} \).
5. Application to a Life Insurance Contract When the \((H)\)-Hypothesis Does Not Hold.

In this section, we study the pseudo local risk-minimizing strategies of the payments described in Section 2.3.1 when we remove the \((H)\)-hypothesis.

In Section 5.1, we first introduce some preliminary results. In Section 2.1, we study the impact of the enlargement of filtration on the financial market and in Section 5.3, its impact on the minimal martingale measure. Finally, in Section 5.4, we study the pseudo locally risk-minimizing strategies in themselves.

As in the previous section, we assume the \((P,\mathcal{F})\)-hazard process \(\Gamma_t\) of \(\tau\) is continuous. We also introduce the following condition.

**Condition 5.1.** For any \(\mathcal{F}\)-stopping time \(\theta\), \(P(\tau = \theta) = 0\). \(^6\)

According to Blanchet-Scalliet and Jeanblanc \([30]\), under this condition, for any \((P,\mathcal{F})\)-martingale \(m_t\), the stopped process

\[
\tilde{m}_{t \wedge \tau} = m_{t \wedge \tau} + \int_0^{t \wedge \tau} e^{\Gamma_s} d[m_s,Y]_s
\]

is a \((P,\mathcal{G})\)-martingale. We call \(\tilde{m}_{t \wedge \tau}\) the \((P,\mathcal{G})\)-martingale part of \(m_{t \wedge \tau}\). We keep this notation in the rest of the chapter. In particular, here, the \((P,\mathcal{G})\)-martingale part of the square integrable \((P,\mathcal{F})\)-martingale \(Y\) can be written

\[
\tilde{Y}_{t \wedge \tau} = Y_{t \wedge \tau} + \int_0^{t \wedge \tau} e^{\Gamma_s} d[Y_s,Y]_s
\]

### 5.1. Preliminary results.

Before studying the Föllmer-Schweizer decomposition of our payments, we introduce some preliminary results.

**Proposition 5.2.** Let \(m_t\) be an \((P,\mathcal{F})\)-martingale such that \(\langle m.,m.\rangle_{t \wedge \tau}^{P,\mathcal{F}}\) exists then \(\langle m.,m.\rangle_{t \wedge \tau}^{P,\mathcal{G}}\) and \(\langle m.,m.\rangle_{t \wedge \tau}^{P,\mathcal{F}}\) are \(P\)-indistinguishable.

**Proof.** By definition of the sharp bracket, we have that \(r_t := [m.,m.]_t - \langle m.,m.\rangle_t^{P,\mathcal{F}}\) is a \((P,\mathcal{F})\)-(local) martingale. We thus have that

\[
\tilde{r}_{t \wedge \tau} = r_{t \wedge \tau} + \int_0^{t \wedge \tau} e^{\Gamma_s} d[r_s,Y]_s
\]

is locally a \((P,\mathcal{G})\)-martingale. Since \(r_t\) is a finite variation process and \(Y_t\) is continuous, we have \([r.,Y]_t = \sum_{0 \leq s \leq t} \Delta r_s \Delta Y_s = 0\). Accordingly, \(r_{t \wedge \tau} = [m.,m.]_{t \wedge \tau} - \langle m.,m.\rangle_{t \wedge \tau}^{P,\mathcal{F}}\) is a \((P,\mathcal{G})\)-(local) martingale. By definition and uniqueness of the sharp bracket, we thus have \(\langle m.,m.\rangle_{t \wedge \tau}^{P,\mathcal{F}} = \langle m.,m.\rangle_{t \wedge \tau}^{P,\mathcal{G}}\) \(P\)-a.s. \(\square\)

**Proposition 5.3.** Let \(m_t\) be a \((P,\mathcal{F})\)-martingale such that \(\langle m.,m.\rangle_t^{P,\mathcal{F}}\) exists, then

\(\langle \tilde{m}_{t \wedge \tau},\tilde{m}_{t \wedge \tau}\rangle_t^{P,\mathcal{G}}\) and \(\langle m_{t \wedge \tau},m_{t \wedge \tau}\rangle_t^{P,\mathcal{G}}\) are \(P\)-indistinguishable.

\(^6\)Notice that in the previous section, this condition was implied by the continuity of the hazard process and the \((H)\)-hypothesis. See Proposition 4.10.
Proof. we have \([\tilde{m}_{\Lambda T}, \tilde{m}_{\Lambda T}] = [m_{\Lambda T}, m_{\Lambda T}]\) since \(\int_0^{T \wedge T} e^{\Gamma_s} d[m_., Y]_s\) is a continuous finite variation process. By the definition of the sharp bracket and its uniqueness, we have the result.

**Corollary 5.4.** Let \(m_t\) be a \((P, F)\)-martingale such that \([m_., m_.]_t^{P, F}\) exists, then \([m_{\Lambda T}, m_{\Lambda T}]_t^{P, F}\) and \([\tilde{m}_{\Lambda T}, \tilde{m}_{\Lambda T}]_t^{P, G}\) are \(P\)-indistinguishable.

Proof. Combine Proposition 5.3 and Proposition 5.2.

**Corollary 5.5.** Let \(m_t\) and \(n_t\) be two square integrable \((P, F)\)-martingales strongly \((P, F)\)-orthogonal. Then \(\tilde{m}_{\Lambda T}\) and \(\tilde{n}_{\Lambda T}\) are \((P, G)\)-strongly orthogonal.

Proof. We have to show that \([\tilde{m}_{\Lambda T}, \tilde{n}_{\Lambda T}]_t^{P, G} = 0 P\)-a.s. for all \(t\). Since, by hypothesis, \(m_t\) and \(n_t\) are strongly \((P, F)\)-orthogonal, we know that \([m_., n_.]_t^{P, F} = 0\). Thanks to the previous corollary, we have that \([\tilde{m}_{\Lambda T}, \tilde{m}_{\Lambda T}]_t^{P, G} = [m_{\Lambda T}, m_{\Lambda T}]_t^{P, F} = [m_., m_.]_t^{P, F} = 0\).

5.2. The financial market. In this section, we want to check that on \(\{\tau > t\}\), the dynamics of the financial market is not “too much” altered by the enlargement of filtration. First, we want to check that \(X_{T \wedge T} \in S^2_{\text{loc}}(P, G)\). Second, we have to study if the structure condition still holds under the new filtration.

**Proposition 5.6.** \(X_{T \wedge T} \in S^2_{\text{loc}}(P, G)\) and its \((P, G)\)-canonical decomposition is given by \(X_{T \wedge T} = X_0 + \tilde{N}_{T \wedge T} + \tilde{A}_{T \wedge T}\) where

\[
\tilde{N}_{T \wedge T} = N_{T \wedge T} + \int_0^{T \wedge T} e^\Gamma_s d[N_., Y]_s
\]

\[
\tilde{A}_{T \wedge T} = A_{T \wedge T} - \int_0^{T \wedge T} e^\Gamma_s d[N_., Y]_s
\]

Proof. Since \(N_t\) is locally a martingale, we know that \(\tilde{N}_{T \wedge T} = N_{T \wedge T} + \int_0^{T \wedge T} e^\Gamma_s d[N_., Y]_s\) is locally a \((P, G)\)-martingale. Furthermore, since \(Y_t\) is continuous, \([N_., Y]_t\) is continuous and so is \(\int_0^{T \wedge T} e^\Gamma_s d[N_., Y]_s\). Since any continuous process is also locally square integrable and since \(N_t\) is locally square integrable, it implies \(\tilde{N}_{T \wedge T}\) is (locally) a locally square integrable martingale and thus a locally square integrable martingale.

On the other hand, since \(\int_0^{T \wedge T} e^\Gamma_s d[N_., Y]_s\) is continuous, its (finite) variation is also continuous and thus locally square integrable. Since the variation of \(A_t\) is locally square integrable, it implies \(\tilde{A}_{T \wedge T}\) is a predictable finite variation process whose variation is locally square integrable.

Before studying if the structure condition holds, we need the following proposition.

**Proposition 5.7.** \(\tilde{A}_{T \wedge T} \ll \langle \tilde{N}_., \tilde{N}_. \rangle_{T \wedge T}^{P, G}\).

Proof. Since \(Y_t\) is a square integrable \((P, F)\)-martingale, its \((P, F)\)-Galtchouk-Kunita-Watanabe decomposition exists with respect to each \(N_i^\circ\) for \(i = 1, \ldots, s\). In other words, for
each $i = 1, \ldots, s$, there exists an $(P, \mathbb{F})$-predictable process $h^i_t \in L^2(N^i)$ such that:

$$Y_t = Y_0 + \int_0^t h^i_u dN^i_u + L^Y_t$$

for each $t$ and such that, for each $i = 1, \ldots, s$, $L^Y_i \in \mathcal{M}_0^2(P, \mathbb{F})$ and is $(P, \mathbb{F})$-strongly orthogonal to $N^i_t$. We thus have $\langle N^i, Y^j \rangle^P = \int_0^t h^i_u d\langle N^i, N^j \rangle^P_u$.

On the other hand, by definition of $\tilde{A}^i_{t \wedge \tau}$, we can write

$$\tilde{A}^i_{t \wedge \tau} = A^i_{t \wedge \tau} - \int_0^{t \wedge \tau} e^{\Gamma_u} d\langle N^i, Y \rangle_u$$

$$= \int_0^{t \wedge \tau} \alpha^i_u d\langle N^i, N^i \rangle^P_u - \int_0^{t \wedge \tau} e^{\Gamma_u} d\langle N^i, Y \rangle^P_u$$

$$= \int_0^{t \wedge \tau} (\alpha^i_u - e^{\Gamma_u h^i_u}) d\langle N^i, N^i \rangle^P_u$$

Thanks to Proposition 5.2, we know that, on $\{ \tau > t \}$, $\langle N^i, N^i \rangle^P = \langle N^i, N^i \rangle^P_t$. So eventually, we have

$$\tilde{A}^i_{t \wedge \tau} = \int_0^{t \wedge \tau} (\alpha^i_u - e^{\Gamma_u h^i_u}) d\langle N^i, N^i \rangle^P_u$$

Since $\langle N^i, N^i \rangle^P_{t \wedge \tau} = \langle N^i, N^i \rangle^P_t$, we can easily see that $\langle N^i, N^i \rangle^P_{t \wedge \tau} \ll \tilde{B}_{t \wedge \tau}$ for all $i, j = 1, \ldots, s$ and that the predictable density of $\langle N^i, N^j \rangle^P_{t \wedge \tau}$ with respect to $\tilde{B}_{t \wedge \tau}$ is equal to $\sigma^{ij}_t$ i.e.

$$\langle N^i, N^i \rangle^P_{t \wedge \tau} = \int_0^{t \wedge \tau} \sigma^{ij}_u d\tilde{B}_u$$

P-a.s. for all $t$ and all $i, j = 1, \ldots, s$. We can then write

$$\tilde{A}^i_{t \wedge \tau} = \int_0^{t \wedge \tau} \tilde{\gamma}^i_u dB_u$$

where $\tilde{\gamma}^i = \tilde{\alpha}^i \sigma^{ii}$ P-a.s. for all $t$ and all $i = 1, \ldots, s$. We denote by $\tilde{\gamma}$ the s-dimensional vector whose elements are $\tilde{\gamma}^i$ for $i = 1, \ldots, s$.

We can now show that the structure condition still holds after the enlargement of filtration (at least on $\{ \tau > t \}$).

**Proposition 5.8.** If $X_t$ satisfies the structure condition for $\mathbb{F}$ and if Conditions 2.3 and 5.1 hold, then $X_{t \wedge \tau}$ satisfies the structure condition for $\mathbb{G}$ i.e. there exists an $\mathbb{R}^s$-valued $\mathbb{G}$ predictable process $\tilde{\lambda} \in L^2_{loc}(\tilde{N}_{\wedge \tau})$ such that, on $\{ \tau > t \}$,

$$\sigma_t \tilde{\lambda}_t = \tilde{\gamma}_t$$
PROOF. Since $Y_t$ is a square integrable martingale, we can find its $(P,F)$-Galtchouk-Kunita-Watanabe decomposition with respect to the $s$-dimensional vector $N_t$, i.e. there exists an $s$-dimensional predictable process $g_t \in L^2(N)$ such that

$$Y_t = Y_0 + \int_0^t g_u dN_u + L^Y_t$$

$P$-a.s. for all $t$, and such that $L^Y_t \in \mathcal{M}^2_0(P,F)$ and is $(P,F)$-strongly orthogonal to $I^2(N)$, the stable subspace generated by $N$. We can thus write for each $i = 1, \ldots s$:

$$\langle Y, N^i \rangle_t^{PF} = \sum_{j=1}^{s} \int_0^t g_{ju} \langle N^j, N^i \rangle_t^{PF} + \langle L^Y, N^i \rangle_t^{PF}$$

$$= \sum_{j=1}^{s} \int_0^t g_{ju} \langle N^j, N^i \rangle_t^{PF}$$

$$= \int_0^t \left( \sum_{j=1}^{s} g_{ju} \sigma_{ji} \right) dB_u$$

On the other hand, we already saw $Y_t$ could be decomposed in the following way for each $i = 1, \ldots s$:

$$Y_t = Y_0 + \int_0^t h_{iu} dN^i_u + L^Y_t$$

where $h_{iu} \in L^2(N^i)$ and $L^Y_t \in \mathcal{M}^2_0(P,F)$ and is $(P,F)$-strongly orthogonal to $N^i_t$. We can thus write for each $i = 1, \ldots s$:

$$\langle Y, N^i \rangle_t^{PF} = \int_0^t h_{iu} \langle N^i, N^i \rangle_t^{PF}$$

$$= \int_0^t h_{iu} \sigma_{ii} dB_t$$

Accordingly, if we denote by $\tilde{h}_t$ the $s$-predictable process whose elements are $\tilde{h}^i_t = h^i_t \sigma^i_t$ for $i = 1, \ldots s$, we can write $\tilde{h}_t = \sigma_t g_t$.

Finally, on $\{ \tau > t \}$, since $\tilde{\gamma}^i_t = \tilde{\alpha}^i_t \sigma^i_t = \alpha^i_t \sigma^i_t - \epsilon^i_t h^i_t \sigma^i_t$, we have

$$\tilde{\gamma} = \sigma_t \tilde{\lambda}_t - \epsilon^i_t \sigma_t g_t$$

where $\tilde{\lambda}_t = \tilde{\lambda}_t - \epsilon^i_t g_t$.

It remains to show that $\tilde{\lambda} \in L^2_\text{loc}(\tilde{N},\mathcal{A})$. By hypothesis $\tilde{\lambda}_t \in L^2_\text{loc}(N)$. By definition of the GKW decomposition, $g_t \in L^2_\text{loc}(N)$ and since $\epsilon^i_t$ is locally bounded, we can even say that $\epsilon^i_t g_t \in L^2_\text{loc}(N)$. Accordingly, $\tilde{\lambda}_t \in L^2_\text{loc}(N)$. Now, thanks to Corollary 5.4, we know
\( \langle N_{i \wedge \tau}, N_{i \wedge \tau} \rangle_t^{P,E} = \langle \tilde{N}_{i \wedge \tau}, \tilde{N}_{i \wedge \tau} \rangle_t^{P,G} \) for each \( i = 1, \ldots s \). We can then conclude that \( \tilde{\lambda}_t \in L^2_{\text{loc}}(\tilde{\mathcal{N}}_{\wedge \tau}) \). \( \square \)

5.3. The minimal martingale measure. As in Section 4.3, we study the impact of the enlargement of filtration on the minimal martingale measure. When they exist, we denote by \( \hat{P}^F \) and \( \hat{P}^G \) the minimal martingales measures for, respectively, the filtration \( F \) and \( G \). We denote by \( \hat{Z}^F_t := \frac{d \hat{P}^F}{d P}|_{\mathcal{F}_t} \), the restriction, to the filtration \( \mathcal{F} \), of the Radon-Nikodym derivative of \( \hat{P}^F \) with respect to \( P \) and by \( \hat{Z}^G_t := \frac{d \hat{P}^G}{d P}|_{\mathcal{G}_t} \), the restriction to the filtration \( \mathcal{G} \), of the Radon-Nikodym derivative of \( \hat{P}^G \) with respect to \( P \). When the \((H)\)-hypothesis does not hold, these densities do not necessarily coincide. We have the following candidate for the minimal martingale measure with respect to the enlarged filtration \( \mathcal{G} \).

**Proposition 5.9.** When the minimal martingale measure \( \hat{P}^G \) exists, its density is given, on \( \{ \tau > t \} \), by

\[
\hat{Z}^G_{t \wedge \tau} = \mathcal{E}_t \left( -\int_0^t \tilde{\lambda}'_u d \tilde{N}_{u \wedge \tau} \right)
\]

Furthermore, if \( X_t \) is continuous, we have

\[
\hat{Z}^G_{t \wedge \tau} = \frac{\hat{Z}^F_{t \wedge \tau}}{\mathcal{E}_{t \wedge \tau} \left( -\int_0^t \left( \lambda_u - \tilde{\lambda}_u \right)' d N_u \right)}
\]

**Proof.** Since \( X_{t \wedge \tau} \) satisfies the structure condition, we directly know that if the minimal martingale measure exists, then on \( \{ \tau > t \} \), we have

\[
\hat{Z}^G_{t \wedge \tau} = \mathcal{E}_t \left( -\int_0^t \tilde{\lambda}'_u d N_u \right)
\]

As far as the second equation is concerned, when \( X_t \) is continuous, we can write

\[
\hat{Z}^G_{t \wedge \tau} = e^{-\frac{1}{2} \int_0^t \tilde{\lambda}'_u d \langle \tilde{N}_{\wedge \tau}, \tilde{N}_{\wedge \tau} \rangle_u^{P,G} \tilde{\lambda}_u - f_0^t \tilde{\lambda}_u d \tilde{N}_{u \wedge \tau}}
\]

\[
= e^{-\frac{1}{2} \int_0^t \tilde{\lambda}'_u d \langle N_{\wedge \tau}, N_{\wedge \tau} \rangle_u^{P,P} \tilde{\lambda}_u - f_0^t \tilde{\lambda}_u d N_{u \wedge \tau} - f_0^t \tilde{\lambda}_u d N_{u \wedge \tau} - f_0^t \tilde{\lambda}_u d N_{u \wedge \tau} - f_0^t \tilde{\lambda}_u d N_{u \wedge \tau} + f_0^t \tilde{\lambda}_u d N_{u \wedge \tau} + f_0^t \tilde{\lambda}_u d N_{u \wedge \tau} + f_0^t \tilde{\lambda}_u d N_{u \wedge \tau}}
\]

\[
= e^{-\frac{1}{2} \int_0^t \tilde{\lambda}'_u d \langle N_{\wedge \tau}, N_{\wedge \tau} \rangle_u^{P,P} \tilde{\lambda}_u - f_0^t \tilde{\lambda}_u d N_{u \wedge \tau} - f_0^t \tilde{\lambda}_u d N_{u \wedge \tau} + f_0^t \tilde{\lambda}_u d N_{u \wedge \tau} + f_0^t \tilde{\lambda}_u d N_{u \wedge \tau} + f_0^t \tilde{\lambda}_u d N_{u \wedge \tau} + f_0^t \tilde{\lambda}_u d N_{u \wedge \tau}}
\]

\[
= \hat{Z}^F_{t \wedge \tau} e^{\frac{1}{2} f_0^t \tilde{\lambda}_u d \langle N_{\wedge \tau}, N_{\wedge \tau} \rangle_u^{P,P} \tilde{\lambda}_u + f_0^t \tilde{\lambda}_u d N_{u \wedge \tau} + f_0^t \tilde{\lambda}_u d N_{u \wedge \tau}} + f_0^t \tilde{\lambda}_u d N_{u \wedge \tau} + f_0^t \tilde{\lambda}_u d N_{u \wedge \tau} + f_0^t \tilde{\lambda}_u d N_{u \wedge \tau} + f_0^t \tilde{\lambda}_u d N_{u \wedge \tau}}
\]

In the second equality, we use the fact that \( \langle N_{\wedge \tau}, N_{\wedge \tau} \rangle_t^{P,E} = \langle \tilde{N}_{\wedge \tau}, \tilde{N}_{\wedge \tau} \rangle_t^{P,G} \) and in the fourth, the fact that \( Y_t \) can be written as \( Y_t = Y_0 + \int_0^t g'_u d N_u + L_\Gamma^Y \). Eventually, we get the
result if we notice
\[ \mathcal{E}_t \left( - \int_0^t e^{\Gamma_u} g_u' dN_{u \wedge \tau} \right) = e^{-\frac{1}{2} \int_0^t e^{\Gamma_u} \left( \frac{dN_{u \wedge \tau}}{u} \right) P_e^{u} e^{\Gamma_u} g_u' dN_{u \wedge \tau} } \]

From now on, we assume \( \hat{Z}_t^G \in L^2(P) \) so that, on \( \{ \tau > t \} \), \( \hat{Z}_t^G \) is indeed the density of the minimal martingale measure \( \hat{P}_G \). An important point to notice is that the restriction on \( \mathbb{F} \) of this minimal martingale measure is not equal to \( \hat{Z}_t^F \). The minimal martingale measure \( \hat{P}_G \) is thus not simply an extension of \( \hat{P}_F \) but a genuine different martingale measure.

We now introduce the following definition.

**Definition 5.10.** When \( X_t \) is continuous, we define the \( \mathbb{F} \)-adapted process \( \Upsilon_t \) by
\[ e^{\Upsilon_t} = \mathcal{E}_t \left( - \int_0^t \left( \lambda_u - \tilde{\lambda}_u \right)' dN_u \right) \]

Notice that since \( \mathcal{E}_t \left( - \int_0^t \left( \lambda_u - \tilde{\lambda}_u \right)' dN_u \right) > 0 \) when \( X_t \) is continuous, the process \( \Upsilon_t \) is here well-defined.

### 5.4. The pseudo locally risk-minimizing strategies.

We now study the pseudo-locally risk-minimizing strategies of our insurance payments when \( X_t \) is continuous. We still need the following result.

**Proposition 5.11.** When \( \Gamma_t \) and \( X_t \) are continuous, the process \( 1_{\{\tau > t\}} e^{(\Gamma_t + \Upsilon_t)} \) can be written as

\[ 1_{\{\tau > t\}} e^{(\Gamma_t + \Upsilon_t)} = 1 - \int_0^t e^{(\Gamma_u + \Upsilon_u)} dM_u + \int_0^t 1_{\{\tau > u\}} e^{(\Gamma^2_u + \Upsilon_u)} d\tilde{Y}_{u \wedge \tau} \]

or
\[ 1_{\{\tau > t\}} e^{(\Gamma_t + \Upsilon_t)} = 1 - \int_0^t e^{(\Gamma_u + \Upsilon_u)} g_u' d\tilde{N}_{u \wedge \tau} \]

(5.2)

\[ \text{or} \]

(5.3)

\[ 1_{\{\tau > t\}} e^{(\Gamma_t + \Upsilon_t)} = 1 - \int_0^t e^{(\Gamma_u + \Upsilon_u)} dM_u + \int_0^t 1_{\{\tau > u\}} e^{(\Gamma^2_u + \Upsilon_u)} d\tilde{L}_{u \wedge \tau} \]

(5.3)

where \( \tilde{L}_{t \wedge \tau} = L_{t \wedge \tau} + \int_0^{t \wedge \tau} e^{\Gamma_u} d \left[ Y, L^Y \right]_u \).

**Proof.** On one hand, we can prove (or see Blanchet-Scalliet) that, when \( \Gamma_t \) is continuous, we have
\[ 1_{\{\tau > t\}} e^{\Gamma_t} = 1 - \int_0^t e^{\Gamma_u} dM_u + \int_0^t 1_{\{\tau > u\}} e^{\Gamma^2_u} d\tilde{Y}_u \]
On the other hand, by definition, we have that

\[ e^{\Upsilon_t} = \mathcal{E}_t \left( - \int_0^t \left( \lambda_u - \tilde{\lambda}_u \right)' dN_u \right) = 1 - \int_0^t e^{(\Gamma_u + \Upsilon_u)} g_u' dN_u \]

Using the integration by parts formula, we can find Equation (5.2)

\[
1_{\{\tau > t\}} e^{(\Gamma_{\tau} + \Upsilon_{\tau})} = 1 - \int_0^t e^{(\Gamma_u + \Upsilon_u)} dM_u + \int_0^t 1_{\{\tau > u\}} e^{(\Gamma_u^2 + \Upsilon_u)} d\tilde{Y}_{u\wedge \tau} - \int_0^t 1_{\{\tau > u\}} e^{(\Gamma_u^2 + \Upsilon_u)} g_u' dN_u \\
- \int_0^t 1_{\{\tau > u\}} e^{\Gamma_u^2} e^{(\Gamma_u + \Upsilon_u)} g_u' d\left[ \tilde{Y}_{\wedge \tau}, N \right]_u \\
= 1 - \int_0^t e^{(\Gamma_u + \Upsilon_u)} dM_u + \int_0^t 1_{\{\tau > u\}} e^{(\Gamma_u^2 + \Upsilon_u)} d\tilde{Y}_{u\wedge \tau} - \int_0^t 1_{\{\tau > u\}} e^{(\Gamma_u^2 + \Upsilon_u)} g_u' dN_u\wedge \tau \\
- \int_0^t 1_{\{\tau > u\}} e^{(\Gamma_u^2 + \Upsilon_u)} g_u' d\left[ \int_0^{u\wedge \tau} e^{\Gamma_s} d[Y_s, N_s] \right] \\
= 1 - \int_0^t e^{\Gamma_u + \Upsilon_u} dM_u + \int_0^t 1_{\{\tau > u\}} e^{(\Gamma_u^2 + \Upsilon_u)} d\tilde{Y}_{u\wedge \tau} - \int_0^t 1_{\{\tau > u\}} e^{(\Gamma_u^2 + \Upsilon_u)} g_u' d\tilde{Y}_{u\wedge \tau} \\
where in the second equality, we use the fact \( \int_0^{u\wedge \tau} e^{\Gamma_s} d[Y_s, N_s] = 0 \) since \( [Y_s, N_s] \) is a continuous finite variation process.

To find Equation (5.3), we have to notice that

\[ \tilde{Y}_{t\wedge \tau} = Y_0 + \int_0^t g_u' d\tilde{N}_{u\wedge \tau} + \tilde{L}_{t\wedge \tau} \]

\[ 5.4.1. \text{Payment at the term of the contract.} \text{ We first need the following result.} \]

\[ \text{Proposition 5.12. If we denote by } \hat{\mathcal{P}}^G \text{ and } \hat{\mathcal{P}}^F \text{ the minimal martingale measure with respect to } \mathcal{G} \text{ and to } \mathcal{F}, \text{ we have the following relationship} \]

\[ E^{\hat{\mathcal{P}}^G} \left[ 1_{\{\tau > T\}} \frac{g_T}{B_T} \mid \mathcal{G}_t \right] = 1_{\{\tau > t\}} e^{(\Gamma_{\tau} + \Upsilon_{\tau})} E^{\hat{\mathcal{P}}^F} \left[ e^{-(\Gamma_T + \Upsilon_T)} \frac{g_T}{B_T} \mid \mathcal{F}_t \right] \]
PROOF. We have

\[
E^{\hat{P}_F}\left[\mathbb{1}_{\{\tau>T\}} \frac{g_T}{B_T} \big| G_t \right] = \frac{E^{\hat{P}_F}\left[\hat{Z}_T^g \mathbb{1}_{\{\tau>T\}} \frac{g_T}{B_T} \big| G_t \right]}{\hat{Z}_t^g} \\
= \mathbb{1}_{\{\tau>t\}} \frac{E^{\hat{P}_F}\left[\hat{Z}_T^g e^{-\Gamma_T} \mathbb{1}_{\{\tau>T\}} \frac{g_T}{B_T} \big| F_t \right]}{\hat{Z}_t^g e^{-\Gamma_t}} \\
= \mathbb{1}_{\{\tau>t\}} \frac{E^{\hat{P}_F}\left[\hat{Z}_T^g e^{-\Gamma_T} \mathbb{1}_{\{\tau>T\}} \frac{g_T}{B_T} \big| F_t \right]}{\hat{Z}_t^g e^{-(\Gamma_t + \Upsilon_t)}} \\
= \mathbb{1}_{\{\tau>t\}} e^{(\Gamma_t + \Upsilon_t)} \mathbb{E}^{\hat{P}}\left[\frac{e^{-\Gamma_T} g_T}{B_T} \big| F_t \right]
\]

□

The Föllmer-Schweizer decomposition of \(\mathbb{1}_{\{\tau>T\}} \frac{g_T}{B_T}\) is given in the following proposition.

**Proposition 5.13.** If, for \(g_T, F_T\)-adapted and \(P\)-square integrable, the \((\hat{P}_F, \mathbb{F})\)-Galtchouk-Kunita-Watanabe decomposition of the \((\hat{P}_F, \mathbb{F})\)-martingale \(U_t^g := E^{\hat{P}_F}\left[\frac{g_T}{B_T} e^{-(\Gamma_T + \Upsilon_T)} \big| F_t \right]\) is written

\[
E^{\hat{P}_F}\left[\frac{g_T}{B_T} e^{-(\Gamma_T + \Upsilon_T)} \big| F_t \right] = U_0^g + \int_0^T \phi_u dX_u + L_{T}^{U^g}
\]

Then the \((P, G)\)-Föllmer-Schweizer decomposition of \(\mathbb{1}_{\{\tau>T\}} \frac{g_T}{B_T}\) is given by

\[
\mathbb{1}_{\{\tau>T\}} \frac{g_T}{B_T} = U_0^g + \int_0^T \mathbb{1}_{\{\tau>u\}} e^{(\Gamma_u + \Upsilon_u)} \phi_u dX_u + \tilde{L}_T^g
\]

where

\[
\tilde{L}_t^g = \int_0^t \mathbb{1}_{\{\tau>u\}} e^{(\Gamma_u + \Upsilon_u)} d\tilde{L}_u^{U^g} + \int_0^t \mathbb{1}_{\{\tau>u\}} V_{u-}^{U^g} e^{(\Gamma_u^2 + \Upsilon_u)} d\tilde{Y}_u
\]

\[
- \int_0^t V_{u-}^{U^g} e^{(\Gamma_u + \Upsilon_u)} dM_u
\]

and

\[
V_t^{U^g} = U_t^g
\]

What is interesting here is that, again, finding the locally risk-minimizing strategies of our contingent claims comes down to finding the locally risk-minimizing strategy of a modified financial claims. Notice that, somewhat surprisingly, the solution involved the Galtchouk-Kunita-Watanabe decomposition of a modified claim but under the minimal martingale measure \(\hat{P}_F\) i.e. under the minimal martingale measure of the non-modified financial market.
PROOF. Our goal is to find the \( \hat{P}^G \)-Galtchouk-Kunita-Watanabe of \( E^{\hat{P}^G} \left[ 1_{\{\tau>T\}} \frac{g_T}{B_T} | G_t \right] \). According to Proposition 5.12, we have

\[
E^{\hat{P}^G} \left[ 1_{\{\tau>T\}} \frac{g_T}{B_T} | G_t \right] = 1_{\{\tau>t\}} e^{-(\Gamma_t + \Upsilon_t)} E^{\hat{P}^F} \left[ e^{-\left(\Gamma_T + \Upsilon_T\right)} g_T B_T | F_t \right]
\]

Let us define \( U^g_t := E^{\hat{P}^F} \left[ e^{-\left(\Gamma_T + \Upsilon_T\right)} g_T B_T | F_t \right] \). By hypothesis, its \( \hat{P}^F, \mathbb{F} \)-Galtchouk-Kunita-Watanabe decomposition with respect to \( X_t \) is given by

\[
U^g_t = U^g_0 + \int_0^t 1_{\{\tau>u\}} e^{-(\Gamma_u + \Upsilon_u)} dX_u + L_t^{U^g}
\]

Using the integration by parts formula and the integral representation of \( 1_{\{\tau>t\}} e^{(\Gamma_t + \Upsilon_t)} \) given in Proposition 5.11, we have

\[
E^{\hat{P}^G} \left[ 1_{\{\tau>T\}} \frac{g_T}{B_T} | G_t \right] = 1_{\{\tau>t\}} e^{(\Gamma_t + \Upsilon_t)} U^g_t
\]

\[
= \int_0^t 1_{\{\tau>u\}} e^{(\Gamma_u + \Upsilon_u)} dU^g_u
- \int_0^t U^g_{u-} e^{(\Gamma_u + \Upsilon_u)} dM_u + \int_0^t 1_{\{\tau>u\}} U^g_{u-} e^{(\Gamma_u + \Upsilon_u)} d\bar{L} Y_u^{\wedge \tau}
+ \left[ U^g, 1_{\{\tau>\cdot\}} e^{(\Gamma + \Upsilon)} \right]_t
\]

As far as the square bracket is concerned, we have

\[
\left[ U^g, 1_{\{\tau>\cdot\}} e^{(\Gamma + \Upsilon)} \right]_t = U^g_u + \int_0^t 1_{\{\tau>u\}} e^{(\Gamma_u + \Upsilon_u)} d\left[ X_u, \bar{L} Y_u^{\wedge \tau} \right]_u
+ \int_0^t 1_{\{\tau>u\}} e^{(\Gamma_u + \Upsilon_u)} d\left[ L^{U^g} U^g, \bar{L} Y_u^{\wedge \tau} \right]_u
= U^g_u + \int_0^t 1_{\{\tau>u\}} e^{(\Gamma_u + \Upsilon_u)} d\left[ U^g_t, \bar{Y}_{u}^{\wedge \tau} \right]_u
= U^g_u + \int_0^t 1_{\{\tau>u\}} e^{(\Gamma_u + \Upsilon_u)} d\left[ U^g_t, \bar{Y}_{u}^{\wedge \tau} \right]_u
- \int_0^t 1_{\{\tau>u\}} e^{(\Gamma_u + \Upsilon_u)} d\left[ L^{U^g} U^g, g_s d\bar{N}_{s \wedge \tau} \right]_u
= U^g_u + \int_0^t 1_{\{\tau>u\}} e^{(\Gamma_u + \Upsilon_u)} d\left( \int_0^T e^{\Gamma_s} d \left[ L^{U^g}, Y_s \right] \right)
In the second equality, we use the fact \( [X, \tilde{L}^Y_{\wedge \tau}]_t = \langle N, L^Y \rangle^P \) and in the fourth that 
\[
\left[ L^{10}_t, \int_0^1 g_s d\tilde{N}_{s\wedge \tau} \right]_u = \left[ L^{10}_t, \int_0^1 g_s dN_s \right]_u = 0. \]
Eventually, we get
\[
E^\tilde{P}^G \left[ 1_{\{\tau > T\}} \frac{g_T}{B^T} | \mathcal{G}_t \right] = U^g_t + \int_0^T 1_{\{\tau > u\}} e^{(\Gamma_u + \Upsilon_u)} e\|^g_u dX_u + \tilde{L}_t^g
\]
where \( \tilde{L}_t^g = \int_0^T 1_{\{\tau > u\}} e^{(\Gamma_u + \Upsilon_u)} d\tilde{L}^{10}_u - \int_0^T U^g_u e^{(\Gamma_u + \Upsilon_u)} dM_u + \int_0^T 1_{\{\tau > u\}} U^g_u e^{(\Gamma_u + \Upsilon_u)} d\tilde{L}^{10}_u. \)
Notice \( \tilde{L}_t^g \) is indeed a \((\tilde{P}^G, \mathbb{G})\)-martingale since by definition of the minimal martingale measure all \((P, \mathbb{G})\)-martingales \((\tilde{P}^G, \mathbb{G})\)-strongly orthogonal to \( \tilde{N}_t \) remain \((\tilde{P}^G, \mathbb{G})\)-martingales. We still have to show that \( \tilde{L}_t^g \) is \((\tilde{P}^G, \mathbb{G})\)-strongly orthogonal to \( X_t \) on \( \{ \tau > t \} \). We have to show that
\[
\left\langle \tilde{L}_t^g, X \right\rangle^{\tilde{P}^G, \mathbb{G}}_{t \wedge \tau} = 0. \]
We have
\[
\left\langle \tilde{L}_t^g, X \right\rangle^{\tilde{P}^G, \mathbb{G}}_{t \wedge \tau} = -\int_0^T U^g_u e^{(\Gamma_u + \Upsilon_u)} d\left\langle M, X \right\rangle^\tilde{P}^G \mathbb{G} + \int_0^T 1_{\{\tau > u\}} e^{(\Gamma_u + \Upsilon_u)} d\left\langle \tilde{L}^{10}_u, X \right\rangle^\tilde{P}^G \mathbb{G} + \int_0^T 1_{\{\tau > u\}} U^g_u e^{(\Gamma_u + \Upsilon_u)} d\left\langle \tilde{L}^{10}_u, X \right\rangle^{\tilde{P}^G, \mathbb{G}}
\]
On one hand, since \( X_t \) is continuous, we obviously have \( \left\langle M, X \right\rangle^\tilde{P}^G \mathbb{G} = 0 \). On the other hand,
\[
\left\langle \tilde{L}^{10}_u, X \right\rangle^{\tilde{P}^G, \mathbb{G}} = \left\langle \tilde{L}^{10}_u, \tilde{N}_t \right\rangle^{\tilde{P}^G, \mathbb{G}} = \left\langle \tilde{L}^{10}_u, \tilde{N}_t \right\rangle^{P, \mathbb{G}} = 0. \]
We can follow the same argument for
\[
\left\langle \tilde{L}^{10}_u, X \right\rangle^{\tilde{P}^G, \mathbb{G}}.
\]
\[\square\]

5.4.2. Payment at the surrender time. We first need the following result.

**Proposition 5.14.** Let us denote by \( \tilde{P}^G \) and \( \tilde{P}^F \) the minimal martingale measure with respect to \( \mathbb{G} \) and to \( \mathbb{F} \). If \( h_t \) is a \((\mathcal{F})\)-predictable process, we have the following relationship
\[
E^{\tilde{P}^G} \left[ \int_t^T h_u dH_u | \mathcal{G}_t \right] = 1_{\{\tau > t\}} e^{(\Gamma_t + \Upsilon_t)} E^{\tilde{P}^G} \left[ \int_t^T h_u d\tilde{F}_u | \mathcal{F}_t \right]
\]
where \( \tilde{F}_t = 1 - e^{(\Gamma_t + \Upsilon_t)}. \)
Proof. We have

\[
E^{Pg} \left[ \int_t^T h_u dH_u | G_t \right] = \frac{E^P \left[ \hat{Z}_T^G \int_t^T h_u dH_u | G_t \right]}{\hat{Z}_t^G}.
\]

\[
= 1_{\{\tau > t\}} e^{\gamma_t} \frac{E^P \left[ \hat{Z}_T^G \int_t^T h_u dH_u | F_t \right]}{\hat{Z}_t^F}.
\]

Since on \( \{\tau > t\} \), \( \hat{Z}_t^G = \hat{Z}_t^F e^{-\gamma_t} \).

Let us now assume \( h_u \) is a bounded stepwise \( F \)-predictable process. We can write \( h_u = \sum_{i=0}^n h_{t_i} 1_{\{t_i < u \leq t_{i+1}\}} \) where \( t = t_0 < t_1 < \cdots < u < \cdots < t_n = T \) and \( h_{t_i} \) is a bounded \( F_{t_i} \)-measurable random variable for \( i = 0, \ldots, n \). We have in this case

\[
E^P \left[ \hat{Z}_T^G \int_t^T h_u dH_u | F_t \right] = \sum_{i=0}^n E^P \left[ \hat{Z}_T^G h_{t_i} 1_{\{t_i < u \leq t_{i+1}\}} | F_t \right] \frac{\hat{Z}_t^F}{\hat{Z}_i^F}.
\]

\[
= \sum_{i=0}^n E^P \left[ \hat{Z}_T^G h_{t_i} 1_{\{t_i < u \leq t_{i+1}\}} | F_t \right] \frac{\hat{Z}_t^F}{\hat{Z}_i^F}.
\]

\[
= \sum_{i=0}^n E^P \left[ h_{t_i} 1_{\{t_i < u \leq t_{i+1}\}} \hat{Z}_T^G | G_{t_i} \right] \frac{\hat{Z}_t^F}{\hat{Z}_i^F}.
\]

\[
= \sum_{i=0}^n E^P \left[ h_{t_i} 1_{\{t_i < u \leq t_{i+1}\}} \hat{Z}_T^G | F_{t_i+1} \right] \frac{\hat{Z}_t^F}{\hat{Z}_i^F}.
\]

\[
= \sum_{i=0}^n E^P \left[ h_{t_i} 1_{\{t_i < u \leq t_{i+1}\}} \hat{Z}_T^G | F_t \right] e^{-\gamma_{t_i}} \frac{\hat{Z}_t^F}{\hat{Z}_i^F}.
\]

\[
= \sum_{i=0}^n E^P \left[ h_{t_i} 1_{\{t_i < u \leq t_{i+1}\}} \hat{Z}_T^G | F_t \right] e^{-\gamma_{t_{i+1}}} \frac{\hat{Z}_t^F}{\hat{Z}_i^F}.
\]
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\[
= \sum_{i=0}^{n} E^{\hat{P}} \left[ h_i 1_{\{\tau > t_i\}} e^{-\Upsilon t_i} \big| \mathcal{F}_t \right] - \sum_{i=0}^{n} E^{\hat{P}} \left[ h_i 1_{\{\tau > t_i+1\}} e^{-\Upsilon t_{i+1}} \big| \mathcal{F}_t \right]
\]

\[
= E^{\hat{P}} \left[ \sum_{i=0}^{n} h_i \left\{ e^{-(\Gamma t_i + \Upsilon t_i)} - e^{-(\Gamma t_{i+1} + \Upsilon t_{i+1})} \right\} \big| \mathcal{F}_t \right]
\]

\[
= E^{\hat{P}} \left[ \int_{t}^{T} h_u d\tilde{F}_u \big| \mathcal{F}_t \right]
\]

When \( h_u \) is a general (bounded) \( \mathbb{F} \)-predictable process, we can approximate this process by a suitable sequence of stepwise processes. In this case, we would obtain, under the sign of conditional expectation, a sequence of sums that would converge to the integral \( \int_{t}^{T} h_u d\tilde{F}_u \) in the sense of Itô. At the price of a much more cumbersome proof, we could even generalize this proposition to an unbounded \( h_u \) (\( \mathbb{F} \)-predictable) process. \( \square \)

We can now study the pseudo locally risk-minimizing strategies of the discounted cumulative payment process \( A_t = \int_{0}^{T} R_u \frac{B_u}{B_t} dH_u \) which is assumed to be square integrable with respect to \( P \).

**Proposition 5.15.** If, for \( R_u, \mathbb{F} \)-predictable and \( P \)-square integrable, the \((\hat{P}^F, \mathbb{F})\)-Galtchouk-Kunita-Watanabe decomposition of the \((\hat{P}^F, \mathbb{F})\)-martingale \( E^{\hat{P}} \left[ \int_{0}^{T} R_u \frac{B_u}{B_t} d\tilde{F}_u \big| \mathcal{F}_t \right] \) is written

\[
E^{\hat{P}} \left[ \int_{0}^{T} R_u \frac{B_u}{B_t} d\tilde{F}_u \big| \mathcal{F}_t \right] = U^R_0 + \int_{0}^{t} \psi_u dX_u + L^R_T
\]

then the \((P, \mathbb{G})\)-Föllmer-Schweizer decomposition of \( \int_{0}^{T} R_u \frac{B_u}{B_t} dH_u \) is given by

\[
\int_{0}^{T} R_u \frac{B_u}{B_t} dH_u = U^R_0 + \int_{0}^{T} 1_{\{\tau > u\}} e^{-(\Gamma_u + \Upsilon_u)} \psi_u dX_u + \tilde{L}^R_T
\]

where

\[
\tilde{L}^R_T = \int_{0}^{T} \left( R_u \frac{B_u}{B_t} - V^U_{u-} e^{-(\Gamma_u + \Upsilon_u)} \right) dM_u
\]

\[
+ \int_{0}^{T} 1_{\{\tau > u\}} e^{-(\Gamma_u + \Upsilon_u)} d\tilde{L}^U_T - \int_{0}^{t} 1_{\{\tau > u\}} e^{-(\Gamma_u + \Upsilon_u)} d\tilde{L}^U_T \int_{0}^{T} R_u \frac{B_u}{B_t} - e^{-(\Gamma_u + \Upsilon_u)} V^U_{u-} d\tilde{L}_u
\]

and

\[
V^U_T = E^{\hat{P}} \left[ \int_{t}^{T} R_u \frac{B_u}{B_t} d\tilde{F}_u \big| \mathcal{F}_t \right]
\]
where $\tilde{F}_t = 1 - e^{-(\Gamma_t + \Upsilon_t)}$. We thus have

$$E^{\tilde{G}} \left[ \int_0^T \frac{R_u}{B_u} dH_u \mid G_t \right] = 1_{\{\tau > t\}} e^{(\Gamma_t + \Upsilon_t)} E^{\hat{P}} \left[ \int_0^t \frac{R_u}{B_u} d\tilde{F}_u \mid \mathcal{F}_t \right]$$

As far as the term (I) is concerned, we can follow the same argument as in Proposition 5.13. It leads to

$$(I) = U_t^R + \int_0^t 1_{\{\tau > u\}} e^{(\Gamma_u + \Upsilon_u)} \xi_u R_u dX_u + \int_0^t 1_{\{\tau > u\}} e^{(\Gamma_u + \Upsilon_u)} d\tilde{L}_u^R$$

$$- \int_0^t U_t^R e^{(\Gamma_u + \Upsilon_u)} dM_u + \int_0^t 1_{\{\tau > u\}} U_t^R e^{(\Gamma_u + \Upsilon_u)} d\tilde{L}_u^Y$$

where $U_t^R = E^{\hat{P}} \left[ \int_0^T \frac{R_u}{B_u} d\tilde{F}_t \mid \mathcal{F}_t \right]$.

As far as the term (II) is concerned, in Appendix 1, we show that it can be written as

$$(II) = \int_0^t \left[ \frac{R_u}{B_u} + e^{(\Gamma_u + \Upsilon_u)} \left( \int_0^u \frac{R_s}{B_s} d\tilde{F}_s \right) \right] dM_u$$

$$- \int_0^t 1_{\{\tau > u\}} \Gamma_u \left[ \frac{R_u}{B_u} + e^{(\Gamma_u + \Upsilon_u)} \left( \int_0^u \frac{R_s}{B_s} d\tilde{F}_s \right) \right] d\tilde{L}_u^Y$$

Since $V_t^{UR} = U_t^R - \int_0^t \frac{R_u}{B_u} d\tilde{F}_u$. So eventually, we have

$$E^{\hat{P}} \left[ \int_0^T \frac{R_u}{B_u} dH_u \mid G_t \right] = U_t^R + \int_0^t 1_{\{\tau > u\}} e^{(\Gamma_u + \Upsilon_u)} \xi_u R_u dX_u + \int_0^t 1_{\{\tau > u\}} e^{(\Gamma_u + \Upsilon_u)} d\tilde{L}_u^R$$

$$+ \int_0^t \left[ \frac{R_u}{B_u} - e^{(\Gamma_u + \Upsilon_u)} V_t^{UR} \right] dM_u$$

$$- \int_0^t 1_{\{\tau > u\}} \Gamma_u \left[ \frac{R_u}{B_u} - e^{(\Gamma_u + \Upsilon_u)} V_t^{UR} \right] d\tilde{L}_u^Y$$

As in Proposition 5.13, we can easily show $\tilde{L}_t^R$ is $(\hat{P}^{\tilde{G}}, \mathcal{G})$-strongly orthogonal to $X_t$. □
We can find similar results for the payment \( \int_0^T 1_{\{\tau > u\}} dC_u \) if we notice we have the following equality:

\[
E^\hat{\mathbb{P}}_G \left[ \int_t^T 1_{\{\tau > u\}} dC_u \mid G_t \right] = 1_{\{\tau > t\}} e^{(\Gamma_t + \Upsilon_t)} E^\hat{\mathbb{P}}_G \left[ \int_t^T e^{-(\Gamma_u + \Upsilon_u)} dC_u \mid \mathcal{F}_t \right]
\]

This is left to the reader.

6. Conclusion.

In this chapter, we studied the locally risk-minimizing strategies for insurance contracts with a surrender option. We distinguished two different cases. In the first one, we assumed the \((H)\)-hypothesis held. In the second case, we removed this assumption. In each case, we studied the impact of an enlargement of filtration on the financial market and the minimal martingale measure.

Prior to that, an another important contribution of this chapter consisted of showing how the local risk-minimization theory could be extended to payment processes.

Appendix 1.

**Proposition.** We have

\[
\int_0^t \frac{R_u}{B_u} dH_u - 1_{\{\tau > t\}} e^{(\Gamma_t + \Upsilon_t)} \int_0^t \frac{R_u}{B_u} d\tilde{F}_u = \int_0^t \left[ \frac{R_u}{B_u} + e^{(\Gamma_u + \Upsilon_u)} \left( \int_0^u \frac{R_s}{B_s} d\tilde{F}_s \right) \right] dM_u
\]

\[\quad - \int_0^t 1_{\{\tau > u\}} e^{\Gamma_u} \left[ \frac{R_u}{B_u} + e^{(\Gamma_u + \Upsilon_u)} \left( \int_0^u \frac{R_s}{B_s} d\tilde{F}_s \right) \right] d\tilde{L}_Y\]

**Proof.** First, recall we have the different representations

\[
1_{\{\tau > t\}} e^{(\Gamma_t + \Upsilon_t)} = 1 - \int_0^t e^{(\Gamma_u + \Upsilon_u)} dM_u + \int_0^t 1_{\{\tau > u\}} e^{(\Gamma_u^2 + \Upsilon_u)} d\tilde{Y}_{u \wedge \tau}
\]

\[\quad - \int_0^t 1_{\{\tau > u\}} e^{(\Gamma_u^2 + \Upsilon_u)} g_u' d\tilde{N}_{u \wedge \tau}\]

and

\[
e^{\Upsilon_t} = 1 - \int_0^t e^{\Gamma_u + \Upsilon_u} g_u' dN_u\]

We need to know the integral representation of \( e^{-\Upsilon_t} \) and \( \tilde{F}_t \). Using Itô’s formula for continuous semimartingales on the function \( f(x) = x^{-1} \), we have

\[
e^{-\Upsilon_t} = 1 - \int_0^t e^{-\Upsilon_u^2} d\Upsilon_u + \int_0^t e^{-\Upsilon_u^2} d\langle e^{\Upsilon_u}, e^{\Upsilon_u} \rangle_u
\]

\[\quad = 1 + \int_0^t e^{\Gamma_u - \Upsilon_u} g_u dN_u + \int_0^t e^{\Gamma_u - \Upsilon_u} g_u' d\langle N, N \rangle_u g_u'\]

As far as \( \tilde{F}_t \) is concerned, we have

\[\square\]
\[\tilde{F}_t = 1 - e^{-(\Gamma_t + \Upsilon_t)}\]
\[= - \int_0^t e^{-\Upsilon_u}d\Gamma_u - \int_0^t e^{-\Gamma_u}d\Upsilon_u - [e^{-\Gamma}, e^{-\Upsilon}]_t\]
\[= \int_0^t e^{-\Upsilon_u}dF_u - \int_0^t e^{-\Gamma_u}d\Upsilon_u + [Y, e^{-\Upsilon}]_t\]

Replacing Equation (6.1) in the square bracket, we get \([Y, e^{-\Upsilon}]_t = \int_0^t e^{\Gamma_u - \Upsilon_u}g'_u d(N, N)g'_u\). Thus it leads to

\[\tilde{F}_t = \int_0^t e^{-\Upsilon_u}dF_u - \int_0^t e^{-\Gamma_u}g'_u dN_u\]

Since \(F_t = Y_t + D_t\), we can also write

\[\tilde{F}_t = \int_0^t e^{-\Upsilon_u}dY_u + \int_0^t e^{-\Upsilon_u}dD_u - \int_0^t e^{-\Gamma_u}g'_u dN_u\]
\[(6.2)\]
\[= \int_0^t e^{-\Upsilon_u}dD_u + \int_0^t e^{-\Gamma_u}dL^Y_u\]

Let us now study the term \(1_{\{\tau > t\}} e^{(\Gamma_t + \Upsilon_t)} \int_0^t \frac{R_u}{B_u} d\tilde{F}_u\). Using the integration by parts formula, we have

\[1_{\{\tau > t\}} e^{(\Gamma_t + \Upsilon_t)} \int_0^t \frac{R_u}{B_u} d\tilde{F}_u = - \int_0^t \left( \int_0^u \frac{R_s}{B_s} d\tilde{F}_s \right) e^{(\Gamma_u + \Upsilon_u)} dM_u\]
\[+ \int_0^t 1_{\{\tau > u\}} e^{(\Gamma_u + \Upsilon_u)} \left( \int_0^u \frac{R_s}{B_s} d\tilde{F}_s \right) dN_{u\wedge \tau}\]
\[+ \int_0^t 1_{\{\tau > u\}} e^{(\Gamma_u + \Upsilon_u)} \frac{R_u}{B_u} d\tilde{F}_u\]
\[+ \int_0^t \frac{R_u}{B_u} d \left[ \tilde{F}, 1_{\{\tau > \cdot\}} e^{(\Gamma + \Upsilon)} \right]_u\]
Replacing Equation (6.2), we get

\[
1_{\{\tau > t\}} e^{(\Gamma_t + \Upsilon_t)} \int_0^t \frac{R_u}{B_u} d\tilde{F}_u = - \int_0^t \left( \int_0^u \frac{R_s}{B_s} d\tilde{F}_s \right)_- e^{(\Gamma_u + \Upsilon_u)} dM_u \\
+ \int_0^t 1_{\{\tau > u\}} e^{(\Gamma_u + \Upsilon_u)} \left( \int_0^u \frac{R_s}{B_s} d\tilde{F}_s \right)_- d\tilde{L}^Y_{u \wedge \tau} \\
+ \int_0^t 1_{\{\tau > u\}} e^{\Gamma_u} \frac{R_u}{B_u} dD_u + \int_0^t 1_{\{\tau > u\}} e^{\Gamma_u} \frac{R_u}{B_u} dL^Y_u \\
+ \int_0^t \frac{R_u}{B_u} d \left[ \int_0^t e^{-\Upsilon_u} dL^Y_u \int_0^t 1_{\{\tau > u\}} e^{(\Gamma_u + \Upsilon_u)} dY_{u \wedge \tau} \right]_u \\
- \int_0^t \frac{R_u}{B_u} d \left[ \int_0^t e^{-\Upsilon_u} dL^Y_u \int_0^t 1_{\{\tau > u\}} e^{(\Gamma_u + \Upsilon_u)} g_u dN_{u \wedge \tau} \right]_u
\]

Since the last square bracket is null, eventually, we have

\[
1_{\{\tau > t\}} e^{(\Gamma_t + \Upsilon_t)} \int_0^t \frac{R_u}{B_u} d\tilde{F}_u = \int_0^t \frac{R_u}{B_u} d\Lambda_u - \int_0^t \left( \int_0^u \frac{R_s}{B_s} d\tilde{F}_s \right)_- e^{(\Gamma_u + \Upsilon_u)} dM_u \\
+ \int_0^t 1_{\{\tau > u\}} e^{\Gamma_u} \left[ \frac{R_u}{B_u} + e^{(\Gamma_u + \Upsilon_u)} \left( \int_0^u \frac{R_s}{B_s} d\tilde{F}_s \right)_- \right] d\tilde{L}^Y_{u \wedge \tau}
\]
Part 3

Life Insurance Contracts and the Longevity Risk.
CHAPTER 5

Risk-Minimization with Systematic mortality risk.

1. Introduction.

Systematic mortality risk is, by now, widely recognized as one of the important risks a life insurance company faces. Unfortunately, the management of this risk poses serious problems. On one hand, its systematic nature does not allow an insurer to diversify this risk away. On the other hand, it is, at this time, still difficult to buy an efficient protection from a reinsurer or to transfer this risk to the financial market. Accordingly, this systematic risk represents an important source of incompleteness and it is important to closely study the hedging strategies an insurer should follow to mitigate its effects.

Different approaches have been developed to tackle the problem of hedging in an incomplete market. In this chapter, we focus on the risk-minimizing hedging strategies developed by Föllmer and Sondermann in [44] and extended in Møller [68]. Actually, the application of the risk-minimization theory to insurance contracts with systematic mortality risk has already been studied in Dahl and Møller [39]. Under some simplifying assumptions, these authors found closed-form solutions for the risk-minimizing strategies and the cost process (i.e. the unhedgeable part) of some insurance payments. As far as the cost process is concerned, they found it was mainly due to the unpredictability of the random time of death and to the random evolution of a single point of the stochastic survival probability distribution function. As far as the risk-minimizing strategies are concerned, Dahl and Møller showed they are given by relatively complex expressions without, at first sight, any particular meaning. However, a close look at their solutions reveals that these strategies actually take a very simple and intuitive form. Indeed, in their model, simple algebraic manipulations show that the risk-minimizing strategies can be written as the average of risk-minimizing strategies of purely financial claim, weighted by the stochastic survival probabilities. The main motivation of this chapter is to investigate if these conclusions would hold in a more general setting or if they are intrinsically related to the particular assumptions made by Dahl and Møller in [39].

Our theoretical setting is more general than Dahl and Møller’s in, at least, three respects. Firstly, as far as the financial market is concerned, Dahl and Møller assume that only two assets are traded, a saving account and a single zero-coupon bond with maturity $T$. The instantaneously risk-free rate is modelled as a time-homogeneous single factor affine model of the CIR type. Since the term structure is modelled through a single factor model, bonds of all maturities are perfectly correlated to the traded $T$-bond. Bonds of any maturities can then be perfectly synthesized with the help of the two traded assets. Their financial market, given the price of the $T$-bond, is thus a complete market. In this chapter, we only assume the
(discounted) financial assets prices follow an arbitrary $s$-dimensional local martingale. We do not assume the financial market is complete.

Secondly, Dahl and Møller assume the insurance payments, given the times of death of their policyholders, are deterministic. Since the financial market is complete, these deterministic payments can be perfectly hedged with the help of two traded assets. In this chapter, the insurance payments, given the time of death, can be stochastic and depend on the evolution of the financial market. Furthermore, since our financial market is not necessarily complete, these payments can not necessarily be perfectly hedged.

Finally, in Dahl and Møller, the probability distribution function of the random time of death is described through its intensity. Indeed, in their model, the (stochastic) survival probability distribution function can be derived from this intensity, basically as bond prices of all maturities can be derived from the short rate. In Dahl and Møller, this intensity is assumed to be stochastic and to follow a time-inhomogeneous single factor model of the CIR type also\(^1\). In such a model, the (stochastic) survival probability distribution function is thus driven by a single factor as the term structure of interest rate is. In other words, the survival probabilities at two different points are perfectly correlated with each others as interest rates at two different maturities are. Put again differently, the evolution of the whole probability distribution function can be summarized through the evolution of a single point of this probability distribution function. We follow a different course in this thesis. We actually describes the whole survival probability distribution function as a stochastic process that takes its values in the space of finite measure but without giving the specific form of this probability function or the specific form of the intensity of the random time of death. This method is very general since it allows, for examples, for non-Brownian models, for multi-factors (and even possibly for an uncountable infinite number of factors) models and even for discontinuities in the process describing the survival probability distribution function. Moreover, our approach differs from Dahl and Møller’s in that it allows to distinguish two situations. In the first one, the so called \((H)\)-hypothesis is assumed to hold whereas in the second situations we relax this assumption. To my knowledge, this \((H)\)-hypothesis actually holds implicitly in every current stochastic mortality models, including Dahl and Møller’s, of course.

These very weak assumptions on the financial market and the dynamics of the probability distribution function of the random time of death, are actually already sufficient to derive interesting results that generalize Dahl and Møller’s. Our main conclusions are the following. As far as the risk-minimizing strategies are concerned, we show that we can always write them as an average of risk-minimizing strategies of purely financial claims, weighted by the stochastic survival probabilities. The important corollary of this is that, finding the risk-minimizing strategy of a life insurance contract can be separated in two independent problems, one related to the modelling of the survival probabilities and one related to the hedging of purely financial claims. As far as the cost process is concerned, the conclusion depends on whether or not we assume the \((H)\)-hypothesis holds. When this hypothesis holds, we find a result similar to Dahl and Møller’s. The differences come from the fact that, firstly, in our model, the cost process depends on the movement of the whole survival probability distribution function and

\(^1\)Under the real measure as well as under the martingale measure.
not on the movement of a single point of this function. This is explained by the fact we do not necessarily assume a single factor model for the dynamics of the probability distribution function. Secondly, we have an additional term related to the incompleteness of the financial market which does not appear in Dahl and Møller since their financial market is complete. When the \((H)\)-hypothesis does not hold, we have in addition, a new systematic term related to the stochastic mortality which does not appear in the cost process of Dahl and Møller.

Technically speaking, the crucial point is to notice that the expected insurance payments and the stochastic probability distribution of death can be understood as Banach space-valued processes. We show that the stochastic integral defined in Gravereaux and Pellaumail \([46]\) with respect to such Banach space-valued processes, can be useful in life insurance. In particular, we give an integration-by-parts formula for Banach space-valued martingales that appears to be one of the key elements to find the risk-minimizing strategies.

The present chapter is organized as follows. In Section 2, we introduce the general theoretical setting. More precisely, in Section 2.1, we describe our financial market model and in Section 2.2, we describe our model for the random time of death and its stochastic probability distribution function. In Section 2.3, we describe the combined model. Finally, in Section 2.4, we present the insurance payments we wish to hedge. In Section 3, we give a brief review of the risk-minimization theory. In Section 4 and Section 5, we study in details the risk-minimizing strategies of our insurance payments. In the first section, we assume the \((H)\)-hypothesis holds; in the second one, we remove this assumption.

2. The Theoretical Framework.

2.1. The financial market. Let \((\Omega^F, \mathcal{F}^F, P^F)\) be a complete probability space. A perfect frictionless financial market is defined on this space. We assume there is one locally risk free asset denoted by \(B_t = B(t, \omega)\) and \(s\) risky assets \(S_i^t = S_i^t(t, \omega), i = 1, \ldots s\) following real-valued càdlàg stochastic processes. The price of the locally risk free asset is assumed to follow a strictly positive, continuous process of finite variation. The discounted values of these assets are denoted by

\[
X^i_t = \frac{S^i_t}{B_t}, i = 1, \ldots s
\]

Let \(\mathcal{F} = (\mathcal{F}_t)_{t \geq 0}\) be the filtration generated by the stochastic processes \(S^i_t\) for \(i = 1, \ldots s\) and \(B_t\). We assume this filtration respects the usual hypothesis.

We assume furthermore there is no arbitrage opportunity in this financial market with respect to the filtration \(\mathcal{F}\). This assumption is equivalent to the existence of at least one probability measure \(Q^F\), equivalent to \(P^F\), such that the discounted prices \(X^i\) are \((\mathcal{F}, Q^F)\)-local martingales for each \(i = 1, \ldots s\).

2.2. The random time of death. Let us introduce another probability space \((\Omega^I, \mathcal{F}^I, Q^I)\). We study a portfolio consisting of a single policyholder. The random time of death of this policyholder is defined on \(\Omega^I\) and denoted by \(\tau = \tau(\omega)\). We define the stochastic process \(H_t\) by \(H_t = 1_{\{t \leq \tau\}}\). The filtration generated by the process \(H_t\) is denoted by \(\mathcal{H} = (\mathcal{H}_t)_{t \geq 0}\) where \(\mathcal{H}_t = \sigma (H_s, 0 \leq s \leq t)\).
We assume that the insurer can observe an additional set of information relevant to the estimation of the distribution of $\tau$. We assume this information comes from the observation of a set of some unspecified stochastic processes also defined on $\Omega^I$. We denote $\mathcal{J} = (\mathcal{J}_t)_{t \geq 0}$ the filtration generated by these unspecified stochastic processes and we assume this filtration respects the usual hypothesis, and assume the following condition holds.

**CONDITION 2.1.** $\tau$ is not a $\mathcal{J}$-stopping time.

This filtration could be generated, for example, by the number and the timing of deaths in a given population or generated by some socio-economic factors related to the evolution of the mortality.

As in Jeanblanc and Rutkowski [58], we can define the so called $(Q, \mathcal{J})$-hazard process $\Gamma_t$ of $\tau$ in the following way:

$$F_t := 1 - e^{-\Gamma_t}$$

where $F_t = Q(\tau \leq t | \mathcal{J}_t)$. For simplicity, we assume $F_0 = 0$. We also assume $Q(\tau > t) > 0$ for all $t \geq 0^2$. Thanks to Condition 2.1, it implies $F_t < 1$ for all $t \geq 0$ and $\Gamma_t = -\ln(1 - F_t)$ is thus indeed well defined.

$F_t$ is a $(Q, \mathcal{J})$-submartingale and, accordingly, $F_t$ admits a unique Doob-Meyer decomposition. We write $F_t = D_t + Z_t$ where $D_t$ is an $\mathcal{J}$-predictable increasing process and $Z_t$ a $(Q, \mathcal{J})$-martingale.

The information related to the mortality is denoted by $\mathbb{I} = (\mathbb{I}_t)_{t \geq 0}$ where $\mathbb{I}_t = \mathcal{H}_t \lor \mathcal{J}_t$. The $(Q^I, \mathbb{I})$-compensator $\Lambda_t$ of $H_t$ is given by

$$\Lambda_t = \int_0^{t \wedge \tau} \frac{1}{1 - F_u} dD_u$$

We denote the compensated process $M^Q_t = H_t - \Lambda_t$.

For any time $t \geq 0$, we can also define the regular conditional distribution of $\tau$ (with respect to $\mathcal{J}$) $Q_t(du) = Q_t(\omega, du)$ as a modification of the following conditional expectation:

$$Q_t(u) := E^{Q^I} [1_{\{\tau \leq u\}} | \mathcal{J}_t] = Q(\tau \leq u | \mathcal{J}_t)$$

On one hand, for a given $t \geq 0$, $Q_t(\cdot)$ defines a random probability measure on $([0, T], \mathcal{B}([0, T]))$ (where $T$ is the term of an insurance contract). On the other hand, for a given $u \in [0, T]$, the stochastic process $Q_t(u)$ is a $(Q^I, \mathcal{J})$-martingale. Let us also denote

$$S_t(u) := Q(\tau > u | \mathcal{J}_t) = 1 - Q_t(u)$$

Alternatively, $Q_t(\cdot)$ and $S_t(\cdot)$ can also be seen as stochastic processes taking their values in the space of probability measures or, more generally, in the space of signed and finite measures on $([0, T], \mathcal{B}([0, T]))$. Endowed with the appropriate norms, these spaces are Banach spaces. $Q_t(\cdot)$ or at least to the term of the contract $T$. 
and $S_t(\cdot)$ can thus be seen as Banach space-valued stochastic processes. Let us now introduce the following definition.

**Definition 2.2.** Let $E$ be a Banach space and let $E^*$ be the dual of $E$. We say that an $E$-valued process $G_t$ is an $E$-valued martingale if and only if, for any (deterministic) $h \in E^*$, the action of the operator $T_h$ associated with $h$, on $G_t$, denoted by $T_h(G_t) = (h, G_t)$, is a real-valued martingale.

Actually, $Q_t(\cdot)$ and $S_t(\cdot)$ are even Banach space-valued martingales. Indeed, the dual of the space of signed and finite measures is the space $BM([0,T])$ of bounded measurable functions on $[0,T]$. If $h(u) \in BM([0,T])$, then the action of the operator associated with $h(u)$, on $Q_t(\cdot)$ is given by

$$T_h(Q_t) = (h, Q_t) = \int_0^T h(u)Q_t(du)$$

which is a real-valued martingale.

In [46], Gravereaux and Pellaumail defined a stochastic integral for Banach space-valued stochastic processes. In Appendix 1, based on the definition of these authors, we give an integration-by-parts formula for Banach space-valued processes. This result will be at the heart of our results on the risk-minimizing strategies. The remark that $Q_t(\cdot)$ can be seen as a Banach space-valued martingale, is thus essential in this chapter.

Before continuing, we have a last point to underline. In this section, we introduced a first enlargement of filtration by adding $\mathbb{H}$ to $\mathbb{J}$ to form the filtration $\mathbb{I}$. In general, a local martingale with respect to a given filtration is not necessarily a local martingale with respect to a larger one. In our setting, it means that the (local) $(Q^I, \mathbb{I})$-martingales are not necessarily (local) $(Q^I, \mathbb{J})$-martingales. In particular, the process $Z_t$ or, for a given $u \in [0,T]$, the process $Q_t(u)$ are not necessarily $(Q^I, \mathbb{I})$-martingales. Let us introduce the following condition:

**Condition 2.3.** $\tau$ avoids all $\mathbb{J}$-stopping times i.e. for any $\mathbb{J}$-stopping time $\theta$, $Q(\tau = \theta) = 0$.

According to Blanchet-Scalliet and Jeanblanc [30], under this condition, for any $(Q^I, \mathbb{J})$-martingale $m_t$, the stopped process

$$(2.2) \quad \hat{m}_{t\wedge \tau} = m_{t\wedge \tau} + \int_0^{t\wedge \tau} e^{r_s} d [m., Z]_s$$

is a $(Q^I, \mathbb{I})$-martingale. We call $\hat{m}_{t\wedge \tau}$ the $(Q^I, \mathbb{I})$-martingale part of $m_{t\wedge \tau}$. In particular, the $(Q^I, \mathbb{I})$-martingale part of the $(Q^I, \mathbb{J})$-martingale $Z$ can be written

$$\hat{Z}_{t\wedge \tau} = Z_{t\wedge \tau} + \int_0^{t\wedge \tau} e^{r_s} d [Z., Z]_s$$

and for each $u \in [0,T]$, the $(Q^I, \mathbb{I})$-martingale part of the $(Q^I, \mathbb{J})$-martingale $Q_t(u) := Q(\tau \leq u | \mathcal{F}_t)$ can be written:

$$\hat{Q}_{t\wedge \tau}(u) = Q_{t\wedge \tau}(u) + \int_0^{t\wedge \tau} e^{r_s} d [Q.(u), Z.]_s$$
We also have
\[ \hat{S}_{t \wedge \tau}(u) = S_{t \wedge \tau}(u) + \int_0^{t \wedge \tau} e^{\Gamma_u} d[S(u), Z_u]. \]

Let us introduce the following definition.

**Definition 2.4.** We say the \((H)\)-hypothesis holds under \(Q\) between \(\mathcal{J}\) and \(\mathcal{I}\) if any \((Q^I, \mathcal{I})\)-local martingale is a \((Q^I, \mathcal{I})\)-local martingale.

In this chapter, we will distinguish two situations. Section 4 deals with the risk-minimization of life insurance contracts when this \((H)\)-hypothesis holds and Section 5 deals with the same problem when this \((H)\)-hypothesis does not hold.

**2.3. The combined model.** The combined model consists in the following probability space \((\Omega, \mathcal{F}, \mathcal{G}, Q)\) where \(Q\) is the product measure of \(Q^F\) and \(Q^I\). We define the filtration \(\mathcal{G} = (\mathcal{G}_t)_{t \geq 0}\) where \(\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{I}_t\). Under the product measure \(Q\), the filtrations \(\mathcal{F}\) and \(\mathcal{I}\) are independent.

We have here a second enlargement of filtration from \(\mathcal{F}\) to \(\mathcal{G}\). Fortunately, if the independence of \(\mathcal{F}\) and \(\mathcal{I}\) under \(Q\) holds, there is no real issue. Indeed, this independence implies the (local) martingale properties of the \(\mathcal{F}\)-adapted and \(\mathcal{I}\)-adapted processes are preserved under this enlargement of filtration. In other words, all the local \((Q^F, \mathcal{F})\)-martingales and all the local \((Q^I, \mathcal{I})\)-martingales are local \((Q, \mathcal{G})\)-martingales. In particular, it means the \((Q^I, \mathcal{I})\)-compensator of the process \(H_t\) is indistinguishable of its \((Q, \mathcal{G})\)-compensator. It also means the discounted prices of the financial securities are local \((Q, \mathcal{G})\)-martingales. If the financial market is initially arbitrage-free then it remains arbitrage-free after this enlargement of filtration.

**2.4. The payments.** In this section, we describe the payments of the life insurance contracts, we wish to hedge.

**2.4.1. The insurer’s payments.** A large variety of life insurance contracts can be modelled as a combination of the following three building blocks. The first one is the payoff the insurer has to pay at the term of the contract \(T\). This payoff is assumed to be a \(\mathcal{F}_T\)-measurable random variable and is denoted by \(g(T, \omega)\). At the term of the contract, the insurer has to pay:

\[ g(T, \omega)1_{\{\tau > T\}} \]

The second building block is the amount the insurer has to pay when the policyholder dies before the term \(T\). This amount is denoted by \(1_{\{0 < \tau \leq T\}} R(\tau, \omega)\) where \((R(t, \omega))_{t \geq 0}\) is assumed to be an \(\mathcal{F}\)-predictable stochastic process. We have

\[ 1_{\{0 < \tau \leq T\}} R(\tau, \omega) = \int_0^T R(u, \omega) dH_u \]

The third building block is the payoffs the insurer has to pay as long as the policyholder is alive. We model these payoffs through their cumulative value up to time \(t\), denoted by \(C_t = C(t, \omega)\). This process is assumed to be a right-continuous increasing square integrable \(\mathcal{F}\)-adapted process. This cumulative payoff is then given by:
\[ C(T, \omega)1_{\{\tau > T\}} + C(\tau, \omega)1_{\{0 < \tau \leq T\}} = \int_0^T (1 - H_u)dC(u, \omega) \]

where we assume \( C(0, \omega) = 0 \) and \( C(T, \omega) = C(T, \omega) \).

For each \( u \in [0, T] \), let us consider the conditional expectation \( \tilde{U}^C_t \) of the form:

\[ \tilde{U}^C_t(u) = \mathbb{E}_Q \left[ \int_0^u \frac{1}{B_s}dC_s \mid \mathcal{F}_t \right]. \]

It can be shown that there exists a càdlàg process \( U^C_t(\cdot) \) taking its values in the space of \( \mathcal{B}([0, T]) \)-measurable function and such that, for each \( u \), \( U^C_t(u) \) is a version of \( \tilde{U}^C_t(u) \). Furthermore, since \( U^C_t(u) \) is increasing with \( u \), \( U^C_t(\cdot) \) can also be seen as a stochastic process taking its value in the space of finite (signed) measures on \( ([0, T], \mathcal{B}([0, T])) \). In other words, \( U^C_t(\cdot) \) is a Banach space-valued process. We can also easily see that it is even a Banach space-valued martingale since, for any \( u \in [0, T] \), the process \( \tilde{U}^C_t(u) \) is a \((Q, \mathcal{G})\)-martingale. In the following, we will always consider the version \( U^C_t(u) \) of the conditional expectation \( \mathbb{E}_Q \left[ \int_0^u \frac{1}{B_s}dC_s \mid \mathcal{F}_t \right] \).

2.4.2. The policyholder’s payments. We assume the policyholder pays premiums periodically at \( N \) fixed dates \( t_i \) with \( i = 0, \ldots, N - 1 \), as long as he is alive. We denote by \( P(t_i, \omega) \) the value of the premium paid at time \( t_i \) and assume \( P(t_i, \omega) \) is \( \mathcal{F}_{t_i} \)-measurable for each \( i = 0, \ldots, N - 1 \). The policyholder’s payments is then given by:

\[
\sum_{i=0}^{N-1} P(t_i, \omega)1_{\{\tau > t_i\}}
\]


We will here recall the main results of the theory of risk-minimization. These results are due to Föllmer and Sondermann [44] for a single payoff and were extended by Møller [68] to payment processes. We mainly follow Schweizer [78] for the presentation.

The financial market is defined as in Section 2.1. We will here consider an arbitrary filtration \( \mathcal{G} \) larger than the filtration \( \mathcal{F} \) generated by the financial assets prices. We assume a specific martingale measure \( Q \) has been chosen so that \( X \) is an \((Q, \mathcal{G})\)-local martingale. This condition holds indeed in the enlarged model described in Section 2.3.

**Definition 3.1.** A trading strategy \( \rho \) is a pair of processes \((\varepsilon, \eta)\) where \( \eta \) is a 1-dimensional real-valued \( \mathcal{G} \)-adapted process and \( \varepsilon \) is an \( s \)-dimensional real-valued \( \mathcal{G} \)-predictable process.

The process \( \varepsilon_t \) represents the amount of risky assets held at time \( t \) and the process \( \eta_t \) is the discounted amount invested in the risk free asset.

**Definition 3.2.** The value process \( V_t(\rho) \) of a trading strategy \( \rho \), is defined by \( V_t(\rho) = \varepsilon_t X_t + \eta_t \) for \( 0 \leq t \leq T \).

This value process represents the discounted value of the insurer’s financial portfolio following the trading strategy \( \rho \). This portfolio is not necessarily self-financed. We define by \( L^2(X, Q) \) the following space.
Definition 3.3. \( L^2(X, Q, G) \) is the space of \( \mathbb{R}^s \)- and \( G \)-predictable processes \( \varepsilon \) such that
\[
\|\varepsilon\|_{L^2(X, Q, G)} = \left( \mathbb{E}^Q \left[ \int_0^T \varepsilon_u' d[X]_u \varepsilon_u \right] \right)^{1/2} < \infty.
\]

We also have the following important lemma due to Schweizer:

Lemma 3.4. Suppose \( X \) is a \((Q, G)\)-local martingale. For any \( \varepsilon \in L^2(X, Q, G) \), the process \( \int_0^T \varepsilon_u dX_u \in M^2_0(Q, G) \). Moreover, the space \( I^2(X, Q, G) = \{ \int \varepsilon dX \mid \varepsilon \in L^2(X, Q, G) \} \) is a stable subspace of \( M^2_0(Q, G) \).

Proof. See Schweizer [78]. \( \square \)

Definition 3.5. An RM-strategy \( \rho \) is any pair \((\varepsilon, \eta)\) where \( \varepsilon \in L^2(X, Q, G) \) and \( \eta \) is a càdlàg adapted process such that the value process \( V_t(\rho) \) is right continuous and square integrable.

The liabilities of the insurer towards a policyholder are modelled as a process \( A_t \). The process \( A_t \) is assumed to be càdlàg, \( G \)-adapted and square integrable. The process \( A_t \) represents the discounted value of the cumulative payments up to time \( t \).

Definition 3.6. The cumulative cost process \( C_t(\rho) \) of a RM-strategy \( \rho \), is defined by
\[
C_t(\rho) = V_t(\rho) - \int_0^t \varepsilon_u dX_u + A_t.
\]

The initial cumulative cost process \( C_0(\rho) \) is given by \( C_0(\rho) = V_0(\rho) + A_0 \). The first term \( V_0(\rho) \) represents the initial amount the insurer has to invest to create his financial portfolio. The second term is the initial payment the insurer has to pay to the policyholder. Usually, \( A_0 \) will be negative and instead of a payment will represent the initial premium paid by the policyholder.

Definition 3.7. A strategy is said to be mean self financing if its cumulative cost process is a martingale.

The risk process is defined as follows:

Definition 3.8. The risk process of an RM-strategy \( \rho \) is defined by
\[
R_t(\rho) = \mathbb{E}^Q \left[ (C_T(\rho) - C_t(\rho))^2 \mid G_t \right].
\]

Following Møller [68], we restrict our attention to strategies which are 0-admissible in the following sense:

Definition 3.9. A strategy \( \rho \) is said to be 0-admissible if and only if: \( V_T(\rho) = 0 \), \( Q \)-a.s.

The theory of risk minimization aims at finding an 0-admissible RM-strategy that minimizes the risk process \( R_t(\rho) \) in the following sense.

Definition 3.10. An RM-strategy \( \rho \) is called risk-minimizing if and only if \( R_t(\rho) \leq R_t(\tilde{\rho}) \) \( Q \)-a.s. for every \( t \in [0, T] \) and for any RM-strategy \( \tilde{\rho} \) which is an admissible continuation of \( \rho \) from \( t \) on, in the sense that \( V_T(\rho) = V_T(\tilde{\rho}) \) \( Q \)-a.s., \( \tilde{\varepsilon}_u = \varepsilon_u \) for \( u \leq t \) and \( \tilde{\eta}_u = \eta_u \) for \( u < t \).

To solve this problem, we need the following result:

Lemma 3.11. Any risk minimizing RM-strategy is also mean self financing.
Proof. See Møller [68] for a proof.

This lemma leads directly to the following one:

**Lemma 3.12.** The discounted value of the risk minimizing RM-strategy \( \rho^* \) is given by:

\[
V_t(\rho^*) = E^Q [A_T - A_t | G_t]
\]

**Proof.** Indeed since by Lemma 3.11, the cost process is a martingale, we have

\[
C_t(\rho^*) = E^Q [C_T(\rho^*) | G_t]
\]

\[
V_t(\rho^*) - \int_0^t \varepsilon_u dX_u + A_t = E^Q \left[ 0 - \int_0^T \varepsilon_u dX_u + A_T | G_t \right]
\]

Since the stochastic integral \( \int \varepsilon_u dX_u \) is a martingale by Lemma 3.4, we have the result.

The risk-minimizing discounted amount invested in the risk free asset is thus given by:

\[
\eta_t^* = E^Q [A_T - A_t | G_t] - \varepsilon_t^* X_t
\]

We now have to determine the amount of risky assets \( \varepsilon^* \). Schweizer shows the solution of this problem is closely related to the so-called Galtchouk-Kunita-Watanabe decomposition of \( A_T \). Since \( I^2(X, Q, G) \) is a stable subspace of \( M_0^2(Q, G) \), any \( A_T \) square integrable random variable can be uniquely written as:

\[
A_T = E^Q [A_T | G_0] + \int_0^T \varepsilon_u^A dX_u + L_T^A
\]

where \( \varepsilon_u^A \in L^2(X, Q, G) \) and \( L_T^A \in M_0^2(Q, G) \), is \( Q \)-strongly orthogonal to \( I^2(X, Q, G) \). With these notation, we can give the solution of our risk minimizing problem in the following lemma:

**Lemma 3.13.** The unique risk minimizing RM-strategy \( \rho^* = (\varepsilon^*, \eta^*) \) of \( A \) is given by \( \varepsilon^* = \varepsilon^A \) and \( \eta_t^* = E^Q [A_T - A_t | G_t] - \varepsilon_t^* X_t \). The cumulative cost process of \( \rho^* \) is given by

\[
C_t(\rho^*) = E^Q [A_T | G_0] + L_t^A = C_0(\rho^*) + L_t^A
\]

and the value process of \( \rho^* \) is given by

\[
V_t(\rho^*) = E^Q [A_T - A_t | G_t]
\]

**Proof.** See Schweizer [78] for the single payoff case or Møller [68] for the extension to payment processes.

As Schweizer noticed in [78], we do not have to assume \( X \) is (locally) square integrable for these results to hold.

### 4. Application to a Life Insurance Contract When the \((H)\)-Hypothesis Holds.

In this section, we assume the following hypothesis holds:

**Condition 4.1.** The \((H)\)-hypothesis holds under \( Q^I \) between the filtrations \( J \) and \( \mathbb{I} \).

In our setting, we have the following equivalence:
Lemma 4.2. The $(H)$-hypothesis under $Q^I$ is equivalent to

\[ Q(\tau \leq t | \mathcal{J}_\infty) = Q(\tau \leq t | \mathcal{J}_t) \]

for any $t \geq 0$.

Proof. See Jeanblanc and Rutkowski [58]. □

The corollary of this lemma is that, under the $(H)$-hypothesis, $F_t$ and thus $\Gamma_t$ are increasing and accordingly, of finite variation.

By definition, when this $(H)$-hypothesis holds then any square integrable $(Q^I, \mathbb{J})$-martingale $m_t$ is a square integrable $(Q^I, \mathbb{I})$-martingale. If, in addition to the $(H)$-hypothesis, we assume Condition 2.3 holds then $\hat{m}_{t \wedge \tau}$ the $(Q^I, \mathbb{I})$-martingale part of $m_{t \wedge \tau}$ given by Equation (2.2) is indistinguishable of $m_{t \wedge \tau}$. In particular, we have:

\[ \hat{Q}_{t \wedge \tau}(u) = Q_{t \wedge \tau}(u) \]

As far as $Z_t$ is concerned, we have the following proposition:

Proposition 4.3. If Conditions 4.1 and 2.3 hold then the $(Q^I, \mathbb{J})$-martingale part $Z_t$ of the Doob-Meyer decomposition of the $(Q^I, \mathbb{J})$-submartingale $F_t$ is equal to 0 for all $t \geq 0$ a.s.

Proof. Conditions 2.3 and 4.1 implies $\langle Z, Z \rangle_t = 0$ for all $t > 0$ a.s. since $\hat{Z}_{t \wedge \tau} = Z_{t \wedge \tau}$. In turn, this implies $Z_t = 0$ for all $t > 0$ a.s. □

In addition of being increasing, $F_t$ is even $\mathbb{J}$-predictable since $F_t = D_t$. We can even say a little more thanks to the following proposition.

Proposition 4.4. If Condition 4.1 holds then Condition 2.3 holds if and only if the $(Q, \mathbb{J})$-hazard process $\Gamma_t$ is continuous.

Proof. First, we have

\[
E^Q[1_{\{\tau < t\}} | \mathcal{J}_t] = E^Q[\lim_{\epsilon \to 0} 1_{\{\tau \leq t - \epsilon\}} | \mathcal{J}_t] = \lim_{\epsilon \to 0} E^Q[1_{\{\tau \leq t - \epsilon\}} | \mathcal{J}_{t-\epsilon}] = \lim_{\epsilon \to 0} (1 - e^{-\Gamma_{t-\epsilon}}) = 1 - e^{-\Gamma_t}
\]

where $\epsilon > 0$. In the third equality, we use the fact that when the $(H)$-hypothesis holds, we have for all $s > t$,

\[
E^Q[1_{\{\tau \leq t\}} | \mathcal{J}_s] = E^Q[E^Q[1_{\{\tau \leq t\}} | \mathcal{J}_\infty] | \mathcal{J}_s] = E^Q[E^Q[1_{\{\tau \leq t\}} | \mathcal{J}_t] | \mathcal{J}_s] = E^Q[1_{\{\tau \leq t\}} | \mathcal{J}_t]
\]
where in the second equality we use Lemma 4.2. Let us denote by \( \theta \) an arbitrary \( \mathbb{J} \)-stopping time. We have:

\[
Q(\tau = \theta) = E^Q \left[ E^Q \left[ 1_{\{\tau \leq \theta \}} - 1_{\{\tau < \theta \}} \mid J_\theta \right] \right] \\
= E^Q \left[ (1 - e^{-\Gamma_\theta}) - (1 - e^{-\Gamma_{\theta-}}) \right] \\
= E^Q \left[ e^{-\Gamma_{\theta-}} \{ 1 - e^{-\Delta \Gamma_\theta} \} \right]
\]

On one hand, if \( \Gamma_t \) is continuous, we have \( Q(\tau = \theta) = 0 \) for any \( \mathbb{J} \)-stopping time \( \theta \). On the other hand, since \( \Gamma_t \) is increasing, \( e^{-\Gamma_{\theta-}} - e^{-\Gamma_\theta} \) is a positive random variable and \( Q(\tau = \theta) = 0 \) only if \( e^{-\Gamma_{\theta-}} - e^{-\Gamma_\theta} = 0 \). We thus have \( \Delta \Gamma_\theta = 0 \) for any \( \mathbb{J} \)-stopping time \( \theta \). □

Accordingly, when Conditions 4.1 and 2.3 hold \( F_t \) is continuous and increasing. Thanks to the uniqueness of the Doob-Meyer decomposition, it confirms the fact \( F_t = D_t \).

As far as the \((Q^I, \mathbb{I})\)-compensator \( \Lambda_t \) of \( H_t \) is concerned, we have the following proposition:

**Proposition 4.5.** If \( \Gamma_t \) is continuous and increasing, the \((Q^I, \mathbb{I})\)-compensator \( \Lambda_t \) of \( H_t \) is given by \( \Lambda_t = \Gamma_t \wedge \tau \).

**Proof.** The proof can be found in Jeanblanc and Rutkowski [58]. We give it here for the sake of completeness. Replacing \( D_t = F_t \) in Equation (2.1), we have:

\[
\Lambda_t = \int_0^{t \wedge \tau} e^{\Gamma_u} d(1 - e^{-\Gamma_u}) \\
= \int_0^{t \wedge \tau} d\Gamma_u
\]

The second equality depends on \( \Gamma_t \) being a finite variation continuous process. □

We denote the compensated process \( M_t^Q = H_t - \Gamma_t \wedge \tau \). We can now study the risk-minimizing strategy of each payment described in Section 2.4.

**4.1. Payment at the term of the contract.** We assume the insurer pays an amount \( g_T \) at the term \( T \) of the contract if the policyholder is still alive. The discounted payoff is then given by \( 1_{\{\tau > T \}} g_T \) where \( g_T \) is assumed to be \( \mathcal{F}_T \)-measurable and \( Q^F \)-square integrable.

**Proposition 4.6.** The value of the risk minimizing portfolio \( V_t^g(\rho^*) \) is given by

\[
V_t^g(\rho^*) = 1_{\{\tau > t \}} \frac{S_t(T)}{S_t(t)} E^Q \left[ \frac{g_T}{B_T} \mid \mathcal{F}_t \right]
\]

and the risk-minimizing strategy \( \rho^*,g = (\varepsilon^*,g, \eta^*,g) \) is given by

\[
E^Q \left[ \frac{g_T}{B_T} 1_{\{\tau > T \}} \mid \mathcal{G}_t \right] = V_0^g(\rho^*) + \int_{u \in [0,t]} \varepsilon_{u,g}^* dX_u + L_t^g
\]
where
\[ \varepsilon_u^{*,g} = 1_{\{\tau > u\}} e^{\Gamma_u - S_u - (T)\varepsilon_u^g} \]
and
\[
L_t^g = - \int_{u}^{t} e^{\Gamma_u - S_u - (T)} E^{Q} \left[ \frac{g_T}{B_T} | \mathcal{F}_u \right] \, dM_u^Q \\
+ \int_{u}^{t} 1_{\{\tau > u\}} e^{\Gamma_u - E^{Q} \left[ \frac{g_T}{B_T} | \mathcal{F}_u \right]} \, dS_u(T) \\
(4.2) \\
+ \int_{u}^{t} 1_{\{\tau > u\}} e^{\Gamma_u - S_u - (T)} dL_t^U^g
\]

The processes \( \varepsilon_t^g \) and \( L_t^U^g \) are given by the Galtchouk-Kunita-Watanabe decomposition of the purely financial contingent claim:
\[
E^{Q} \left[ \frac{g_T}{B_T} | \mathcal{F}_t \right] = E^{Q} \left[ \frac{g_T}{B_T} | \mathcal{F}_0 \right] + \int_{u}^{t} \varepsilon_u^g dX_u + L_t^U^g
\]

The discounted amount invested in risk free asset \( \eta_t^{*,g} \) is given by:
\[
\eta_t^{*,g} = 1_{\{\tau > t\}} \frac{S_t(T)}{S_t(t)} E^{Q} \left[ \frac{g_T}{B_T} | \mathcal{F}_t \right] - \varepsilon_t^{*,g} X_t
\]

The risk-minimizing strategy of this insurance payment is given by the product of the risk-minimizing strategy of a purely financial contingent claim \( \frac{g_T}{B_T} \) and the conditional probability at time \( t \) of survival up to time \( T \). Also the part that cannot be hedged through this risk minimizing strategy (the cost process) is equal to the sum of three terms. The first one is related to the unpredictability of the time of death, the second one is related to the random evolution of the \( T \)-survival probability and the third one is related to the incompleteness of the financial market.

**Proof.** The discounted value of the risk-minimizing strategy is given by:
\[
V_t^g(\rho^*) = E^{Q} \left[ 1_{\{\tau > T\}} \frac{g_T}{B_T} | \mathcal{G}_t \right] \\
= E^{Q'} \left[ 1_{\{\tau > T\}} | \mathcal{J}_t \cap \mathcal{H}_t \right] E^{Q''} \left[ \frac{g_T}{B_T} | \mathcal{F}_t \right] \\
= 1_{\{\tau > t\}} E^{Q'} \left[ 1_{\{\tau > T\}} | \mathcal{J}_t \cap \{\tau > t\} \right] E^{Q''} \left[ \frac{g_T}{B_T} | \mathcal{F}_t \right] \\
= 1_{\{\tau > T\}} \frac{S_t(T)}{S_t(t)} E^{Q''} \left[ \frac{g_T}{B_T} | \mathcal{F}_t \right]
\]

The risk-minimizing strategy \( \rho_t^{*,g} \) is given by the \((Q,G)\)-Galtchouk-Kunita-Watanabe of the \((Q,G)\)-square integrable martingale \( E^{Q} \left[ \frac{g_T}{B_T} 1_{\{\tau > T\}} | \mathcal{G}_t \right] \). Using Theorem 1 of Blanchet-Scalliet and Jeanblanc [30] and using the fact that (1) \( Z_t \equiv 0 \), (2) \( \Gamma_t \) is continuous and (3)
Since the process $U$ have:

\[ E^Q \left[ \frac{g}{B_T} 1_{\sigma > T} \mid \mathcal{G}_t \right] = E^Q \left[ \frac{g}{B_T} 1_{\sigma > T} \mid \mathcal{G}_0 \right] + \int_0^t 1_{\sigma > u} e^{r_u - dm^g_u} \]

(4.3)

\[ - \int_{u:0,t} e^{r_u - m^g_{u-} dM^Q_u} \]

where $m^g_t := E^Q \left[ 1_{\sigma > T} \frac{g}{B_T} \mid \mathcal{F}_t \land \mathcal{J}_t \right]$. Let us denote $U^g_t := E^{Q^F} \left[ \frac{g}{B_T} \mid \mathcal{F}_t \right]$. Thanks to the $Q$-independence between $\mathcal{F}$ and $\mathcal{J}$ and using the integration-by-parts formula, we have:

\[ m^g_t = S_t(T)U^g_t \]
\[ = \int_{u:0,t} S_u - (T) du^g_u + \int_{u:0,t} U^g_{u-} dS_u(T) + [S.(T),U^g]_t \]
\[ = m^g_0 + \int_{u:0,t} S_u - (T) du^g_u + \int_{u:0,t} U^g_{u-} dS_u(T) \]

Since the process $U^g_t$ is a square integrable $(Q^F,\mathcal{F})$-martingale, we can write its $(Q^F,\mathcal{F})$-GKW decomposition:

\[ U^g_t = U^g_0 + \int_{u:0,t} \varepsilon^g u dX_u + L^U^g_t \]

where $\varepsilon^g_t$ is an $\mathcal{F}$-predictable process in $L^2(X,Q^F,\mathcal{F})$ and $L^U^g_t$ is a square integrable $(Q^F,\mathcal{F})$-martingale strongly $(Q^F,\mathcal{F})$-orthogonal to $\mathcal{I}^2(X,Q^F,\mathcal{F})$. Replacing this equation in $m^g_t$, we have:

\[ m^g_t = m^g_0 + \int_{u:0,t} S_u - (T) \varepsilon^g u dX_u + \int_{u:0,t} S_u - (T) dL^U^g_t \]

(4.4)

\[ + \int_{u:0,t} U^g_{u-} dS_u(T) \]

Replacing Equation (4.4) in Equation (4.3), we have the result. Notice Equation (4.1) gives us the Galtchouk-Kunita-Watanabe of $E^Q \left[ \frac{g}{B_T} 1_{\sigma > T} \mid \mathcal{G}_t \right]$. Indeed, thanks to the $(H)$-hypothesis between $\mathcal{J}$ and $\mathcal{I}$ and the $Q$-independence between $\mathcal{F}$ and $\mathcal{I}$, $L^g_t$ defined by Equation (4.2) is a $(Q,G)$-square integrable martingale and is $Q$-strongly orthogonal to $\mathcal{I}^2(X,Q,G)$. \( \square \)

4.2. Payment at the time of death. We assume that the insurer pays an amount $R_t$ at the time of death $\tau$ if it happens before the term of the contract $T$. The discounted payoff is then given by $\int_0^T \frac{R_u}{B_u} dH_u$ where $\frac{R_u}{B_u}$ is assumed to be $\mathcal{F}$-predictable and $Q^F$-square integrable.

**Proposition 4.7.** The value of the risk minimizing portfolio $V^R_t(\rho^*)$ is given by

\[ V^R_t(\rho^*) = 1_{\tau > T} \frac{1}{S_t(t)} \int_{u:0,T} E^Q \left[ \frac{R_u}{B_u} \mid \mathcal{F}_t \right] S_t(du) \]

and the risk-minimizing strategy $\rho^{*,R} = (\varepsilon^{*,R}_t, \eta^{*,R}_t)$ is given by
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\( E^Q \left[ \int_{u[0,T]} \frac{R_u}{B_u} dH_u \mid \mathcal{G}_t \right] = V_0^R (\rho^*) + \int_{u[0,t]} \varepsilon_u^{*,R} dX_u + L^R_t \)

where

\( \varepsilon_u^{*,R} = 1_{\{\tau > u\}} e^{\Gamma_u^-} \int_{s[u,T]} \varepsilon_u^{UR}(s) Q_{u-}(ds) \)

and

\( L^R_t = \int_{u[0,t]} \left( \frac{R_u}{B_u} - e^{\Gamma_u^-} \int_{s[u,T]} \frac{R_s}{B_s} dH_s \mid \mathcal{F}_u \cup \mathcal{F}_u \right) dM^Q_u \)

\( + \int_{u[0,t]} 1_{\{\tau > u\}} e^{\Gamma_u^-} \int_{s[u,T]} \left( E^Q \left[ \frac{R_s}{B_s} \mid \mathcal{F}_u \right] - Q_{u-}(ds) \right) dQ_u \)

\( + \int_{u[0,t]} 1_{\{\tau > u\}} e^{\Gamma_u^-} \int_{s[u,T]} Q_{u-}(ds) dL^R_t \)

where \( \varepsilon_u^{UR}(s) \) and \( L^R_t \) for \( s > u \), are given by the following infinite family of G-K-W decompositions of the purely financial contingent claims \( U^R_t(s) := E^Q \left[ \frac{R_s}{B_s} \mid \mathcal{F}_t \right] \):

\( U^R_t(s) = U^R(s) + \int_{u[0,t]} \varepsilon_u^{UR}(s) dX_u + L^R_t(s) \)

The discounted amount invested in risk free asset is given by:

\( \eta_*^{*,R} = 1_{\{\tau > t\}} \frac{1}{S_t(t)} \int_{u[t,T]} E^Q \left[ \frac{R_u}{B_u} \mid \mathcal{F}_t \right] S_t(du) - \varepsilon_t^{*,R} X_t \)

Here the risk-minimizing strategy is given by the weighted average of the risk minimizing-strategies of the purely financial contingent claim \( \frac{R_s}{B_s} \). The weights are given by the conditional probabilities of death. Also the part that cannot be hedged through this risk-minimizing strategy (the cost process) is equal to the sum of three terms. The first one is related to the unpredictability of the time of death, the second one is related to the random evolution of the whole stochastic probability distribution function of the time of death and the third one is related to the incompleteness of the financial market. Notice in the paper of Dahl and Möller, a term similar to the second one appears. But in their paper, this term depends on the random evolution of a single probability (i.e. a single maturity). Here this term depends on the random evolution of the whole probability measure (i.e. on all maturities from \( t \) to \( T \)). This difference comes from the fact their model of the stochastic probability measure is a single factor model whereas our is a possibly infinite factors model. In a single factor model, the probabilities associated to different maturities are perfectly correlated and the movement of the whole curve can be summarized through the movement of a probability of a single maturity.
Proof. The discounted value of the risk minimizing strategy is given by:

\[
V_t^R(\rho^*) = E^Q \left[ \int_{u\geq t} \frac{R_u}{B_u} dH_u \mid \mathcal{G}_t \right]
\]

\[
= 1_{\{\tau > u\}} \frac{E^Q \left[ \int_{u\geq t} \frac{R_u}{B_u} dH_u \mid \mathcal{F}_t \right]}{S_t(t)}
\]

\[
= 1_{\{\tau > u\}} \frac{1}{S_t(t)} \int_{u \geq t} E^Q \left[ \frac{R_u}{B_u} \mid \mathcal{F}_t \right] S_t(du)
\]

In the second and third equalities, we use the \(Q\)-independence between \(\mathcal{F}\) and \(\mathcal{J}\).

The risk-minimizing strategy \(\rho^{*,R}\) is given by the Galtchouk-Kunita-Watanabe of the \((Q, G)\)-square integrable martingale \(E^Q \left[ \int_{u\geq 0} \frac{R_u}{B_u} dH_u \mid \mathcal{G}_t \right]\). Using Theorem 1 of Blanchet-Scalliet and Jeanblanc [30] and using the fact that (1) \(Z_t = 0\) and (2) \(\Gamma_t\) is continuous and \(m^R_{t,T} = \hat{m}^R_{t,T}\), we can write the following decomposition:

\[
E^Q \left[ \int_{u\geq 0} \frac{R_u}{B_u} dH_u \mid \mathcal{G}_t \right] = E^Q \left[ \int_{u\geq 0} \frac{R_u}{B_u} dH_u \mid \mathcal{G}_0 \right] + \int_0^t 1_{\{\tau > u\}} e^{\Gamma_u} - dm^R_u
\]

\[
+ \int_{u \geq t} \left( \frac{R_u}{B_u} - e^{\Gamma_u} - E^Q \left[ \int_{s \geq u} \frac{R_s}{B_s} dH_s \mid \mathcal{F}_u \right] \right) dM^Q_u
\]

(4.7)

where \(m^R_t = E^Q \left[ \int_{0 \leq u \leq T} \frac{R_u}{B_u} d(1 - e^{-\Gamma_u}) \mid \mathcal{F}_t \right]\). We show in Appendix 2, the process \(m^R_t\) can be written as:

\[
m^R_t = m^R_0 + \int_{s \geq 0} \int_{s \geq u} e^{\Gamma_u} (u) Q_{s-}(du) dX_s
\]

(4.8)

\[
+ \int_{s \geq 0} \int_{s \geq u} E^Q \left[ \frac{R_u}{B_u} \mid \mathcal{F}_s \right] dQ_{s-}(du)
\]

\[
+ \int_{s \geq 0} \int_{s \geq u} Q_{s-}(du) dL^U_{s-}(u)
\]

Replacing Equation (4.8) in Equation (4.7), we have the result. Thanks to the \((H)\)-hypothesis between \(\mathcal{J}\) and \(\mathcal{I}\) and the \(Q\)-independence between \(\mathcal{F}\) and \(\mathcal{I}\), \(L^R_t\) defined by Equation (4.6) is a \((Q, G)\)-square integrable martingale and is \(Q\)-strongly orthogonal to \(\mathcal{T}_t(X, Q, G)\). Thus, Equation (4.5) gives us indeed the \((Q, G)\)-Galtchouk-Kunita-Watanabe decomposition of \(E^Q \left[ \int_{u\geq 0} \frac{R_u}{B_u} dH_u \mid \mathcal{G}_t \right]\).

4.3. Payment up to the time of death. We assume the insurer pays an accumulated amount \(C_u\) up to the time of death of the policyholder. The discounted payoff is then given by \(\int_{u\geq 0} \frac{1_{\{\tau > u\}}}{B_u} dC_u\), where \(C_t\) is assumed \(\mathcal{F}\)-adapted and where \(\int_{u\geq 0} \frac{1}{B_u} dC_u\) is assumed square \(Q\)-integrable.

Proposition 4.8. The value of the risk-minimizing portfolio \(V_t^C(\rho^*)\) is given by
V_t^C(\rho^*) = 1_{\{\tau>t\}} \frac{1}{S_t(t)} \int_{u,[t,T]} S_t(u) U_t^C(du)

and the risk minimizing strategy \(\rho^{*,C} = (\varepsilon_t^{*,C}, \eta_t^{*,C})\) is given by

\[E^Q \left[ \int_{[0,T]} 1_{\{\tau>u\}} \frac{1}{B_u} dC_u \mid G_t \right] = V_0^C(\rho^*) + \int_{[0,t]} \varepsilon_u^{*,C} dX_u + L_t^C\]

where

\[\varepsilon_u^{*,C} = 1_{\{\tau>u\}} e^{\Gamma_u} \int_{[u,T]} S_u(s) \varepsilon_u^{*,C}(ds)\]

and

\[L_t^R = - \int_{[0,t]} e^{\Gamma_u} E^Q \left[ \int_{[u,T]} 1_{\{\tau>s\}} \frac{1}{B_s} dC_s \mid F_u \sqcup J_u \right] dM^Q_u\]

\[+ \int_{[0,t]} 1_{\{\tau>u\}} e^{\Gamma_u} \int_{[u,T]} U_u^C(ds) dS_u(s)\]

\[+ \int_{[0,t]} 1_{\{\tau>u\}} e^{\Gamma_u} \int_{[u,T]} S_u(s) dL_u^C(ds)\]

and where \(\varepsilon_u^U(s)\) and \(L_u^C(s)\) for \(s > u\), are given by the infinite family of G-K-W decompositions of the purely financial contingent claims \(U_t^C(s) := E^Q \left[ \int_{[0,s]} \frac{1}{B_u} dC_u \mid F_t \right]\)

\[U_t^C(s) = U_0^C(s) + \int_{[0,t]} \varepsilon_u^U(s) dX_u + L_t^C(s)\]

The discounted amount invested in risk free asset is given by:

\[\eta_t^{*,C} = 1_{\{\tau>t\}} \frac{1}{S_t(t)} \int_{[u,[t,T]} S_t(u) U_t^C(du) - \varepsilon_t^{*,C} X_t\]

Here, the risk-minimizing strategy is given by a weighted average of survival probabilities where the weights are given by the risk-minimizing strategies of the purely financial contingent claim \(\int_{[0,s]} \frac{1}{B_u} dC_u\). Again, the part that cannot be hedged through this risk minimizing strategy (the cost process) is equal to the sum of three terms. The first one is related to the unpredictability of the time of death, the second one is related to the random evolution of the whole stochastic survival probability distribution function and the third one is related to the incompleteness of the financial market.
Proof. The discounted value of the risk-minimizing strategy is given by:

\[ V_C^t(\rho^*) = E^Q \left[ \int_{u \in ]t,T]} \frac{1_{\{t > u\}}}{B_u} dC_u | \mathcal{G}_t \right] \]

\[ = 1_{\{t > t\}} \frac{1}{S_t(t)} E^Q \left[ \int_{u \in ]t,T]} \frac{1_{\{t > u\}}}{B_u} dC_u | \mathcal{F}_t \vee \mathcal{J}_t \right] \]

\[ = 1_{\{t > t\}} \frac{1}{S_t(t)} \int_{u \in ]t,T]} S_t(u) U_C^t(du) \]

where \( U_C^t(u) = E^Q \left[ \int_{[0,u]} \frac{1_{\{t > u\}}}{B_u} dC_s | \mathcal{F}_t \right] \). In the second and third equalities, we use the \( Q \)-independence between \( \mathcal{F} \) and \( \mathcal{J} \).

The risk-minimizing strategy \( \rho^*,C \) is given by the Galtchouk-Kunita-Watanabe of the \((Q, \mathcal{G})\)-square integrable martingale \( E^Q \left[ \int_{u \in ]0,T]} \frac{1_{\{t > u\}}}{B_u} dC_u | \mathcal{G}_0 \right] \). Using Theorem 1 of Blanchet-Scalliet and Jeanblanc [30] and using the fact that (1) \( Z_t = 0 \), (2) \( \Gamma_t \) is continuous and (3) \( m^C_t \wedge \tau = \hat{m}^C_t \wedge \tau \), we can write the decomposition:

\[ E^Q \left[ \int_{u \in ]0,T]} \frac{1_{\{t > u\}}}{B_u} dC_u | \mathcal{G}_t \right] = E^Q \left[ \int_{u \in ]0,T]} \frac{1_{\{t > u\}}}{B_u} dC_u | \mathcal{G}_0 \right] + \int_0^t 1_{\{t > u_0\}} e^{\Gamma_u - dm^C_u} \]

\[ = \int_{u \in ]0,t]} \left( e^{\Gamma_u - E^Q \left[ \int_{s \in ]u,T]} \frac{1_{\{t > s\}}}{B_s} dC_s | \mathcal{F}_u \vee \mathcal{J}_u \right] \right) dM^Q_u \]

where \( m^C_t = E^Q \left[ \int_{[0,T]} \frac{e^{-\Gamma_u}}{B_u} dC_u | \mathcal{F}_t \vee \mathcal{J}_t \right] \). We show in Appendix 3 that the process \( m^C_t \) can be written as:

\[ m^C_t = J^C_0 + \int_{s \in ]0,t]} \left( \int_{u \in ]s,T]} S_{s-}(u) \epsilon^C_s(du) \right) dX_s \]

\[ + \int_{s \in ]0,t]} \left( U^C_{s-}(du) ds(u) + \int_{s \in ]0,t]} \left( U^C_{s-}(du) dL^C_s(u) \right) \right) \]

Replacing Equation (4.10) in Equation (4.9), we have the result. Again, we have indeed the \((Q, \mathcal{G})\)-Galtchouk Kunita Watanabe of \( E^Q \left[ \int_{u \in ]0,T]} \frac{1_{\{t > u\}}}{B_u} dC_u | \mathcal{G}_t \right] \).

5. Application to a Life Insurance Contract When the \((H)\)-Hypothesis Does Not Hold.

In this section, we drop the assumption that the \((H)\)-hypothesis holds under \( Q^I \) between the filtration \( \mathcal{J} \) and \( \mathcal{I} \) but we keep Condition 2.3. In this case, the process \( Z_t \) in the Doob-Meyer decomposition of \( F_t \) is not necessarily equal to 0. We are in the general setting described in Section 2.2.
5.1. Payment at the term of the contract. We assume the insurer pays an amount $g_T$ at the term $T$ of the contract if the policyholder is still alive. The discounted payoff is then given by $1_{\{T>T\}} \frac{B_T}{B_T}$ where $g_T$ is assumed to be $\mathcal{F}_T$-measurable and $Q$-square integrable.

Proposition 5.1. The value of the risk minimizing portfolio $V_t^g(\rho^*)$ is given by

$$ V_t^g(\rho^*) = 1_{\{T>t\}} \frac{S_t(T)}{S_t(t)} E^Q \left[ \frac{g_T}{B_T} | \mathcal{F}_t \right] $$

and the risk-minimizing strategy $\rho^g = (\varepsilon^g, \eta^g)$ is given by

$$ E^Q \left[ \frac{g_T}{B_T} 1_{\{T>T\}} | \mathcal{G}_t \right] = V_0^g(\rho^*) + \int_{u \in [0,t]} \varepsilon^{*,g}_u dX_u + \hat{L}_t^g $$

where

$$ \varepsilon^{*,g}_u = 1_{\{T>u\}-} e^{\Gamma_u S_u-(T)} \varepsilon_u^g $$

$$ \hat{L}_t^g = - \int_{u \in [0,t]} e^{\Gamma_u S_u-(T)} E^Q \left[ \frac{g_T}{B_T} | \mathcal{F}_u \right] dM^Q_u + \int_{u \in [0,t]} 1_{\{T>u\}-} e^{2\Gamma_u S_u-(T)} E^Q \left[ \frac{g_T}{B_T} | \mathcal{F}_u \right] d\hat{Z}_u $$

$$ + \int_{u \in [0,t]} 1_{\{T>u\}-} e^{\Gamma_u S_u-(T)} E^Q \left[ \frac{g_T}{B_T} | \mathcal{F}_u \right] d\hat{S}_u(T) $$

$$ + \int_{u \in [0,t]} 1_{\{T>u\}-} e^{\Gamma_u S_u-(T)} dL^g_u $$

where $\varepsilon_u^g$ and $L^g_u$, are given by the G-K-W decomposition of the purely financial contingent claim:

$$ E^Q \left[ \frac{g_T}{B_T} | \mathcal{F}_t \right] = E^Q \left[ \frac{g_T}{B_T} | \mathcal{F}_0 \right] + \int_{u \in [0,t]} \varepsilon_u^g dX_u + L^g_u $$

The discounted amount invested in risk free asset $\eta^{*,g}$ is given by:

$$ \eta^{*,g}_t = 1_{\{T>t\}} \frac{S_t(T)}{S_t(t)} E^Q \left[ \frac{g_T}{B_T} | \mathcal{F}_t \right] - \varepsilon^{*,g}_t X_t $$

Proof. The value of the risk-minimizing strategy $V_t^g(\rho^*)$ remains unchanged when we drop the $(H)$-hypothesis. We can thus follow the same proof than in Proposition 4.6. As far as the GKW decomposition of the $(Q, \mathcal{G})$-martingale $E^Q \left[ \frac{g_T}{B_T} 1_{\{T>T\}} | \mathcal{G}_t \right]$ is concerned, using Theorem 1 of Blanchet-Scalliet and Jeanblanc [30], we can write
\[ E^Q \left[ \frac{g_T}{B_T} 1_{\{\tau > T\}} \mid \mathcal{G}_t \right] = E^Q \left[ \frac{g_T}{B_T} 1_{\{\tau > T\}} \mid \mathcal{G}_0 \right] + \int_0^t 1_{\{\tau > u\}} e^{\Gamma u} d\hat{m}_u^g \]

\[ - \int_{u \mid [0,t]} e^{\Gamma u} m_u^g dM^Q_u \]

\[ + \int_{u \mid [0,t]} 1_{\{\tau > u\}} e^{2\Gamma u} m_u^g d\tilde{Z}_u \]

where, on \( \{ \tau > t \} \), \( \hat{m}_u^g \) is the \( \mathcal{I} \)-martingale part of

\[ m_t^g = E^Q \left[ \frac{g_T}{B_T} 1_{\{\tau > T\}} \mid \mathcal{F}_t \vee \mathcal{J}_t \right] \]

\[ = S_t(T) U_t^g \]

given, on \( \{ \tau > t \} \), by

\[ \hat{m}_t^g = m_t^g + \int_0^t e^{\Gamma s} d[m_\cdot, Z_\cdot]_s \]

We saw in the proof of Proposition 4.6 that

\[ m_t^g = m_0^g + \int_{u \mid [0,t]} S_u(T) dU_t^g + \int_{u \mid [0,t]} U_u^g dS_u(T) \]

On \( \{ \tau > t \} \), we thus have

\[ \hat{m}_t^g = m_t^g + \int_0^t S_u(T) e^{\Gamma u} d[U^g, Z_\cdot]_u + \int_0^t U_u^g e^{\Gamma u} d[S(T), Z_\cdot]_u \]

\[ = m_t^g + \int_0^t U_u^g e^{\Gamma u} d[S(T), Z_\cdot]_u \]

\[ = m_0^g + \int_{u \mid [0,t]} S_u(T) dU_u^g + \int_{u \mid [0,t]} U_u^g d[S(T) + \int_0^u e^{\Gamma s} d[S(T), Z_\cdot]_s] \]

\[ = m_0^g + \int_{u \mid [0,t]} S_u(T) dU_u^g + \int_{u \mid [0,t]} U_u^g d\tilde{S}_u(T) \]

In the second equality, we use the fact \( U^g \) and \( Z \) are strongly orthogonal. Using the decomposition for \( E^Q \left[ \frac{g_T}{B_T} \mid \mathcal{F}_u \right] \) as in Proposition 4.6, we get the result.

The important point to notice here is that the risk minimizing strategy \( \rho^{*,g} = (\varepsilon^{*,g}_t, \eta^{*,g}_t) \) remains unchanged when we drop the \( (H) \)-hypothesis. Only the cost process is modified.

### 5.2. Payment at the time of death

The risk minimizing strategy of the payment at the time of death is given by
Proposition 5.2. The value of the risk minimizing portfolio $V_t^R(\rho^*)$ is given by

$$V_t^R(\rho^*) = 1_{\{\tau > t\}} \frac{1}{S_t(t)} \int_{u[t,T]} EQ \left[ \frac{R_u}{B_u} | F_t \right] S_t(du)$$

and the risk minimizing strategy $\rho^{*,R} = (\varepsilon^{*,R}_t, \eta^{*,R}_t)$ is given by

$$EQ \left[ \int_{u[0,T]}^t \frac{R_u}{B_u} dH_u | G_t \right] = V_0^R(\rho^*) + \int_{u[0,t]} \varepsilon^{*,R}_u dX_u + \hat{L}_t^R$$

where

$$\varepsilon^{*,R}_u = 1_{\{\tau > u\}} e^{\Gamma_u - \int_{s[u,T]} \varepsilon^{*,R}_u(s) Q_u - (ds)}$$

and

$$\hat{L}_t^R = \int_{u[0,t]} \left( \frac{R_u}{B_u} - e^{\Gamma_u - EQ \left[ \int_{s[u,T]}^t \frac{R_s}{B_s} dH_s | F_u \vee J_u \right] } \right) e^{\Delta \Gamma_u} dM^Q_u$$

$$- \int_{u[0,t]} 1_{\{\tau > u\}} e^{\Gamma_u - \int_{s[u,T]} \varepsilon^{*,R}_u(s) Q_u - (ds)} \left( \frac{R_u}{B_u} - e^{\Gamma_u - EQ \left[ \int_{s[u,T]}^t \frac{R_s}{B_s} dH_s | F_u \vee J_u \right] } \right) d\hat{Z}_u$$

$$+ \int_{u[0,t]} 1_{\{\tau > u\}} e^{\Gamma_u - \int_{s[u,T]}^t EQ \left[ \frac{R_s}{B_s} | F_u \right] - d\hat{Q}_u(ds)}$$

$$+ \int_{u[0,t]} 1_{\{\tau > u\}} e^{\Gamma_u - \int_{s[u,T]}^t Q_u - (ds)} dL^U_R(s)$$

where $\varepsilon^{uR}_u(s)$ and $L^U_R(s)$ for $s > u$, are given by the following infinite family of G-K-W decompositions of the purely financial contingent claims $U^R_t(s) := EQ \left[ \frac{R_t}{B_t} | F_t \right] :$

$$U^R_t(s) = U_0^R(s) + \int_{u[0,t]} \varepsilon^{uR}_u(s) dX_u + L^U_R(s)$$

The discounted amount invested in risk free asset is given by:

$$\eta^{*,R}_t = 1_{\{\tau > t\}} \frac{1}{S_t(t)} \int_{u[t,T]} EQ \left[ \frac{R_u}{B_u} | F_t \right] S_t(du) - \varepsilon^{*,R}_t X_t$$

Proof. The value of the risk-minimizing strategy $V_t^R(\rho^*)$ remains unchanged when we remove the $(H)$-hypothesis. We can thus follow basically the same proof than in Proposition 4.7. As far as the GKW decomposition of the square integrable $(Q,G)$-martingale $EQ \left[ \int_{u[0,T]}^t \frac{R_u}{B_u} dH_u | G_t \right]$ is concerned, using Theorem 1 of Blanchet-Scalliet and Jeanblanc [30],
we can write

\[
E^Q \left[ \int_{u:[0,T]} \frac{R_u}{B_u} dH_u | G_t \right] = E^Q \left[ \int_{u:[0,T]} \frac{R_u}{B_u} dH_u | G_0 \right] + \int_0^t 1_{\{\tau > u\}} e^{\Gamma_u - \Delta u} \tilde{m}^R_t d\tilde{M}_u^Q
\]

\[+ \int_{u:[0,T]} \left( \frac{R_u}{B_u} - e^{\Gamma_u - \Delta u} E^Q \left[ \int_{s:[u,T]} \frac{R_s}{B_s} dH_s | F_u \cup J_u \right] \right) e^{\Delta u} dM_u^Q \]

(5.1) \[- \int_{u:[0,T]} 1_{\{\tau > u\}} e^{\Gamma_u - \Delta u} \left( \frac{R_u}{B_u} - e^{\Gamma_u - \Delta u} E^Q \left[ \int_{s:[u,T]} \frac{R_s}{B_s} dH_s | F_u \cup J_u \right] \right) d\tilde{Z}_u \]

where \( \tilde{m}^R_t \) is the \( (Q, \mathbb{I}) \)-martingale part of \( m^R_t = E^Q \left[ \int_{[0,T]} \frac{R_s}{B_s} d(1 - e^{-\Gamma_s}) | F_t \cup J_t \right] \) given by Equation (2.2). In Appendix 2, we show that:

\[
m^R_t = m^R_0 + \int_{s:[0,t]} \int_{u:[s,T]} \varepsilon^R_s(u) Q_{s-}(du) dX_s
\]

\[+ \int_{s:[0,t]} \int_{u:[s,T]} U^R_{s-}(u) dQ_s(du) + \int_{s:[0,t]} \int_{u:[s,T]} Q_{s-}(du) dL^R_{s-}(u) \]

So, on \( \{ \tau > t \} \), we have

\[
\tilde{m}^R_t = m^R_t + \int_0^t e^{\Gamma_s} d[m^R_t, Z.]_s
\]

\[= m^R_t + \int_0^t \int_{u:[s,T]} U^R_{s-}(u) e^{\Gamma_s} d[Q_s(du), Z.]_s \]

\[= m^R_0 + \int_{s:[0,t]} \int_{u:[s,T]} \varepsilon^R_s(u) Q_{s-}(du) dX_s + \int_{s:[0,t]} \int_{u:[s,T]} Q_{s-}(du) dL^R_{s-}(u) + \int_{s:[0,t]} \int_{u:[s,T]} U^R_{s-}(u) dQ_s(du) + \int_{s:[0,t]} \int_{u:[s,T]} Q_{s-}(du) dL^R_{s-}(u) \]

Eventually, on \( \{ \tau > t \} \), the decomposition of \( \tilde{m}^R_t \) is thus given by:

\[
\tilde{m}^R_t = m^R_0 + \int_{s:[0,t]} \int_{u:[s,T]} \varepsilon^R_s(u) Q_{s-}(du) dX_s + \int_{s:[0,t]} \int_{u:[s,T]} Q_{s-}(du) dL^R_{s-}(u) + \int_{s:[0,t]} \int_{u:[s,T]} U^R_{s-}(u) dQ_s(du) \]
Replacing this last equation in Equation (5.1), we get the result.

As in the previous proposition, the risk-minimizing strategy remains unchanged when we drop the \((H)\)-hypothesis.

### 5.3. Payment up to the time of death.

**Proposition 5.3.** The value of the risk-minimizing portfolio \(V_t^C(\rho^*)\) is given by

\[
V_t^C(\rho^*) = 1_{\{\tau > t\}} \frac{1}{S_t(t)} \int_{u\in[t,T]} S_t(u) U_t^C(du)
\]

and the risk minimizing strategy \(\rho_t^{*,C} = (\varepsilon_t^{*,C}, \eta_t^{*,C})\) is given by

\[
E^Q \left[ \int_{u\in[0,T]} \frac{1_{\{\tau > u\}}}{B_u} dC_u | G_t \right] = V_0^C(\rho^*) + \int_{u\in[0,t]} \varepsilon_u^{*,C} dX_u + \hat{L}_t^C
\]

with

\[
\varepsilon_u^{*,C} = 1_{\{\tau > u\}} e^\Gamma_u \int_{s\in[u,T]} S_u(s) \varepsilon_u^{UC}(ds)
\]

and

\[
\hat{L}_t^R = -\int_{u\in[0,t]} e^\Gamma_u E^Q \left[ \int_{s\in[u,T]} \frac{1_{\{\tau > s\}}}{B_s} dC_s | F_u \vee J_u \right] e^{\Delta \Gamma_u} dM_u^Q
\]

\[
+ \int_{u\in[0,t]} 1_{\{\tau > u\}} e^\Gamma_u E^Q \left[ \int_{s\in[u,T]} \frac{1_{\{\tau > s\}}}{B_s} dC_s | F_u \vee J_u \right] e^{\Gamma_u} d\hat{Z}_u
\]

\[
+ \int_{u\in[0,t]} 1_{\{\tau > u\}} e^\Gamma_u \int_{s\in[u,T]} U_u^{UC}(ds) d\hat{S}_u(s)
\]

\[
+ \int_{u\in[0,t]} 1_{\{\tau > u\}} e^\Gamma_u \int_{s\in[u,T]} S_u(s) dL_u^{UC}(ds)
\]

and where \(\varepsilon_u^{UC}(s)\) and \(L_u^{UC}(s)\) for \(s > u\), are given by the infinite family of G-K-W decompositions of the purely financial contingent claims \(U_t^C(s) := E^Q \left[ \int_{s\in[0,T]} \frac{1_{\{\tau > s\}}}{B_s} dC_u | F_t \right] \)

\[
U_t^C(s) = U_0^C(s) + \int_{u\in[0,t]} \varepsilon_u^{UC}(s) dX_u + L_t^{UC}(s)
\]

The discounted amount invested in risk free asset is given by:

\[
\eta_t^{*,C} = 1_{\{\tau > t\}} \frac{1}{S_t(t)} \int_{u\in[t,T]} S_t(u) U_t^C(du) - \varepsilon_t^{*,C} X_t
\]
Proof. Again, as for the other payments, the value of the risk-minimizing strategy \( V_t^C(\rho^*) \) remains unchanged when we remove the \((H)\)-hypothesis. We can thus follow basically the same proof than in Proposition 4.8. As far as the GKW decomposition of the \((Q,G)\)-martingale \( E^Q \left[ \int_{u:0,T} \frac{1_{\{\tau > u\}}}{B_u} dC_u | \mathcal{G}_t \right] \) is concerned, using Theorem 1 of Blanchet-Scalliet and Jeanblanc [30], we can write

\[
E^Q \left[ \int_{u:0,T} \frac{1_{\{\tau > u\}}}{B_u} dC_u | \mathcal{G}_t \right] = E^Q \left[ \int_{u:0,T} \frac{1_{\{\tau > u\}}}{B_u} dC_u | \mathcal{G}_0 \right] + \int_0^t 1_{\{\tau > u\}} e^{\Gamma_u} \hat{m}^C_u \\
- \int_{u:0,t} e^{\Gamma_u} E^Q \left[ \int_{u:s,T} \frac{1_{\{\tau > s\}}}{B_s} dC_s | \mathcal{F}_u \cup \mathcal{J}_u \right] e^\Delta \hat{m}^C_u dM^Q_u \\
+ \int_{u:0,t} 1_{\{\tau > u\}} e^{\Gamma_u} E^Q \left[ \int_{u:s,T} \frac{1_{\{\tau > s\}}}{B_s} dC_s | \mathcal{F}_u \cup \mathcal{J}_u \right] d\tilde{Z}_u
\]

(5.2)

where \( \hat{m}^C_t \) is the \( \mathbb{I}\)-martingale part of \( m^C_t = E^Q \left[ \int_{[0,T]} \frac{\epsilon^{\gamma \Gamma_u} - 1}{B_u} dC_u | \mathcal{F}_t \cup \mathcal{J}_t \right] \) given by Equation (2.2). In Appendix 3, we show that

\[
m^C_t = m^C_0 + \int_{s:0,T} \int_{u:s,T} S_{s-u} \xi^C_s(u) dX_s + \int_{s:0,T} S_{s-u} dL^C_s(u) + \int_{s:0,T} U^C_s(u) dS_s(u)
\]

So on \( \{\tau > t\} \), we have

\[
\hat{m}^C_t = m^C_t + \int_{s:0,T} e^{\Gamma_u} d[m^C, Z]_s
\]

\[
= m^C_t + \int_{s:0,T} e^{\Gamma_u} d \left[ U^C_{s-u}(du) S_s(u), Z \right]_s
\]

\[
= m^C_t + \int_{s:0,T} e^{\Gamma_u} d \left[ U^C_{s-u}(du) S_s(u), Z \right]_s
\]

\[
= m^C_t + \int_{s:0,T} \int_{u:s,T} U^C_{s-u}(du) d \left[ S_s(u), Z \right]_s
\]

\[
= m^C_t + \int_{s:0,T} \int_{u:s,T} U^C_{s-u}(du) d \left[ e^{\Gamma_u} d \left[ S_s(u), Z \right]_v \right]
\]

\[
= m^C_t + \int_{s:0,T} \int_{u:s,T} U^C_{s-u}(du) \left[ \int_{v:s,T} S_{s-u} \xi^C_s(u) dX_s + \int_{s:0,T} S_{s-u} dL^C_s(u) + \int_{s:0,T} S_{s-u} dL^C_s(u) \right]
\]

\[
+ \int_{s:0,T} U^C_{s-u}(du) \left[ S_s(u) + \int_{v:s,T} e^{\Gamma_u} d \left[ S_s(u), Z \right]_v \right]
\]
Eventually, on \( \{ \tau > t \} \), the decomposition of \( \hat{m}_t^C \) is given by:

\[
\hat{m}_t^C = m_0^C + \int_{s \in [0,t]} \left( \int_{u \in [s,T]} S_{s-}(u)\varepsilon^U_s(du) \right) dX_s
\]

\[
+ \int_{s \in [0,t]} \int_{u \in [s,T]} S_{s-}(u)dU_s^C(du)
\]

\[
+ \int_{s \in [0,t]} \int_{u \in [s,T]} U_s^C(du)d\hat{S}_s(u)
\]

Replacing this equation in Equation (5.2), we get the result. \( \square \)

As in the previous propositions, the risk-minimizing strategy remains unchanged when we drop the \((H)\)-hypothesis.

6. Conclusion.

In this chapter, we studied the risk-minimizing strategies of a single life insurance policy. Our main conclusion is that these strategies can be written as an average of purely financial strategies, weighted by the stochastic survival probabilities.

The fact that we focused on a single life insurance policy might seem rather limited. However, we can easily find the risk-minimizing strategies for a portfolio of policies if we assume that the random times of death of the different policyholders are independent conditionally on \( J \). In this case, the risk-minimizing strategies for the portfolio are simply given by the sum of the risk-minimizing strategies of each policy as described here (i.e. as if they were considered individually).

Appendix 1.

**Proposition. Integration by parts formula.**

Let us consider two Banach spaces \( F \) and \( E \) and a bilinear bounded application \((\cdot,\cdot) : F \times E \rightarrow \mathbb{R}\). Let \( G_t \) (resp. \( H_t \)) be an \( E \)-valued (resp. \( F \)-valued) càdlàg stochastic process. We assume \((\Delta H_t, \Delta G_t) = 0, \forall t \geq 0 Q\text{-a.s.}\)

If \( G_t \) is an \( E \)-valued local \( Q \)-martingale, \( H_t \) an \( F \)-valued local \( Q \)-martingale, then, if \( (H_t,G_t) \) is a real-valued local \( Q \)-martingale, we have:

\[
(H_t,G_t) = (H_0,G_0) + \int_{[0,t]} (H_{s-},dG_s) + \int_{[0,t]} (dH_s,G_{s-})
\]

\( Q\text{-a.s.}, \) where the stochastic integrals are defined as in Gravereaux and Pellaumail [46].

**Proof.** In [46], Gravereaux and Pellaumail defined a stochastic integral for Banach space-valued processes. More precisely, for three Banach spaces \( E,F,B \) and a bilinear bounded application \((\cdot,\cdot) : F \times E \rightarrow B\), they define a \( B \)-valued stochastic integral for \( E \)-valued càdlàg stochastic integrators and \( F \)-valued càglàd stochastic integrands, as a sequence of elementary
stochastic integrals converging in ucp. For \( X_t \) a \( E \)-valued càdlàg stochastic process and \( Y_t \) a \( F \)-valued càglàd stochastic process, they denote their \( B \)-valued càdlàg stochastic process \( Z_t \) as

\[
Z_t = \int_{[0,t]} (Y_s, dX_s)
\]

In our proposition, we assume \( B = \mathbb{R} \) equipped with the usual norm. Let us define the real-valued process \( K_t \):

\[
K_t := (H_0, G_0) + \lim_{n \to \infty} \sum_i \left( H_{T^n_{i+1}} - H_{T^n_i}, G_{T^n_{i+1}} - G_{T^n_i} \right)
\]

where the stochastic integrals are defined as in \([46]\). We have to show that \( K_t = 0 \) \( \forall t \geq 0 \), \( Q \)-a.s. We prove this in 4 steps.

1) Thanks to the bilinearity of \((\cdot, \cdot)\) and the definition of Graveraux and Pellaumail of the stochastic integrals as converging sequence in ucp of elementary stochastic integrals, we can show this process \( K_t \) is the limit in ucp of

\[
K_t = (H_0, G_0) + \lim_{n \to \infty} \sum_i \left( H_{T^n_{i+1}} - H_{T^n_i}, G_{T^n_{i+1}} - G_{T^n_i} \right)
\]

where \((\sigma_n)_{n \geq 1}\) is a sequence of random partitions \( 0 = T^n_0 \leq T^n_1 \leq \cdots \leq T^n_i \leq \cdots \leq T^n_{k_n} \) tending to the identity. We can show, as in the usual integration by parts formula for real-valued stochastic processes, that this limit tends to a finite variation process.

2) Furthermore, we have that \( \Delta K_t = \Delta (H_t, G_t) - (H_{t-}, \Delta G_t) - (\Delta H_t, G_{t-}) \). Thanks to the bilinearity of the operator \((\cdot, \cdot)\), simple algebraic manipulations lead to

\[
\Delta K_t = (\Delta H_t, \Delta G_t)
\]

Since by assumption \((\Delta H_t, \Delta G_t) = (H_0, G_0) \forall t \geq 0 \) \( Q \)-a.s., \( K_t \) is thus, a.s., a continuous process.

3) We can prove that a stochastic integral with respect to a Banach-valued local martingale integrator and for a càglàd integrand, is a real-valued local martingale. Since \((H_t, G_t)\) is, by assumption, a local \( Q \)-martingale, \( K_t \) is thus a real-valued local \( Q \)-martingale.

4) Since a predictable finite-variation real-valued local martingale is constant, we have that \( K_t = (H_0, G_0) \) and we can thus conclude that

\[
(H_t, G_t) = (H_0, G_0) + \int_{[0,t]} (H_s, dG_s) + \int_{[0,t]} (dH_s, G_{s-})
\]

\( \square \)

**Appendix 2.**

**Proposition.** The Galtchouk-Kunita-Watanabe decomposition of

\[
m_t^R = E^Q \left[ \int_{[0,T]} \frac{R_u}{B_u} d \left( 1 - e^{-\Gamma_u} \right) |\mathcal{F}_t \cap \mathcal{J}_t \right]
\]
is given by:
\[
m_t^R = m_0^R + \int_{s_0}^s \left[ \int \varepsilon_s^R(u) Q_{s-}(du) \right] dX_s \\
+ \int_{s_0}^s \int \varepsilon_s^R(u) dQ_s(du) + \int_{s_0}^s \int Q_{s-}(du) dL^R_s(u)
\]

**Proof.** We can rewrite \( m_t^R \) as:
\[
m_t^R = \int_{[0,t]} R_u \left( 1 - e^{-\Gamma_u} \right) - \int_{[0,t]} R_u Q_t(du) + E^Q \left[ \int_{[0,T]} R_u Q_t(du) | \mathcal{F}_t \cup \mathcal{J}_t \right]
\]

Let us focus on \( J_t^R := E^Q \left[ \int_{[0,T]} R_u Q_t(du) | \mathcal{F}_t \cup \mathcal{J}_t \right] \). Thanks to the \( Q \)-independence between \( \mathcal{F} \) and \( \mathcal{J} \), we can write:
\[
J_t^R = \int_{[0,T]} U_t^R(u) Q_t(du)
\]

The crucial point here is to notice \( Q_t(\cdot) \) can be seen as a process taking its values in the space of finite signed measures on \([0,T], B([0,T]) \). This space equipped with the usual norm is a Banach space. The process \( U_t^R(\cdot) \) can also be seen as a process taking its values in the space of bounded Borel measurable functions on \([0,T], BM([0,T]) \), equipped with the supremum norm. This space is also a Banach space. For \( h(\cdot) \in BM([0,T]) \) and \( \mu(\cdot) \) a finite measure on \([0,T], B([0,T]) \), the application \( (h, \mu) \) defined as \( (h, \mu) = \int_0^T h(u) \mu(du) \), is thus a bilinear bounded application. Accordingly, we have that
\[
J_t^R = (U_t^R, Q_t)
\]

and the result of Appendix 1 applies. We can thus write the process \( J_t^R \) as:
\[
J_t^R = J_0^R + \int_{[0,t]} (dU_t^R, Q_{s-}) + \int_{[0,t]} (U_t^R, dQ_s)
\]

From now on, we use the obvious following notation:
\[
J_t^R = J_0^R + \int_{[s,0]} \int_{[u,0]} Q_{s-}(du) dU_t^R(u) + \int_{[s,0]} \int_{[u,0]} U_t^R(u) dQ_s(du)
\]

Since for each \( u \in [0,T]\), the process \( U_t^R(u) \) is a \((Q, \mathcal{F})\)-square integrable martingale, we can find its \((Q, \mathcal{F})\)-GKW decomposition for each \( u \in [0,T] \). We can write:
\[
U_t^R(u) = U_0^R(u) + \int_{[s,0]} \varepsilon_s^R(u) dX_s + L_t^R(u)
\]

where for each \( u \in [0,T], \varepsilon_t^R(u) \) is an \( \mathcal{F} \)-predictable process in \( L^2(X,Q^F) \) and \( L_t^R(u) \) is a square integrable \((Q^F, \mathcal{F})\)-martingale strongly orthogonal to \( T^2(X,Q^F, \mathcal{F}) \) and where for all
$u < t$, we have $\varepsilon^U_t(u) = 0$ and $L^U_t(u) = L^U_a(u)$. Replacing this decomposition in $J^R_t$, we expect to get

$$J^R_t = J^R_0 + \int_{s\in[0,t]} \int_{u\in[0,T]} Q_{s-}(du)\varepsilon^R_s(u) dX_s$$

$$+ \int_{s\in[0,t]} \int_{u\in[0,T]} Q_{s-}(du) dL^R_s(u)$$

$$+ \int_{s\in[0,t]} \int_{u\in[0,T]} U^R_s(u) dQ_s(du)$$

$$= J^R_0 + \int_{s\in[0,t]} \left[ \int_{u\in[s,T]} \varepsilon^R_u(u) Q_{s-}(du) \right] dX_s$$

$$+ \int_{s\in[0,t]} \int_{u\in[s,T]} Q_{s-}(du) dL^R_s(u)$$

$$+ \int_{s\in[0,t]} \int_{u\in[s,T]} U^R_s(u) dQ_s(du)$$

$$+ \int_{s\in[0,t]} \int_{u\in[s,0]} U^R_s(u) dQ_s(du)$$

since $\varepsilon^R_u(u) = 0$ and $L^U_s(u) = L^U_u(u)$ for all $u \leq s$. The last term can also be written as:

$$\int_{s\in[0,t]} \int_{u\in[0,s]} U^R_{s-}(u) dQ_s(du) = \int_{u\in[0,t]} \int_{u\in[0,s]} U^R_u(u) dQ_s(du)$$

$$= \int_{u\in[0,t]} U^R_u(u) \int_{s\in[u,s]} dQ_s(du)$$

$$= \int_{u\in[0,t]} \frac{R_u}{B_u} [Q_t(du) - Q_u(du)]$$

In the first equality, we use the fact that $U^R_{s-}(u) = U^R_u(u)$ for any $u \leq s$. In the second equality, we assume that we can safely interchange the order of integration. Eventually, we have for $m^R_t$:

$$m^R_t = m^R_0 + \int_{s\in[0,t]} \int_{u\in[s,T]} \varepsilon^R_s(u) Q_{s-}(du) dX_s$$

$$+ \int_{s\in[0,t]} \int_{u\in[s,T]} U^R_{s-}(u) dQ_s(du) + \int_{s\in[0,t]} \int_{u\in[s,T]} Q_{s-}(du) dL^R_s(u)$$

This completes the proof. □
Appendix 3.

**Proposition.** The Galtchouk-Kunita-Watanabe decomposition of 
\[ m^C_t = E^Q \left[ \int_{[0,T]} \frac{e^{-\Gamma_u}}{B_u} dC_u | \mathcal{F}_t \vee \mathcal{J}_t \right] \]
is given by:

\[
m^C_t = m^C_0 + \int_{s:[0,t]} S_{s-}(u) \nu^C_s (du) \] \[dX_s + \int_{s:[0,t]} \int_{u:s,T} S_{s-}(u) dL^U_{s} (du) + \int_{s:[0,t]} \int_{u:s,T} U^C_{s-} (du) dS_s (u) \]

**Proof.** Let us denote \( U^C_t (u) = E^Q \left[ \int_{[0,u]} \frac{1}{B_u} dC_u | \mathcal{F}_t \right] \). For a given \( u \in [0,T] \), \( U^C_t (u) \) is a square integrable \((Q^F, \mathcal{F})\)-martingale. For each \( t \in [0,T] \), \( U^C_t (\cdot) \) is a finite measure on \([0,T], \mathcal{B}([0,T])\). We can rewrite \( m^C_t \) as:

\[
m^C_t = \int_0^t \frac{e^{-\Gamma_u}}{B_u} dC_u + E^Q \left[ \int_{[t,T]} \frac{e^{-\Gamma_u}}{B_u} dC_u | \mathcal{F}_t \vee \mathcal{J}_t \right]
= \int_0^t \frac{e^{-\Gamma_u}}{B_u} dC_u + E^Q \left[ \int_{[t,T]} S_t(u) \frac{dC_u}{B_u} | \mathcal{F}_t \vee \mathcal{J}_t \right]
= \int_0^t \frac{e^{-\Gamma_u}}{B_u} dC_u \int_0^t \frac{S_t(u)}{B_u} dC_u + E^Q \left[ \int_{[t]} \frac{S_t(u)}{B_u} dC_u | \mathcal{F}_t \vee \mathcal{J}_t \right]

Let us first study the two first terms, we have:

\[
\int_0^t \frac{e^{-\Gamma_u}}{B_u} dC_u \int_0^t \frac{S_t(u)}{B_u} dC_u = - \int_{u:[0,t]} [S_t(u) - S_u(u)] U^C_t (du)
\]

Let us now focus on \( J^C_t := E^Q \left[ \int_{[0,T]} \frac{S_t(u)}{B_u} dC_u | \mathcal{F}_t \vee \mathcal{J}_t \right] \). Thanks to the \( Q \)-independence between \( \mathcal{F} \) and \( \mathcal{J} \), we can write

\[
J^C_t = \int_{[0,T]} S_t(u) U^C_t (du)
\]

The crucial point here is to notice that \( S_t(\cdot) \) is a process taking its value in the space of bounded Borel measurable functions \( BM([0,T]) \) and \( U^C_t (\cdot) \) is a process taking its values in the space of finite signed measures on \([0,T]\). As explained in the previous appendix, these spaces, equipped with the appropriate norms, are Banach spaces and the application \((h, \mu) = \int_0^T h(u) \mu (du)\) for \( h \in BM([0,T]) \) and \( \mu \) a finite measure, is thus a bilinear bounded application. We can thus write

\[
J^C_t = (S_t, U^C_t)
\]
Thanks to Appendix 1, we have:

\[ J_t^C = (S_t, U_t^C) \]

\[ = J_0^C + \int_{s \in [0,t]} (S_{s-}, dU_s^C) + \int_{s \in [0,t]} (dS_s, U_s^C) \]

From now on, we use the following obvious notation:

\[ J_t^C = J_0^C + \int_{s \in [0,t]} \int_{u \in [0,T]} S_{s-}(u)dU_s^C(du) + \int_{s \in [0,t]} \int_{u \in [0,T]} U_s^C(du)dS_s(u) \]

Since, for any \( B \in \mathcal{B}([0,T]) \), the process \( U_t^C(B) \) is a \((Q^F, \mathcal{F})\)-square integrable martingale, we can find its \((Q^F, \mathcal{F})\)-GKW decomposition. For any \( B \in \mathcal{B}([0,T]) \), we can write

\[ U_t^C(B) = U_0^C(B) + \int_{s \in [0,t]} \varepsilon_s^U(B)dX_s + L_t^U(B) \]

where \( \varepsilon_t^U(B) \) is an \( \mathcal{F} \)-predictable process in \( L^2(X, Q^F) \) and \( L_t^U(B) \) is a square integrable \((Q^F, \mathcal{F})\)-strongly orthogonal to \( T^2(X, Q^F, \mathcal{F}) \). The processes \( \varepsilon_t^U(\cdot) \) and \( L_t^U(\cdot) \) are stochastic processes taking their values in the space of finite and signed measures. Accordingly, we expect to get

\[ J_t^C = J_0^C + \int_{s \in [0,t]} \left[ \int_{u \in [0,T]} S_{s-}(u)\varepsilon_s^U(du) \right] dX_s \]

\[ + \int_{s \in [0,t]} \int_{u \in [0,T]} S_{s-}(u) dL_s^U(du) \]

\[ + \int_{s \in [0,t]} \int_{u \in [0,T]} U_s^C(du)dS_s(u) \]

\[ = J_0^C + \int_{s \in [0,t]} \left[ \int_{u \in [s,T]} S_{s-}(u)\varepsilon_s^U(du) \right] dX_s \]

\[ + \int_{s \in [0,t]} \int_{u \in [0,s]} S_{s-}(u)dL_s^U(du) \]

\[ + \int_{s \in [0,t]} \int_{u \in [s,T]} U_s^C(du)dS_s(u) \]

\[ + \int_{s \in [0,t]} \int_{u \in [0,s]} U_s^C(du)dS_s(u) \]
In the second equality, we use the fact that $\varepsilon_s^{UC}(u) = 0$ and $L_s^{UC}(u) = L_u^{UC}(u)$ for all $u \leq s$. As far as the last term is concerned, we have:

$$
\int_{s:[0,t]} \int_{u:[0,s]} UC_s^{-1}(du) dS_s(u) = \int_{s:[0,t]} \int_{u:[0,s]} dS_s(u) U_t^{UC}(du) \\
= \int_{u:[0,t]} \int_{s:[u,t]} dS_s(u) U_t^{UC}(du) \\
= \int_{u:[0,t]} [S_t(u) - S_u(u)] U_t^{UC}(du)
$$

In the first equality, we use the fact that for $t \geq s$, $U_s^{UC}(u) = U_t^{UC}(u)$ for all $u \leq s$. In the second equality, we assume that we can safely interchange the order of integration. Eventually, we have for $m_t^{C}$:

$$
m_t^{C} = m_0^{C} + \int_{s:[0,t]} \left[ \int_{u:[s,T]} S_{s-}^{-}(u) \varepsilon_s^{UC}(du) \right] dX_s \\
+ \int_{s:[0,t]} \int_{u:[s,T]} S_{s-}^{-}(u) dL_s^{UC}(du) \\
+ \int_{s:[0,t]} \int_{u:[s,T]} UC_s^{-1}(du) dS_s(u)
$$

since $J_0^{C} = m_0^{C}$. □
Heath-Jarrow-Morton Modelling of Longevity Bonds.

1. Introduction.

A zero-coupon longevity bond is a financial security whose single payout occurs at maturity and is equal to the value at this time, of a so-called survivor index. The value at any times \( t \), of a survivor index is given by the proportion of people still alive at time \( t \) in an initially given population. We can distinguish two broad approaches in the literature to the modelling of longevity bond prices. A first approach is known as the “intensity” approach. In a sense, it is the counterpart to the “short-rate” approach in the interest rate term structure literature. Typical examples of these intensity models are, for examples, Biffis [29] (2005), Dahl [37] (2004), Luciano and Vigna [64] (2005), Schrager [74] (2006) and Hainaut and Devolder [49] (2007). The second approach consists of adapting the Heath-Jarrow-Morton (HJM) (see [50] (1992)) methodology to the modelling of longevity bond prices. Cairns et al. [36] (2006) is, to our knowledge, the first paper to exploit this idea. These authors studied different specifications for the prices of the longevity bonds. In particular, they described the term structure of longevity bond prices as the product of two independent HJM models, one related to the the term structure of risk-free zero-coupon bonds and and a second one related to the survival probabilities. Miltersen and Persson [67] (2005) extended Blake et al.’s results by removing this independence. However, they assumed the payouts of their longevity bonds at maturity were defined on the survival of a single individual. Bauer [18] (2006) introduced the fact that the actual payouts of the longevity bonds depend on the proportion of individuals alive in a given population and not on a single individual.

Unfortunately, none of these papers properly defines the prices of the longevity bonds they are supposed to study. The trouble comes from the fact that they fail to take into account the effect of the actual mortality in the population on the prices of their longevity bonds. The first and, probably, the main goal of this chapter is, therefore, to describe a coherent theoretical setting to which we can rigorously apply the HJM methodology to the modelling of longevity bond prices. For this, it is necessary to introduce an explicit model of the (actual) number of deaths in a given population, through a more or less general (marked) point process. Modelling this mortality explicitly is fundamental to offering a coherent and rigorous definition of the longevity bond prices.

A second important objective of this chapter is to describe a more realistic financial market. We extend the existing literature in various directions. Firstly, we take into account an additional effect of the actual mortality on the longevity bond prices. We can distinguish two main effects of this actual mortality. The first is a purely mechanical effect. If 5% of the population
of the survivor index on which a longevity bond is defined dies, then, everything else being equal, the price of the longevity bond will also decline by 5%. This simple mechanical and proportional effect clearly appears for example, in the pricing formula of longevity bonds in the intensity models. This first effect can obviously only depend on the actual mortality in the population of the survivor index on which the longevity bond is defined. However, there is also a possible much more subtle second effect which has (to my knowledge) never been taken into account either in the intensity models or in the HJM models. We know from financial economics theory that the price at time $t$ of a longevity bond with maturity $T$, should depend on the survival probabilities up to time $T$, possibly adjusted for the risk, in the population on which the survivor index is defined. *A priori*, there is no reason to believe these (risk-adjusted) survival probabilities might not also depend on the actual mortality in the population. Investors, if they are rational, should update their estimates of the survival probabilities (even under the real measure) if they observe a mortality which is not in line with their previous expectations. They could even modify their correction for risk if the actual mortality makes them think they are bearing more (or less) systematic risk than they previously thought. Unlike the first effect, this second effect does not necessarily only depend on the actual mortality in the survivor index on which the longevity bond is defined; it may also depend on the mortality in other parts of the population. To be more concrete, let us assume we have a longevity bond defined on 40-year-old individuals and assume the time to maturity is 20 years. The price of this bond should depend not only on the actual current mortality of 40-year-old individuals, but also on the actual current mortality of 45, 50, 55, etc-year-old individuals because investors can use this information to update their expected future survival probabilities of the current 40-year-old population. This second effect does not appear in the existing intensity models because these models assume, at least implicitly, the random times of death are conditionally independent with respect to a certain filtration (which does not include the actual mortality) and to the risk-adjusted measure (the risk-neutral martingale measure). To me, these assumptions are not obvious.

Secondly, we simultaneously consider different survivor indices, each defined on a different age group of the overall population. We can thus define a longevity bond term structure on each of these indices. In other words, unlike the existing literature, we do not consider a single longevity bond term structure but multiple longevity bond term structures. This is a natural extension since longevity bonds defined on several indices would allow the insurance companies and pension funds to better tailor their assets to their liabilities. As the longevity bonds market matures, more and more longevity bonds defined on different survivor indices, will naturally be issued. Since each longevity bond depends on the actual mortality in the whole population, complex dependencies could appear between the prices of the different longevity bonds. In particular, it is interesting to study the condition of absence of arbitrage between the different longevity bond term structures.

Finally, we extend the existing literature by allowing the different survivor indices to be heterogeneous. In the “intensity” models, the survivor index is always defined on individuals of the same age and who are homogeneous (identical forces of mortality). We do not make these assumptions here. In particular, our survivor indices can be defined on individuals of different ages. As we said, it is very likely that longevity bonds defined on different survivor indices
will be issued in the future but the number of different survivor indices will probably remain relatively low. In order to cover a large proportion of the insured population with a limited number of indices, it might be interesting to simply define larger survivor indices. For example, one longevity bond could be defined on the population currently between 20 and 35 years old, another on the population currently between 35 and 45 years old, etc. Survivor indices could be defined on larger or smaller age groups according to the needs of the insurance companies or the pension funds. Obviously, the population of such survivor indices would be heterogenous from the point of view of mortality. Our model allows such “heterogenous” longevity bonds to be dealt with.

The third aim is to study the asset allocation problem for portfolios of endowments and annuities. In particular, we choose to study the “risk-minimizing” strategies of these liabilities. This theory has already been applied with success to life insurance by Møller [68](2001), Riesner [72] (2006) and Dahl and Møller [39] (2006), for example. In particular, Dahl and Møller deal with the risk-minimizing strategies of life insurance liabilities with longevity risk. However, these authors assume only risk-free zero-coupon bonds are available for trading. In this thesis, we allow for trading in longevity bonds. An important point to underline is that we do not assume that the traded longevity bonds are defined on the same age groups than the insurer’s liabilities. We thus introduce different types of basis risks.

Finally, as a by-product of the asset allocation problem, we also study the market value of portfolios of endowments and annuities when longevity bonds are traded in the financial market. In this situation, the market values of these liabilities can be directly inferred from the observable prices of the longevity bonds, without assuming the completeness of the financial market.

This chapter is organized as follows. In Section 2, we give a critical review of the literature and show that previous models actually do not define properly either the prices or the payoffs of the longevity bonds. In Section 3, we study in detail how to apply the HJM framework to the modelling of longevity bond prices. More precisely, Section 3.1 introduces the theoretical setting on which this whole chapter rests. In particular, we describe the random times of death in the population as a marked point process. The assumptions about the distribution of these random times are very weak. We only assume they cannot occur simultaneously and that the times of death are totally inaccessible stopping times. In particular, our model encompasses all the “intensity” models defined on a Brownian filtration, as in Dahl [37] (2004), Biffis [29] (2005), Dahl and Moller [39] (2006), Luciano and Vigna [64] (2005), Schrager [74] (2006). Notice it does not, however, encompass Hainaut and Devolder’s model [49] (2007) since this model is defined on a more general “Lévy” filtration. In Section 3.2, we define the survivor indices on which the longevity bonds will depend, and in Section 3.3, the prices of the risk-free zero-coupon bonds. For the sake of generality, we assume these prices can also depend on the development of the actual mortality in the overall population. In this way, we introduce, another possible dependence between the risk-free zero-coupon bonds and the longevity bonds that does not appear in the existing literature. In Section 3.4, we define the longevity bond prices. We believe our definition is more coherent than the previous ones appearing in the literature. In particular, in order to introduce the second effect of the actual mortality on the longevity bond prices described above, we assume the forward rates of the longevity bonds
depend on the actual mortality in the whole population. In Section 3.5, we then derive the no-arbitrage condition in this general financial market. As in the traditional HJM model, this no-arbitrage condition can be expressed as a constraint on the drift of the different forward rates. Since we are modelling multiple term structures, this condition ensures there is not only no arbitrage opportunity for each term structure but also between the different term structures. In Section 3.6, we give alternative representations of the zero-coupon bond prices and longevity bond prices. In this way, we make the connection between the HJM approach and the “intensity” approach for modelling the longevity bonds. The results suggest that the basic pricing formula of the “intensity” approach could be extended to more general intensity models. In Section 3.7, we study the well-known forward martingale measures and recall their main properties. Then we describe the exact counterpart to these forward martingale measures for the longevity bonds that we call the longevity forward martingale measures. We show the main properties of the classical forward martingale measures are preserved for the longevity forward martingale measures.

In Section 4, we consider the asset allocation problem for endowments and annuities portfolios using the risk minimization theory. We recall the main results of this theory in Section 4.1. In Section 4.2 and Section 4.3, we study the risk-minimizing strategies for these portfolios when some, but not all, the longevity bonds are traded. Sections 4.2 and 4.3 differ in the sense that in the second one, we assume the policyholders of a given insurance company, represent only a fraction of the population on which a survivor index is defined, whereas in the first one, we assume the policyholders represents the whole of this population. As a by-product of the risk-minimizing strategies, we, in Sections 4.2 and 4.3, also give the market values of endowments and annuities portfolios when longevity bonds are traded.

2. A Critique of the literature.

In this section, we study how the Heath-Jarrow-Model methodology (see [50] (1992)) has been applied to the modelling of longevity bonds prices. To our knowledge, three papers treat this topic: Miltersen and Persson’s [67] (2005), Cairns et al.’s [36] (2006) and Bauer’s [18] (2006). Unfortunately, a detailed study of these papers shows that none of them properly tackles the problem they are supposed to study. All these papers actually do not properly define either the prices or the payoffs of the longevity bonds. In the following subsections, we study each of the three papers mentioned above, and underline their main troubles.

2.1. Miltersen and Persson’s Paper. In [67], Miltersen and Persson study longevity bonds defined on a single individual. In other words, a longevity bond with maturity $T$ pays 1 at time $T$ if a given individual is alive at time $T$ and pays 0 if this individual dies before time $T$. They denote by $\tau$ the time of death of this individual and $Pe(t,T,x+t-t_0)$ the price at time $t$ of a longevity bond with maturity $T$. For the sake of simplicity, we will simply write $Pe(t,T)$ in the following. In this paper, these authors define for all $t \leq T$, the price of their longevity bonds by the following equation (see middle of page 7 in [67]):

$$Pe(t,T) = e^{-\int_t^T f(t,s)+\mu(t,s;x+s-t_0)ds}$$
where \( f(t, s) \) is the \( s \)-(risk-free) forward rate at time \( t \) and \( \mu(t, s; x + s - t_0) \) is a so-called (stochastic) \( s \)-forward force of mortality at time \( t \) for an individual of age \( x \) at time \( t \) (see the original paper for more details).

This definition implies that the price of such a longevity bond is always strictly positive. However, the price of this longevity bond should be null for all \( t \geq \tau \) since, in this case, we know with certainty that the payoff is null at maturity. We thus have: \( \text{Pe}(t, T) = 0 \) for all \( t \geq \tau \). Accordingly, Miltersen and Persson should have defined the price of such a longevity bond as

\[
\text{Pe}(t, T) = \begin{cases} 0 & \text{for } t > \tau \\ e^{-\int_t^T f(t, s) + \mu(t, s; x + s - t_0) ds} & \text{for } t \leq \tau \end{cases}
\]

At first sight, this simple modification seems innocuous but we will see that it is actually fundamental. Intuitively, by neglecting this indicator function, they miss a potential jump of size -100%. When the payoff of the longevity bonds depends on a single individual as in their example, Miltersen and Persson derive the no-arbitrage condition in their model. If the no-arbitrage condition holds under \( Q \), the discounted price \( \text{Pe}^*(t, T) = \frac{\text{Pe}(t, T)}{B_t} \) should be a (local) \( Q \)-martingale for all \( T \). Top of the page 8 in [67], the authors give the dynamics of this discounted price based on their definition of \( \text{Pe}(t, T) \):

\[
d\text{Pe}^*(\cdot, T)_t = \text{Pe}^*(t, T) \left( \mu(t; x + t - t_0) - \int_t^T (\nu_f(t, s) + \nu_\mu(t, s; x + s - t_0)) ds \right) dt + \int_t^T \sigma_f(t, s) + \sigma_\mu(t, s; x + s - t_0) dW_t
\]

where \( W_t \) is a Brownian motion under the pricing measure \( Q \). To be a (local) martingale, the drift of this SDE should be equal to 0. Based on the drift of this equation, they should have written their no-arbitrage condition as

\[
\int_t^T (\nu_f(t, s) + \nu_\mu(t, s; x + s - t_0)) ds - \mu(t, x + t - t_0) = \frac{1}{2} \left( \int_t^T (\sigma_f(t, s) + \sigma_\mu(t, s; x + s - t_0)) ds \right)^2
\]

But actually, they did not. After soad-hoc argument (they study the price of another contingent claim), they eventually conclude the no-arbitrage condition is not given by the previous formula but by the following one (see the last but one formula from the bottom of page 8 in [67]):

\[
\int_t^T (\nu_f(t, s) + \nu_\mu(t, s; x + s - t_0)) ds = \frac{1}{2} \left( \int_t^T \sigma_f(t, s) + \sigma_\mu(t, s; x + s - t_0) ds \right)^2
\]

This is indeed the correct no arbitrage condition but they should have found it directly from the condition on the drift of \( \frac{\text{Pe}(t, T)}{B_t} \). By using the modified definition of \( \text{Pe}(t, T) \), they would have directly found the right condition since, in this case, the drift of \( \text{Pe}^*(t, T) \) would have
directly given the correct no arbitrage condition. Indeed, if we consider the modified definition in Equation (2.1), we can write using the integration-by-parts formula
\[
dPe^*(t, T) = e^{-\int_t^T f(t, s) + \mu(t, s; x + s - t_0) ds} \frac{dM_t}{B_t} \quad \{\tau > t\}
+ \frac{1_{\{\tau > t \}_-}}{B_t} e^{-\int_t^T f(t, s) + \mu(t, s; x + s - t_0) ds} \Bigg( e^{-\int_t^T f(t, s) + \mu(t, s; x + s - t_0) ds} - 1_{\{\tau > t \}_-} \Bigg)
+ \left[ 1_{\{\tau > t \}}, \frac{e^{-\int_t^T f(t, s) + \mu(t, s; x + s - t_0) ds}}{B_t} \right]
\]
The square bracket is null since the exponential term is continuous. Denote \( \int_0^t 1_{\{\tau > s \}_-} \mu(s, x) ds \) the Q-compensator\(^1\) of \( 1_{\{\tau \leq t\}} \), we can then write:
\[
dPe^*(t, T) = Pe^*(t, T)_- \left( \mu(t; x + t - t_0) - \mu(t, x) - \int_t^T (\nu_f(t, s) + \nu_\mu(t, s; x + s - t_0)) ds \right) dt
+ \frac{1}{2} \left( \int_t^T \sigma_f(t, s) + \sigma_\mu(t, s; x + s - t_0) ds \right)^2 dW_t
- Pe^*(t, T)_- dM_t^Q
\]
where \( M_t^Q = 1_{\{\tau \leq t\}} - \int_0^t 1_{\{\tau > s \}_-} \mu(s, x) ds \) is a Q-martingale by definition of the compensator. For the no-arbitrage condition to hold, we need:
\[
\int_t^T (\nu_f(t, s) + \nu_\mu(t, s; x + s - t_0)) ds + \mu(t, x + t - t_0) = \frac{1}{2} \left( \int_t^T (\sigma_f(t, s) + \sigma_\mu(t, s; x + s - t_0)) ds \right)^2
\]
If we let \( T \to t \), this condition leads to \( \mu(t; x) - \mu(t, x + t - t_0) = 0 \). Replacing this equality in Equation (2.4), we directly find the correct no-arbitrage condition (Equation (2.3)) without having to rely on other arguments. This simple example shows that, though the authors do correctly model the actual mortality (here in their simple setting whether the individual is alive or not), they do not properly take it into account in their definition of the longevity bonds prices.

2.2. Cairns, Blake and Dowd’s Paper. Let us now study Cairns et al.’s paper [36]. In this paper, the payoff of a longevity bond at maturity is given by the value, at this time, of a survivor index, denoted by \( S(T, x) \). Indeed, they explicitly state (see Section 2.1.2 in [36]) that “the \( (T, x) \)-bond pays the amount \( S(T, x) \) at time \( T \)”.

On one hand, in Equation (2.1) of their paper, the authors define the survivor index as
\[
S(T, x) = e^{-\int_0^T \mu(t, x + t) dt}
\]
where \( \mu(t, x + t) \) is a (possibly stochastic) sport force of mortality. The value of such a survivor index is necessarily strictly positive. On another hand, in Equation (2.2) of their paper, the

\(^1\)In other words the intensity process of \( 1_{\{\tau \leq t\}} \) is given by \( 1_{\{\tau > t \}_-} \mu(t, x) \).
HJM Modelling of Longevity Bonds.

The authors write
\[ EP \left[ Y_x(T) \mid \mathcal{M}_T \right] = S(T, x) \]

where \( \mathcal{M}_T \) is the information "generated by the evolution of the term structure of mortality [...] up to time \( t' \)" and where \( Y_x(T) \) is a random variable taking the value 1 if a given individual is alive at time \( T \) and 0 if not. The authors are not really prolix on how the filtration \( (\mathcal{M}_t)_{0 \leq t \leq T} \) is defined but we can safely say that the authors, at least implicitly, assume the random variable \( Y_x(T) \) is not measurable with respect to \( \mathcal{M}_T \). Indeed, otherwise, the index could take the value 0, which would contradict the first definition (Equation (2.5)). In other words, the information set \( \mathcal{M}_T \) does not include whether the given individual is alive or not at time \( T \).

Alternatively, if we consider a population with initially \( n_0(x) \) individuals with all the same age, we can denote the actual proportion of individual alive in this population at time \( T \) by \( I_T(x) := \frac{\sum_{i=1}^{n_0(x)} Y^i_x(T)}{n_0(x)} \) where \( Y^i_x(T) \) is a random variable taking the value 1 if the \( i \)th individual is alive at time \( T \) and 0 if not. We can then write:
\[ EP \left[ I_T(x) \mid \mathcal{M}_T \right] = \frac{EP \left[ \sum_{i=1}^{n_0(x)} Y^i_x(T) \mid \mathcal{M}_T \right]}{n_0(x)} = \frac{\sum_{i=1}^{n_0(x)} EP \left[ Y^i_x(T) \mid \mathcal{M}_T \right]}{n_0(x)} = \frac{n_0(x) S(T, x)}{n_0(x)} = S(T, x) \]

where in the third equality we assume that the population is homogeneous from the point of view of mortality i.e. \( EP \left[ Y^i_x(T) \mid \mathcal{M}_T \right] = EP \left[ Y^j_x(T) \mid \mathcal{M}_T \right] = S(T, x) \) for all \( i, j = 1, \ldots n_0(x) \). Since we know the random variable \( Y^i_x(T) \) are not measurable with respect to \( \mathcal{M}_T \), we also know \( S(T, x) \) cannot be equal to \( I_T(x) \). These equations clearly show \( S(T, x) \) can be understood as the expected (given some information \( \mathcal{M}_T \) at time \( T \)) proportion of individuals alive at time \( T \).

But why choose as a payoff, the EXPECTED proportion at time \( T \) when, at time \( T \), you know the ACTUAL proportion of individuals alive up to time \( T \)? Moreover, this definition of the payoff of a longevity bond contradicts the one made in the intensity models literature (see for example Dahl [37] (2004), Biffis [29] (2005), Luciano and Vigna [64] (2005), Schrager [74] (2006), etc.) where they explicitly consider the actual proportion of the population still alive at time \( T \) as their payoff, i.e. \( I_T(x) \) instead of \( S(T, x) \). Notice that we have
\[ E \left[ I_T(x) \right] = E \left[ E \left[ I_T(x) \mid \mathcal{M}_T \right] \right] = E \left[ S(T, x) \right] \]

However, it does not mean \( I_T(x) = S(T, x) \).

Again, one may say it is just a matter of definition. But actually it is not true since here, Cairns et al. 's definition leads to non-sensical conclusions. For example, assume the survival
probabilities are deterministic, then in this case $S(T, x)$ is deterministic. According to their definition, the payoff of a longevity bond is also deterministic (since it is equal to this $S(T, x)$). In this case, according to their definition, a longevity bond is thus just a simple risk free zero-coupon bond. It is not realistic: the payoff of a longevity bond is random whether the survival probabilities are deterministic or not.

To conclude, we clearly see the payoffs of Cairns et al.’s longevity bonds are not correctly defined. The trouble comes from the fact the authors do not properly take into account the actual mortality in the definition of their longevity bond payoffs. Actually, they do not even model this actual mortality at all.

2.3. Bauer’s Paper. As far as Bauer’s paper [18] is concerned, even though he claims to take into account, in his longevity bonds payoffs, the actual proportion of individuals alive in a given population it is not true. In order to explicitly model this actual mortality, one has to necessarily introduce a more or less general point process (as Milthersen and Persson did actually). However, nowhere does Bauer explicitly describe such point process and accordingly nowhere does he explicitly model the actual proportion of the population still alive.

Moreover, page 4 in [18], Bauer denotes by $T_{\tau}^\tau_{\tau}p_{x\tau}(T)$ the “realized proportion of the population still alive at time $\tau$ and $x\tau$ years old at time $\tau$, who are still alive at time $T$”. We can for example consider $T_{\tau}^{x\tau}p_{x\tau}(T)$ which is, according to this definition, the actual proportion in a given population of $x_0$ years old at time 0, who are still alive at time $T$. But at p.12 in [18], Bauer writes

$$\frac{\partial}{\partial t} (t_{x_0}^{(t)}) = -\tilde{\mu}_t(0, t, x_0)t_{x_0}^{(f)}$$

The solution of this differential equation is then given by

$$t_{x_0}^{(t)} = e^{-\int_0^t \tilde{\mu}_s(0, s, x_0) ds}$$

where $\tilde{\mu}_s(0, s, x_0)$ is simply the spot force of mortality at time $s$ (see the second point in Definition 3.4 page 7 in [18]). Accordingly, Bauer uses the exact same definition than in Cairns et al’s paper (see Equation (2.5) above). Again, Bauer makes the same confusion than Cairns et al.

By the way, Bauer himself recognizes his notation are very “cryptic”. Actually, all these complicated notation are not necessary and they simply translate the fact that his model is not well-defined from the start.

To conclude, we clearly see none of these three papers properly applies the HJM methodology to the modelling of longevity bonds prices. As far as Cairns et al.’s and Bauer’s papers are concerned, the authors failed to define the payoff of a longevity bond as the actual proportion of individual alive in a given population at the bond’s maturity. The reason for this is simple: they do not model this actual mortality at all but instead model its expectation. As far as Milthersen and Persson’s paper is concerned, the authors are well aware they have to model the actual mortality. Unfortunately, they did not properly take it into account in their definition of the longevity bond prices.
3. The HJM Methodology for Longevity Bonds.

3.1. The theoretical setting. As explained in the introduction, the payout of a longevity bond at maturity $T$ is given by the value at time $T$, of a given survivor index. The value of a survivor index at any time $t$, is given by the proportion of people still alive at time $t$, in an initial given age group of an overall population. In this chapter, we study survivor indices defined for different age groups of the same overall population. In order to model the values of these survivor indices, in a realistic way, we have to explicitly model the number of deaths in the overall population and, more precisely, the number of deaths in each of the different age groups. Moreover, in Section 4, we study the asset allocation problem of an insurance company in the presence of different basis risks. In particular, there exists an important basis risk coming from the fact that, even for a given age group, the policyholders of a given insurance company represent only a fraction of the population on which the survivor index is defined. In order to study, in a rigorous manner, the asset allocation problem in the presence of such basis risk, we have to explicitly model the number of deaths not only in a given age group but also in a given insurance company. We formalize these ideas in this section.

We consider an overall population consisting of policyholders of different ages and splitted up among different insurance companies. Let us define a measurable space $(K, \mathcal{K})$ where $K$ represents the set of all these insurance companies. We assume the size of this set is finite. $\mathcal{K}$ is the set $\mathcal{P}(K)$ of all subsets of $K$. Let us also define a measurable space $(X, \mathcal{X})$ where $X$ is the subset of $\mathbb{R}_+$ representing the range of admissible values at time 0, for the age of the policyholders. For example, we can consider a finite space $X = \{x_1, x_2, \ldots, x_a\}$ where the $x_i$ are the different ages of the policyholders at time 0, or we can consider an infinite space $X = [x_{\text{min}}, x_{\text{max}}]$. The $\sigma$-algebra $\mathcal{X}$ is assumed to be the Borel $\sigma$-algebra: $\mathcal{X} = \mathcal{B}(X)$.

Let us define a probability space $(\Omega, \mathcal{F}, P)$. We assume a $d$-dimensional process $W_t$ and a $E$-marked point process $(T_n, Z_n)_{n \geq 1}$ are defined on this space. See, for example, Brémaud [33] (1981) for an introduction to marked point processes. $(T_n)_{n \geq 1}$ is a sequence of real-valued random variables representing the times of death of the policyholders in the whole population. $(Z_n)_{n \geq 1}$ is a sequence of random variables taking their values in the space $E$. Given the death of a policyholder at time $T_n$, the random variable $Z_n$ describes the age of this policyholder and the insurance company to which he or she belongs.

For each $A \in \mathcal{E}$, we define the counting process $N(t, A)$ by

$$N(t, A) = \sum_{n=1}^{\infty} 1_{\{T_n \leq t\}} 1_{\{Z_n \in A\}}$$

for $t \geq 0$. It represents the number of deaths up to time $t$ associated with the set $A$. For each $A \in \mathcal{E}$, we denote by $n_0(A)$ the initial number of policyholders belonging to the set $A$. Since we necessarily have $N(t, A) \leq n_0(A) < \infty$, this marked point process is bounded and is thus nonexplosive and integrable.

We define the filtration $\mathcal{G} = (\mathcal{G}_t)_{t \geq 0}$ as the one generated by $W_t$ and $(T_n, Z_n)_{n \geq 1}$ i.e. $\mathcal{G}_t = \sigma(W_s : 0 \leq s \leq t) \vee \sigma(N(s, A) : 0 \leq s \leq t, \forall A \in \mathcal{E})$. We assume $W_t$ is a $d$-dimensional $(P, \mathcal{G})$-Brownian motion. For each $A \in \mathcal{E}$, we denote by $\Lambda(t, A)$ the $(P, \mathcal{G})$-compensator of the
counting process \( N(t, A) \) i.e. \( \Lambda(t, A) \) is the unique \((P, \mathcal{G})\)-predictable increasing process such that \( q(t, A) := N(t, A) - \Lambda(t, A) \) is a \((P, \mathcal{G})\)-martingale. For each \( A \in \mathcal{E} \), the compensator \( \Lambda(t, A)(\omega) = 0 \) a.s. on the set \{ \( (t, \omega) : N(t, A)(\omega) = n_0(A) \) \}.

We also assume there exists a \((P, \mathcal{G})\)-kernel intensity for \( N(t, \cdot) \) i.e. there exists a measure-valued process \( \nu_t(\cdot) \) is such that \( \Lambda(t, A) = \int_0^t \int_A \nu_s(\cdot)ds \) for each \( A \in \mathcal{E} \) and each \( t \geq 0 \).

Notice that the assumptions on the distribution of the times of death in the population are rather weak since we only assume that they are totally inaccessible stopping times and there is no simultaneous death in the population. For example, we do not assume the policyholders in a given set \( A \in \mathcal{E} \), necessarily have homogeneous intensities of mortality. To the best of my knowledge, it generalizes all the existing intensity models defined on a Brownian filtration, described in the literature. In these models, for all \( k \in K \) and for a given age \( x \in X \), the intensity kernel \( \nu_t((x, k)) \) can be written as

\[
\nu_t((x, k)) = \left[n_0((x, k)) - N(t, (x, k))\right] \mu(t, x + t)
\]

where \( \mu(t, x + t) \) is the “force of mortality” at time \( t \) for an individual of age \( x + t \). For example, in Dahl [37] (2004), this force of mortality follows an extended Cox-Ingersoll-Ross model

\[
d\mu(t, x + t) = \left(\beta^\mu(t, x) - \gamma^\mu(t, x)\mu(t, x + t)\right)dt + \rho^\mu(t, x)\sqrt{\mu(t, x + t)}dW_t
\]

Luciano and Vigna [64] (2005) suggest various non-reverting models such as

\[
d\mu(t, x + t) = a\mu(t, x + t)dt + \sigma dW_t
\]
or

\[
d\mu(t, x + t) = a\mu(t, x + t)dt + \sqrt{\mu(t, x + t)}dW_t
\]

3.2. The survivor indices. In this chapter, we want to introduce multiple survivor indices and define a term structure of longevity bonds on each of these survivor indices. As already explained, the payout of a longevity bond at maturity, is given by the value of a survivor index at this time. The value of this survivor index at any times \( t \), is given by the proportion of people still alive at time \( t \), from an initial age group of the overall population. Accordingly, we are particularly interested in counting processes of the form \( N(t, B \times K) \) where \( B \in \mathcal{X} \), since it represents the number of deaths in the age group \( B \) of the overall population. In order to define several survivor indices, we consider a finite number of disjoints subsets of \( X \) denoted \( B_m \) with \( m = 1, \ldots M \), such that each \( B_m \in \mathcal{X} \). For each \( B_m \), \( m = 1, \ldots M \), we can now define the value of a survivor index \( I_t(B_m) \) as

\[
I_t(B_m) = \frac{n_0(B_m \times K) - N(t, B_m \times K)}{n_0(B_m \times K)}
\]

where \( n_0(B_m \times K) \) is the initial number of policyholders in the age group \( B_m \) of the overall population. We thus have defined a \( M \)-dimensional survivor index. At time 0, we obviously have \( I_0(B_m) = 1 \) for each \( m = 1, \ldots M \).

In the following sections, we will define a term structure of longevity bonds for each index. We will then derive not only the no-arbitrage condition between the risk-free zero-coupon bonds

\footnote{They also suggest to adding a jump component to these models.}
3.3. The risk-free zero-coupon bonds. In this section, we define the prices of the risk-free zero-coupon bonds. As usual in the HJM framework, we assume there is a zero-coupon bond for each maturity $T \in [0, T^*]$. The price at time $t$ of a zero-coupon bond of maturity $T \geq t$, is denoted $B(t, T)$. Following the HJM methodology, we assume the dynamics of the zero-coupon bond prices can be described through the dynamics of the instantaneous forward rates $f_n(t, s)$ defined by

$$f_n(t, s) = -\frac{\partial \ln B(t, y)}{\partial y} \bigg|_{y=s}$$

(3.1)

So for each $t$, we have

$$B(t, T) = e^{-\int_t^T f_n(t, s) ds}$$

In this chapter, we assume the forward rates follow, for each $s \geq t$, the stochastic differential equation

$$f_n(t, s) = f_n(0, s) + \int_0^t \alpha_n(u, s) du + \int_0^t \sigma_n(u, s) dW_u$$

$$+ \int_0^t \int_E \zeta_n(u, s, e) N(du, de)$$

(3.2)

where $\alpha_n(t, s), \sigma_n(t, s)$ are respectively a 1-dimensional and a $d$-dimensional $\mathcal{G}$-predictable processes and where $\zeta_n(t, s, e)$ is a 1-dimensional $\mathcal{G}$-predictable $E$-marked process. We assume all the required technical integrability conditions hold such that the bond prices are well defined.

Here, we assume the prices of these bonds not only depend on the $(P, \mathcal{G})$-Brownian motion $W_t$ as it is usual in the literature, but can also depend on the actual mortality (through the integral with respect to $N(t, e)$ in the dynamics of the forward rates). This assumption is maybe not very realistic but we prefer to keep it for two reasons. First of all, we will need similar calculations for the longevity bonds and it is easier for us to introduce these calculations at this point. Secondly, removing this assumption does not lead to significantly easier technical results. If this is considered as clearly too unrealistic, the reader can very easily remove this assumption by setting $\zeta_n(t, s, e) = 0$ for all $t, s$ and $e$.

The following proposition gives the dynamics of the zero-coupon bond prices:

**Proposition 3.1.** If the forward rates follow Equation (3.2), then the price of a zero-coupon bond with maturity $T$ is given by the solution of the stochastic differential equation

$$dB(t, T) = B(t, T) \left\{ \left[ f_n(t, t) - \alpha_n^*(t, T) + \frac{1}{2} \left[ \sigma_n^*(t, T) \sigma_n^*(t, T) \right] \right] dt \right. - \sigma_n^*(t, T) dW_t + \int_E \psi_n^*(t, T, e) N(dt, de) \right\}$$

(3.3)
where

\[
\begin{align*}
\alpha^*_n(t, T) &= \int_t^T \alpha_n(t, s) ds \\
\sigma^*_n(t, T) &= \int_t^T \sigma_n(t, s) ds \\
\psi^*_n(t, T, e) &= e^{-\zeta^*_n(t, T, e)} - 1 \\
\zeta^*_n(t, T, e) &= \int_t^T \zeta_n(t, s, e) ds
\end{align*}
\]

Proof. See Appendix 1.

3.4. The longevity bonds. In this section, we define the prices of the longevity bonds. In addition to the risk-free term structure, we assume that there are \(M\) longevity bond term structures, one for each survivor index. A zero-coupon \((T, B_m)\)-longevity bond is a financial security that pays, at time \(T\), the value of the survivor index associated with the age group \(B_m\):

\[
I_T(B_m) = \frac{n_0(B_m \times K) - N(T, B_m \times K)}{n_0(B_m \times K)}
\]

As usual in the HJM setting, we assume there is, for each \(B_m, m = 1, \ldots, M\), a zero-coupon longevity bond for each maturity \(T \in [0, T^*]\). The price at time \(t\) of such a zero-coupon longevity bond with maturity \(T\), is \(P(t, T, B_m)\). Following the HJM methodology, we assume the dynamics of zero-coupon longevity bond prices can be described through the dynamics of some forward rates. More precisely, we define the price at time \(t\) of a \((T, B_m)\)-longevity bond as

\[
P(t, T, B_m) = I_t(B_m)e^{-\int_t^T f_l(t, s, B_m) ds}
\]

where the processes \(f_l(t, s, B_m)\) are called longevity forward rates and are described below in more detail. Notice that these longevity forward rates depend on the survivor index on which the longevity bond is defined. This definition has several interesting features. First, it ensures that the price at maturity of any longevity bond is equal to the value of its payout i.e. \(P(T, T, B_m) = I_T(B_m)\) for each \(T\) and each \(B_m\). Secondly, we have

\[
\frac{P(t, U, B_m)}{P(t, T, B_m)} = e^{-\int_t^U f_l(t, s, B_m) ds}
\]

Accordingly, these forward rates can then be written as

\[
f(t, s, B_m) = -\frac{\partial \ln P(t, y, B_m)}{\partial y} \bigg|_{y=s}
\]

This equation shows that the definition of the longevity forward rates is the exact counterpart to the definition of the risk-free forward rates (see Equation (3.1)). Thirdly, it allows to separate the effects of the actual mortality on the price of a longevity bond, that we discussed in the introduction. As we already explained, the first is a purely mechanical and proportional effect due to the actual mortality in the associated survivor index. The term \(I_t(B_m)\) in Equation (3.4)
translates this first effect. We can then introduce, independently of the first, an additional effect by allowing the second term \( e^{-\int_T^T f_i(t, s, B_m)ds} \) to depend on the actual mortality in the whole population. More precisely, we assume the longevity forward rates \( f_i(t, s, B_m) \) are described for each \( s \geq t \), by the equation

\[
f_i(t, s, B_m) = f_i(0, s, B_m) + \int_0^t \alpha_i(u, s, B_m)du + \int_0^t \sigma_i(u, s, B_m)dW_u
\]

where, for each \( B_m, \ m = 1 \ldots M \), \( \alpha_i(t, s, B_m) \) and \( \sigma_i(t, s, B_m) \) are respectively a 1-dimensional and a \( d \)-dimensional \( \mathbb{G} \)-predictable process, and where \( \zeta_i(t, s, e) \) is a 1-dimensional \( \mathbb{G} \)-predictable \( \mathbb{E} \)-marked process. Notice that we let the forward rates depend on the actual mortality in the whole set \( E \) and not only in the subset \( B_m \times K \). In other words, this definition also allows the price of a longevity bond to depend on the actual mortality in the whole population and not only on the actual mortality in the associated survivor index.

Here again, we assume that all the required technical integrability conditions hold such that the longevity bond prices are well defined. In the following, we will denote \( e^{-\int_0^T f_i(t, s, B_m)ds} \) by \( Z(t, T, B_m) \).

The following proposition gives the dynamics of the zero-coupon longevity bond prices.

**Proposition 3.2.** If the longevity forward rates follow Equation (3.5) then the price of a zero-coupon \((T, B_m)\)-longevity bond is given by the solution of the stochastic differential equation

\[
dP(t, T, B_m) = P(t, T, B_m) \left\{ \begin{array}{l}
\left[ f_i(t, t, B_m) - \alpha_i^*(t, T, B_m) \right] dt \\
+ \frac{1}{2} \sigma_i^*(t, T, B_m)^\top \sigma_i^*(t, T, B_m) dt \\
- \sigma_i^*(t, T, B_m)dW_t + \int_E \psi_i^*(t, T, B_m, e)N(dt, de)
\end{array} \right.
\]

where

\[
\begin{align*}
\alpha_i^*(t, T, B_m) &= \int_t^T \alpha_i(t, s, B_m)ds \\
\sigma_i^*(t, T, B_m) &= \int_t^T \sigma_i(t, s, B_m)ds \\
\psi_i^*(t, T, B_m, e) &= e^{-\zeta_i^*(t, T, B_m, e)} - 1 \\
\psi_i^*(t, T, B_m, e) &= \psi_i^*(t, T, B_m, e) - 1_{\{e \in B_m \times K\}} \frac{e^{-\zeta_i^*(t, T, B_m, e)}}{n_0(B_m \times K) - N(t, B_m \times K)} \\
\zeta_i^*(t, T, B_m, e) &= \int_t^T \zeta_i(t, s, B_m, e)ds
\end{align*}
\]

**Proof.** Following the same lines as in the proof of Proposition 3.1, we can easily show that for each \( B_m, \ m = 1 \ldots M \) and each \( T \in [0, T^*] \), the process \( Z(t, T, B_m) \) is the solution of
the stochastic differential equation

\[
dZ(t,T,B_m) = Z(t-T,B_m) \left\{ \begin{array}{l}
[f_l(t,t,B_m) - \alpha_l^*(t,T,B_m)]dt \\
+ \frac{1}{2}\sigma_l^*(t,T,B_m)^2 dt \\
- \sigma_l^*(t,T,B_m) dW_t + \int_E \psi^*_Z(t,T,B_m,e) N(dt,de)
\end{array} \right.
\]

where \( \psi^*_Z(t,T,B_m,e) = e^{-G(t,T,B_m,e)} - 1 \). Using the integration-by-parts formula for stochastic processes, we can find the price of a zero-coupon longevity bond \( P(t,T,B_m) \) is given by

\[
dP(t,T,B_m) = I_{t-}(B_m) dZ(t,T,B_m) + Z(t-T,B_m) dI_t(B_m) + d[I_t(B_m), Z(\cdot,T,B_m)]_t
\]

The square bracket is given by

\[
d[N(\cdot,B_m \times K), Z(\cdot,T,B_m)]_t = Z(t-T,B_m) \int_{B_m \times K} \psi^*_Z(t,T,B_m,e) N(dt,de)
\]

Eventually we get

\[
dP(t,T,B_m) = P(t-T,B_m) \left\{ \begin{array}{l}
[f_l(t,t,B_m) - \alpha_l^*(t,T,B_m)]dt \\
+ \frac{1}{2}\sigma_l^*(t,T,B_m)^2 dt \\
- \sigma_l^*(t,T,B_m) dW_t + \int_E \psi^*_Z(t,T,B_m,e) N(dt,de)
\end{array} \right.
\]

\[\square\]

3.5. The no-arbitrage condition. In this section, we examine the condition for the no-arbitrage hypothesis to hold in the financial market described above. This condition ensures there is no-arbitrage (1) between the risk-free zero-coupon bonds, (2) between the longevity bonds associated with the same survivor index, (3) between the longevity bonds and the risk-free zero-coupon bonds and (4) between the longevity bonds associated with different survivor indices. We assume there is a locally risk-free asset \( B_t \) which grows at rate \( r_t = f_\alpha(t,t) \). We first need the following theorem.
Theorem 3.3. Any probability measure $Q$ equivalent to $P$ on $(\Omega, \mathcal{F})$ has a density $\eta_t = \frac{dQ}{dP} |_{\mathcal{G}_t}$ that is solution of
\[
\frac{d\eta_t}{dt} = \eta_t (-\beta^W_t dW_t + \int_E (\beta^N(t,e) - 1) q(dt, de) ) , \eta_0 = 1
\]
for a $d$-dimensional $\mathcal{G}$-predictable process $\beta^W_t$ and a 1-dimensional $\mathcal{G}$-predictable $E$-marked process $\beta^N(t,e) > 1$ and such that $E^P [\eta_{T*}] = 1$. The processes $W^Q_t$ and $q^Q(t,A)$ defined below, are respectively a $d$-dimensional $(Q, \mathcal{G})$-Brownian motion and, for each $A \in \mathcal{E}$, a $(Q, \mathcal{G})$-local martingale:
\[
W^Q_t = W_t + \int_0^t \beta^W_u du
\]
\[
q^Q(t,A) = q(t,A) - \int_0^t \int_A (\beta^N(u,e) - 1) \nu_u(de) du
\]
\[
= N(t,A) - \int_0^t \int_A \beta^N(u,e) \nu_u(de) du
\]
for all $A \in \mathcal{E}$.


The next proposition gives the no-arbitrage condition.

Proposition 3.4. There is no arbitrage if there exist an $\mathcal{G}$-predictable process $\beta^W_t$ and an $\mathcal{G}$-predictable $E$-marked process $\beta^N(t,e) > 1$, such that for each $T$ and each $B_m$, $m = 1, \ldots, M$, we have simultaneously
\[
\alpha^*_{n}(t,T) = \frac{1}{2} \left[ \sigma^*_{n}(t,T) \sigma^*_{n}(t,T) \right] + \sigma^*_{n}(t,T) \beta^W_t + \int_E \psi^*_{n}(t,T,e) \beta^N(t,e) \nu_t(de)
\]
and
\[
\alpha^*_{l}(t,T,B_m) = \frac{1}{2} \left[ \sigma^*_{l}(t,T,B_m) \sigma^*_{l}(t,T,B_m) \right] + \sigma^*_{l}(t,T,B_m) \beta^W_t + \int_E \psi^*_{l}(t,T,B_m,e) \beta^N(t,e) \nu_t(de)
\]
\[
+ \frac{\int_{B_m \times K} \beta^N(t,e) \nu_t(de)}{(\eta_0(B_m \times K) - N(t_-, B_m \times K))}
\]

Proof. For the no-arbitrage condition to hold, we have to find the conditions of existence of an equivalent measure, $Q$, such that $B(t,T)/\mathcal{B}_t$ and $P(t,T,B_m)/\mathcal{B}_t$ are $(Q, \mathcal{G})$-local martingales for each $T \in [t, T^*]$ and each $B_m$ simultaneously. We can write the discounted prices
of these assets as
\[
d\left( \frac{B(t,T)}{B_t} \right) = \frac{B(t_-,T)}{B_t} \left\{ \left[ f_n(t,t) - r_t - \alpha_n^*(t,T) + \frac{1}{2} \left[ \sigma_n^*(t,T)^T \sigma_n^*(t,T) \right] \right] dt \right. 
\left. + \left[ \sigma_n^*(t,T) \beta^W_t + \int_E \psi_n^*(t,T,e) \beta^N(t,e) \nu_t(de) \right] dt \right. 
\left. - \sigma_n^*(t,T) dW_t^Q + \int_E \psi_n^*(t,T,e) q^Q(dt,de) \right\}
\]
and
\[
d\left( \frac{P(t,T,B_m)}{B_t} \right) = \frac{P(t_-,T,B_m)}{B_t} \left\{ \left[ f_t(t,t,B_m) - r_t - \alpha_t^*(t,T,B_m) \right] dt 
\left. + \frac{1}{2} \sigma_t^*(t,T,B_m)^T \sigma_t^*(t,T,B_m)dt \right. 
\left. + \left[ \sigma_t^*(t,T,B_m) \beta^W_t + \int_E \psi_t^*(t,T,B_m,e) \beta^N(t,e) \nu_t(de) \right] dt \right. 
\left. - \sigma_t^*(t,T,B_m) dW_t^Q + \int_E \psi_t^*(t,T,B_m,e) q^Q(dt,de) \right\}
\]
For these prices to be \((Q,G)\)-local martingales, the drift should be equal to 0 \(P\)-a.s for all \(T\) and all \(B_m\). Since \(f_n(t,t) = r_t\), the first condition is immediate. As far as the second condition is concerned, \(\beta^W_t\) and \(\beta^N(t,e)\) should be such that
\[
\alpha_t^*(t,T,B_m) = f_t(t,t,B_m) - r_t + \frac{1}{2} \left[ \sigma_t^*(t,T,B_m)^T \sigma_t^*(t,T,B_m) \right]
+ \sigma_t^*(t,T,B_m) \beta^W_t + \int_E \psi_t^*(t,T,B_m,e) \beta^N(t,e) \nu_t(de)
\]
where
\[
\psi_t^*(t,T,B_m,e) = \psi_Z^*(t,T,B_m,e) - 1_{e \in B_m \times K} \left( n_0(B_m \times K) - N(t, B_m \times K) \right)
\]
If we let \(T \to t\), the drift condition leads to the equality
\[
f_t(t,t,B_m) - r_t = - \int_E \psi_t^*(t,t,B_m,e) \beta^N(t,e) \nu_t(de)
\]
(3.7)
\[
= \frac{\int_{B_m \times K} \beta^N(t,e) \nu_t(de)}{(n_0(B_m \times K) - N(t, B_m \times K)_-)}
\]
By rearranging the different terms, we get the result. 

**Remark 3.5.** The assumption that the drifts of the forward rates \(f_n(\cdot, \cdot)\) and the drifts of the longevity forward rates \(f_t(\cdot, \cdot, B_m)\) are absolutely continuous with respect to the Lebesgue measure, implies the \((P,G)\)-compensator of \(N(t, \cdot)\) is itself absolutely continuous. In other words, this assumption implies the \((P,G)\)-kernel intensity of \(N(t, \cdot), \nu_t(de)\) exists. It justifies the assumption we made in Section 3.1 on the existence of this kernel intensity.
Notice in the previous proof, we show the absence of arbitrage leads to the equality

\[ f_l(t, t, B_m) - r_t = \frac{\int_{B_m \times K} \beta^N(t, e) \nu_t(de)}{(n_0(B_m \times K) - N(t, B_m \times K))} \]

The right-hand term is the average \((Q, G)\)-intensity of mortality for the individuals in the category age \(B_m\) at time \(t\). The left-hand term is the spread of the longevity forward rates over the instantaneous risk-free rate when \(T \rightarrow t\). This equality generalizes the result on page 4 of Miltersen and Persson [67] (2005) where, when \(T \rightarrow t\), they show that what they called the forward force of mortality rate is equal to the spot force of mortality rate.

The no-arbitrage condition leads to the following representation for the forward rates:

**Proposition 3.6.** If there is no arbitrage, the forward rates of the zero-coupon bonds are given by

\[ f_n(s, u) = f_n(0, u) + \int_0^s \sigma_n(v, u) \sigma^*_n(v, u) dv - \int_0^s \int_E \zeta_n(v, u, e) \psi^*_n(v, u, e) \beta^N(v, e) \nu_v(de) dv + \int_0^s \sigma_n(v, u) dW_Q^v + \int_0^s \int_E \zeta_n(v, u, e) q^Q(de, de) \]

and the longevity forward rates of the longevity bonds are given for each \(B_m\), by

\[ f_l(s, u, B_m) = f_l(0, u, B_m) + \int_0^s \sigma_l(v, u, B_m) \sigma_l^*(v, u, B_m) dv - \int_0^s \int_E \zeta_l(v, u, B_m, e) \psi_l^*(v, u, B_m, e) \beta^N(v, e) \nu_v(de) dv + \int_0^s \sigma_l(v, u, B_m) dW_Q^v + \int_0^s \int_E \zeta_l(v, u, B_m, e) q^Q(de, de) \]

**Proof.** Deriving the no-arbitrage condition with respect to \(T\), we also find the equivalent set of conditions:

\[ \alpha_n(t, T) = \sigma_n(t, T) \left( \sigma_n^*(t, T) + \beta^W_t \right) - \int_E \zeta_n(t, T, e) e^{-\zeta^*_n(t, T, e)} \beta^N(t, e) \nu_t(de) \]

\[ \alpha_l(t, T, B_m) = \sigma_l(t, T, B_m) \left( \sigma_l^*(t, T, B_m) + \beta^W_t \right) - \int_E \zeta_l(t, T, B_m, e) e^{-\zeta^*_l(t, T, B_m, e)} \beta^N(t, e) \nu_t(de) + \int_{B_m \times K} \frac{\zeta_l(t, T, B_m, e) e^{-\zeta^*_l(t, T, B_m, e)}}{(n_0(B_m \times K) - N(t, B_m \times K))} \beta^N(t, e) \nu_t(de) \]

Replacing these conditions in Equations (3.2) and (3.5), we get the result. \(\square\)
The no-arbitrage condition leads also to the following representation for the risk-free zero-coupon bond prices and the zero-coupon longevity bond prices:

**Proposition 3.7.** If there is no arbitrage, the price of a risk-free zero-coupon bond with maturity $T$ is given by:

$$
\begin{align*}
dB(t,T) &= B(t_-,T) \left\{ \left[ r_t - \sigma_n^*(t,T) \beta_t^W - \int_E \psi_n^*(t,T,e) (\beta_t^N(t,e) - 1) \nu_t(de) \right] dt \right. \\
& \left. - \sigma_n^*(t,T) dW_t + \int_E \psi_n^*(t,T,e) q(dt,de) \right\}
\end{align*}
$$

or

$$
\begin{align*}
dB(t,T) &= B(t_-,T) \left\{ r_t dt - \sigma_n^*(t,T) dW_t^Q + \int_E \psi_n^*(t,T,e) q^Q(dt,de) \right\}
\end{align*}
$$

and the price of a zero-coupon $(T, B_m)$-longevity bond is given by:

$$
\begin{align*}
dP(t,T,B_m) &= P(t_-,T,B_m) \left\{ \left[ r_t - \sigma_l^*(t,T,B_m) \beta_t^W - \int_E \psi_l^*(t,T,B_m,e) (\beta_t^N(t,e) - 1) \nu_t(de) \right] dt \right. \\
& \left. - \sigma_l^*(t,T,B_m) dW_t + \int_E \psi_l^*(t,T,B_m,e) q(dt,de) \right\}
\end{align*}
$$

or

$$
\begin{align*}
dP(t,T,B_m) &= P(t_-,T,B_m) \left\{ r_t dt - \sigma_l^*(t,T,B_m) dW_t^Q + \int_E \psi_l^*(t,T,B_m,e) q^Q(dt,de) \right\}
\end{align*}
$$

**Proof.** Straightforward. Replace the no-arbitrage conditions of Proposition 3.4 and Equation (3.7) in Equations (3.3) and (3.6).

These equations give the equilibrium risk premiums for the risk-free zero-coupon bonds and the zero-coupon longevity bonds. These risk premiums consist of two parts: one is related to the risk coming from the Brownian motion and the other to the risk coming from the actual mortality. In particular, for the longevity bonds, we can study in more detail, the risk premium related to the actual mortality. Since we know the actual mortality can have two effects on the price of the longevity bonds, we can expect to also have two separate risk premiums for this mortality risk. Indeed, we can write:

$$
- \int_E \psi_l^*(t,T,B_m,e) (\beta_t^N(t,e) - 1) \nu_t(de) = \int_{B_m \times K} \frac{(\beta_t^N(t,e) - 1) \nu_t(de)}{(n_0(B_m \times K) - N(t_-,B_m \times K))} - \int_E \psi_n^*(t,T,B_m,e) \delta(t,B_m) (\beta_t^N(t,e) - 1) \nu_t(de)
$$

(3.8)

where $\delta(t,B_m) = \left(1 - \frac{1_{(t \in B_m \times K)}}{(n_0(B_m \times K) - N(t_-,B_m \times K))}\right)$. If, as in the intensity models, we assume the actual mortality has only a mechanical effect on the prices of the longevity bonds or in other words, if we assume the longevity forward rates do not depend on the actual mortality, then we can put $\zeta(u,s,B_m,e) = 0$. In this case, the second term vanishes. Accordingly, the first term is the risk premium due to the mechanical effect of the actual mortality. Intuitively, this risk premium simply corresponds to the average $(Q,G)$-intensity of mortality for the age group on which the longevity bond is defined. It also appears in the existing intensity models. The second term is thus an additional risk premium due to the second effect of the actual mortality, which does not appear in the existing intensity models. It is important to notice that in the
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In the general case, we have this additional risk premium. If, in order to find the value of $\beta^N(\cdot, \cdot)$, the risk premium of a longevity bond is estimated, the estimate does not necessarily correspond to the term $\frac{\int_{B_m \times K} (\beta^N(t, e) - 1) \nu_1(de)}{(n_0(B_m \times K) - N(t, B_m \times K))}$, as the “intensity” models suggest, but to a much more complex expression.

### 3.6. Alternative representations of zero-coupon bond prices and longevity bond prices

This section aims to underline the connections between the intensity models and our HJM model. In Proposition 3.8, we first recall a well-known result in the traditional HJM model, namely, that the price of a risk-free zero-coupon bond, in our HJM framework, can also be written as in the short-rate models. In Proposition 3.9, we show that we can derive a similar result for the longevity bonds: the price of a longevity bond, in our HJM model, can also be written as in the intensity models. We obtain a more general form that suggests current intensity models could be extended to more general settings. As a by-product, it also strengthens our definition of the longevity bond prices. We assume here the discounted prices are $(Q, \mathcal{G})$-martingales.

**Proposition 3.8.** The price at time $t$ of a zero-coupon bond of maturity $T$, $B(t, T)$ is given by

$$B(t, T) = E^Q \left[ e^{-\int_t^T r_u du} | \mathcal{G}_t \right]$$

**Proof.** The proof is straightforward and can be found in any textbook. We give it for the sake of completeness. Using the no-arbitrage condition, the discounted value of a zero-coupon bond with maturity $T$ is a $(Q, \mathcal{G})$-martingale that can be written as

$$d \left( \frac{B(t, T)}{B_t} \right) = \frac{B(t, T)}{B_t} \left\{ -\sigma_n^*(t, T) dW^Q_t + \int E \psi_n^*(t, T, e) q^Q(dt, de) \right\}$$

So we have

$$\frac{B(t, T)}{B_t} = E^Q \left[ \frac{B(T, T)}{B_T} | \mathcal{G}_t \right]$$

where $B(T, T) = 1$. \hfill $\Box$

The price of a longevity bond can also be written as in the intensity models. We have the following proposition:

**Proposition 3.9.** The price at time $t$ of a longevity bond with maturity $T$ associated with the index $I_T(B_m)$ is given by

$$(3.9) \quad P(t, T, B_m) = I_t(B_m) E^Q \left[ e^{-\int_t^T r_u + \kappa_u(B_m) du} | \mathcal{G}_t \right]$$

where $\kappa_u(B_m) = \frac{\int_{B_m \times K} e^{-\xi^*(u, T, B_m, e) D^N(u, e)} \nu_u(de)}{(n_0(B_m \times K) - N(u, B_m \times K))}$.\hfill $\Box$
Proof. Using the no-arbitrage condition, for each \( B_m \) and each \( T \), the process \( Z(t,T,B_m) \) can be written as

\[
dZ(t,T,B_m) = Z(t-,T,B_m) \left\{ \begin{array}{l}
[r_t + \kappa_t(B_m)] dt - \sigma_t^*(t,T)dW_t^Q \\
+ \int_E \psi_Z^*(t,T,B_m,e)q^Q(dt,de)
\end{array} \right\}
\]

Accordingly, the process \( \frac{Z(t,T,B_m)}{B_t} e^{-\int_0^t \kappa_u(B_m)du} \) is a \((Q, \mathcal{G})\)-martingale and we thus have

\[
\frac{Z(t,T,B_m)}{B_t} e^{-\int_0^t \kappa_u(B_m)du} = E^Q \left[ \frac{Z(T,T,B_m)}{B_T} e^{-\int_0^T \kappa_u(B_m)du} | \mathcal{G}_t \right]
\]

Since \( Z(T,T,B_m) = 1 \) and \( P(t,T,B_m) = I_t(B_m)Z(t,T,B_m) \), we have the result. \( \square \)

This pricing formula looks like the existing ones in the literature on intensity models (see Biffis [29] (2005), Dahl [37] (2004), etc.). In these models, the price of a longevity bond is given by the proportion of individuals alive in the survivor index multiplied by an expectation, under a martingale measure \( Q \), of a modified discounting term \( r_s + \mu^Q(s, x + s) \), where \( \mu^Q \) is the \( Q \)-force of mortality. However, there are some substantial differences.

First of all, in the existing intensity models, the random times of death are assumed to be conditionally \( Q \)-independent with respect to a filtration that does not include these times of death and which is, accordingly, strictly smaller than our filtration \( \mathcal{G} \). Mathematically speaking, this assumption implies that the expectations that appear in the pricing formula of these intensity models are taken with respect to this smaller filtration and not with respect to \( \mathcal{G} \), as happens here. Intuitively speaking, as we explained in the introduction, these models are unable to take into account both effects of the actual mortality on the price of the longevity bonds. In these models, the actual mortality can only affect the prices of the longevity bonds through the proportion \( I_t(B_m) \).

Secondly, if, as in the intensity models, we assume the longevity forward rates cannot depend on the actual mortality, we can put \( \zeta^i_t(u,T,B_m,e) = 0 \). In this case, we have

\[
\kappa_u(B_m) = \frac{\int_{B_m \times K} \beta^N(u,e) \nu_u(de)}{(n_0(B_m \times K) - N(u, B_m \times K))}
\]

Intuitively, \( \kappa_u(B_m) \) represents the average \((Q, \mathcal{G})\)-intensity of mortality in the age group \( B_m \). This is similar to the existing “intensity” models except for two things. First, in these intensity models, the policyholders are assumed to be homogeneous (identical force of mortality) but, as we already explained, we do not make this assumption in this chapter: the members of the age group \( B_m \) are not necessarily homogeneous. For example, the group \( B_m \) could represent the policyholders between 30 and 40 years old. Even for a given age, we do not actually assume that the population is necessarily homogeneous. Accordingly, our formula extends the existing one to the pricing of longevity bonds on heterogeneous population and shows that what is important is the average intensity of mortality in the chosen population. For example, if we assume that in the set \( B_m \times K \) there is a finite number of (disjoint) homogeneous groups of policyholders \((B_m \times K)_i\), where \( B_m \times K = \bigcup_i (B_m \times K)_i \) with \( P \)-intensity of mortality \( \lambda_u \left( (B_m \times K)_i \right) \), and if \( \beta^N(u,e) \) is constant on each of these sets, then the \( Q \)-intensity of mortality is given by...
\[ \lambda^Q_u \left( (B_m \times K)_i \right) = \beta^N \left( u, (B_m \times K)_i \right) \lambda_u \left( (B_m \times K)_i \right). \]

\( \kappa_u(B_m) \) can then be written as the weighted average of these \( Q \)-intensities:

\[ \kappa_u(B_m) = \sum_i \frac{n_0 \left( (B_m \times K)_i \right) - N \left( u, (B_m \times K)_i \right)}{n_0(B_m \times K) - N(u, B_m \times K)} \lambda^Q_u \left( (B_m \times K)_i \right) \]

Second, in these models, the \( P \)-intensity of mortality of a policyholder, given he or she is alive (the force of mortality), and the change of measure parameter \( \beta(\cdot) \) are not allowed to depend on the past actual mortality. We do not make these assumptions here.

Thirdly, the term \( e^{-\zeta\left(uT,B_m,e\right)} \) does not appear in the existing intensity models literature. It plays the role of an additional change of measure specific to each longevity bond category and each maturity. Indeed, in the proof of the last proposition, we showed that the process \( \frac{Z(tT,B_m)}{Z(0,T,B_m)} e^{-\int_0^t \kappa_u(B_m)du} \) is a \((Q, \mathbb{G})\)-martingale. For each \( T \) and each \( B_m \), we can define a measure \( Q^{Z,T,B_m} \) equivalent to \( Q \) such that its density \( \eta^{Z,T,B_m}_t = \frac{dQ^{Z,T,B_m}}{dQ} |\mathcal{G}_t \) for \( t \leq T \), is given by

\[ \eta^{Z,T,B_m}_t = \frac{Z(t, T, B_m)e^{-\int_0^t \kappa_u(B_m)du}}{Z(0,T,B_m)} \]

We can also write this density as

\[ d\eta^{Z,T,B_m}_t = \eta^{Z,T,B_m}_t \left( -\sigma_t(t, T) dW_t^Q + \int_E \psi_Z(t, T, B_m, e) q^Q(dt, de) \right) \]

with \( \eta^{Z,T,B_m}_0 = 1 \). According to Theorem 3.3, for each \( A \in \mathcal{E} \), the process

\[ q^{Q^{Z,T,B_m}}(t, A) = N(t, A) - \int_{A} \int_0^t (1 + \psi_Z(t, T, B_m, e)) \beta^N(u, e) \nu_u(de)du \]

\[ = N(t, A) - \int_{A} \int_0^t e^{-\zeta\left(uT,B_m,e\right)} \beta^N(u, e) \nu_u(de)du \]

is a \((Q^{Z,T,B_m}, \mathbb{G})\)-local martingale. Accordingly, \( \kappa_u(B_m) \) should be understood as the average \((Q^{Z,T,B_m}, \mathbb{G})\)-intensity of mortality in the age group \( B_m \). This additional change of measure translates the additional risk premium we discussed in Equation (3.8).

Notice that, in the HJM methodology, the prices of the bonds are given \textit{a priori}. In other words, we do not use Equation (3.9) to derive the prices of the longevity bonds. It is only a mathematical identity implied by the no-arbitrage condition. However, it suggests current intensity models could be extended to much more general models without losing the form of their basic pricing formula.

### 3.7. Forward martingale measures and longevity forward martingale measures

In the pricing of interest rate derivatives, it is often useful to rewrite a problem under, what is called in the financial literature, a forward martingale measure instead of the spot martingale measure \( Q \). In Section 3.7.1, we give the form of these forward martingale measures in our setting and show some of their well-known properties. In Section 3.7.2, we present new martingale measures that are the exact counterparts to these forward measures, but for longevity bonds. We call these new measures the longevity forward martingale measures. We show they
basically exhibit similar properties to the traditional forward measures. We assume here the discounted prices of the zero-coupon bonds and the discounted prices of the longevity bonds are \((Q, G)\)-martingales.

3.7.1. \(T\)-forward martingale measure. As in the traditional HJM model, we can, for each \(T\), define a measure \(Q_T\), equivalent to \(Q\), such that its density \(\eta_T = \frac{dQ_T}{dQ}\mid_G\) is given for \(t \leq T\), by

\[
\eta_T^T = \frac{e^{-\int_0^t r_u \, du} B(t, T)}{B(0, T)}
\]

We can also write this density as a solution of the differential equation

\[
d\eta_T^T = \frac{1}{B(0, T)} d \left( e^{-\int_0^t r_u \, du} B(t, T) \right) = \eta_T^T \left( -\sigma^*_n(t, T) dW_t^Q + \int_E \psi_n^*(t, T, e) q^Q(dt, de) \right)
\]

with \(\eta_0^T = 1\). In other words, the density \(\eta_T^T\) can be written as the following Doléans exponential:

\[
\eta_T^T = \mathcal{E} \left( -\sigma^*_n(t, T) dW_t^Q + \int_E \psi_n^*(t, T, e) q^Q(dt, de) \right)
\]

As in the traditional HJM model, we have the two following well known properties:

**Proposition 3.10.** Under the forward martingale measure \(Q^T\), the forward rate \(f(t, T)\) is a \((Q^T, G)\)-martingale.

**Proof.** Using Theorem 3.3, we know the process \(W_t^{Q^T}\) is a \(d\)-dimensional \((Q^T, G)\)-Brownian motion and \(q^{Q^T}(t, A)\) is a \((Q^T, G)\)-local martingale where these processes are defined by:

\[
W_t^{Q^T} = W_t^Q + \int_0^t \sigma_n^*(u, T) \, du
\]

\[
q^{Q^T}(t, A) = q^Q(t, A) - \int_0^t \int_A \psi_n^*(u, T, e) \beta^N(u, e) \nu_u(de) \, du
\]

The forward rate \(f(t, T)\) can then be written as

\[
f_n(t, T) = f_n(0, T) + \int_0^t \sigma_n(v, T) dW_v^{Q^T} + \int_0^t \int_E \zeta_n(v, T, e) q^{Q^T}(dv, de)
\]

which is indeed a \((Q^T, G)\)-martingale. \(\square\)

It is well-known from financial economics that the usefulness of these forward martingale measures resides in the following proposition:
Proposition 3.11. For any $\mathcal{G}_T$-measurable random variable $H_T$, we have

$$E^Q \left[ \frac{B_t}{B_T} H_T | \mathcal{G}_t \right] = B(t,T) E^{Q_T} [H_T | \mathcal{G}_t]$$

Proof. We have

$$E^{Q_T} [H_T | \mathcal{G}_t] = \frac{E^Q \left[ \eta_T^T H_T | \mathcal{G}_t \right]}{E^Q \left[ \eta_T^T | \mathcal{G}_t \right]} = \frac{E^Q \left[ e^{-\int_0^T r_u du} B(T,T) H_T | \mathcal{G}_t \right]}{e^{-\int_0^T r_u du} B(t,T)}$$

Thanks to the previous proposition, we can obtain another formula for the price of a $(T,B_m)$-longevity bond.

Proposition 3.12. The price at time $t$ of a $(T,B_m)$-longevity bond can be written as

$$P(t,T,B_m) = B(t,T) E^{Q_T} [I_T(B_m) | \mathcal{G}_t]$$

or

$$P(t,T,B_m) = I_t(B_m) B(t,T) E^{Q_T} \left[ e^{-\int_t^T \kappa_u(B_m) du} | \mathcal{G}_t \right]$$

Proof. Using Proposition 3.11 with $H_T = I_T(B_m)$, we have

$$E^{Q_T} [I_T(B_m) | \mathcal{G}_t] = \frac{E^Q \left[ I_T(B_m) e^{-\int_t^T \kappa_u(B_m) du} | \mathcal{G}_t \right]}{B(t,T)}$$

For the second equality, using Proposition 3.9, we have for $H_T = e^{-\int_t^T \kappa_u(B_m) du}$

$$P(t,T,B_m) = I_t(B_m) E^Q \left[ e^{-\int_t^T \kappa_u(B_m) du} | \mathcal{G}_t \right] = I_t(B_m) B(t,T) E^{Q_T} \left[ e^{-\int_t^T \kappa_u(B_m) du} | \mathcal{G}_t \right]$$

3.7.2. $(T,B_m)$-longevity forward martingale measure. In this section, we give the counterpart to the forward martingale measures but for the longevity bonds. Similarly to the previous subsection, for each $B_m$ and each $T$, we can define a measure $Q^{T,B_m}$ equivalent to $Q$ such that its density $\eta_t^{T,B_m} = \frac{dQ^{T,B_m}}{dQ} | \mathcal{G}_t$ is given for $t \leq T$ by

$$\eta_t^{T,B_m} = \frac{e^{-\int_0^t r_u du} P(t,T,B_m)}{P(0,T,B_m)}$$
We can also write this density as a solution of the following differential equation

\[
d\eta^T_{t,T,B^m} = \frac{1}{P(0,T,B^m)} d \left( e^{-\int_0^t r_u du} P(t,T,B^m) \right) \\
= \frac{P(t_-,T,B^m)e^{-\int_0^t r_u du}}{P(0,T,B^m)} \left( -\sigma^*_t(t,T,B^m) dW_t^Q + \int_E \psi^*_t(t,T,B^m,e) q^Q(dt,de) \right) \\
= \eta^T_{t-} \left( -\sigma^*_t(t,T,B^m) dW_t^Q + \int_E \psi^*_t(t,T,B^m,e) q^Q(dt,de) \right)
\]

with \( \eta^T_{0,T,B^m} = 1 \). Again, the density \( \eta^T_{t,T,B^m} \) can be written as the following Doléans exponential:

\[
\eta^T_{t,T,B^m} = \mathcal{E} \left( -\sigma^*_t(t,T,B^m) dW_t^Q + \int_E \psi^*_t(t,T,B^m,e) q^Q(dt,de) \right)
\]

We have the following property which is the counterpart to Proposition 3.10 for longevity forward rates:

**Proposition 3.13.** Under the forward martingale measure \( Q^{T,B^m} \), the longevity forward rate \( f_t(t,T,B^m) \) is a \( (Q^{T,B^m},\mathcal{G}) \)-martingale.

**Proof.** Using Theorem 3.3, we know the process \( W^Q_{t,T,B^m} \) is a \( d \)-dimensional \( (Q^{T,B^m},\mathcal{G}) \)-Brownian motion and \( q^{Q,T,B^m}(t,A) \) is a \( (Q^{T,B^m},\mathcal{G}) \)-local martingale where these processes are defined by:

\[
W^Q_{t,T,B^m} = W^Q_t + \int_0^t \sigma^*_t(u,T,B^m) du \\
q^{Q,T,B^m}(t,A) = q^Q(t,A) - \int_0^t \int_A \psi^*_t(u,T,B^m,e) \beta^N(u,e) \nu_u(de) du \\
= N(t,A) - \int_0^t \int_A (1 + \psi^*_t(u,T,B^m,e)) \beta^N(u,e) \nu_u(de) du
\]

The longevity forward rates of the \((T,B^m)\)-longevity bond can then be written as

\[
f_t(t,T,B^m) = f_t(0,T,B^m) + \int_0^t \sigma_t(v,T,B^m) dW^Q_{v,T,B^m} + \int_0^t \int_E \zeta_t(v,T,B^m,e) q^{Q,T,B^m}(dv,de)
\]

which shows \( f_t(t,T,B^m) \) is indeed a \( (Q^{T,B^m},\mathcal{G}) \)-martingale. \( \square \)

Notice that the previous proposition was not obvious \textit{a priori} since the definition of a longevity bond price given by Equation (3.4) is different from the traditional HJM definition of a zero-coupon bond price.

The following proposition is the counterpart to Proposition 3.11 for the longevity forward martingale measures:

**Proposition 3.14.** For any \( \mathcal{G}_T \)-measurable random variable \( H_T \)

\[
E^Q \left[ H_T (n_0(B_m \times K) - N(T,B^m \times K)) e^{-\int_T^T r_u du} | \mathcal{G}_t \right] = n_0(B_m \times K) P(t,T,B^m) E^{Q,T,B^m} [H_T | \mathcal{G}_t]
\]
PROOF. We have

$$E^{Q_{T,Bm}}[H_T | \mathcal{G}_t] = \frac{E^Q\left[ e^{-\int_0^T r_u du} P(t,T,B_m) H_T | \mathcal{G}_t \right]}{P(0,T,B_m) \cdot \frac{e^{-\int_0^T r_u du} P(t,T,B_m)}{P(0,T,B_m)}} = \frac{1}{n_0(B_m \times K) \cdot P(t,T,B_m)} E^Q\left[ (n_0(B_m \times K) - N(T, B_m \times K)) e^{-\int_0^T r_u du} H_T | \mathcal{G}_t \right]$$

□

This proposition shows that the problem of finding the value of a security paying a stochastic payout $H_T$ to each surviving member of a survivor index basically comes down to finding the expectation of this payout under the associated longevity forward martingale measure.


The aim of this section is to study the asset allocation problem for endowment and annuity portfolios when longevity bonds are traded in the financial market. Longevity bonds give insurers the opportunity, at least theoretically, to perfectly match their liabilities and reduce their risk to zero. However, this situation would seldom occur in real life. Most of the time, an insurer would bear some basis risks since the payouts of the longevity bonds traded in the financial market would not exactly match either the payouts of the liabilities the insurer wants to hedge, or their timings. We can distinguish at least 3 sources of possible basis risks:

1. the maturities of the traded longevity bonds do not match the maturities of the insurance contracts;
2. the age groups of the survivor indices on which the traded longevity bonds are defined, do not match the age of the policyholders;
3. the policyholders of an insurance company is only a fraction of the whole population on which the survivor indices are defined.

To study the asset allocation problem with such basis risks, we chose to apply the risk minimization theory. This theory was developed by Föllmer and Sondermann [44] (1986) for a single payoff and extended by Möller [68] (2001) for payment processes. It has already been applied in life insurance by Möller [68] (2001), Riesner [72] (2006) and Dahl and Möller [39] (2006) for example. In particular, Dahl and Möller only studied the risk-minimizing strategies of life insurance portfolios with longevity risk, when risk-free zero-coupon bonds are traded. Unlike them, we assume longevity bonds are also traded.

In Section 4.1, we give a brief review of the main results of risk minimization theory (see the original papers for more details). In Section 4.2, we study the asset allocation for portfolios of pure endowments and annuities when the set of policyholders of these portfolios exactly corresponds to the population of one of the survivor indices on which a longevity bond is defined. As a by-product of the risk-minimizing strategies calculation, we derive the market values of such portfolios of pure endowments and annuities in, respectively, Propositions 4.2 and 4.5. In Section 4.3, under an additional assumption, we study the asset allocation for
portfolios of pure endowments and annuities when the set of policyholders of these portfolios only represents a fraction of the population of one of these survivor indices. We also give the market values of such portfolios in Propositions 4.11 and 4.14.

4.1. A review of the risk-minimization theory. Let us consider an arbitrary filtration \( G \). \( X \) is the \( s \)-dimensional vector of the discounted prices of the financial securities used for trading. \( X \) corresponds to a vector of \( s^1 \) discounted prices of risk-free zero-coupon bonds and of \( s^2 \) zero-coupon longevity bonds. We assume a specific martingale measure \( Q \) has been chosen so that \( X \) is a (local) martingale under \( Q \).

A trading strategy \( \rho \) is a pair of processes \((\varepsilon, \eta)\) where \( \eta \) is a 1-dimensional real-valued \( G \)-adapted process and \( \varepsilon \) is an \( s \)-dimensional real-valued \( G \)-predictable process. The process \( \varepsilon_t \) represents the amount of risky assets held at time \( t \), and the process \( \eta_t \) is the discounted amount invested in the instantaneously risk-free asset. The discounted value process \( V_t(\rho) \) of a trading strategy \( \rho \), is defined by \( V_t(\rho) = \varepsilon_t X_t + \eta_t \) for \( 0 \leq t \leq T^* \). This value process represents the discounted value of the insurer’s financial portfolio following the trading strategy \( \rho \). This portfolio is not necessarily self-financed.

The liabilities of the insurer are modelled as a process \((A_t)_{0 \leq t \leq T^*}\). The process \( A_t \) is assumed to be càdlàg, \( G \)-adapted and square integrable. The process \( A_t \) represents the discounted value of the cumulative payments up to time \( t \).

The cumulative cost process \( C_t(\rho) \) of a strategy \( \rho \), is defined by \( C_t(\rho) = V_t(\rho) - \int_0^t \varepsilon_u dX_u + A_t \). It represents the discounted value of the portfolio reduced by the discounted trading gains and including the discounted net payments to the policyholders. The initial cumulative cost process \( C_0(\rho) \) is given by \( C_0(\rho) = V_0(\rho) + A_0 \). The first term \( V_0(\rho) \) represents the initial amount an insurer needs to create a financial portfolio. The second term is the initial payment the insurer has to pay to the policyholders. A strategy \( \rho \) is called risk-minimizing, if it minimizes the risk process \( R_t(\rho) \) defined by \( R_t(\rho) = E^Q [(C_{T^*}(\rho) - C_t(\rho))^2 | G_t] \), for every \( t \) and for every strategy such that \( V_{T^*}(\rho) = 0 \), \( Q \) – a.s.

Föllmer and Sondermann [44] (1986) for a single payoff and Möller [68] (2001) for a payment process, show that the solution of this problem is closely related to the so-called Galtchouk-Kunita-Watanabe decomposition of the square integrable martingale \( E [A_{T^*} | G_t] \):

(4.1) \[ E [A_{T^*} | G_t] = E^Q [A_{T^*} | G_0] + \int_0^t \varepsilon^A_u dX_u + L^A_t Q \quad a.s. \]

where \( \varepsilon^A_t \) is an \( s \)-dimensional predictable process and \( L^A_t \) is a zero-mean martingale strongly orthogonal to (the stable subspace generated by) \( X \). With these notation, we can give the solution of the risk minimization problem in the following theorem:

**Theorem 4.1.** The unique risk-minimizing RM-strategy \( \rho^* = (\varepsilon^*, \eta^*) \) of \( A_t \) is given by \( \varepsilon^* = \varepsilon^A \) and \( \eta^*_t = E^Q [A_{T^*} - A_t | G_t] - \varepsilon^*_X_t \). The cumulative cost process of \( \rho^* \) is given by \( C_t(\rho^*) = E^Q [A_{T^*} | G_0] + L^A_t = C_0(\rho^*) + L^A_t \) and the value process of \( \rho^* \) is given by \( V_t(\rho^*) = E^Q [A_{T^*} - A_t | G_t] \).
From a more practical point of view, the predictable integrand $\varepsilon^A_t$ of the Galtchouk-Kunita-Watanabe can be found in the following way. If we denote $J_t(\rho^*) = E^Q[ A^*_t | \mathcal{G}_t ]$, we can write

$$
\langle J_t, X^j \rangle_t = \left< E^Q[ A^*_t | \mathcal{G}_0 ] + \sum_{i=1}^s \int_0^t \varepsilon^i_u dX^i_u + L_t, X^j \right>_t \\
= \sum_{i=1}^s \int_0^t \varepsilon^i_u d\langle X^i, X^j \rangle_u
$$

Accordingly, the integrand $\varepsilon^A_t = (\varepsilon^1_t, \varepsilon^2_t, \ldots, \varepsilon^s_t)$ can be found by solving the system of equations

$$
\begin{pmatrix}
d \langle J_t, X^1 \rangle_t \\
d \langle J_t, X^2 \rangle_t \\
\vdots \\
d \langle J_t, X^s \rangle_t
\end{pmatrix} =
\begin{pmatrix}
d \langle X^1, X^1 \rangle_t & d \langle X^2, X^1 \rangle_t & \cdots & d \langle X^s, X^1 \rangle_t \\
d \langle X^1, X^2 \rangle_t & d \langle X^2, X^2 \rangle_t & \cdots & d \langle X^s, X^2 \rangle_t \\
\vdots & \vdots & \ddots & \vdots \\
d \langle X^1, X^s \rangle_t & d \langle X^2, X^s \rangle_t & \cdots & d \langle X^s, X^s \rangle_t
\end{pmatrix}
\begin{pmatrix}
\varepsilon^1_t \\
\varepsilon^2_t \\
\vdots \\
\varepsilon^s_t
\end{pmatrix}
$$

4.2. Risk-minimization for the population of a survivor index. In this section, we study the risk-minimizing strategies of a portfolio of pure endowments and annuities, subscribed by a set of policyholders of the type $B_m \times K$ whereas in the next section, we want to hedge the mortality risk of a set of policyholders of the type $B_m \times k$ where $k \in K$.

4.2.1. Pure endowments. In this section, we want to find the risk-minimizing hedging strategy of a portfolio of pure endowments with maturity $T$. We assume this portfolio is subscribed by a set of policyholders of the type $B_m \times K$ where $B_m$ is the age group on which a survivor index is defined. The discounted payment at time $T$ is given by

$$
A_T = \frac{g_T( n_0(B_m \times K) - N(T, B_m \times K) )}{B_T}
$$

where $g_T$ is deterministic. To derive the risk-minimizing strategies, we first need the following propositions. The first one gives the market value of such an endowments portfolio.

**Proposition 4.2.** The discounted value of the risk-minimizing portfolio is given by

$$
V^q_t(\rho^*) = n_0(B_m \times K) g_T P(t, T, B_m)
$$

**Proof.** Straightforward since $g_T$ is deterministic.

$$
V^q_t(\rho^*) = E^Q \left[ g_T \left( n_0(B_m \times K) - N(T, B_m \times K) \right) e^{-\int_t^T r_s ds} | \mathcal{G}_t \right] \\
= n_0(B_m \times K) g_T \frac{1}{B_t} E^Q \left[ \left( n_0(B_m \times K) - N(T, B_m \times K) \right) e^{-\int_t^T r_s ds} | \mathcal{G}_t \right] \\
= n_0(B_m \times K) g_T \frac{P(t, T, B_m)}{B_t}
$$
PROPOSITION 4.3. The martingale representation of $J_t^g(\rho^*)$ is given by

$$
J_t^g(\rho^*) = n_0(B_m \times K)g_T P(0,T,B_m) - n_0(B_m \times K)g_T \int_0^t \frac{P(s-T,B_m)}{B_s} \sigma_t^g(s,T,B_m) dW^Q_s + n_0(B_m \times K)g_T \int_0^t \int_E \frac{P(s-T,B_m)}{B_s} \psi_t^g(s,T,B_m,e) q^Q(ds,de)
$$

PROOF. We have $J_t^g(\rho^*) = V_t^g(\rho^*)$ and

$$
V_t^g(\rho^*) = n_0(B_m \times K)g_T \frac{P(t,T,B_m)}{B_t}
$$

$$
= n_0(B_m \times K)g_T P(0,T,B_m) + n_0(B_m \times K)g_T \int_0^t d \left( \frac{P(s,T,B_m)}{B_s} \right)
$$

We get the result since

$$
dP(s,T,B_m) = P(s-T,B_m) \left\{ r_s ds - \sigma_t^g(s,T,B_m) dW^Q_s + \int_E \psi_t^g(s,T,B_m,e) q^Q(ds,de) \right\}
$$

The risk-minimizing strategy depends on the available securities. If we assume that a $(T,B_m)$-longevity bond exists then we can perfectly hedge this portfolio with a simple “buy and hold” strategy by buying $n_0(B_m \times K)g_T$ longevity bonds with maturity $T$. So as already explained, the interesting situations are when:

1. there are longevity bonds defined on the same age group $B_m$ but not for the maturity $T$;
2. there are longevity bonds with the same maturity $T$ but not defined on the same age group;
3. there are no longevity bonds defined on the same age group or the same maturity.

Obviously, even though we assume we cannot directly trade $(T,B_m)$-longevity bonds, we nevertheless assume we can find good estimates of their values and volatilities. This is a standard assumption (at least implicitly) when the HJM methodology is applied. From now on, all the propositions will directly treat the third case above.

We assume we can trade in $s^1$ risk-free zero-coupon bonds of different maturities denoted $B(t,T_i)$ with $i = 1, \ldots, s^1$, and in $s^2$ longevity bonds of different maturities and different age groups denoted $P(t,T_i,B_i)$ with $i = 1, \ldots, s^2$, such that $B_i \in (B_1, \ldots, B_M)$. $s^1$ and $s^2$ are such that $s^1 + s^2 = s$. The risk-minimizing strategies are given by the following propositions:
**Proposition 4.4.** The risk-minimizing strategy \( \rho^* = (\varepsilon_t^*, \eta_t^*) = ((\varepsilon_t^1, \ldots, \varepsilon_t^s), \eta_t^*) \) is given by the solution of the system of equations

\[
\begin{pmatrix}
\frac{d \langle J, X^1 \rangle_t}{d} \\
\frac{d \langle J, X^2 \rangle_t}{d} \\
\vdots \\
\frac{d \langle J, X^s \rangle_t}{d}
\end{pmatrix}
= 
\begin{pmatrix}
\frac{d \langle X^1, X^1 \rangle_t}{d} & \frac{d \langle X^1, X^2 \rangle_t}{d} & \cdots & \frac{d \langle X^1, X^s \rangle_t}{d} \\
\frac{d \langle X^2, X^1 \rangle_t}{d} & \frac{d \langle X^2, X^2 \rangle_t}{d} & \cdots & \frac{d \langle X^2, X^s \rangle_t}{d} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{d \langle X^s, X^1 \rangle_t}{d} & \frac{d \langle X^s, X^2 \rangle_t}{d} & \cdots & \frac{d \langle X^s, X^s \rangle_t}{d}
\end{pmatrix}
\begin{pmatrix}
\varepsilon_t^1 \\
\varepsilon_t^2 \\
\vdots \\
\varepsilon_t^s
\end{pmatrix}
\]

where

\[
\frac{d \langle X^i, X^j \rangle_t}{d} = \frac{B(t_-, T_i) B(t_-, T_j)}{B_t} \left\{ \begin{array}{c}
\int_E \psi_n^*(t, T_i, e) \psi_n^*(t, T_j, e) \beta^N(t, e) \nu_t(de) dt
\end{array} \right\}
\]

for all \( i, j = 1, \ldots, s^1, \)

\[
\frac{d \langle X^i, X^j \rangle_t}{d} = \frac{P(t_-, T_i, B_i) P(t_-, T_j, B_j)}{B_t} \left\{ \begin{array}{c}
\int_E \psi_l^*(t, T_i, B_i) \psi_l^*(t, T_j, B_j) \beta^N(t, e) \nu_t(de) dt
\end{array} \right\}
\]

for all \( i, j = s^1 + 1, \ldots, s, \)

\[
\frac{d \langle X^i, X^j \rangle_t}{d} = \frac{B(t_-, T_i) P(t_-, T_j, B_j)}{B_t} \left\{ \begin{array}{c}
\int_E \psi_n^*(t, T_i, e) \psi_n^*(t, T_j, B_j) \beta^N(t, e) \nu_t(de) dt
\end{array} \right\}
\]

for all \( i = 1, \ldots, s^1 \) and \( j = s^1 + 1, \ldots, s \)

\[
\frac{d \langle J, X^i \rangle_t}{d} = n_0(B_m \times K) g_T \frac{P(t_-, T, B_m) B(t_-, T_i)}{B_t} \Phi(t, T, B_m, i)
\]

where

\[
\Phi(t, T, B_m, i) = \left\{ \begin{array}{c}
\int_E \psi_l^*(t, T, B_m, e) \psi_n^*(t, T_i, e) \beta^N(t, e) \nu_t(de) dt
\end{array} \right\}
\]

for all \( i = 1, \ldots, s^1, \) and

\[
\frac{d \langle J, X^i \rangle_t}{d} = n_0(B_m \times K) g_T \frac{P(t_-, T, B_m) P(t_-, T_i, B_i)}{B_t} \Psi(t, T, B_m, i)
\]

where

\[
\Psi(t, T, B_m, i) = \left\{ \begin{array}{c}
\int_E \psi_l^*(t, T, B_m, e) \psi_l^*(t, T_i, B_i) \beta^N(t, e) \nu_t(de) dt
\end{array} \right\}
\]

for all \( i = s^1 + 1, \ldots, s. \)
The discounted amount to invest in the risk-free asset is given by

\[ \eta^*_t = n_0(B_m \times K) g_T \frac{P(t,T,B_m)}{B_t} - \sum_{i=1}^{s^1} \varepsilon^i_t \frac{B(t,T_i)}{B_t} - \sum_{i=s^1+1}^{s} \varepsilon^i_t \frac{P(t,T_i,B_i)}{B_t} \]

**Proof.** Use Theorem 4.1 and Propositions 4.2 and 4.3 as well as the rules of the sharp bracket. \(\square\)

4.2.2. Annuities. In this section, we want to find the risk-minimizing strategy of a portfolio of continuous annuities paid, up to a given time \(T\), to the surviving members of a survivor index. If we assume the survivor index is defined on the age group \(B_m\), then the cumulated discounted payment at time \(T\), \(A_T\), is given by

\[ A_T = \int_0^T a(u) \left( n_0(B_m \times K) - N(u,B_m \times K) \right) \frac{B_t}{B_u} du \]

Again, to find the risk-minimizing strategies, we need the following propositions. The first one gives the market value of these annuities.

**Proposition 4.5.** The discounted value \(V^a_t(\rho^*)\) of the risk-minimizing portfolio is

\[ V^a_t(\rho^*) = n_0(B_m \times K) \int_t^T a(u) \frac{P(t,u,B_m)}{B_t} du \]

**Proof.** We have

\[ V^a_t(\rho^*) = E^Q [A_T - A_t | G_t] \]

\[ = E^Q \left[ \int_t^T a(u) \left( n_0(B_m \times K) - N(u,B_m \times K) \right) \frac{B_t}{B_u} du \right] | G_t \]

\[ = n_0(B_m \times K) \int_t^T a(u) \frac{1}{B_t} E^Q \left[ \frac{B_t}{B_u} \left( n_0(B_m) - N(u,B_m \times K) \right) \right] | G_t \]

\[ = n_0(B_m \times K) \int_t^T a(u) \frac{P(t,u,B_m)}{B_t} du \]

\(\square\)

**Proposition 4.6.** The martingale representation of \(J^a_t(\rho^*)\) is

\[ J^a_t(\rho^*) = n_0(B_m \times K) \int_0^T a(u) \frac{P(0,u,B_m)}{B_0} du \\
- n_0(B_m \times K) \int_0^t \sigma^*_a(s,T,B_m) dW^Q_s \\
+ n_0(B_m \times K) \int_0^t \int_E \psi^*_a(s,T,B_m,e) q^Q(ds,de) \]
where
\[
\sigma^*_a(s, T, B_m) = \left( \int_s^T a(u) \frac{P(s, u, B_m)}{B_s} \sigma^*_l(s, u, B_m) du \right)
\]
\[
\psi^*_a(s, T, B_m, e) = \left( \int_s^T a(u) \frac{P(s, u, B_m)}{B_s} \psi^*_l(s, u, B_m, e) du \right)
\]

**Proof.** See Appendix 2. □

The risk-minimizing strategies are then given by:

**Proposition 4.7.** The risk-minimizing strategy \( \rho^* = (\varepsilon^*_t, \eta^*_t) = ( (\varepsilon^*_1, \ldots, \varepsilon^*_s), \eta^*_t) \) is given by Proposition 4.4 where

\[
d \langle J, X^i \rangle_t = n_0(B_m \times K) B(t, T_i) B_t \Phi_a(t, T, B_m, i)
\]

where
\[
\Phi_a(t, T, B_m, i) = \left\{ \begin{array}{l}
\sigma^*_a(t, T, B_m) \sigma^*_l(t, T_i) \ dt + \\
\int_E \psi^*_a(t, T, B_m, e) \psi^*_l(t, T_i, B_i, e) \beta^N(t, e) \nu_t(de) \ dt
\end{array} \right. 
\]

for all \( i = 1, \ldots, s^1 \), and

\[
d \langle J, X^i \rangle_t = n_0(B_m \times K) P(t, T_i, B_i) B_t \Psi_a(t, T, B_m, i)
\]

where
\[
\Psi_a(t, T, B_m, i) = \left\{ \begin{array}{l}
\sigma^*_a(t, T, B_m) \sigma^*_l(t, T_i, B_i) \ dt + \\
\int_E \psi^*_a(t, T, B_m, e) \psi^*_l(t, T_i, B_i, e) \beta^N(t, B_i, e) \nu_t(de) \ dt
\end{array} \right. 
\]

for all \( i = s^1 + 1, \ldots, s \).

The discounted amount to invest in the risk-free asset is

\[
\eta^*_t = n_0(B_m \times K) \int_t^T a(u) P(t, u, B_m) du - \sum_{i=1}^{s^1} \varepsilon^*_i B(t, T_i) B_t - \sum_{i=s^1+1}^s \varepsilon^*_i P(t, T_i, B_i) B_t
\]

**Proof.** Use Theorem 4.1 and Propositions 4.5 and 4.6 as well as the rules of the sharp bracket. □

### 4.3. Risk-minimization for the portfolio of an insurance company.

In the previous section, we showed how to hedge pure endowments and annuities associated with a survivor index. However, in real life, the portfolio of an insurance company represents only a fraction of the population of a survivor index. In this section, we study the risk-minimizing strategies for the portfolio of a given insurance company \( k \in K \).
In order to apply the risk-minimization theory as in the previous section, we need to know the dynamics of the process $P^k(t, T, B_m)$ defined by

$$P^k(t, T, B_m) = \mathbb{E}^Q \left[ \frac{n_0(B_m \times k) - N_T(B_m \times k)}{n_0(B_m \times k)} e^{-\int_t^T r_u \, du} | \mathcal{G}_t} \right]$$

In the general model described above, we cannot say much about this process and cannot go much further. However, by introducing the following assumption, we can find interesting and useful relationships between the longevity bond price $P(t, T, B_m)$, and the process $P^k(t, T, B_m)$.

**CONDITION 4.8.** For each $t$, we assume the following equality holds:

$$\frac{\int_{B_m \times k} \beta^N(t, e) \, \nu_t(de)}{(n_0(B_m \times k) - N(t, B_m \times k))} = \frac{\int_{B_m \times K} \beta^N(t, e) \, \nu_t(de)}{(n_0(B_m \times K) - N(t, B_m \times K))}$$

Intuitively, this condition means that the average $Q$-intensity of mortality of the policyholders of the company $k$ in the age group $B_m$ is equal to the average $Q$-intensity of mortality of the whole population in the same age group. Notice also, this condition holds, at least implicitly, in any of the existing intensity models. Indeed, for a given age $x \in X$, this condition becomes

$$\frac{\beta^N(t, (x \times k)) \, \nu_t((x \times k))}{(n_0(B_m \times k) - N(t, B_m \times k))} = \frac{\sum_{i=1}^K \beta^N(t, (x \times k)) \, \nu_t((x \times k))}{(n_0(B_m \times K) - N(t, B_m \times K))}$$

and since in these intensity models, we have $\beta^N(t, (x \times k)) = \beta^N(t, x)$ and $\nu_t((x \times k)) = (n_0(B_m \times k) - N(t, B_m \times k)) \mu(t, x)$, we finally get:

$$\frac{(n_0(B_m \times k) - N(t, B_m \times k)) \beta^N(t, x) \mu(t, x)}{(n_0(B_m \times k) - N(t, B_m \times k))} = \frac{\sum_{i=1}^K \beta^N(t, x) \, (n_0(B_m \times k) - N(t, B_m \times k)) \mu(t, x)}{(n_0(B_m \times K) - N(t, B_m \times K))}$$

$$\beta^N(t, x) \, \mu(t, x) = \beta^N(t, x) \, \mu(t, x)$$

It must be emphasized that only the average $Q$-intensity intervenes in Condition 4.8. We have now the following proposition:

**PROPOSITION 4.9.** If Condition 4.8 holds then

$$P^k(t, T, B_m) = I_t^k(B_m) Z(t, T, B_i)$$

or

$$P^k(t, T, B_m) = \frac{I_t^k(B_m)}{I_t(B_m)} P(t, T, B_i)$$

**PROOF.** Let us assume longevity bonds of all maturities are defined on an additional index

$$I_t^k(B_m) = \frac{n_0(B_m \times k) - N_T(B_m \times k)}{n_0(B_m \times k)}$$

As in Equation (3.4), we define the forward rates $f_t^k(t, s, B_m)$ as

$$P^k(t, T, B_m) = I_t^k(B_m) e^{-\int_t^T f_t^k(t, s, B_m) \, ds}$$
where we assume

\[ f^k_t (t, s, B_m) = f^k_t (0, s, B_m) + \int_0^t \alpha^k_t (u, s, B_m) du + \int_0^t \sigma^k_t (u, s, B_m) dW_u \]

\[ + \int_0^t \int_E \zeta^k_t (u, s, B_m, e) N (du, de) \]

We can write

\[ P^k_t (t, T, B_m) = I^k_t (B_m) e^{-\int_0^T f_t (t, s, B_m) ds - \int_0^T f^\Delta^k_t (t, s, B_m) ds} \]

where \( f^\Delta^k_t (t, s, B_m) = f^k_t (t, s, B_m) - f^k_t (0, s, B_m) \). We could extend Proposition 3.4 with this additional term structure. We would obtain a similar no-arbitrage condition. In particular, we would obtain the equalities

\[ f^k_t (t, t, B_m) - r_t = \frac{\int_{B_m \times K} \beta^N (t, e) \nu_t (de)}{(n_0 (B_m \times K) - N (t_- , B_m \times K))} \]

and

\[ f^k_t (t, t, B_m) - r_t = \frac{\int_{B_m \times k} \beta^N (t, e) \nu_t (de)}{(n_0 (B_m \times k) - N (t_- , B_m \times k))} \]

If Condition 4.8 holds then \( f_t (t, t, B_m) - r_t = f^k_t (t, t, B_m) - r_t \) which implies \( f^\Delta^k_t (t, t, B_m) = 0 \) for all \( t \in [0, T^*] \). In turn, this leads to \( f^\Delta^k_t (s, t, B_m) = 0 \) for all \( s \leq t \). Eventually, we have

\[ P^k_t (t, T, B_m) = I^k_t (B_m) e^{-\int_0^T f_t (t, s, B_m) ds} = \frac{I^k_t (B_m) Z (t, T, B_m)}{I_t (B_m) P (t, T, B_m)} \]

Intuitively, due to Condition 4.8, the price of a security defined on a fraction of a given survivor index is equal to the price of the associated longevity bond times the ratio of the current proportions of survivors. Without Condition 4.8, this simple proportional rule does not necessarily hold. Since Condition 4.8 holds in all the existing intensity models, this simple rule also works well in these cases. However, Proposition 4.9 suggests we could extend these intensity models to more general settings and preserve this property.

As a by-product, Condition 4.8 also leads to the following result. We give it here because we believe it is interesting in its own right but it will not be used later.

**Proposition 4.10.** If Condition 4.8 holds then the ratio

\[ \frac{I^k_t (B_m)}{I_t (B_m)} \]

is a \((Q^{T, B_m}, \mathbb{G})\)-martingale.
PROOF. On one hand, using Proposition 3.14 with $H_T = \frac{I^k_t(B_m)}{I_t(B_m)}$, we have:

$$P^k(t, T, B_m) = E^Q \left[ e^{-\int_t^T r_u \, du} I^k_T(B_m) \middle| G_t \right]$$

$$= P(t, T, B_m) E^{Q^{T, B_m}} \left[ \frac{I^k_T(B_m)}{I_T(B_m)} \middle| G_t \right]$$

On the other hand, we have thanks to the previous proposition, $P^k(t, T, B_m) = \frac{I^k_t(B_m)}{I_t(B_m)} P(t, T, B_m)$. This gives

$$\frac{I^k_t(B_m)}{I_t(B_m)} = E^{Q^{T, B_m}} \left[ \frac{I^k_T(B_m)}{I_T(B_m)} \middle| G_t \right]$$

□

We can now focus on the risk-minimizing strategies of an insurer’s portfolio. From now on, we will assume Condition 4.8 holds.

4.3.1. Pure endowments. The discounted payment at time $T$ is now

$$A_T = \frac{g_T (n_0(B_m \times k) - N(T, B_m \times k))}{B_T}$$

where $g_T$ is deterministic. As in the previous section, we have several propositions. The first one also gives the market value of the insurer’s portfolio when Condition 4.8 holds.

**Proposition 4.11.** The discounted value of the risk-minimizing portfolio is given by

$$V^g_t(\rho^*) = n_0(B_m \times k) g_T \frac{I^k_t(B_m)}{I_t(B_m)} \frac{P(t, T, B_m)}{B_t}$$

**Proof.** Since $g_T$ is deterministic, we have

$$V^g_t(\rho^*) = E^Q \left[ g_T (n_0(B_m \times k) - N(T, B_m \times k)) e^{-\int_0^t r_s \, ds} \middle| G_t \right]$$

$$= n_0(B_m \times k) g_T \frac{1}{B_t} E^Q \left[ \frac{(n_0(B_m \times k) - N(T, B_m \times k))}{n_0(B_m \times k)} e^{-\int_0^t r_s \, ds} \middle| G_t \right]$$

$$= n_0(B_m \times k) g_T \frac{I^k_t(B_m)}{I_t(B_m)} \frac{P(t, T, B_m)}{B_t}$$

□

**Proposition 4.12.** The martingale representation of $J^g_t(\rho^*)$ is given by

$$J^g_t(\rho^*) = n_0(B_m \times k) g_T P(0, T, B_m)$$

$$- n_0(B_m \times K) g_T \int_0^t \left( \frac{I^k_s(B_m)}{I_s(B_m)} \frac{P(s, T, B_m)}{B_s} \right) \sigma^*_l(s, T, B_m) \, dW^Q_s$$

$$+ n_0(B_m \times K) g_T \int_0^t \int_E \left( \frac{I^k_s(B_m)}{I_s(B_m)} \frac{P(s, T, B_m)}{B_s} \right) \psi^*_l(s, T, B, \epsilon) q^Q(ds, de)$$
where

\[
\psi^{s,k}_t(s, T, B_m, e) = \psi^*_Z(s, T, B_m, e) - 1_{\{e \in B_m \times k\}} \frac{e^{-\zeta^*_t(s, T, B_m, e)}}{n_0(B_m \times k) - N(s_-, B_m \times k)}
\]

**Proof.** Again, we have here \( J^t_t(\rho^*) = V^t_t(\rho^*) \) and since \( P^k(0, T, B_m) = P(0, T, B_m) \), we can write

\[
V^t_t(\rho^*) = n_0(B_m \times k) g_T P(0, T, B_m)
+ n_0(B_m \times k) g_T \int_0^t \left( \frac{P^k(s, T, B_m)}{B_s} \right) ds
\]

Following the same argument as for \( P(s, T, B_m) \), we can prove

\[
P^k(s, T, B_m) = P^k(s_-, T, B_m) \left\{ r_s ds - \sigma^*_t(s, T, B_m) dW^Q_s + \int_E \psi^{s,k}_t(s, T, B_m, e) q^Q(ds, de) \right\}
\]

where \( \psi^{s,k}_t(s, T, B_m, e) \) is given above.

We can now give the risk-minimizing strategy. As in the previous section, we assume we can trade in risk-free zero-coupon bonds of \( s^1 \) different maturities and in longevity bonds of \( s^2 \) different maturities with \( s^1 + s^2 = s \).

**Proposition 4.13.** If Condition 4.8 holds, the risk-minimizing strategy \( \rho^* = (\varepsilon^*_t, \eta^*_t) = ((\varepsilon^*_t, \ldots, \varepsilon^*_t), \eta^*_t) \) is given by Proposition 4.4 except we have

\[
d \langle J, X^i \rangle_t = n_0(B_m \times k) g_T \frac{I^k_t(B_m)}{I_{t-}(B_m)} \frac{P(t_-, T, B_m) B(t_-, T_1)}{B_t} \Phi^k(t, T, B_m, i)
\]

where

\[
\Phi^k(t, T, B_m, i) = \left\{ \begin{array}{c}
\sigma^*_t(t, T, B_m) \sigma^*_n(t, T_1) dt + \\
\int_E \psi^{s,k}_t(s, T, B_m, e) \psi^*_n(t, T_1, e) \beta^N(t, e) \nu_t(de) dt
\end{array} \right\}
\]

for all \( i = 1, \ldots, s^1 \), and

\[
d \langle J, X^i \rangle_t = n_0(B_m \times k) g_T \frac{I^k_t(B_m)}{I_{t-}(B_m)} \frac{P(t_-, T, B_m) P(t_-, T_1, B_t)}{B_t} \Psi^k(t, T, B_m, i)
\]

where

\[
\Psi^k(t, T, B_m, i) = \left\{ \begin{array}{c}
\sigma^*_t(t, T, B_m) \sigma^*_n(t, T_1, B_t) dt + \\
\int_E \psi^{s,k}_t(s, T, B_m, e) \psi^*_n(t, T_1, B_t, e) \beta^N(t, e) \nu_t(de) dt
\end{array} \right\}
\]

for all \( i = s^1 + 1, \ldots, s \).

The discounted amount to invest in the risk-free asset is

\[
\eta^*_t = n_0(B_m \times k) g_T \frac{I^k_t(B_m)}{I_{t-}(B_m)} \frac{P(t, T, B_m)}{B_t} - \sum_{i=1}^{s^1} \varepsilon^*_i \frac{B(t, T_1)}{B_t} - \sum_{i=s^1+1}^s \varepsilon^*_i \frac{P(t, T_1, B_t)}{B_t}
\]
Proof. Again, use Theorem 4.1 and Propositions 4.11 and 4.12 as well as the rules of the sharp bracket.

4.3.2. Annuities. In this section, we study the risk-minimizing strategy of a portfolio of continuous annuities paid, up to a given time \( T \), to each surviving policyholder in the age group \( B_m \) of a given insurer \( k \). The cumulated discounted payments \( A_t \) at time \( t \), is then given by

\[
A_t = \int_0^t a(u) \left( n_0(B_m \times k) - N(u, B_m \times k) \right) du
\]

The following proposition gives the market value of the annuity portfolio when Condition 4.8 holds:

**Proposition 4.14.** The discounted value \( V_t^a(\rho^*) \) of the risk-minimizing portfolio is given by

\[
V_t^a(\rho^*) = n_0(B_m \times k) \frac{I_t^k(B_m)}{I_t(B_m)} \int_t^T a(u) \frac{P(t, u, B_m)}{B_t} du
\]

Proof. We have

\[
V_t^a(\rho^*) = E^Q [A_T - A_t | G_t]
\]

\[
= E^Q \left[ \int_t^T a(u) \left( n_0(B_m \times k) - N(u, B_m \times k) \right) du | G_t \right]
\]

\[
= n_0(B_m \times k) \int_t^T a(u) \frac{1}{B_t} E^Q \left[ \frac{B_t \left( n_0(B_m) - N(u, B_m \times k) \right) n_0(B_m)}{n_0(B_m)} | G_t \right] du
\]

\[
= n_0(B_m \times k) \int_t^T a(u) \frac{I_t^k(B_m)}{I_t(B_m)} \frac{P(t, u, B_m)}{B_t} du
\]

**Proposition 4.15.** The martingale representation of \( J_t^a(\rho^*) \) is

\[
J_t^a(\rho^*) = n_0(B_m \times k) \int_0^T a(u) \frac{P(0, u, B_m)}{B_0} du
\]

\[
- n_0(B_m \times k) \int_0^t \frac{I_s^k(B_m)}{I_s(B_m)} \sigma_a^{s,k}(s, T, B_m) dW_s^Q
\]

\[
+ n_0(B_m \times k) \int_0^t \int_E \frac{I_s^k(B_m)}{I_s(B_m)} \psi_a^{s,k}(s, T, B_m, e) q^Q(ds, de)
\]

where

\[
\sigma_a^{s,k}(s, T, B_m) = \int_s^T a(u) \frac{P(s-, u, B_m)}{B_s} \sigma_t^{s,k}(s, u, B_m) du
\]

\[
\psi_a^{s,k}(s, T, B_m, e) = \int_s^T a(u) \frac{P(s-, u, B_m)}{B_s} \psi_t^{s,k}(s, u, B_m, e) du
\]
PROOF. The proof is similar that in Appendix 2.

As far as the risk-minimizing strategy is concerned, we have:

**Proposition 4.16.** If Condition 4.8 holds, the risk-minimizing strategy \( \rho^* = (\varepsilon^*_t, \eta^*_t) = ((\varepsilon^*_1, \ldots, \varepsilon^*_s)_t, \eta^*_t) \) is given by Proposition 4.7 except we have

\[
d \langle J, X^i \rangle_t = n_0(B_m \times k) \frac{I^k_t(B_m)}{I_t(B_m)} B_t \Phi^k_a(t, T, B_m, i)
\]

where

\[
\Phi^k_a(t, T, B_m, i) = \left\{ \begin{array}{c}
\sigma^*_a(t, T, B_m) \sigma^*_n(t, T_i)' dt + \\
\int_E \psi^*_a(t, T, e) \psi^*_n(t, T_i, e) \beta^N(t, e) \nu_t(de) dt
\end{array} \right. 
\]

for all \( i = 1, \ldots, s^1 \), and

\[
d \langle J, X^i \rangle_t = n_0(B_m \times k) \frac{I^k_t(B_m)}{I_t(B_m)} B_t \Psi^k_a(t, T, B_m, i)
\]

where

\[
\Psi^k_a(t, T, B_m, i) = \left\{ \begin{array}{c}
\sigma^*_a(t, T, B_m) \sigma^*_n(t, T_i, B_i) dt + \\
\int_E \psi^*_a(t, T, e) \psi^*_n(t, T_i, B_i, e) \beta^N(t, e) \nu_t(de) dt
\end{array} \right. 
\]

for all \( i = s^1 + 1, \ldots, s \).

The discounted amount to invest in the risk-free asset is given by:

\[
\eta^*_t = n_0(B_m \times k) \frac{I^k_t(B_m)}{I_t(B_m)} \frac{P(t, u, B_m)}{B_t} a(u) du - \sum_{i=1}^{s^1} \varepsilon^*_i B(t, T_i) - \sum_{i=s^1+1}^{s} \varepsilon^*_i P(t, T_i, B_i) B_t
\]

**Proof.** Again, it suffices to use Theorem 4.1 and Propositions 4.14 and 4.15 as well as the rules of the sharp bracket.

**Appendix 1.**

**Proposition.** If the forward rates follow Equation (3.2), then the price of a zero-coupon bond with maturity \( T \) is given by the solution of the differential equation

\[
\begin{align*}
&dB(t, T) = B(t, T) \left\{ \left[ f_n(t, u) - \alpha^*_n(t, T) + \frac{1}{2} \left[ \sigma^*_n(t, T) \sigma^*_n(t, T) \right] \right] dt \\
&- \sigma^*_n(t, T) dW_t + \int_E \psi^*_n(t, T, e) N(dt, de) \right\}
\end{align*}
\]
where

\[
\alpha_n^*(t, T) = \int_t^T \alpha_n(t, s) ds \\
\sigma_n^*(t, T) = \int_t^T \sigma_n(t, s) ds \\
\psi_n^*(t, T, e) = e^{-\zeta_n^*(t, T, e)} - 1 \\
\zeta_n^*(t, T, e) = \int_t^T \zeta_n(t, s, e) ds
\]

**Proof.** Let us first find the dynamics of \( J_n(t, T) = \int_t^T f_n(t, s) ds \). We have

\[
J_n(t, T) = \int_t^T f_n(0, s) ds + \int_t^T \left( \int_0^t \alpha_n(u, s) du \right) ds + \int_t^T \left( \int_0^t \sigma_n(u, s) dW_u \right) ds \\
+ \int_t^T \int_0^t \int_E \zeta_n(u, s, e) N(du, de) ds \\
= \int_t^T f_n(0, s) ds + \int_0^t \left( \int_0^T \alpha_n(u, s) ds \right) du + \int_0^t \left( \int_0^T \sigma_n(u, s) ds \right) dW_u \\
+ \int_0^t \int_E \int_t^T \zeta_n(u, s, e) ds N(du, de) \\
- \int_0^t f_n(0, s) ds - \int_0^t \left( \int_0^t \alpha_n(u, s) ds \right) du - \int_0^t \left( \int_0^t \sigma_n(u, s) ds \right) dW_u \\
- \int_0^t \int_E \int_u^T \zeta_n(u, s, e) ds N(du, de)
\]

On the other hand,

\[
\int_0^t f_n(s, s) ds = \int_0^t f_n(0, s) ds + \int_0^t \left( \int_0^s \alpha_n(u, s) du \right) ds + \int_0^t \left( \int_0^s \sigma_n(u, s) dW_u \right) ds \\
+ \int_0^t \int_0^s \int_E \zeta_n(u, s, e) N(du, de) ds \\
= \int_0^t f_n(0, s) ds + \int_0^t \left( \int_0^t \alpha_n(u, s) ds \right) du + \int_0^t \left( \int_0^t \sigma_n(u, s) ds \right) dW_u \\
+ \int_0^t \int_E \int_t^t \zeta_n(u, s, e) ds N(du, de)
\]
which leads to

\[
J_n(t, T) = \int_0^T f_n(0, s) ds - \int_0^T f_n(s, s) ds + \int_0^T \alpha_n^*(u, T) du + \int_0^T \sigma_n^*(u, T) dW_u
+ \int_0^T \int_{\mathcal{E}} \zeta_n^*(u, T, e) N(du, de)
\]

Using Itô’s lemma for semimartingale on the function \( f(x) = \exp(-x) \), we can find the dynamic of \( B(t, T) = e^{-J_n(t, T)} \). We obtain

\[
 dB(t, T) = -e^{-J_n(t_- , T)} dJ_n(t, T) + \frac{1}{2} e^{-J_n(t_- , T)} (d\langle J_n(t, T), J_n(t, T) \rangle_c + e^{-J_n(t, T)} - e^{-J_n(t_- , T)})
\]

\[
= -e^{-J_n(t_- , T)} \left\{ dJ_n(t, T) + \frac{1}{2} \left[ \sigma_n^*(t, T)' \sigma_n^*(t, T) \right] dt + \left( e^{-\Delta J_n(t, T)} - 1 \right) \right\}
\]

\[
= B(t_, T) \left\{ -dJ_n(t, T) + \frac{1}{2} \left[ \sigma_n^*(t, T)' \sigma_n^*(t, T) \right] dt + \int_{\mathcal{E}} (e^{-\zeta_n(t_, T, e)} - 1) N(dt, de) \right\}
\]

So eventually, we get

\[
 dB(t, T) = B(t_, T) \left\{ \left[ f_n(t, t) - \alpha_n^*(t, T) + \frac{1}{2} \left[ \sigma_n^*(t, T)' \sigma_n^*(t, T) \right] \right] dt - \sigma_n^*(t, T) dW_t + \int_{\mathcal{E}} (e^{-\zeta_n(t_, T, e)} - 1) N(dt, de) \right\}
\]

\[
\square
\]

Appendix 2.

**Proposition.** The martingale representation of \( J_t^p(\rho^*) \) is given by

\[
J_t^p(\rho^*) = \int_0^T a(u)n_0(B_m \times K) P(0, u, B_m) \frac{P(0, u, B_m)}{B_0} du
- \int_0^T n_0(B_m \times K) \left( \int_s^T a(u) \frac{P(s-, u, B_m)}{B_s} \sigma_l^*(s, u, B_m) du \right) dW_s^Q
+ \int_0^T n_0(B_m \times K) \left( \int_s^T a(u) \frac{P(s-, u, B_m)}{B_s} \int_{\mathcal{E}} \psi_l^*(s, u, B_m, e) du \right) q^Q(ds, da)
\]
PROOF. We have

\[
J_t^n = n_0(B_m \times K)E^Q \left[ \int_0^T a(u) \frac{1}{B_u} \left( n_0(B_m \times K) - N(u, B_m \times K) \right) du \big| G_t \right]
\]

\[
= n_0(B_m \times K) \int_0^T a(u) \frac{1}{B_u} \left( n_0(B_m \times K) - N(u, B_m \times K) \right) du
\]

\[
+ n_0(B_m \times K) \int_t^T a(u) \frac{1}{B_u} E^Q \left[ \frac{B_t \left( n_0(B_m \times K) - N(u, B_m \times K) \right)}{n_0(B_m \times K)} \right] du
\]

\[
= n_0(B_m \times K) \int_0^T a(u) \frac{1}{B_u} \left( n_0(B_m \times K) - N(u, B_m \times K) \right) du
\]

\[
+ n_0(B_m \times K) \int_t^T a(u) \frac{P(t, u, B_m)}{B_t} du
\]

Let us focus on the second term \((I) = n_0(B_m \times K) \int_t^T a(u) \frac{P(t, u, B_m)}{B_t} du\). Since the price of a longevity bond is given by

\[
P(t, u, B_m) = P(0, u, B_m)
\]

\[
- \int_0^t \frac{P(s, u, B_m)}{B_s} \sigma^*_t (s, u, B_m) dW^Q_s
\]

\[
+ \int_0^t \frac{P(s, u, B_m)}{B_s} \int_E \psi^*_t (s, u, B_m, e) q^Q(ds, de)
\]

where \(\psi^*_t (s, u, B_m, e) = (e^{-\zeta^*_t (s,u,B_m,e)} - 1) - 1_{e \in B_m \times K} \frac{e^{-\zeta^*_t (s,u,B_m,e)}}{(n_0(B_m \times K) - N(s, B_m \times K))})\), we can write

\[
(I) = \int_t^T a(u) n_0(B_m \times K) \frac{P(0, u, B_m)}{B_0} du
\]

\[
- \int_t^T a(u) n_0(B_m \times K) \left( \int_0^t \frac{P(s, u, B_m)}{B_s} \sigma^*_t (s, u, B_m) dW^Q_s \right) du
\]

\[
+ \int_t^T a(u) n_0(B_m \times K) \left( \int_0^t \frac{P(s, u, B_m)}{B_s} \int_E \psi^*_t (s, u, B_m, e) q^Q(ds, de) \right) du
\]
(I) \[ \begin{align*} &\int_0^T a(u)n_0(B_m \times K) \frac{P(0, u, B_m)}{B_0} du \\ &- \int_0^t n_0(B_m \times K) \left( \int_s^T a(u) \frac{P(s-, u, B_m)}{B_s} \sigma^*_t(s, u, B_m) du \right) dW_t^Q \\ &+ \int_0^t n_0(B_m \times K) \left( \int_s^T a(u) \frac{P(s-, u, B_m)}{B_s} \int_E \psi^*_t(s, u, B_m) du \right) q^Q(ds, de) \\ &- \int_0^t a(u)n_0(B_m \times K) \frac{P(0, u, B_m)}{B_0} du \\ &+ \int_0^t n_0(B_m \times K) \left( \int_s^T a(u) \frac{P(s-, u, B_m)}{B_s} \sigma^*_t(s, u, B_m) du \right) dW_t^Q \\ &- \int_0^t n_0(B_m \times K) \left( \int_s^T a(u) \frac{P(s-, u, B_m)}{B_s} \int_E \psi^*_t(s, u, B_m, e) du \right) q^Q(ds, de) \\ \end{align*} \]

Then, by changing the order of integration again, the three last terms (II) can be written as

(II) \[ \begin{align*} &- \int_0^t a(u)n_0(B_m \times K) \frac{P(0, u, B_m)}{B_0} du \\ &+ \int_0^t n_0(B_m \times K)a(u) \left( \int_0^u \frac{P(s-, u, B_m)}{B_s} \sigma^*_t(s, u, B_m) dW_t^Q \right) du \\ &- \int_0^t n_0(B_m \times K)a(u) \left( \int_0^u \frac{P(s-, u, B_m)}{B_s} \int_E \psi^*_t(s, u, B_m, e) du \right) q^Q(ds, de) \\ &= - \int_0^t a(u)n_0(B_m \times K) \frac{P(u, u, B_m)}{B_u} du \\ &= - \int_0^t a(u)n_0(B_m \times K) - N(u, B_m \times K) du \\ \end{align*} \]

Eventually, we get

\[ \begin{align*} J^a_t &= \int_0^T a(u)n_0(B_m \times K) \frac{P(0, u, B_m)}{B_0} du \\ &- \int_0^t n_0(B_m \times K) \left( \int_s^T a(u) \frac{P(s-, u, B_m)}{B_s} \sigma^*_t(s, u, B_m) du \right) dW_t^Q \\ &+ \int_0^t n_0(B_m \times K) \left( \int_s^T a(u) \frac{P(s-, u, B_m)}{B_s} \int_E \psi^*_t(s, u, B_m, e) du \right) q^Q(ds, de) \\ &\square \]
Part 4

Non-Life Insurance Contracts and the Inflation Risk.
CHAPTER 7

Risk-Minimization with Inflation and Interest Rate Risk.

1. Introduction.

This chapter aims at studying the asset allocation problem of a non-life insurance company. The academic literature on this topic is rather scarce. Indeed, most portfolio selection problems in actuarial mathematics concern the optimal asset allocation for life insurance portfolios or pension plans. This is due to the fact that these contracts are long-term contracts more subject to investment risk than the short-term non-life contracts. There are however good arguments to study the asset allocation problem in non-life insurance as well. Indeed, the cost of claims is highly sensitive to inflation, which, in turn, is itself correlated with the financial assets returns and in particular, with the interest rates movements. Moreover, the reserve an insurer should maintain, computed as the expected discounted claims, is subject to interest rate risk through the variation in the discounting term and indirectly through the variation in expected outstanding claims due to the inflation. The application of ALM techniques in non-life insurance is thus justified from a theoretical point of view.

The ALM methods based on duration and convexity have been studied in non-life insurance by Ahlgrim and al [5]. More precisely, they studied the effective durations and convexities of non-life insurance liabilities. These notions extend the classical concepts of duration and convexity when the future cash flows are interest rates sensitive. In their paper, they assume the interest rates and the inflation are stochastic and correlated. Furthermore, they assume a fraction of the claim costs is inflation sensitive. They showed that the effective durations and convexities based on these assumptions are substantially lower than those measured with traditional approaches i.e. without stochastic inflation and interest rates. Their analysis underlines the importance of introducing stochastic interest rates and inflation when dealing with ALM in non-life insurance.

Even though Ahlgrim et al. [5] introduce stochastic interest rates and inflation, their approach relies nonetheless on static ALM techniques. Dynamic portfolio selection in continuous time for non-life insurance businesses is rather rare. We can cite the paper of Browne [35], those of Hipp and Plum [52, 53] and Korn [60]. In [35], Browne derives the financial policy that minimizes the probability of ruin and the one that maximizes the expected exponential utility of terminal wealth when the cumulative cost of claims follows a Brownian motion with drift. For each objective, he derives the optimal policy when the insurer can invest in a single risky asset and when he can invest in a risky asset and a risk free asset. The risky asset follows a geometric Brownian motion and the risk free asset grows at a constant risk free rate. Hipp and Plum [52] derive the asset allocation that minimizes the probability of ruin when
the cost of claims follows a compound Poisson process and when the insurer can invest in a single risky asset following a geometric Brownian motion. In [53], Hipp and Plum study the accumulated discounted expected utility of wealth as the objective function. They find the optimal financial policy of an insurer with a cost of claims following a compound Poisson process and a compound Cox process (generated by a finite state space Markov process) when the insurer can invest in a risky financial asset following a geometric Brownian motion. In [60], Korn extends the work of Browne by introducing the possibility of a market crash. He finds the best deterministic financial policy that maximizes the worst-case exponential utility from terminal wealth. As in Browne, the cost of claims follows a Brownian motion with drift and the financial market consists of a risky asset following a geometric Brownian motion subject to a single sudden downward jump, and a risk free asset. In a very interesting paper, Delong [41] introduces another objective. He defines a linear-quadratic function that penalizes deviations of the insurance surplus process below a reserve for outstanding claims (computed under prudential basis) and rewards deviations above the reserve. He assumes the cost of claims follows a compound Poisson process and considers a financial market with a risk free asset (constant rate) and 2 risky assets following geometric Brownian motions. Unfortunately, these papers do not take into account the inflation risk and in most case, they do not assume a stochastic interest rates term structure.

In order to try to overcome these shortcomings, we choose to follow Møller who applied the risk-minimization theory in [68] to non-life insurance problems. This theory has been developed by Föllmer and Sondermann [44] for a single payoff and extended by Møller [68] for a payment process. We extend the work of Møller in two directions. Firstly, we introduce a financial market with stochastic interest rates and stochastic inflation. Secondly, as far as the cumulative cost of claims is concerned, we study an arbitrary (increasing) process (non necessarily Markov) adapted to the filtration of a general marked point process. However, we are also slightly less general than Møller in the sense that he allows the distribution of the cost of the claims to depend on the value of a financial index whereas in our model, the size of claims depends on the financial market through the mechanism of inflation only.

This chapter is organized as followed. In Section 2.1, we describe the financial market developed by Jarrow and Yildirim [57]. It is a general continuous time HJM model with inflation. In Section 2.2, we describe the claims model. It is based on a general marked point process (with absolutely continuous compensator though) as suggested by Arjas [9]. In Section 3.1, we give a brief review of the risk minimization theory and in Section 3.2, we solve the risk minimization problem in the general framework described in Section 2. In Section 4 and Section 5, we then apply systematically this result to 4 specific models of insurance claims. Section 4 treats two collective models based on the compound Poisson process. In the first one, the number of policies is constant whereas in the second one the number of policies is stochastic and follows a point process with an inhomogeneous and stochastic intensity. Section 5 treats two individual models where claims are notified at a random time and settled through time according to a deterministic function in the first one and according to a Beta-Stacy process in the second one.
2. The Model.

2.1. The financial market. We use the model developed by Jarrow and Yildirim [57]. Hughston [55] also described a similar model. In their paper, these authors exploit an analogy between the inflation and an exchange rate. They distinguish two “economies”. One is a “nominal economy” and the other is a “real economy”. For each one, an interest rate curve of the HJM-type is described. The inflation plays the role of an exchange rate between these two economies. Let us describe briefly this model. See the original paper for more details.

Let $W$ be a $d$-dimensional Brownian motion defined on a probability space $(\Omega, F, F, P)$. We assume $F = (F_t)_{0 \leq t < T^*}$ is the $P$-completed version of the filtration generated by $W$.

The nominal interest rate term structure. We assume there exists for each maturity $T \in [0, T^*]$, a nominal 0-coupon bond whose price at time $t \in [0, T]$ is denoted $P_n(t, T)$. As in the traditional HJM model, we assume the nominal forward rates follow:

$$df_n(t, T) := \alpha_n(t, T)dt + \sigma_n(t, T)dW(t)$$

where $\alpha_n(t, T)$ and $\sigma_n(t, T)$ are some $F$-adapted stochastic processes. By definition, the price $P_n(t, T)$ at time $t$ of a nominal 0-coupon bond with maturity $T$ is given by

$$P_n(t, T) := e^{-\int_t^T f_n(u, u)du}$$

We assume we can invest in an instantaneously risk free asset (in nominal terms) whose price at time $t$ is denoted $B_n(t)$. The price of this asset evolves as

$$dB_n(t) = n(t)B_n(t)dt$$

with $B_n(0) = 1$ and where $n(t)$ is an instantaneously nominal risk free rate. It can be shown that we have $n(t) = f_n(t, t)$.

The real interest rate term structure. Similarly, we assume there exists a real 0-coupon bond for each maturity $T \in [0, T^*]$ whose price is denoted $P_r(t, T)$. The real interest rates curve is described through the following real forward rates:

$$df_r(t, T) := \alpha_r(t, T)dt + \sigma_r(t, T)dW(t)$$

The price $P_r(t, T)$ at time $t$ of a real 0-coupon bond with maturity $T$ is then given by

$$P_r(t, T) := e^{-\int_t^T f_r(u, u)du}$$

We assume there is an instantaneously risk free asset (in real terms) whose price is denoted by $B_r(t)$. The price of this asset is given by

$$dB_r(t) = r(t)B_r(t)dt$$

where $r(t)$ is the instantaneously real risk free rate. We also have $r(t) = f_r(t, t)$. The difference $n(t) - r(t)$ is a risk premium for bearing the inflation risk.
The price index. As far as the value of the price index \( I(t) \) is concerned, its evolution is described by the following SDE:

\[
dI(t) \ := \ I(t) \left[ \mu_I(t)dt + \sigma_I(t)dW(t) \right]
\]

where \( \mu_I(t) \) and \( \sigma_I(t) \) are some arbitrary \( \mathbb{F} \)-adapted stochastic processes.

The no-arbitrage opportunity condition. As usual in financial economics, Jarrow and Yildirim assume that the no-arbitrage hypothesis holds. This assumption implies there exists at least one measure \( Q^F \) equivalent to \( P^F \), under which the following discounted asset prices are \( (Q^F, \mathbb{F}) \)-martingales:

\[
\frac{I(t)B_r(t)}{B_n(t)}, \quad \frac{P_n(t,T)}{B_n(t)}, \quad \frac{I(t)P_r(t,T)}{B_n(t)}
\]

Such a measure \( Q^F \) is called a martingale measure.

Let us define

\[
\sigma^*_n(t,T) := \int_t^T \sigma_n(t,u)du \quad \alpha^*_n(t,T) := \int_t^T \alpha_n(t,u)du
\]

\[
\sigma^*_r(t,T) := \int_t^T \sigma_r(t,u)du \quad \alpha^*_r(t,T) := \int_t^T \alpha_r(t,u)du
\]

Jarrow and Yildirim showed that the no arbitrage condition implies the following constraints on the drift of the inflation and the drifts of the forward rates:

\[
\mu_I(t) = n(t) - r(t) + \sigma_I(t)\beta(t)
\]

\[
\alpha^*_n(t,T) = \sigma^*_n(t,T) \left[ \frac{1}{2} \sigma^*_n(t,T)' + \beta(t) \right]
\]

\[
\alpha^*_r(t,T) = \sigma^*_r(t,T) \left[ \frac{1}{2} \sigma^*_r(t,T)' + \beta(t) - \sigma_I(t)' \right]
\]

where \( \beta(t) \) is the \( d \)-dimensional market price of risk associated to the Brownian motion \( W \). In other words, for any stochastic process \( \beta(t) \) such that these constraints hold, the measure \( Q^F \) defined by \( \frac{dQ^F}{dP^F} := \mathcal{E}_{T^*} \left( -\int_0^T \beta(u)du \right) \) is an equivalent martingale measure. The process \( W^Q(t) \) defined by \( W^Q(t) = W(t) + \int_0^t \beta(u)du \) is a \( (Q^F, \mathbb{F}) \)-Brownian motion.

Jarrow and Yildirim gave the following dynamics:

\[
dI(t) = I(t) \left[ (n(t) - r(t) + \sigma_I(t)\beta(t)) dt + \sigma_I(t)dW(t) \right]
\]

\[
dP_n(t,T) = P_n(t,T) \left[ (n(t) - \sigma^*_n(t,T)\beta(t)) dt - \sigma^*_n(t,T)dW(t) \right]
\]

\[
dP_r(t,T) = P_r(t,T) \left[ (r(t) + \sigma^*_r(t,T)\sigma_I(t)' - \sigma_r^*(t,T)\beta(t)) dt - \sigma_r^*(t,T)dW(t) \right]
\]

We can also write:
where \( W^Q(t) \) is a \( d \)-dimensional Brownian motion under \( Q^F \).

**Treasury Inflation Protected securities (TIPS).** In Section 3.2, we will assume that we can invest in two different categories of securities. The first category is the nominal 0-coupon bonds. The second category is the 0-coupon Treasury Inflation Protected Securities (TIPS). A 0-coupon TIPS of maturity \( T \) pays one real unit at time \( T \). In other words, it pays \( I(T) \) units in nominal terms. Jarrow and Yildirim showed the price \( Tp(t, T) \) at time \( t \) of such security is given by

\[
Tp(t, T) = E^{Q^F} \left[ \left. I(T) \frac{B_n(t)}{B_n(T)} \right| \mathcal{F}_t \right] = I(t)P_t(t, T)
\]

Using the results of the previous subsection, we can easily show the discounted price of a nominal 0-coupon bond with maturity \( T \) and the discounted price of a 0-coupon TIPS with maturity \( T \) follow the SDEs:

\[
d \left( \frac{P_n(t, T)}{B_n(t)} \right) = -\frac{P_n(t, T)}{B_n(t)} \sigma_n^*(t, T)dW^Q(t)
\]

\[
d \left( \frac{Tp(t, T)}{B_n(t)} \right) = -\frac{Tp(t, T)}{B_n(t)} \sigma_{Tp}^*(t, T)dW^Q(t)
\]

where \( \sigma_{Tp}^*(t, T) = \sigma^*(t, T) - \sigma_I(t) \).

**2.2. The Insurance model.** We follow some ideas developed in Arjas [9]. Let us consider another probability space \( (\Omega^I, \mathcal{F}^I, Q^I) \). We assume a marked point process \( (T_n, Z_n)_{n \geq 1} \) is defined on this space. \( (T_n)_{n \geq 1} \) is a sequence of random times. \( (Z_n)_{n \geq 1} \) is a sequence of random variables taking their values in a Borel space \( (E, \mathcal{E}) \) and associated to the sequence \( (T_n)_{n \geq 1} \).

For a detailed description of marked point processes, see Brémaud [33], Jacobsen [56] or Last and Brandt [61]. For each \( B \in \mathcal{E} \), we define the counting process \( N(t, B) = \sum_{n=1}^{\infty} 1_{T_n \leq t} Z_n \in B \). The filtration \( \mathbb{I} \) is assumed to be the natural filtration of this marked point process i.e. \( \mathbb{I} = (\mathbb{I}_t)_{0 \leq t \leq T} \) where \( \mathbb{I}_t = \sigma(N(s, B), 0 \leq s \leq t, \forall B \in \mathcal{E}) \). For each \( B \in \mathcal{E} \), we denote by \( p(t, B) \) the \((Q^I, \mathbb{I})\)-compensator of \( N(t, B) \) and \( q(t, B) \) the compensated counting process defined as \( q(t, B) := N(t, B) - p(t, B) \).

The cumulative payments paid by the insurer to the policyholders are described by a process \( (S_t)_{0 \leq t \leq T^*} \) defined on the space \( \Omega^I \). The payments are expressed in real units. We denote by \( \Lambda^S_t \) the \((Q^I, \mathbb{I})\)-compensator \( S_t \) and assume the following conditions hold:
CONDITION 2.1. $S_t$ is a square integrable, increasing$^1$, right-continuous stochastic process adapted to the filtration $\mathcal{I}$.

CONDITION 2.2. $\Lambda^S_t$ is absolutely continuous with respect to the Lebesgue measure i.e. $\Lambda^S_t = \int_0^t \lambda^S_u du$ for an $\mathcal{I}$-predictable process $\lambda^S_u$.

We note $M^S_t$ the compensated process: $M^S_t = S_t - \Lambda^S_t$.

We do not assume $Q^I$ is necessarily the physical measure.

2.3. The combined model. The combined model consists in the following probability space $(\Omega^F \times \Omega^I, \mathcal{F} \otimes \mathcal{F}^I, Q)$ where $Q$ is the product measure of $Q^F$ and $Q^I$. We define the filtration $G = (G_t)_{0 \leq t \leq T}$, where $G_t = \mathcal{F}_t \vee \mathcal{I}_t$. Under the product measure $Q$, the filtrations $\mathcal{F}$ and $\mathcal{I}$ are independent. It implies the martingale property of the different processes is preserved under this enlargement of filtration. In other words, all the $(Q^F, \mathcal{F})$-martingales and all the $(Q^I, \mathcal{I})$-martingales are $(Q, G)$-martingales$^2$. Accordingly, the discounted prices of the financial securities are $(Q, G)$-martingales and the $(Q^I, \mathcal{I})$-compensators of the processes $N(t, B)$ and $S_t$ are also their $(Q, G)$-compensators (for any $B \in \mathcal{E}$).

In the previous section, we expressed the cumulative insurance payments in real terms. We can now express the cumulative discounted nominal payments process $A_t$. This process is given by

$$A_t = \int_0^t \frac{I(u)}{B_n(u)} dS_u$$


3.1. A review of the risk-minimization theory. Let us consider an arbitrary filtration $\mathcal{G}$. We denote by $X$ an $s$-dimensional vector of discounted prices of the financial securities used for trading. $X$ corresponds to a vector of discounted prices of 0-coupon nominal bonds and of 0-coupon TIPS. We assume a specific martingale measure $Q$ has been chosen so that $X$ is a (local) martingale under $Q$.

A trading strategy $\rho$ is a pair of processes $(\varepsilon, \eta)$ where $\eta$ is a 1-dimensional real-valued $\mathcal{G}$-adapted process and $\varepsilon$ is an $s$-dimensional real-valued $\mathcal{G}$-predictable process. The process $\varepsilon_t$ represents the amount of risky assets held at time $t$ and the process $\eta_t$ is the discounted amount invested in the instantaneously risk free asset. The discounted value process $V_t(\rho)$ of a trading strategy $\rho$, is defined by $V_t(\rho) = \varepsilon_t X_t + \eta_t$ for $0 \leq t \leq T^*$. This value process represents

$^1$We could easily extend our results to a finite variation process (not necessarily increasing) since it can be written as the difference of two increasing processes. Thanks to the linearity of the Galtchouk Kunita Watanabe, the risk-minimizing strategies of this process is also the difference of the risk-minimizing strategies of the increasing processes.

$^2$In particular, the sharp bracket with respect to the filtration $\mathcal{F}$, of a given process is indistinguishable of the sharp bracket with respect to the filtration $\mathcal{G}$ of the same process. In the following, we will not distinguish these different versions of the sharp bracket.
the discounted value of the insurer’s financial portfolio following the trading strategy $\rho$. This portfolio is not necessarily self-financed.

The liabilities of the insurer are modelled as a process $(A_t)_{0 \leq t < T^*}$. The process $A_t$ is assumed to be càdlàg, $\mathbb{G}$-adapted and square integrable. The process $A_t$ represents the discounted value of the cumulative payments up to time $t$.

The cumulative cost process $C_t(\rho)$ of a strategy $\rho$, is defined by $C_t(\rho) = V_t(\rho) - \int_0^t \varepsilon_u dX_u + A_t$. It represents the discounted value of the portfolio reduced by the discounted trading gains and added discounted net payments to the policyholders. The initial cumulative cost process $C_0(\rho)$ is given by $C_0(\rho) = V_0(\rho) + A_0$. The first term $V_0(\rho)$ represents the initial amount the insurer needs to create his financial portfolio. The second term is the initial payment the insurer has to pay to the policyholder. Usually, $A_0$ will be negative and instead of a payment will represent the initial premium paid by the policyholder. A strategy $\rho$ is called risk-minimizing, if it minimizes the risk process $R_t(\rho)$ defined by $R_t(\rho) = E^Q \left[ (C_{T^*}(\rho) - C_t(\rho))^2 \mid \mathcal{G}_t \right]$, for every $t$ and for every strategy such that $V_{T^*}(\rho) = 0$, $Q$-a.s.

Föllmer and Sondermann [44] for a single payoff and Møller [68] for a payment process, showed that the solution of this problem is closely related to the so-called Galtchouk-Kunita-Watanabe decomposition of the square integrable martingale $E^Q [A_{T^*} \mid \mathcal{G}_t]$:

$$E^Q [A_{T^*} \mid \mathcal{G}_t] = E^Q [A_{T^*} \mid \mathcal{G}_0] + \int_0^t \varepsilon_u^A dX_u + L_t^A$$

where $\varepsilon_t^A$ is an $s$-dimensional predictable process and $L_t^A$ is a zero-mean martingale strongly orthogonal to (the stable subspace generated by) $X$. With these notation, we can give the solution of the risk minimization problem in the following theorem:

**Theorem 3.1.** The unique risk-minimizing RM-strategy $\rho^* = (\varepsilon^*, \eta^*)$ of $A_t$ is given by $\varepsilon^* = \varepsilon^A$ and $\eta_t^* = E^Q [A_{T^*} - A_t \mid \mathcal{G}_t] - \varepsilon_t^* X_t$. The cumulative cost process of $\rho^*$ is given by $C_t(\rho^*) = E^Q [A_{T^*} \mid \mathcal{G}_0] + L_t^A = C_0(\rho^*) + L_t^A$ and the value process of $\rho^*$ is given by $V_t(\rho^*) = E^Q [A_{T^*} - A_t \mid \mathcal{G}_t]$.

From a more practical point of view, the predictable integrand $\varepsilon_t^A$ of the Galtchouk-Kunita-Watanabe can be found in the following way. If we denote $J_t = E^Q [A_{T^*} \mid \mathcal{G}_t]$, we can write:

$$\langle J_t, X^j \rangle_t = \left\langle E^Q [A_{T^*} \mid \mathcal{G}_0] + \sum_{i=1}^s \int_0^t \varepsilon_u^i dX_u^i + L_t, X^j \right\rangle_t$$

$$= \sum_{i=1}^s \int_0^t \varepsilon_u^i d\langle X^i, X^j \rangle_u$$

Accordingly, the integrand $\varepsilon_t^A = (\varepsilon_1^A, \varepsilon_2^A, \ldots, \varepsilon_s^A)$ can be found by solving the following ordinary system of equations:
\[
\begin{pmatrix}
  d \langle J, X_1 \rangle_t \\
  d \langle J, X_2^1 \rangle_t \\
  \vdots \\
  d \langle J, X_s \rangle_t \\
\end{pmatrix} =
\begin{pmatrix}
  d \langle X_1, X_1 \rangle_t & d \langle X_1, X_2^1 \rangle_t & \cdots & d \langle X_1, X_s \rangle_t \\
  d \langle X_2^1, X_1 \rangle_t & d \langle X_2^1, X_2^1 \rangle_t & \cdots & d \langle X_2^1, X_s \rangle_t \\
  \vdots & \vdots & \ddots & \vdots \\
  d \langle X_s, X_1 \rangle_t & d \langle X_s, X_2^1 \rangle_t & \cdots & d \langle X_s, X_s \rangle_t \\
\end{pmatrix}
\begin{pmatrix}
  \varepsilon_1^t \\
  \varepsilon_2^t \\
  \vdots \\
  \varepsilon_s^t \\
\end{pmatrix}
\]

3.2. Risk-Minimization in non-life insurance. In this section we give the risk-minimizing strategy of the cumulative payments process \( A_t \) described in Section 2.3.

**Proposition 3.2.** The value process \( V_t(\rho^*) \) is given by

\[
V_t(\rho^*) = \int_t^{T^*} \frac{Tp(t,u)}{B_n(t)} E^Q \left[ \lambda_u^S | I_t \right] du
\]

**Proof.** Indeed we have

\[
V_t(\rho^*) = E^Q \left[ \int_t^{T^*} \frac{I(u)}{B_n(u)} dS_u | G_t \right] = E^Q \left[ \int_t^{T^*} \frac{I(u)}{B_n(u)} \lambda_u^S du | G_t \right] = \int_t^{T^*} E^Q \left[ \frac{I(u)}{B_n(u)} | G_t \right] E^Q \left[ \lambda_u^S | G_t \right] du
\]

here we use Fubini Theorem and the \( Q \)-independence between the financial market and the insurance payments. Since \( E^Q \left[ \frac{I(u)}{B_n(u)} | G_t \right] = E^Q \left[ \frac{I(u)}{B_n(u)} | F_t \right] = \frac{Tp(t,u)}{B_n(t)} \) and \( E^Q \left[ \lambda_u^S | G_t \right] = E^Q \left[ \lambda_u^S | I_t \right] \), we get the result. \( \square \)

The next proposition is the heart of this chapter. It gives the martingale representation of \( E^Q [A_{T^*} | G_t] \) which will lead directly to the Galtchouk-Kunita Watanabe decomposition.

**Proposition 3.3.** The martingale representation of \( E^Q [A_{T^*} | G_t] \) is given by

\[
E^Q [A_{T^*} | G_t] = E^Q [A_{T^*} | G_0] + \int_0^t \frac{I(u)}{B_n(u)} dM_u^S + \int_0^t \left( \int_0^{T^*} \frac{Tp(u,s)}{B_n(u)} h(s,u,z) ds \right) q(du,dz) - \int_0^t \left( \int_u^{T^*} \frac{Tp(u,s)}{B_n(u)} \sigma_t^* P(u,s) E^{Q^t} [\lambda_u^S | I_u] ds \right) dW_u^Q
\]

(3.3)
or
\[
E^Q [A_{T^*} | \mathcal{G}_t] = E^Q [A_{T^*} | \mathcal{G}_0] \\
+ \int_0^t \int_E \left( \frac{I(u)}{B_n(u)} k(u, z) + \int_u^{T^*} \frac{T_p(u, s)}{B_n(u)} h(s, u, z) \, ds \right) q(du, dz) \\
- \int_0^t \left( \int_u^{T^*} \frac{T_p(u, s)}{B_n(u)} \sigma^*_T(u, s) E^{Q^I} [\lambda^S | I_t] \, ds \right) \, dW^Q_u
\]
\[
(3.4)
\]
where \( h(s, u, z) \) is the predictable integrand in the martingale representation \( E^{Q^I} [\lambda^S | I_t] = E[\lambda^S | I_0] + \int_0^t \int_E h(s, u, z)q(du, dz), k(u, z) \) is the predictable integrand in the martingale representation \( M^S_t = M^S_0 + \int_0^t \int_E k(u, z)q(du, dz). \)

PROOF. The somewhat technical proof can be found in Appendix 1. It mainly relies on the absolute continuity of the compensator of \( S_t \) and on the martingale representation property of a marked point process in its natural filtration. \( \square \)

Notice the integral with respect to \( q(du, dz) \) in Equation (3.3) does not appear in Møller’s paper. We now have all the tools to derive the risk-minimizing strategy. We assume we can trade in nominal 0-coupon bonds of \( s^1 \) different maturities and in TIPS of \( s^2 \) different maturities with \( s^1 + s^2 = s \).

PROPOSITION 3.4. The risk-minimizing strategy \( \rho^* = (\varepsilon^*_t, \eta^*_t) = (\varepsilon^1_t, \ldots, \varepsilon^s_t, \eta^*_t) \) is given by the solution of the following system of equations:

\[
\begin{pmatrix}
    d\langle J, X^1_t \rangle \\
    d\langle J, X^2_t \rangle \\
    \vdots \\
    d\langle J, X^s_t \rangle
\end{pmatrix} = \begin{pmatrix}
    d\langle X^1, X^1 \rangle_t \\
    d\langle X^2, X^1 \rangle_t \\
    \vdots \\
    d\langle X^s, X^1 \rangle_t
\end{pmatrix} \begin{pmatrix}
    \varepsilon^1_t \\
    \varepsilon^2_t \\
    \vdots \\
    \varepsilon^s_t
\end{pmatrix}
\]

where
\[
d\langle X^i, X^j \rangle_t = \frac{P_n(t, T_i)P_n(t, T_j)}{B^2_n(t)} \left[ \sigma^*_n(t, T_i) \sigma^*_n(t, T_j)' \right] \, dt
\]
for all \( i, j = 1, \ldots, s^1, \)
\[
d\langle X^i, X^j \rangle_t = \frac{T_p(t, T_i)T_p(t, T_j)}{B^2_n(t)} \left[ \sigma^*_T(p, T_i) \sigma^*_T(p, T_j)' \right] \, dt
\]
for all \( i, j = s^1 + 1, \ldots, s, \) and
\[
d\langle X^i, X^j \rangle_t = \frac{P_n(t, T_i)P_n(t, T_j)}{B^2_n(t)} \left[ \sigma^*_n(t, T_i) \sigma^*_T(p, T_j)' \right] \, dt
\]
for all \( i = 1, \ldots, s^1 \) and \( j = s^1 + 1, \ldots, s. \) Furthermore, we have
\[
(3.5) \quad d\langle J, X^i \rangle_t = \left[ \frac{P_n(t, T_i)}{B^2_n(t)} \left( \int_t^{T^*} T_p(t, s) \sigma^*_T(p, t, s) E^{Q^I} [\lambda^S | I_t] \, ds \right) \sigma^*_n(t, T_i)' \right] \, dt
\]
for all \( i = 1, \ldots, s^1 \) and

\[
(3.6) \quad d \langle J_i, X_i \rangle_t = \left[ \frac{TP(t, T_i)}{B_n^2(t)} \left( \int_t^T TP(t, s) \sigma^{*}_{TP}(t, s) E^{Q^i} \left[ \lambda^S_{s|T_i} \right] \, ds \right) \sigma^{*}_{TP}(t, T_i) \right] dt
\]

for all \( i = s^1 + 1, \ldots, s \).

The discounted amount invested in the risk free asset is given by:

\[
\eta^*_t = \int_t^T \frac{TP(t, u)}{B_n(u)} E^{Q^i} \left[ \lambda^S_{u|T_i} \right] \, du - \sum_{i=1}^{s^1} \varepsilon^i P_n(t, T_i) - \sum_{i=s^1+1}^{s} \varepsilon^i \frac{TP(t, T_i)}{B_n(t)}
\]

As far as the cost process is concerned, we have:

**Proposition 3.5.** The cost process \( C_t(\rho^*) \) is given by

\[
(3.7) \quad C_t(\rho^*) = E^Q \left[ A_{T^*} \mid \mathcal{G}_0 \right] + \int_0^t \int_E \left( \frac{I(u)}{B_n(u)} k(u, z) + \int_u^T \frac{TP(u, s)}{B_n(u)} h(s, u, z) \, ds \right) q(du, dz)
\]

\[
- \int_0^t \left( \int_u^T \frac{TP(u, s)}{B_n(u)} \sigma^{*}_{TP}(u, s) E^{Q^i} \left[ \lambda^S_{s|T_i} \right] \, ds - \sigma^{*}_{RM}(u) \right) dW^Q_u
\]

and the risk process \( R_t(\rho^*) \):

\[
(3.8) \quad R_t(\rho^*) = E^Q \left[ \int_t^T \int_E \left( \frac{I(u)}{B_n(u)} k(u, z) + \int_u^T \frac{TP(u, s)}{B_n(u)} h(s, u, z) \, ds \right)^2 p(du, dz) \mid \mathcal{G}_t \right]
\]

\[
+ E^Q \left[ \int_t^T \left\| \int_u^T \frac{TP(u, s)}{B_n(u)} \sigma^{*}_{TP}(u, s) E^{Q^i} \left[ \lambda^S_{s|T_i} \right] \, ds - \sigma^{*}_{RM}(u) \right\|^2 \, du \mid \mathcal{G}_t \right]
\]

where \( \sigma^{*}_{RM}(u) = \sum_{i=1}^{s^1} \varepsilon^i \frac{P_n(u, T_i)}{B_n(u)} \sigma^{*}_{n}(u, T_i) + \sum_{i=s^1+1}^{s} \varepsilon^i \frac{TP(u, T_i)}{B_n(u)} \sigma^{*}_{TP}(u, T_i) \) and \( \| \| \) is the Euclidean norm.

It is worth noticing that in Equation (3.7), the cost process consists in the sum of two stochastic integrals. The first one measures the part that cannot be hedge due to the policyholder’s claims. The second stochastic integral measures the part that cannot be hedge due to the incompleteness of the financial market (inflation and interest rates).

**Proof.** The cost process is defined by \( C_t(\rho^*) = V_t(\rho^*) - \int_0^T \varepsilon^*_u dX_u + A_t = E^Q \left[ A_{T^*} \mid \mathcal{G}_t \right] - \int_0^T \varepsilon^*_u dX_u \). We immediately get the result thanks to Proposition 3.3 and the dynamics of the asset prices.
As far as the risk process is concerned, using Equation (3.7), we immediately get:

\[
R_t(\rho^*) = E^Q \left( \left( \int_t^{T^*} \int_E \left( \frac{f(u)}{B_n(u)} k(u, z) + \int_u^{T^*} \frac{Tp(u, s)}{B_n(u)} h(s, u, z) \, ds \right) q(du, dz) \right. \\
- \int_t^{T^*} \left. \left( \int_u^{T^*} \frac{Tp(u, s)}{B_n(u)} \sigma_T^p(u, s) E^Q \left[ \mathcal{L}^S_{u} \mid I_u \right] \, ds - \sigma^*_R M(u) \right) \, dW_u^Q \right)^2 \right)_{\mathcal{G}_t}
\]

\[
= E^Q \left[ \int_t^{T^*} \int_E \left( \frac{f(u)}{B_n(u)} k(u, z) + \int_u^{T^*} \frac{Tp(u, s)}{B_n(u)} h(s, u, z) \, ds \right)^2 p(du, dz) \right]_{\mathcal{G}_t}
\]

\[
+ E^Q \left[ \int_t^{T^*} \left\| \int_u^{T^*} \frac{Tp(u, s)}{B_n(u)} \sigma_T^p(u, s) E^Q \left[ \mathcal{L}^S_{u} \mid I_u \right] \, ds - \sigma^*_R M(u) \right\|^2 du \right]_{\mathcal{G}_t}
\]

since the integrals with respect to \( q(t, \cdot) \) and \( W_t^Q \) are \( Q \)-strongly orthogonal.

\[\square\]


In this section and the following one, we study different models for the payments process \( S_t \). In particular in this section, we study two collective models of risk. In the first one, we assume that the payments process follows the well-known compound Poisson process. This model has already been studied in Møller [68]. He even studied a more general model since the size of claims could depend on the value of a financial index. In the second one, we study an extension of this simple model where we assume that the number of policies in the portfolio is stochastic.

4.1. Compound Poisson Process. In this section, we assume that the payment process \( S_t \) follows a square integrable compound Poisson process\(^3\), i.e.

\[
S_t = \sum_{i=1}^{N_t} Z_i
\]

where \( N_t \sim \text{Poi}(\lambda) \) is a Poisson process and \((Z_n)_{n \geq 0}\) are i.i.d. square integrable positive random variables with expectation \( E^Q \left[ Z_n \right] = \mu^Z \). If we denote by \( T_n \) the random times of jump of the process \( N_t \), we can introduce for each \( B \in \mathcal{B}(\mathbb{R}_+) \), the counting process \( N_c(t, B) = \sum_{i=1}^{\infty} 1_{\{T_i \leq t\}} 1_{\{Z_i \in B\}} \). We obviously have \( N_t = N_c(t, \mathbb{R}_+) \). For each \( B \in \mathcal{B}(\mathbb{R}_+) \), the compensator of \( N_c(t, B) \) can be written as \( p_c(t, B) = \int_0^t \int_B \lambda Q^Z(dz) du \) where \( Q^Z(dz) \) is the distribution of the random variables \( Z_n \) under \( Q^1 \). We denote by \( q_c(t, B) \) the compensated process: \( q_c(t, B) = N_c(t, B) - p_c(t, B) \). We can then write:

\[
S_t = \int_0^t \int_{\mathbb{R}_+} z N_c(du, dz)
\]

\(^3\)We could extend the results of this subsection to more general increasing Lévy (additive) processes. Since these processes can be described as a constant (deterministic) drift and a real-marked point process with absolutely continuous compensator and a Lévy measure independent (dependent) of time, these processes fit into the framework defined in Section 2.2.
Applications in Some Collective Models.

The filtration $\mathcal{I}$ is the one generated by the marked point process $N_c(t, B)$. $S_t$ is thus indeed $\mathcal{I}$-adapted. Moreover, its compensator is absolutely continuous and given by $\int_0^t \lambda_u^S du = \int_0^t \int_{\mathbb{R}^+} z \lambda Q^Z(du)dz$.

**Proposition 4.1.** The conditional expectation $E^{Q^I}[\lambda^S_u | \mathcal{I}_t]$ is given by

$$E^{Q^I}[\lambda^S_u | \mathcal{I}_t] = \lambda \mu^Z$$

**Proof.** Obvious since $\lambda^S_u = \int_{\mathbb{R}^+} z \lambda Q^Z(du)dz = \lambda \mu^Z$ is a constant. □

**Proposition 4.2.** The value process $V_t(\rho^*)$ is given by

$$V_t(\rho^*) = \lambda \mu^Z \int_t^{T^*} \frac{T_p(t,u)}{B_n(u)} du$$

**Proof.** Put Equation (4.1) in Equation (3.2). □

**Proposition 4.3.** The martingale representation of $E^Q[A_{T^*} | \mathcal{G}_t]$ is given here by

$$E^Q[A_{T^*} | \mathcal{G}_t] = E^Q[A_{T^*} | \mathcal{G}_0] + \int_0^t \int_{\mathbb{R}^+} \frac{I(u)}{B_n(u)} z q_c(du,dz) - \lambda \mu^Z \int_0^t \left( \int_u^{T^*} \frac{T_p(u,s)}{B_n(u)} \sigma^*_T p(u,s) ds \right) dW^Q_u$$

**Proof.** The integrand $h(s,u,z)$ is equal to 0 since $E^{Q^I}[\lambda^S_u | \mathcal{G}_t] = \lambda \mu^Z$ is a constant. The integrand $k(u,z)$ is obviously equal to $z$ since

$$M_t^S = \int_0^t \int_{\mathbb{R}^+} z N_c(du,dz) - \int_0^t \int_{\mathbb{R}^+} z \lambda Q^Z(du)dz = \int_0^t \int_{\mathbb{R}^+} z q_c(du,dz)$$

□

**Proposition 4.4.** The risk-minimizing strategy is given by Proposition 3.4 where

$$d\langle J_i, X^i \rangle_t = \lambda \mu^Z \left[ \frac{P_n(t,T_i)}{B_n^2(t)} \left( \int_t^{T^*} T_p(t,s) \sigma^*_T p(t,s) ds \right) \sigma^*_T(t,T_i) \right] dt$$

for all $i = 1, \ldots, s^1$. and

$$d\langle J_i, X^i \rangle_t = \lambda \mu^Z \left[ \frac{T_p(t,T_i)}{B_n^2(t)} \left( \int_t^{T^*} T_p(t,s) \sigma^*_T p(t,s) ds \right) \sigma^*_T(t,T_i) \right] dt$$

for all $i = s^1 + 1, \ldots, s$. 
The discounted amount invested in the risk free asset is given by:

\[ \eta^* = \lambda \mu Z \int_t^T \frac{T_p(t,u)}{B_n(t)} du - \sum_{i=1}^{s^1} \epsilon^i P_n(t,T_i) \frac{B_n(t)}{B_n(t)} - \sum_{i=s^1+1}^s \epsilon^i T_p(t,T_i) \frac{B_n(t)}{B_n(t)} \]

**Proof.** Put Equation (4.1) in Equations (3.5) and (3.6). □

Notice that, in this simple setting, the financial strategies do not depend on the liabilities (or only up to the constant \( \lambda \mu Z \)). Mathematically speaking, the financial strategy \( \rho^* \) is \( \mathcal{F} \)-adapted and not \( \mathcal{G} \)-adapted.

**Proposition 4.5.** The cost process is given by

\[ C_t(\rho^*) = E^Q [A_T^* | \mathcal{G}_0] + \int_0^t \int \frac{I(u)}{B_n(u)} z q_e(du,dz) \]

\[ - \lambda \mu Z \int_0^t \left( \int_u^T \frac{T_p(u,s)}{B_n(u)} \sigma^*_p(u,s) ds - \sigma^*_R M(u) \right) dW^Q_u \]

and the risk process \( R_t(\rho^*) : \)

\[ R_t(\rho^*) = \lambda E^Q \| Z \|^2 \int_t^T E^Q \left[ \left( \frac{I(u)}{B_n(u)} \right)^2 \right] | \mathcal{F}_t \] du

\[ + \int_t^T E^Q \left[ \left\| \lambda \mu Z \int_u^T \frac{T_p(u,s)}{B_n(u)} \sigma^*_p(u,s) - \sigma^*_R M(u) \right\|^2 \right] | \mathcal{F}_t \] du

The first stochastic integral in the cost process translates the fact that the insurer cannot hedge perfectly its liabilities due to the unpredictable arrival and size of the claims. The second stochastic integral is related to the incompleteness of the financial market.

**4.2. Stochastic portfolio size.** In the previous section, we could have alternatively assumed that the number of policies in the portfolio was a constant \( k \) and that the claims of each policyholder followed independent compound Poisson processes with an intensity of arrival of claims \( \lambda^i = \frac{\lambda}{k} \). We now extend this model by allowing the number \( k_t \) of policies to be stochastic.

We model \( k_t \) as a point process in the following way:

\[ k_t = k_0 + I_t - \int_0^t 1_{\{k_u > 0\}} dO_u \]

where \( I_t \) and \( O_t \) are simple independent point processes. \( I_t \) describes the cumulative number of new policyholders in the portfolio on the interval \([0,t]\) and \( \int_0^t 1_{\{k_u > 0\}} dO_u \) the cumulative number of policyholders leaving the portfolio on the interval \([0,t]\). We study a model where the size of the portfolio evolves heterogeneously with respect to time. We assume \( I_t \) admits a deterministic intensity \( \lambda^I(t) \) and \( O_t \) a stochastic intensity \( k_t - \lambda^O(t) \) where \( \lambda^O(t) \) is a deterministic function. Accordingly, we assume that the number of policyholders leaving the portfolio depends on its size whereas the arrival of new policyholders does not. We denote by \( \Lambda^I_t \) the
compensator of \(k_t\) and the compensated process \(M^k_t = k_t - \Lambda^k_t\). The compensator \(\Lambda^k_t\) is given by

\[
\Lambda^k_t = \int_0^t \left[ \lambda^I(u) - k_u - \lambda^O(u) \right] du
\]

For each policyholder in the portfolio, we assume that the cost of claims follows an independent compound Poisson process. Let us \((S_n)_{n \geq 1}\) denote the random times of arrival of a claim in the portfolio and let us \((Z_n)_{n \geq 1}\) denote the i.i.d square integrable positive random variables describing the size of the claims. We can then introduce for each \(B \in B(\mathbb{R}_+)\), the counting process \(N_c(t, B) = \sum_{n=1}^{\infty} 1\{S_n \leq t\} 1\{Z_n \in B\}\). Its compensator \(p_c(t, B)\) is written as

\[
p_c(t, B) = \int_0^t \int_B \lambda k_u - Q^I(z) du
\]

where \(Q^I(z)\) is the distribution of the random variables \(Z_n\).

We can then describe the payment process as:

\[
S_t = \int_0^t \int_{\mathbb{R}_+} z N_c(du \times dz)
\]

Notice that we can rewrite our three point processes \(N_c(t, B), I_t\) and \(O_t\) in the form of a single general point process as in Section 2.2. Consider the following measurable space \((E, \mathcal{E})\) where \(E = \{c, i, o\} \times \mathbb{R}_+\) and \(\mathcal{E} = B(\mathbb{R}_+)\). Let us define a point process \((T_n, Y_n)_{n \geq 1}\) where \(Y_n\) are \(E\)-valued random variables associated to the random times \(T_n\). For each \(B' \in \mathcal{E}\) where \(B' = S \times B\), we can then define the general point process:

\[
N(t, B') = \sum_{n=1}^{\infty} 1\{T_n \leq t\} 1\{Y_n \in B'\}
\]

whose compensator is given by

\[
p(t, B') = \int_0^t \sum_{s \in S} \int_B m_u^s(dz) \lambda_u(s) du
\]

where \(\lambda_u(c) = \lambda k_u, m_u^c(dz) = Q^I(dz), \lambda_u(i) = \lambda^I(u), m^i(dz) = \delta_{\{0\}}(dz), \lambda_u(o) = k_u - \lambda^O(u), m^O(dz) = \delta_{\{0\}}(dz)\). In particular, we can write

\[
I_t = N(t, (i, \mathbb{R}_+)) = \sum_{n=1}^{\infty} 1\{T_n \leq t\} 1\{Y_n \in (i, \mathbb{R}_+)\}
\]

\[
O_t = N(t, (o, \mathbb{R}_+)) = \sum_{n=1}^{\infty} 1\{T_n \leq t\} 1\{Y_n \in (o, \mathbb{R}_+)\}
\]

\[
N_c(t, B) = N(t, (c, B)) = \sum_{n=1}^{\infty} 1\{T_n \leq t\} 1\{Y_n \in (c, B)\}
\]
for all $B \in \mathcal{B}(\mathbb{R}_+)$. The filtration $\mathbb{I}$ is the one generated by this process $N(t, B')$. $S_t$ is indeed $\mathbb{I}$-adapted.

In order to determine the risk-minimizing strategy and its value process, we first need the following result:

**Proposition 4.6.** The conditional expectation $E^{Q^I} \left[ \lambda^S_u | \mathcal{I}_t \right]$ is given by:

\[
E^{Q^I} \left[ \lambda^S_u | \mathcal{I}_t \right] = \lambda \mu Z E^{Q^I} \left[ k_u | \mathcal{I}_t \right]
\]

and where

\[
E^{Q^I} \left[ k_u | \mathcal{I}_t \right] = k_t e^{-\int_t^u \lambda^O(v) dv} + \left( \int_t^u \lambda^I(v) e^{\int_t^v \lambda^O(w) dw} dv \right) \left( 1 - e^{-\int_t^u \lambda^O(v) dv} \right)
\]

**Proof.** Since $\lambda^S_u = \lambda k_u - \mu Z$, we have

\[
E^{Q^I} \left[ \lambda^S_u | \mathcal{I}_t \right] = E^{Q^I} \left[ \lambda k_u - \mu Z | \mathcal{I}_t \right] = \lambda \mu Z E^{Q^I} \left[ k_u | \mathcal{I}_t \right]
\]

For the expression of $E^{Q^I} \left[ k_u | \mathcal{I}_t \right]$, see Appendix 2. \qed

**Proposition 4.7.** The value process $V_t(\rho^*)$ is given by

\[
V_t(\rho^*) = \lambda \mu Z \int_0^{T^*} T^*_p(t, u) E^{Q^I} \left[ k_u | \mathcal{I}_t \right] du
\]

**Proof.** Put Equation (4.2) in Equation (3.2). \qed

**Proposition 4.8.** The martingale representation of $E^Q \left[ A_{T^*} | \mathcal{G}_t \right]$ is given by

\[
E^Q \left[ A_{T^*} | \mathcal{G}_t \right] = E^Q \left[ A_{T^*} | \mathcal{G}_0 \right] + \int_0^t \int_{\mathbb{R}_+} I(u) \frac{T_p(t, u)}{B_n(u)} z q_c(du, dz)
\]

\[
+ \lambda \mu Z \int_0^t \left( \int_0^u T_p(u, s) e^{-\int_u^s \lambda^O(v) dv} ds \right) dM^k_u
\]

\[
- \lambda \mu Z \int_0^t \left( \int_0^u T_p(u, s) \sigma^*_T(u, s) E^{Q^I} \left[ k_s | \mathcal{I}_u \right] ds \right) dW^Q_u
\]

**Proof.** In Appendix 3, we show that the martingale representation of $E^{Q^I} \left[ \lambda^S_s | \mathcal{I}_u \right]$ is given by:

\[
dE^{Q^I} \left[ \lambda^S_s | \mathcal{I}_u \right] = \lambda \mu Z e^{-\int_u^s \lambda^O(w) dw} dM^k_u
\]

The integrand $h(s, u, z)$ is then given by

\[
h(s, u, z) = \lambda \mu Z e^{-\int_u^s \lambda^O(w) dw}
\]
Moreover the integrand \( k(u, z) \) in the martingale representation of \( M_t^S \) is obviously equal to \( z \) since

\[
M_t^S = \int_0^t \int_{\mathbb{R}^+} z \, N_c(du, dz) - \int_0^t \int_{\mathbb{R}^+} z \lambda k_u - P(dz)du \\
= \int_0^t \int_{\mathbb{R}^+} z \, q_c(du, dz)
\]

\[\square\]

**Proposition 4.9.** The risk-minimizing strategy is given by Proposition 3.4 where

\[
\begin{align*}
\eta^*_t &= \lambda \mu Z \int_t^{T^*} \frac{P_{n}(t, s)}{B_n(t)} \left( \int_t^s T_p(t, s) \sigma_{T_p}(t, s) E^{Q^t} \left[ k_s \mid I_t \right] \, ds \right) \sigma_{T_p}(t, s) \, dt \\
&\quad - \lambda \mu Z \int_t^{T^*} \frac{P_{n}(t, s)}{B_n(t)} \left( \int_t^s T_p(t, s) \sigma_{T_p}(t, s) E^{Q^t} \left[ k_s \mid I_t \right] \, ds \right) \sigma_{T_p}(t, s) \, dt
\end{align*}
\]

for all \( i = 1, \ldots, s \), and

\[
\begin{align*}
\eta^*_t &= \lambda \mu Z \int_t^{T^*} \frac{P_{n}(t, s)}{B_n(t)} \left( \int_t^s T_p(t, s) \sigma_{T_p}(t, s) E^{Q^t} \left[ k_s \mid I_t \right] \, ds \right) \sigma_{T_p}(t, s) \, dt
\end{align*}
\]

for all \( i = s + 1, \ldots, s \).

The discounted amount invested in the risk free asset is given by:

\[
\begin{align*}
\eta^*_t &= \lambda \mu Z \int_t^{T^*} \frac{P_{n}(t, u)}{B_n(t)} E^{Q^t} \left[ k_u \mid I_t \right] \, du - \sum_{i=1}^{s_1} \varepsilon_i P_{n}(t, T_i) \frac{B_n(t)}{B_n(t)} - \sum_{i=s_1+1}^{s} \varepsilon_i T_p(t, T_i) \frac{B_n(t)}{B_n(t)}
\end{align*}
\]

**Proof.** Put Equation (4.2) in Equations (3.5) and (3.6). \(\square\)

The interesting feature here is that the financial strategy \( \rho^* \) is not anymore \( \mathbb{F} \)-adapted but well \( \mathbb{G} \)-adapted. Indeed, the risk-minimizing strategy depends on the size of the portfolio \( k_t \) (through \( E^{Q^t} \left[ k_s \mid I_t \right] \)).

**Proposition 4.10.** The cost process is given by

\[
C_t(\rho^*) = E^{Q} \left[ A_{T^*} \mid \mathcal{G}_0 \right] + \int_0^t \int_{\mathbb{R}^+} \frac{I(u)}{B_n(u)} \, z \, q_c(du, dz) + \lambda \mu Z \int_0^t \left( \int_u^{T^*} \frac{T_p(u, s)}{B_n(u)} e^{-\int_s^{T^*} \lambda O(v) \, dv} \, ds \right) \, dM_u^k - \lambda \mu Z \int_0^t \left( \int_u^{T^*} \frac{T_p(u, s)}{B_n(u)} \sigma_{T_p}(u, s) E^{Q^t} \left[ k_s \mid I_t \right] \, ds - \sigma_{RM}(u) \right) \, dW_u^Q
\]
The risk process $R_t(\rho^*)$ is given by:

$$
R_t(\rho^*) = \lambda E^{Q^{I_t}} \left[ Z^2 \right] \int_t^{T^*} E^{Q^F} \left[ \left( \frac{I(u)}{B_n(u)} \right)^2 \bigg| \mathcal{F}_t \right] E^{Q^I} \left[ k_u | \mathcal{I}_t \right] du
$$

$$
+ (\lambda \mu Z)^2 \int_t^{T^*} E^{Q^F} \left[ \left( \int_u^{T^*} \frac{T_p(u,s) e^{-\int_s^* \lambda^O(v)dv}}{B_n(u)} ds \right)^2 \bigg| \mathcal{F}_t \right] E^{Q^I} \left[ \lambda^k_u | \mathcal{I}_t \right] du
$$

$$
+ \int_t^{T^*} E^{Q} \left[ \lambda^2 \mu Z \int_u^{T^*} \frac{T_p(u,s)}{B_n(u)} \sigma^R_{T_p}(u,s) E^{Q^I} \left[ k_u | \mathcal{I}_u \right] ds - \sigma^R_{RM}(u) \right] \bigg| \mathcal{G}_t \right] du
$$

where $E^{Q^I} \left[ \lambda^k_u | \mathcal{I}_t \right] = \lambda^I(u) - E^{Q^I} \left[ k_u | \mathcal{I}_t \right] \lambda^O(u)$.

In the cost process of the previous model, we had a term similar to the first stochastic integral of Equation (4.4) but the second integral was absent. Intuitively, these terms mean that the insurer cannot hedge perfectly its liabilities due to the unpredictable arrival and size of the claims (for a given size of his portfolio), and due to the unpredictable evolution of the number of policies in his portfolio. The third stochastic integral is related to the incompleteness of the financial market.

5. Applications in Some Individual Models.

In this section, we study two different models. In both cases, we consider a single policy and assume that a single claim can occur. We also assume that the claim is not paid up-front when reported but is settled through time. In the first model, we assume the claim is settled according to a deterministic function. In the second model, we assume the claim is settled according to a stochastic process. The fact that we study a single policy seems rather limited but keep in mind the risk-minimizing strategy of a portfolio of independent policies is given by the sum of the risk-minimizing strategies of each policy but with respect to their own particular filtration (not the whole filtration generated by the whole portfolio). Under this independence assumption, we can thus easily extend the following results to a portfolio.

5.1. Deterministic claim settlement. We consider a single policy. Let $T$ be the random time of occurrence of a claim. The insurer covers the claims if it occurs in the deterministic interval $[0, L]$. We assume the policyholder notifies the claims at the time of occurrence. The claim amount is a square integrable random variable $Z$ with distribution $Q^Z(dz)$. This amount is settled through time according to an increasing deterministic function $F(t)$ taking its value between $[0, 1]$ with $F(t) = 0$ for $t \leq 0$. We assume there is no upfront payment at the time of notification since $F(0) = 0$.

Here, the general point process of Section 2.2 comes down to $N(t, B) = 1_{\{T \leq t\}} 1_{\{Z \in B\}}$ for all $B \in \mathcal{B}(\mathbb{R}_+)$. The filtration is here $\mathcal{I}_t = \sigma(N(s, B) : 0 \leq s \leq t, B \in \mathcal{B}(\mathbb{R}_+))$. We assume

4By independent policies, we mean that the payment processes expressed in real units and associated to the different policies are independent of each other under $Q^I$. Obviously, expressed in nominal units, they are not anymore independent due to the inflation.
the random time \( T \) has an absolutely continuous distribution \( Q^I(T \leq t) = G(t) \). Let us define \( \lambda(t) = -\frac{d[\ln(1-G(t))]}{dt} \), then the process

\[
q(t, B) = N(t, B) - \int_0^t \int_B 1_{\{T > u\}} \lambda(u) Q^Z(dz)du
\]

is a \((Q^I, I)\)-martingale. At time \( t \), the cumulative payments process \( S_t \) is then given by

\[
S_t = 1_{\{T \leq L\}} 1_{\{T \leq t\}} Z \int_T^L B_n(t) f^*(t, u) du + 1_{\{T \leq L\}} 1_{\{T \leq t\}} Z \int_T^L B_n(t) f(u - T) du
\]

where \( f^*(t, u) = \int_t^{L \wedge u} f(u - v) \lambda(v)e^{-\int_t^v \lambda(w)dw} dv \).

**Proof.** Put Equation (5.1) in Equation (3.2). \( \square \)

**Proposition 5.2.** The value process \( V_t(\rho^*) \) is given by

\[
V_t(\rho^*) = 1_{\{T > t\}} 1_{\{T \leq L\}} \mu^Z \int_t^T \frac{Tp(t, u)}{B_n(t)} f^*(t, u) du + 1_{\{T \leq L\}} 1_{\{T \leq t\}} Z \int_t^T \frac{Tp(t, u)}{B_n(t)} f(u - T) du
\]

where \( f^*(t, u) = \int_t^{L \wedge u} f(u - v) \lambda(v)e^{-\int_t^v \lambda(w)dw} dv \).

**Proof.** Put Equation (5.1) in Equation (3.2). \( \square \)

We can now apply the general result of Proposition 3.3 to find the martingale representation:
PROPOSITION 5.3. The martingale representation of $E^Q [A_{T^*} \mid \mathcal{G}_t]$ is given by

$$
E^Q [A_{T^*} \mid \mathcal{G}_t] = E^Q [A_{T^*} \mid \mathcal{G}_0] + \int_0^t 1_{\{\tau \leq L\}} \int_{\mathbb{R}^+} \left( \int_u^{T^*} \frac{Tp(u, s)}{B_n(u)} (z f(s - u) - \mu^Z f^*(u, s)) \, ds \right) q(du, dz)
$$

$$
- \int_0^t 1_{\{\tau \leq L\}} 1_{\{T > u - \}} \mu^Z \left( \int_u^{T^*} \frac{Tp(u, s)}{B_n(u)} \sigma_{Tp}(u, s) f^*(u, s) \, ds \right) \, dW^Q_u
$$

$$
- \int_0^t 1_{\{\tau \leq L\}} 1_{\{T \leq u - \}} Z \left( \int_u^{T^*} \frac{Tp(u, s)}{B_n(u)} \sigma_{Tp}(u, s) f(s - T) \, ds \right) \, dW^Q_u
$$

where $f^*(u, s) = \int_u^{L^zn} f(s - v) \lambda(v) e^{-\int_u^v \lambda(w) \, dw} \, dv$.

PROOF. In Appendix 5, we show the martingale representation of $E^{Q^I} [\lambda_t^S \mid \mathcal{I}_s]$ is given by

$$
dE^{Q^I} [\lambda_t^S \mid \mathcal{I}_s] = 1_{\{\tau \leq L\}} \int_{\mathbb{R}^+} \left[ z f(t - s) - \mu^Z f^*(s, t) \right] q(ds, dz)
$$

so the integrand $h(s, u, z)$ is given by

$$
h(s, u, z) = 1_{\{u \leq L\}} \left[ z f(s - u) - \mu^Z f^*(u, s) \right]
$$

Moreover, the integrand $k(u, z)$ is equal to 0 because $\int_0^t \frac{I(u)}{B_n(u)} \, dM^S_u = 0$ since $S_t = \Lambda_t^S$. We get immediately the result if we put the integrands $h(s, u, z)$ and $k(u, z)$ in Equation (3.4). □

The risk-minimizing strategy is given by the following proposition.

PROPOSITION 5.4. The risk-minimizing strategy is given by Proposition 3.4 where

$$
d\langle J, X^i_t \rangle = 1_{\{\tau \leq L\}} 1_{\{T > t - \}} \mu^Z \left[ \frac{P_n(t, T_i)}{B_n^2(t)} \left( \int_t^{T^*} Tp(t, s) \sigma_{Tn}^*(t, s) f^*(s, t) \, ds \right) \sigma_{Tn}^*(t, T_i) \right] \, dt
$$

$$
+ 1_{\{T \leq L\}} 1_{\{T \leq t - \}} Z \left[ \frac{P_n(t, T_i)}{B_n^2(t)} \left( \int_t^{T^*} Tp(t, s) \sigma_{Tn}^*(t, s) f(s - T) \, ds \right) \sigma_{Tn}^*(t, T_i) \right] \, dt
$$

for all $i = 1, \ldots, s^1$, and

$$
d\langle J, X^i_t \rangle = 1_{\{\tau \leq L\}} 1_{\{T > t - \}} \mu^Z \left[ \frac{Tp(t, T_i)}{B_n^2(t)} \left( \int_t^{T^*} Tp(t, s) \sigma_{Tn}^*(t, s) f^*(s, t) \, ds \right) \sigma_{Tn}^*(t, T_i) \right] \, dt
$$

$$
= 1_{\{T \leq L\}} 1_{\{T \leq t - \}} Z \left[ \frac{Tp(t, T_i)}{B_n^2(t)} \left( \int_t^{T^*} Tp(t, s) \sigma_{Tn}^*(t, s) f(s - T) \, ds \right) \sigma_{Tn}^*(t, T_i) \right] \, dt
$$

for all $i = s^1 + 1, \ldots, s$. 

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The discounted amount to invest in the risk free asset is given by:

\[ \eta_t^* = \int_t^{T^*} \frac{Tp(t,u)}{B_n(t)} E^{Q^t} [\lambda^S_u | I_t] \, du - \sum_{i=1}^s \xi_i \frac{P_n(t,T_i)}{B_n(t)} - \sum_{i=s+1}^n \xi_i \frac{Tp(t,T_i)}{B_n(t)} \]

**Proof.** Put Equation (5.1) in Equations (3.5) and (3.6). \square

The interesting feature here is that we have two different financial strategies depending on whether the claim has already been reported or not. Accordingly, we distinguish the RBNI (Reserved but not incurred) claims and the RBNS (Reserved but not settled) claims. In other words, the asset allocation of our non-life insurance will depend on the structure of its liabilities.

**Proposition 5.5.** The cost process \( C_t(\rho^*) \) is given by

\[
C_t(\rho^*) = E^Q [A_T^- | G_0] + \int_0^t 1_{\{u \leq L\}} \int_{\mathbb{R}_+} \left( \int_u^{T^*} \frac{Tp(u,s)}{B_n(u)} (zf(s-u) - \mu^Z_f(s,u)) \, ds \right) q(du,dz) - \int_0^t 1_{\{u \leq L\}} \mu^Z \left( \int_u^{T^*} \frac{Tp(u,s)}{B_n(u)} \sigma^*_T \frac{(s-u)}{f(u,s)\sigma^*_T(u,s)} \, ds - \sigma^*_n(u) \right) \, dW^Q_u - \int_0^t 1_{\{T \leq L\}} (T \leq s^-) Z \left( \int_u^{T^*} \frac{Tp(u,s)}{B_n(u)} \sigma^*_T \frac{(s-u)}{f(u,s)\sigma^*_T(u,s)} \, ds - \sigma^*_n(u) \right) \, dW^Q_u
\]

The risk process \( R_t(\rho^*) \) is given by:

\[
R_t(\rho^*) = 1_{\{T > t\}} \int_t^{T^*} 1_{\{u \leq L\}} \int_{\mathbb{R}_+} E^{Q^F} \left[ H_1(u,z,T^*)^2 | F_t \right] Q^Z (dz) e^{-\int_u^{T^*} \lambda(v)dv} \mu(\lambda)du + E^Q \left[ \int_t^{T^*} \| \int_u^{T^*} \frac{Tp(u,s)}{B_n(u)} \sigma^*_T (u,s) E^{Q^t} [\lambda^S_u | I_t] \, ds - \sigma^*_n(u) \|^2 \, du \mid G_t \right]
\]

where \( H_1(u,z,T^*) = \int_u^{T^*} \frac{Tp(u,s)}{B_n(u)} (zf(s-u) - \mu^Z_f(s,u)) \, ds \).

**Proof.** Straightforward due to Proposition 3.5. \square

As in the general model, the cost process consists of two parts. The first one corresponds to the stochastic integral with respect to \( q(du,dz) \) and represents the part that cannot be hedged due to the unpredictable arrival and unpredictable size of the policyholder’s claim. The second part corresponds to the two stochastic integrals with respect to the Brownian motion and represents the part of the claim process that cannot be hedged due to the incompleteness of the financial market. Notice the impact of the incompleteness of the financial market depends on whether the claims has already happened or not.
5.2. **Stochastic claim settlement.** Again, we consider a single policy. We assume the claim occurs and is reported at the random time $T$ (described as in Section 5.1). The insurer covers the claim if it occurs in the deterministic interval $[0, L]$. The total claim amount is a square integrable random variable $Z$ with distribution $Q^Z(dz)$. This amount is settled through time according to a stochastic process $F(t)$ taking its value between $[0, 1]$. In other words, the payment process $S_t$ is given by

$$S_t = 1_{\{T \leq t\}} 1_{\{t \geq T\}} Z F(t - T)$$

We assume the total amount $Z$ is known at the time of notification; $Z$ is thus $\mathcal{G}_T$-measurable. As in the previous example, for each $B \in \mathcal{B}(\mathbb{R}_+)$, we can define a point process $N_Z(t, B) = 1_{\{T \leq t\}} 1_{\{Z \in B\}}$ with a compensator $p_z(t, B) = \int_0^t \int_B 1_{\{T > u\}} \lambda(u) Q^Z(dz) du$

We describe the stochastic process $F(t)$ as a Beta-Stacy process. See Walker and Muliere [82] and Hjort [54] for more details about beta-Stacy process. We describe here the model used by Hjort [54] except that we assume $A_t$ contains no fixed point of discontinuity.

Following Hjort [54], we define $F(t)$ in the following way:

$$F(t) = 1 - \prod_{0 \leq s \leq t} (1 - \Delta A_s)$$

with $A_t$ a beta process: $A_t \sim \text{beta}(c(\cdot), A_0(\cdot))$ where $A_0 = 0$, $c(t)$ is a piecewise continuous non-negative function and $A_0(\cdot)$ is an increasing right-continuous function with $A_0(0) = 0$. According to Walker and Muliere [82], $F(t)$, defined in this way, is a Beta-Stacy process. A beta process $A_t \sim \text{beta}(c(\cdot), A_0(\cdot))$ can also be described as an integral with respect to a real-marked point process $N_A(t, B)$, whose compensator $p_A(t, B)$ is given by $p_A(t, B) = \left( \int_B c(t) \frac{(1-a)^{c(t)-1}}{a} da \right) A_0(dt)$ for each $B \in \mathcal{B}([0,1])$, in the following way:

$$A_t = \int_0^t \int_{[0,1]} a N_A(du, da)$$

Since we assume $A_t$ contains no fixed points of discontinuity, the function $A_0(\cdot)$ is continuous. If furthermore, we assume the $A_0(t)$ is absolutely continuous i.e. $A_0(t) = \int_0^t A_0(u) du$, the beta process $A_t$ can also be described as an increasing additive process with (an inhomogeneous) Lévy measure $\nu^A(da) = c(t) \frac{(1-a)^{c(t)-1}}{a} A_0(t) da$

As far as $S_t$ is concerned, we have:

$$dS_t = 1_{\{T \leq t\}} 1_{\{t \geq T\}} Z dF(t - T)$$

since the measure $A_0(dt)$ is, by assumption, continuous at time 0, there is a.s. no up front payment at time $T$. We can also write $S_t$ as a stochastic integral with respect to the real-marked point process $N_A(t, \cdot)$:

$$dS_t = 1_{\{T \leq L\}} 1_{\{t \leq T\}} Z F(t - T) - \int_{[0,1]} a N_A(d(t - T), da)$$
where $F(t - T) = 1 - F(t - T)$. Indeed we have
\[
F(t) = 1 - \prod_{0 \leq s < t} (1 - \Delta A_s)(1 - \Delta A_t)
\]
so we can write
\[
dF(t) = [1 - F(t_-)] \int_{[0,1]} a \, N_A(dt, da)
\]
The compensator of $S_t$ is indeed also absolutely continuous and is given by:
\[
\lambda^S(t) dt = 1_{\{T \leq L\}} 1_{\{T < t\}} Z \bar{F}(t - T) - Z \bar{F}(t - T) A'_0(t - T) dt
\]

We can also model both marked point processes $N_Z(t, B)$ and $N_A(t, B)$ through a single more general marked point process $N(t, B)$ as in the general framework of Section 2.2. The filtration $\mathcal{F}$ will be the filtration generated by $N(t, B)$ and the process $S_t$ is indeed $\mathcal{F}$-adapted.

In order to determine the risk-minimizing strategy and its value process, we first need the following result:

**Proposition 5.6.** The conditional expectation $E^{Q^I} \left[ \lambda^S_u \mid \mathcal{I}_t \right]$ is given by:
\[
E^{Q^I} \left[ \lambda^S_u \mid \mathcal{I}_t \right] = 1_{\{t \leq L\}} 1_{\{T > t\}} \mu^Z f^*(t, u)
+ 1_{\{T \leq L\}} Z \bar{F}(t - T) f_T(t, u)
\]

where
\[
f_T(t, u) = A'_0(u - T)e^{-(A_0(u - T) - A_0(t - T))}
\]
\[
f^*(t, u) = \int_t^{L \wedge u} f_v(v, u) \lambda(v) e^{-\int_t^v \lambda(w) dw} dw
\]

**Proof.** See Appendix 6.

As far as the value process is concerned, we have:

**Proposition 5.7.** The value process $V_t(\rho^*)$ is given by
\[
V_t(\rho^*) = 1_{\{t \leq L\}} 1_{\{T > t\}} \mu^Z \int_t^{T^*} \frac{Tp(t, u)}{B_n(t)} f^*(t, u) du
+ 1_{\{T \leq L\}} Z \bar{F}(t - T) \int_t^{T^*} \frac{Tp(t, u)}{B_n(t)} f_T(t, u) du
\]

**Proof.** Put Equation (5.2) in Equation (3.2).

We can now apply the general result of Proposition 3.3 to find the following martingale representation.
Proposition 5.8. The martingale representation of $E^Q[A_{T^*} | \mathcal{G}_t]$ is given by

$$
E^Q [A_{T^*} | \mathcal{G}_t] = E^Q [A_{T^*} | \mathcal{G}_0]
+ \int_0^t 1_{\{u \leq L\}} \int_{\mathbb{R}_+} H_1 (u, z, T^*) \ q_Z (du, dz)
+ \int_0^t 1_{\{T \leq u\}} 1_{\{T \leq L\}} Z \int_{[0,1]} a \ H_2 (u, T, T^*) \ q_A (d(u - T), da)
+ \int_0^t 1_{\{T > u\}} 1_{\{u \leq L\}} \mu \int_u^{T^*} \left[ T p(u, s) / B_n(u) - \sigma T^* p(u, s) f^*(u, s) \right] ds \ dW_u^Q
+ \int_0^t 1_{\{T \leq u\}} 1_{\{T \leq L\}} Z \int_u^{T^*} \left[ T p(u, s) / B_n(u) \sigma T^* p(u, s) f_T(u, s) F(u - T) - ds \right] dW_u^Q
$$

where

$$H_1 (u, z, T^*) = \int_u^{T^*} T p(u, s) / B_n(u) \ (z \ f_u (u, s) - \mu Z^* f^*(u, s)) \ ds$$

and

$$H_2 (u, T, T^*) = \frac{I(u)}{B_n(u)} - \int_u^{T^*} T p(u, s) / B_n(u) f_v(u, s) F(u - v) - ds$$

Proof. In Appendix 7, we show the martingale representation of $E^Q^I [\lambda^S \ | \mathcal{I}_u]$ is given by:

$$
dE^Q^I [\lambda^S \ | \mathcal{I}_u] = 1_{\{u \leq L\}} \left[ z \ f_u (u, s) - \mu Z^* f^*(u, s) \right] q_Z (du, dz)
- 1_{\{T \leq L\}} \int_{[0,1]} 1_{\{T < u\}} Z \ F(u - T) - a f_T(u, s) q_A (d(u - T), da)
$$

Moreover

$$\int_0^t \frac{I(u)}{B_n(u)} \ dM^S_u = \int_0^t \frac{I(u)}{B_n(u)} 1_{\{T \leq u\}} 1_{\{T \leq L\}} Z \ d\left( F(t - T) - p^F(t) dt \right)
= \int_0^t 1_{\{T \leq u\}} 1_{\{T \leq L\}} \frac{I(u)}{B_n(u)} Z \int_{[0,1]} a \ q_A (d(u - T), da)$$

We can now easily find the risk-minimizing strategy.
Proposition 5.9. The risk-minimizing strategy is given by Proposition 3.4 where

\[
d(J, X^i_t) = 1_{(t \leq L)} 1_{(T > t -)} \mu^Z \left[ \frac{P_n(t, T_i)}{B_n(t)} \left( \int_t^{T^*} T_p(t, s) \sigma^*_p(t, s) f^*_P(t, s) \, ds \right) \sigma^*_n(t, T_i) \right] \, dt \\
+ 1_{(T \leq L)} 1_{(T \leq t -)} Z \tilde{F}(t - T) \left[ \frac{P_n(t, T_i)}{B_n^2(t)} \left( \int_t^{T^*} T_p(t, s) \sigma^*_p(t, s) f_T(t, s) \, ds \right) \sigma^*_n(t, T_i) \right] \, dt
\]

for all \( i \in \{1, \ldots, s^1\} \) and \( i = s^1 + 1, \ldots, s \).

The discounted amount invested in the risk-free asset is given by:

\[
\eta^* = \int_t^{T^*} \frac{T_p(t, u)}{B_n(t)} \lambda^Q \left[ \lambda^Q \big| \mathcal{G}_t \right] \, du - \sum_{i=1}^{s^1} \epsilon_i \frac{P_n(t, T_i)}{B_n(t)} - \sum_{i=s^1+1}^{s} \epsilon_i \frac{T_p(t, T_i)}{B_n(t)}
\]

As in the previous model, we have two different financial strategies depending on whether or not the claim has already been reported. We can again distinguish the RBNI (Reserved but not incurred) claims and the RBNS (Reserved but not settled) claims.

Proposition 5.10. The cost process \( C_t(\rho^*) \) is given by:

\[
C_t(\rho^*) = E^Q[A_{T^*} | \mathcal{G}_0] \\
+ \int_0^t 1_{(u \leq L)} \int_{\mathbb{R}^+} H_1(u, z, T^*) \, q_Z(du, dz) \\
+ \int_0^t 1_{(T \leq u -)} 1_{(T \leq L)} Z \int_{[0,1]} a H_2(u, T, T^*) \, q_A(d(u - T), da) \\
+ \int_0^t 1_{(T > u -)} 1_{(u \leq L)} \mu^Z \left[ \int_u^{T^*} \frac{T_p(u, s)}{B_n(u)} \sigma^*_p(u, s) f^*_P(u, s) \, ds - \sigma^*_R(u) \right] \, dW^Q_u \\
+ \int_0^t 1_{(T \leq u -)} 1_{(T \leq L)} Z \left[ \int_u^{T^*} \frac{T_p(u, s)}{B_n(u)} \sigma^*_p(u, s) f_T(u, s) \, ds - \sigma^*_R(u) \right] \, dW^Q_u
\]
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The risk process $R_t(\rho^*)$:

$R_t(\rho^*) = 1_{\{T > t\}} \int_t^{T^*} 1_{\{u \leq L\}} \int_{\mathbb{R}_+} E^Q \left[ H_1(u, z, T^*)^2 \bigg| \mathcal{F}_t \right] Q^Z(dz)e^{-\int_t^{T^*} \lambda(u)du} \lambda(u)du$

$+ 1_{\{T > t\}} \int_{L \wedge L}^{T^*} \int_{\mathbb{R}_+} E^{Q^I}[Z^2] \int_{[0,1]} a^2 E^Q \left[ H_2(u, v, T^*)^2 \bigg| \mathcal{F}_t \right] p_A(d(u - v), da) \lambda(v)e^{-\int_t^{T^*} \lambda(u)du} dv$

$+ 1_{\{T \leq t\}} 1_{\{T \leq L\}} \int_{T}^{T^*} Z^2 \int_{[0,1]} a^2 E^Q \left[ H_2(u, T, T^*)^2 \bigg| \mathcal{F}_t \cup F(t - T) \right] p_A(d(u - T), da)$

$+ E^Q \left[ \int_t^{T^*} \int_u^{T^*} \frac{Tp(u, s)}{B_n(u)} \sigma^*_T(u, s) E^{Q^I} [\lambda^S_{Iu} - \sigma^*_RM(u)]_2 - \sigma^*_RM(u) \bigg| \mathcal{G}_t \right]$

where

$H_1(u, z, T^*) = \int_u^{T^*} \frac{Tp(u, s)}{B_n(u)} (z f_u(u, s) - \mu^Z f^*(u, s)) ds$

and

$H_2(u, v, T^*) = \frac{I(u)}{B_n(u)} - \int_u^{T^*} \frac{Tp(u, s)}{B_n(u)} f_v(u, s) F(u - v)_- ds$

Again, the cost process consists in two parts. The first one is related to the policyholder’s claim and corresponds to the two first stochastic integrals. As in the previous model, the integral with respect to $q(du, dz)$ represents the part that cannot be hedged due to the unpredictable arrival and unpredictable total size of the policyholder’s claim. The integral with respect to $q_A(d(u - T), da)$, did not appear in the previous model. It represents the part of the claim process that cannot be hedged due to the unpredictable settlement of the claim. As in the previous model, the two last stochastic integrals represent the part of the claim process that cannot be hedged due the incompleteness of the financial market.

Appendix 1.

PROPOSITION. The martingale representation of $E^Q [A_{T^*} | \mathcal{G}_t]$ is given by

$E^Q [A_{T^*} | \mathcal{G}_t] = E^Q [A_{T^*} | \mathcal{G}_0]$

$+ \int_0^t \frac{I(u)}{B_n(u)} dM_u^S$

$+ \int_0^t \int_E \left[ \int_u^{T^*} \frac{Tp(u, s)}{B_n(u)} h(s, u, z) ds \right] q(du, dz)$

$- \int_0^t \int_u^{T^*} \frac{Tp(u, s)}{B_n(u)} \sigma^*_T(u, s) E^{Q^I} [\lambda^S_{Iu} - \sigma^*_RM(u)]_2 ds dW_u^Q$
or
\[
E^Q \left[ A_{T^*} \mid \mathcal{G}_t \right] = E^Q \left[ A_{T^*} \mid \mathcal{G}_0 \right] + \int_0^t \int_E \left[ \frac{I(u)}{B_n(u)} k(u, z) + \int_u^{T^*} \frac{Tp(u, s)}{B_n(u)} h(s, u, z) ds \right] q(du, dz) \\
- \int_0^t \int_u^{T^*} \frac{Tp(u, s)}{B_n(u)} \sigma_{Tp}^*(u, s) E^{Q^I} \left[ \lambda^S_u \mid \mathcal{I}_u \right] ds dW^Q_u
\]

where \( h(s, u, z) \) is the predictable integrand in the martingale representation \( E^{Q^I} [\lambda^S_u \mid \mathcal{I}_t] = E^{Q^I} [\lambda^S_u \mid \mathcal{I}_0] + \int_0^t \int_E h(s, u, z)q(du, dz) \), \( k(u, z) \) is the predictable integrand in the martingale representation \( M^S_t = M^S_0 + \int_0^t \int_E k(u, z)q(du, dz) \).

**Proof.** We have
\[
E^Q \left[ A_{T^*} \mid \mathcal{G}_t \right] = E^Q \left[ \int_0^{T^*} \frac{I(u)}{B_n(u)} dS_u \mid \mathcal{G}_t \right] = \int_0^t \frac{I(u)}{B_n(u)} dS_u + E^Q \left[ \int_t^{T^*} \frac{I(u)}{B_n(u)} dS_u \mid \mathcal{G}_t \right] = \int_0^t \frac{I(u)}{B_n(u)} dS_u + \int_t^{T^*} \frac{I(t, \cdot)}{B_n(t)} E^{Q^I} \left[ \lambda^S_u \mid \mathcal{I}_t \right] dU_u
\]

thanks to Proposition 3.2. The expression \( \frac{I(t, \cdot)}{B_n(t)} \) corresponds to the discounted price at time \( t \) of a TIPS with maturity \( u \). Using the integration by parts formula, we can write for each \( u \):
\[
\frac{Tp(t, u)}{B_n(t)} E^{Q^I} \left[ \lambda^S_u \mid \mathcal{I}_t \right] = \frac{Tp(0, u)}{B_n(0)} E^Q \left[ \lambda^S_u \mid \mathcal{G}_0 \right] + \int_0^t \frac{Tp(s, u)}{B_n(s)} dE^{Q^I} \left[ \lambda^S_u \mid \mathcal{G}_s \right] + \int_0^t E^Q \left[ \lambda^S_u \mid \mathcal{G}_s \right] - d \left( Tp(s, u) \right) \left( \frac{Tp(s, u)}{B_n(s)} + \frac{Tq(s, \cdot)}{B_n(s)} \right) E^{Q^I} \left[ \lambda^S_u \mid \mathcal{G}_t \right] \bigg|_t
\]

The square bracket is null in the last formula. We saw in Section 2.1 that:
\[
\frac{Tp(t, u)}{B_n(t)} = \frac{Tp(0, u)}{B_n(0)} - \int_0^t \frac{Tp(s, u)}{B_n(s)} \sigma_{Tp(u, s)} dW^Q_u
\]

Furthermore, since \( \lambda^S_u \) is \( \mathbb{I} \)-adapted and \( \mathbb{I} \) and \( \mathcal{F} \) are independent, \( E^Q \left[ \lambda^S_u \mid \mathcal{G}_t \right] = E^{Q^I} \left[ \lambda^S_u \mid \mathcal{I}_t \right] \). This term is obviously a \( (Q^I, \mathbb{I}) \)-martingale for each \( u \). Thanks to the martingale representation property of a marked point process in its own natural filtration (see Brémaud [33]), we can write for each \( u \):
\[
E^{Q^I} \left[ \lambda^S_u \mid \mathcal{I}_t \right] = E^{Q^I} \left[ \lambda^S_u \mid \mathcal{I}_0 \right] + \int_0^t \int_E h(u, v, z)q(du, dz)
\]
for a predictable process $h(u, \cdot, \cdot)$ which is $\sigma(N(t, A), \forall A \in \mathcal{B}(\mathbb{R}_+))$-measurable. We can then write Equation (5.4) as:

$$
\frac{Tp_t u} {B_n(t)} E^Q \left[ \lambda^S_u | \mathcal{G}_t \right] = \frac{Tp_t 0, u} {B_n(0)} E^Q \left[ \lambda^S_u | \mathcal{G}_0 \right]
$$

$$
+ \int_0^t Tp_t s, u \int_E h(u, s, z) q(ds, dz)
$$

$$
- \int_0^t E^Q [\lambda^S_u | \mathcal{G}_s] - \frac{Tp_t s, u} {B_n(s)} \sigma^*_Tp_t (u, s) dW^Q_s
$$

The second integral in Equation (5.3) can then be written as:

$$
\int_T^t \frac{I(t) P_r(t, u)} {B_n(t)} E^Q \left[ \lambda^S_u | \mathcal{G}_t \right] du = \int_T^t \frac{Tp_t 0, u} {B_n(0)} E^Q \left[ \lambda^S_u | \mathcal{G}_0 \right] du
$$

$$
+ \int_T^t \int_0^t \frac{Tp_t s, u} {B_n(s)} \int_E h(u, s, z) q(ds, dz) du
$$

$$
- \int_T^t \int_0^t E^Q [\lambda^S_u | \mathcal{G}_s] - \frac{Tp_t s, u} {B_n(s)} \sigma^*_Tp_t (u, s) dW^Q_s du
$$

By interchanging the order of integration, we have:

$$
\int_T^t \frac{I(t) P_r(t, u)} {B_n(t)} E^Q \left[ \lambda^S_u | \mathcal{G}_t \right] du = \int_T^t \frac{Tp_t 0, u} {B_n(0)} E^Q \left[ \lambda^S_u | \mathcal{G}_0 \right] du
$$

$$
+ \int_T^t \int_0^t \frac{Tp_t s, u} {B_n(s)} \int_E h(u, s, z) q(ds, dz) du
$$

$$
- \int_T^t \int_0^t E^Q [\lambda^S_u | \mathcal{G}_s] - \frac{Tp_t s, u} {B_n(s)} \sigma^*_Tp_t (u, s) dW^Q_s du
$$

$$
= \int_T^t \frac{Tp_t 0, u} {B_n(0)} E^Q \left[ \lambda^S_u | \mathcal{G}_0 \right] du
$$

$$
+ \int_T^t \int_0^t \frac{Tp_t s, u} {B_n(s)} \int_E h(u, s, z) q(ds, dz) du
$$

$$
- \int_T^t \int_0^t E^Q [\lambda^S_u | \mathcal{G}_s] - \frac{Tp_t s, u} {B_n(s)} \sigma^*_Tp_t (u, s) dW^Q_s du
$$

$$
- \int_T^t \int_0^t Tp_t 0, u \frac{B_n(0)} {B_n(s)} E^Q [\lambda^S_u | \mathcal{G}_0] du
$$

$$
- \int_T^t \int_0^t h(u, s, z) q(ds, dz) du
$$

$$
+ \int_T^t \int_0^t E^Q [\lambda^S_u | \mathcal{G}_s] - \frac{Tp_t s, u} {B_n(s)} \sigma^*_Tp_t (u, s) dW^Q_s du
$$
If we denote by \((I)\) the three last lines, and interchange the order of integration of these integrals, we have:

\[
(I) = - \int_0^t \frac{T_p(0,u)}{B_n(0)} E^Q \left[ \lambda^S_u | \mathcal{G}_0 \right] du \\
- \int_0^t \int_0^u \frac{T_p(s,u)}{B_n(s)} \int_E h(u,s,z) q(ds,dz) du \\
+ \int_0^t \int_0^u \frac{T_p(s,u)}{B_n(s)} \sigma^*_T(p)(u,s) E^Q \left[ \lambda^S_u | \mathcal{G}_s \right] - dW^Q_s du \\
= - \int_0^t \frac{T_p(u,u)}{B_n(u)} E^Q \left[ \lambda^S_u | \mathcal{G}_u \right] du \\
= - \int_0^t \frac{I(u)}{B_n(u)} \lambda^S_u du
\]

Since \(E^Q[A_{T^*} | \mathcal{G}_0] = \int_0^t \frac{T_p(0,u)}{B_n(0)} E^Q \left[ \lambda^S_u | \mathcal{G}_0 \right] du\), we eventually get the decomposition for \(E^Q[A_{T^*} | \mathcal{G}_t]\):

\[
E^Q[A_{T^*} | \mathcal{G}_t] = E^Q[A_{T^*} | \mathcal{G}_0] \\
+ \int_0^t \frac{I(u)}{B_n(u)} dM^S_u \\
+ \int_0^t \int_E \left[ \int_u^T \frac{T_p(u,s)}{B_n(u)} h(s,u,z) ds \right] q(du,dz) \\
- \int_0^t \int_u^T \frac{T_p(u,s)}{B_n(u)} \sigma^*_T(p)(u,s) E^{Q^I} \left[ \lambda^S_{s} | \mathcal{I}_u \right] - ds dW^Q_u
\]

where \(dM^S_u = dS_u - \lambda^S_u du\) is a \((Q, \mathcal{G})\)-martingale and \(h(s,u,z)\) is the integrand for the martingale \(E \left[ \lambda^S_u | \mathcal{I}_u \right]\).

Since \(M^S_u\) is also a \((\mathbb{I}, Q^I)\)-martingale, we can also find a predictable process \(k(u,z)\) such that \(M^S_t = M^S_0 + \int_0^t \int_E k(u,z) q(du,dz)\). We thus directly obtain the second representation of \(E^Q[A_{T^*} | \mathcal{G}_t]\).

\[\square\]

**Appendix 2.**

**Proposition.** For \(u > t\), we have:

\[
E^{Q^I}[k_u | \mathcal{I}_t] = \left[ k_t + \int_t^u \lambda^I(v) e^{\int_t^v \lambda^O(w)dw} dv \right] e^{-\int_t^u \lambda^O(v)dv}
\]

or

\[
E^{Q^I}[k_u | \mathcal{I}_t] = k_t e^{-\int_t^u \lambda^O(v)dv} + \left( \frac{\int_t^u \lambda^I(v) e^{\int_t^v \lambda^O(w)dw} dv}{\int_t^u \lambda^O(v) e^{\int_t^v \lambda^O(w)dw} dv} \right) \left( 1 - e^{-\int_t^u \lambda^O(v)dv} \right)
\]
PROOF. We have
\[
E^{Q^t} [k_u | \mathcal{I}_t] = k_t + E^{Q^t} [k_u - k_t | \mathcal{I}_t]
\]
\[
= k_t + \int_t^u \lambda^I(v) dv - \int_t^u \lambda^O(v) E^{Q^t} [k_v | \mathcal{I}_t] dv
\]
We have to find the solution \( f(s) \) of the following equation: 
\[
f(u) = k_t + \int_t^u \lambda^I(v) dv - \int_t^u \lambda^O(v) f(v) dv \text{ or of the following differential equation } f(u) - \frac{1}{\lambda^O(u)} f'(u) + \frac{\lambda^I(u)}{\lambda^O(u)} = 0.
\]
The solution of this equation is of the form 
\[
f(u) = \int_0^u \lambda^I(v) e^{\int_0^v \lambda^O(w) dw} dv + Ce^{-\int_0^u \lambda^O(v) dv}
\]
where \( C \) is a constant. We know that \( f(t) = k_t \). So we have
\[
\frac{\int_0^t \lambda^I(v) e^{\int_0^v \lambda^O(w) dw} dv}{e^{\int_0^t \lambda^O(v) dv}} + Ce^{-\int_0^t \lambda^O(v) dv} = k_t
\]
and thus
\[
C = \left( k_t - \frac{\int_0^t \lambda^I(v) e^{\int_0^v \lambda^O(w) dw} dv}{e^{\int_0^t \lambda^O(v) dv}} \right) e^{\int_0^t \lambda^O(v) dv}
\]
The expectation is then given by:
\[
E^{Q^t} [k_u | \mathcal{I}_t] = \int_0^u \lambda^I(v) e^{\int_0^v \lambda^O(w) dw} dv + \left( k_t - \frac{\int_0^t \lambda^I(v) e^{\int_0^v \lambda^O(w) dw} dv}{e^{\int_0^t \lambda^O(v) dv}} \right) e^{-\int_t^u \lambda^O(v) dv}
\]
Eventually we have:
\[
E^{Q^t} [k_u | \mathcal{I}_t] = k_t e^{-\int_t^u \lambda^O(v) dv} + \left( \int_t^u \lambda^I(v) e^{\int_t^v \lambda^O(w) dw} dv \right) e^{-\int_t^u \lambda^O(v) dv}
\]
We can also rewrite this formula as a weighted average. Let us first notice that
\[
\int_t^u \lambda^O(v) e^{\int_t^v \lambda^O(w) dw} dv = e^{-\int_t^u \lambda^O(v) dv} \int_t^u d \left[ e^{\int_t^v \lambda^O(w) dw} \right]
\]
\[
= e^{\int_t^u \lambda^O(v) dv} - 1
\]
So we can write:
\[
E^{Q^t} [k_u | \mathcal{I}_t] = k_t e^{-\int_t^u \lambda^O(v) dv} + \left( \int_t^u \lambda^I(v) e^{\int_t^v \lambda^O(w) dw} dv \right) \left( e^{\int_t^u \lambda^O(v) dv} - 1 \right) e^{-\int_t^u \lambda^O(v) dv}
\]
\[
= k_t e^{-\int_t^u \lambda^O(v) dv} + \left( \frac{\int_t^u \lambda^I(v) e^{\int_t^v \lambda^O(w) dw} dv}{\int_t^u \lambda^O(v) e^{\int_t^v \lambda^O(w) dw} dv} \right) \left( 1 - e^{-\int_t^u \lambda^O(v) dv} \right)
\]
□
Appendix 3.

**Proposition.** For all \( s > u \), we have

\[
dE^{Q^I}[\lambda_s^S | \mathcal{I}_u] = \lambda Z e^{-\int_u^s \lambda^O \, dw} dM^k_u
\]

**Proof.** Since \( E^{Q^I}[\lambda_s^S | \mathcal{I}_u] = \lambda Z E^{Q^I}[k_s | \mathcal{I}_u] \), we have to find the martingale representation of \( E^{Q^I}[k_s | \mathcal{I}_u] \). Using Itô’s lemma on Equation (4.3) (and the Leibniz rule), we have:

\[
dE^{Q^I}[k_s | \mathcal{I}_u] = \lambda^O(u) e^{-\int_u^s \lambda^O \, dw} k_{u-} du \\
+ \lambda^O(u) e^{-\int_u^s \lambda^O \, dw} \left( \int_u^s \lambda^I(v) e^{\int_u^v \lambda^O \, dw} dv \right) du \\
+ e^{-\int_u^s \lambda^O \, dw} \left( \lambda^O(u) \left( \int_u^s \lambda^I(v) e^{\int_u^v \lambda^O \, dw} dv \right) - \lambda^I(u) e^{\int_u^s \lambda^O \, dw} \right) du \\
+ e^{-\int_u^s \lambda^O \, dw} dM^k_u
\]

\[
= \lambda^O(u) e^{-\int_u^s \lambda^O \, dw} k_{u-} du + \lambda^O(u) e^{-\int_u^s \lambda^O \, dw} \left( \int_u^s \lambda^I(v) e^{\int_u^v \lambda^O \, dw} dv \right) du \\
+ e^{-\int_u^s \lambda^O \, dw} \left( -\lambda^O(u) \left( \int_u^s \lambda^I(v) e^{\int_u^v \lambda^O \, dw} dv \right) - \lambda^I(u) e^{\int_u^s \lambda^O \, dw} \right) du \\
+ e^{-\int_u^s \lambda^O \, dw} dM^k_u \\
= \lambda^O(u) e^{-\int_u^s \lambda^O \, dw} k_{u-} du - \lambda^I(u) e^{-\int_u^s \lambda^O \, dw} du + e^{-\int_u^s \lambda^O \, dw} dM^k_u \]

\[
= -\lambda^O(u) e^{-\int_u^s \lambda^O \, dw} (\lambda^I(u) - k_{u-} \lambda^O(u)) du \\
+ e^{-\int_u^s \lambda^O \, dw} dM^k_u
\]
Using Itô’s formula and the Leibniz rule, we can find the martingale representation of Risk-Minimization with Inflation and Interest Rate Risk. 267

\[ E^{Q^t} \left[ \lambda^S_u \mid I_t \right] = E^{Q^t} \left[ 1_{\{T \leq L\}} 1_{\{T \leq u - \}} Z f(u - T) \mid I_t \right] \]
\[ = 1_{\{T > t\}} E^{Q^t} \left[ 1_{\{T \leq L\}} 1_{\{T \leq u - \}} Z f(u - T) \mid \{T > t\} \right] \]
\[ + 1_{\{T \leq t\}} E^{Q^t} \left[ 1_{\{T \leq L\}} 1_{\{T \leq u - \}} Z f(u - T) \mid T \cup Z \right] \]
\[ = 1_{\{t \leq L\}} 1_{\{T > t\}} \mu \int_t^{L \land u} f(u - v) \lambda(v) e^{- F_0 \int_t^v \lambda(w) dw} dv \]
\[ + 1_{\{T \leq t\}} 1_{\{T \leq u\}} Z f(u - T) \]

\[ \square \]

Appendix 5.

**Proposition.** The martingale representation of \( E^{Q^t} \left[ \lambda^S_t \mid I_s \right] \) is given by
\[ dE^{Q^t} \left[ \lambda^S_t \mid I_s \right] = 1_{\{s \leq L\}} \int_{\mathbb{R}^+} \left[ z f(t - s) - \mu^Z f^*(s, t) \right] q(ds, dz) \]

**Proof.** Thanks to Appendix 4, we know \( E^{Q^t} \left[ \lambda^S_t \mid I_s \right] \) is given by:
\[ E^{Q^t} \left[ \lambda^S_t \mid I_s \right] = 1_{\{s \leq L\}} 1_{\{T > s\}} \mu^Z \int_s^{L \land t} f(t - u) \lambda(u) e^{- F_0 \int_u^t \lambda(v) dv} du \]
\[ + 1_{\{T \leq L\}} 1_{\{T \leq s\}} Z f(t - T) \]

Using Itô’s formula and the Leibniz rule, we can find the martingale representation of \( E^{Q^t} \left[ \lambda^S_t \mid I_s \right] \):
\[ dE^{Q^t} \left[ \lambda^S_t \mid I_s \right] = 1_{\{s \leq L\}} 1_{\{T > s - \}} \lambda(s) \mu^Z \left[ \int_s^{L \land t} f(t - u) \lambda(u) e^{- F_0 \int_u^t \lambda(v) dv} du - f(t - s) \right] ds \]
\[ - 1_{\{s \leq L\}} \mu^Z \left( \int_s^{L \land t} f(t - u) \lambda(u) e^{- F_0 \int_u^t \lambda(v) dv} du \right) dH_s + 1_{\{s \leq L\}} Z f(t - s) dH_s \]
\[ = 1_{\{s \leq L\}} \int_{\mathbb{R}^+} \left[ z f(t - s) - \mu^Z \int_s^{L \land t} f(t - u) \lambda(u) e^{- F_0 \int_u^t \lambda(v) dv} du \right] q(ds, dz) \]

\[ \square \]

Appendix 6.

**Proposition.** We have for \( u \geq t \),
\[ E^{Q^t} \left[ \lambda^S_u \mid I_t \right] = 1_{\{T > t\}} 1_{\{t \leq L\}} \mu^Z f^*(t, u) \]
\[ + 1_{\{T \leq L\}} 1_{\{T \leq t\}} Z \bar{F}(t - T) f_T(t, u) \]

where
\[ f_T(t, u) = e^{A_0(t - T)} A_0(u - T) e^{- A_0(u - T)} \]
\[ f^*(t, u) = \int_t^{L \land u} f_u(v, u) \lambda(v) e^{- F_0 \int_v^u \lambda(w) dw} dv \]
PROOF. We have:

\[
E^{Q^t} \left[ \lambda^S_u \right| I_t \right] = E \left[ 1_{\{T \leq L\}} 1_{\{T \leq u\}} Z \bar{F}(u-T) A'_0(u-T) | I_t \right]
\]

\[
= 1_{\{T > t\}} E^{Q^t} \left[ 1_{\{T \leq L\}} 1_{\{T \leq u\}} Z \bar{F}(u-T) A'_0(u-T) | \{T > t\} \right]
\]

\[
+ 1_{\{T \leq L\}} E^{Q^t} \left[ 1_{\{T \leq L\}} 1_{\{T \leq u\}} Z \bar{F}(u-T) A'_0(u-T) | T \cup Z \cup F(t-T) \right]
\]

\[
= 1_{\{T > t\}} 1_{\{t \leq L\}} \mu Z \left[ \int_t^{\bar{E}^T} E^{Q^t} \left[ \bar{F}(u-v) \right] A'_0(u-v) Q^I (T \in dv | \{T > t\} \right] \right]
\]

\[
+ 1_{\{T \leq L\}} 1_{\{T \leq L\}} Z A'_0(u-T) E^{Q^t} \left[ \bar{F}(u-T) | T \cup F(t-T) \right]
\]

Since \(A_0(t)\) is continuous, we have \(E^{Q^t} \left[ 1 - F(t) \right] = e^{-A_0(t)}\) and since \(T\) admits a deterministic intensity \(\lambda(t)\), we have:

\[
\int_t^{\bar{E}^T} E^{Q^t} \left[ \bar{F}(u-v) \right] A'_0(u-v) Q^I (T \in dv | \{T > t\} \right) = \int_t^{\bar{E}^T} e^{-A_0(u-v)} A'_0(u-v) \lambda(v) e^{-\int u \lambda(w)dw} dv
\]

On the other hand, we have

\[
1 - F(u-T)_- = \prod_{0 \leq s < t} (1 - \Delta A_s) \prod_{t-T < s < u-T} (1 - \Delta A_s)
\]

\[
= \left[ 1 - F(t-T) \right] \prod_{t-T < s < u-T} (1 - \Delta A_s)
\]

so we get

\[
E^{Q^t} \left[ \bar{F}(u-T) | T \cup F(t-T) \right] = \bar{F}(t-T) E \left[ \prod_{t-T < s < u-T} (1 - \Delta A_s) | T \cup F(t-T) \right]
\]

\[
= \bar{F}(t-T) E \left[ \prod_{t-T < s < u-T} (1 - \Delta A_s) \right]
\]

\[
= \bar{F}(t-T) e^{-(A_0(u-T) - A_0(t-T))}
\]

\[
\square
\]

APPENDIX 7.

PROPOSITION. We have:

\[
dE^{Q^t} \left[ \lambda^S_u \right| I_u \right] = 1_{\{t \leq L\}} \int_{\mathbb{R}_+} \left[ z f_t(t,u) - \mu Z f^*(t,u) \right] q_Z(dt,dz)
\]

\[
- 1_{\{T \leq L\}} \int_{[0,1]} 1_{\{T < t\}} Z \bar{F}(t-T)_- a f_T(t,u) q_A (d(t-T),da)
\]
Proof. Using the Leibniz rule, we have
\[
\frac{\partial f^*(t, u)}{\partial t} = \left[ \int_t^{L \wedge u} f_v(v, u) \lambda(t) e^{-\int_v^t \lambda(w) dw} dv \right] - f_t(t, u) \lambda(t) e^{-\int_t^t \lambda(t) dt} \]
\[
= \lambda(t) f^*(t, u) - \lambda(t) f_t(t, u)
\]
and
\[
\frac{\partial f^T(t, u)}{\partial t} = A'_0(t - T) f^T(t, u)
\]
Using Itô’s lemma on Equation (5.2):
\[
dE^Q_I \left[ \lambda_u^S | I_t \right] = 1_{\{T > t- \}} 1_{\{t \leq L\}} \mu^Z (t) f^*(t, u) dt
- 1_{\{t \leq L\}} \mu^Z f^*(t, u) d1_{\{T \leq t\}}
- 1_{\{T > t- \}} 1_{\{t \leq L\}} \mu^Z (t) f_t(t, u) dt
+ 1_{\{T \leq L\}} Z \bar{F}(t - T) f^T(t, u) d1_{\{T \leq t\}}
+ 1_{\{T \leq L\}} 1_{\{T \leq t\}} Z f^T(t, u) d\bar{F}(t - T)
\]
If we notice that
\[
1_{\{t \leq L\}} \mu^Z f^*(t, u) d1_{\{T \leq t\}} = 1_{\{t \leq L\}} \mu^Z \int_{\mathbb{R}^+} f^*(t, u) N(dt, dz)
\]
\[
1_{\{T \leq L\}} Z \bar{F}(t - T) f^T(t, u) d1_{\{T \leq t\}} = 1_{\{t \leq L\}} \int_{\mathbb{R}^+} z f_t(t, u) N(dt, dz)
\]
\[
d\bar{F}(t - T) = - \bar{F}(t - T) - \int_{[0,1]} a N_A(dt, da)
\]
\[
\bar{F}(t - T) A'_0(t - T) dt = \bar{F}(t - T) - \int_{[0,1]} a p_A(d(t - T), da)
\]
We eventually have:
\[
dE^Q_I \left[ \lambda_u^S | I_t \right] = 1_{\{t \leq L\}} \int_{\mathbb{R}^+} \left[ z f_t(t, u) - \mu^Z f^*(t, u) \right] q_Z(dt, dz)
- 1_{\{T \leq L\}} \int_{[0,1]} 1_{\{T \leq t\}} Z \bar{F}(t - T) a f^T(t, u) q_A(d(t - T), da)
\]
Conclusion.

This conclusion is the place to discuss the possible extensions of the results described throughout this thesis and gives some ideas of future researches.

In Part I, we studied the pricing of a guaranteed investment. The pricing was here considered in terms of a guaranteed rate on the premium and a participation rate on the terminal financial surplus. We gave a two-steps procedure to fix these two technical parameters, consistently with the risk management policy of an insurance company. However, this procedure is not anymore valid if we consider periodic bonuses since, in this case, the level of bonus offered to the policyholders would influence the insolvency risk of the insurer. It would be interesting to study how our procedure could be extended in this case.

In Part II, we studied the valuation and the (local) “risk-minimizing” strategies of insurance contracts with a surrender option. However, for the sake of simplicity, we assumed there was no mortality. A natural extension would be therefore to introduce this mortality. In such a case, a general model should allow the surrender probabilities to depend on the survival probabilities of the policyholder. If we assume the survival probabilities are deterministic, we can expect to get results not so different of those described in Part II. However, if we assume the survival probabilities are stochastic (as in Chapter 5), things get a little bit more complicated because we could have to deal with a non-trivial mix of financial and mortality risk. This question is probably worth studying.

In Part III, we studied the systematic mortality risk. More precisely, in Chapter 5, we investigated the form of the risk-minimizing strategy of a single life insurance policy with systematic mortality risk. We can easily extend this result to a portfolio of policies if we assume the time of death of the different policyholders are \( J \)-independent. Indeed, in this case, the risk-minimizing strategy of the portfolio is given by the sum of the individual strategies of the different policies. Finding the risk-minimizing strategy of a portfolio when this conditional independence does not hold, is another interesting further research.

In Chapter 6, we studied the application of the HJM methodology to longevity bonds. In particular, in Section 3.6, we gave a pricing formula for these longevity bonds that suggested that the classical pricing formula found in the intensity-based models could be extended to much more general settings. It would be interesting to study how to bridge this gap.

In Part IV, we studied the risk-minimizing strategies of non-life insurance contracts, when the inflation and the interest rate risks are taken into account. We first gave the form of the risk-minimizing strategy for a general increasing claim process, assuming the financial market and the claim process (expressed in real terms) are independent. Then, we illustrated this result in four different claim models. We could extend these results in at least two ways. First,
we could try to remove the independence between the financial market and the claims. Second, we could look for more realistic claim models than the four studied in Chapter 7.
Bibliography


