"A remark on the number of trading posts in strategic market games"

Koutsougeras, Léonidas

Abstract

In market games the one to one correspondence between commodity types and trading posts would be justified if it were true that the set of equilibria is not affected by the number of trading posts postulated at the outset of the model. We show that this is not true. We develop an example which features equilibria where a commodity is simultaneously exchanged in two trading posts at different prices, i.e., equilibria where the 'law of one price' fails when the one to one correspondence between commodities and trading posts is abandoned. Thus, we conclude that the set of equilibria in market games depends non-trivially on the number of trading posts. This conclusion further suggests that an explanation of the structure of trading posts is necessary.


Référence bibliographique

Koutsougeras, Léonidas. A remark on the number of trading posts in strategic market games. CORE Discussion Papers ; 1999/05 (1999)
A Remark on the Number of Trading Posts in Strategic Market Games

Leonidas C. Koutsougeras

School of Economic Studies, University of Manchester
and
Department of Econometrics, Tilburg University

This draft: January 19, 1999

Abstract

In market games the one to one correspondence between commodity types and trading posts would be justified if it were true that the set of equilibria is not affected by the number of trading posts postulated at the outset of the model. We show that this is not true. We develop an example which features equilibria where a commodity is simultaneously exchanged in two trading posts at different prices, i.e., equilibria where the 'law of one price' fails when the one to one correspondence between commodities and trading posts is abandoned. Thus, we conclude that the set of equilibria in market games depends non-trivially on the number of trading posts. This conclusion further suggests that an explanation of the structure of trading posts is necessary.

Keywords: Trading posts, law of one price.

JEL Classification Number: C72

*The financial support of TMR no ERB4001GT965374 Grant of the European Community is gratefully acknowledged. I would like to thank E. van Damme, F. Germano, P. Madden, J-F. Mertens and H. Polemarchakis for useful discussions. Special thanks are also extended to D. Talman for carefully reading a draft of this paper. I am responsible for any shortcomings.
1 Introduction

Strategic market games have provided an elegant formulation of imperfectly competitive exchange. The standard setup of these models -see [2] [7] [8]- is based on the concept of trading posts where individuals submit orders for purchases and sales of commodities. An allocation rule then distributes commodities to individuals according to their bids as well as the aggregate offers and bids that reach each post. In this way commodity holdings of individuals depend on the whole profile of bid-offer strategies of all agents in each post.

It is noteworthy that market games feature exactly one post for each commodity\(^1\). In this way there is a one to one correspondence between commodity types, trading posts and commodity prices. On a closer look the identification of trading posts with commodity types is a hypothesis, which has one important consequence: all individuals wishing to trade a commodity must do so in one (and the same) trading post, i.e., all bids and offers for a commodity are aggregated at a single trading post. Contrast this with a situation where a commodity may be exchanged, say against a numeraire, in more than one posts. In such a case, individuals would have available a richer variety of strategies, that includes buying a commodity in some trading posts and selling it in others. Another conceivable configuration of net trades, is the partition of the set of agents in the economy into subsets trading a commodity in different posts. Thus, we are compelled to question whether or not the set of market game equilibria depends on the structure of trading posts postulated at the outset.

As the reader may have guessed, what we are about to suggest in this paper is the idea of multiple trading posts for commodities. What we have in mind is a model with possibly more than one trading post where each commodity can be traded against a numeraire and prices as well as net trades in different posts are calculated independently from one another. In this case one may end up with a set of equilibria that cannot be captured by the original model. When trade is perfectly competitive this issue does not arise because the structure of net trades is immaterial in a competitive equilibrium, so the market structure is of no consequence for the set of competitive equilibria.

The issue raised here may be viewed as a robustness test of the market game with respect to the structure of trading posts. The key idea is to abandon the one to one correspondence between commodity types and trading posts. If it turned out that the one to one correspondence between prices and commodity types (i.e., the 'law of one price') is valid regardless of the number of trading posts, then we would have a proof that the model with a single trading post per commodity is robust: the structure of trading posts is immaterial. The implication of this would be that the concept of trading posts does not need further elaboration. If it turned out that the contrary is true, i.e., if the 'law of one price' fails, then we are faced with a meaningful extension of the model. Such a conclusion would suggest, among other things, that some careful clarification of trading posts is necessary. In either case, the investigation of such an extension seems worthwhile.

In this paper we show by means of an example that, in fact, the imperfectly competitive general equilibrium model is not robust in the above sense. Our example features

\(^1\) or one post for each pair of commodities as in [1] or Shapley's window model which is analyzed in [6].
an equilibrium where a commodity is exchanged against another in two trading posts simultaneously, at different prices. As a conclusion, there is an equilibrium strategy configuration in this example\(^2\) that cannot be captured by the market game with a single trading post. Moreover, we show that the allocation produced by this equilibrium strategy profile is not even feasible via a single trading post. Surprisingly enough, in this equilibrium the liquidity constraints are not binding. The emergence of unequal prices in equilibrium is due to the imperfectly competitive conditions of exchange: a contemplated effort of an individual to take advantage of the price difference by shifting orders from one post to another involves price effects. Thus, the primary effect of a shift of an infinitesimal order to another post is accompanied by an opposite effect that comes through the change in prices that this shift will produce. The key idea is that this price effect can be detrimental for the benefits of any shift of orders across posts.

In order to keep things in perspective, the failure of the law of one price that we advocate here should not be confused with the appearance of inconsistent prices in [1]. In that model the authors study a market game with a trading post (and a corresponding price) for each pair of commodities and conclude that the exchange prices between a triple of goods may be inconsistent. Here we discuss the possibility of different prices for a commodity traded in different posts against a numeraire. In order to highlight the difference, the extension of the model in [1] (or Shapley’s window model for that matter) that captures the issue at hand, would be to add more trading posts for some pair of commodities.

Upon departure from the context of a single post per commodity the number of posts becomes ambiguous, because any number would be just as arbitrary. Ideally, we would like trading posts to somehow arise endogenously. However, as the purpose of this paper is to justify the introduction of multiple posts rather than suggest a specific number, we focus on a minimal model with two commodities which can be exchanged for one another in two posts.

We apply our extension to a model suggested in [7]. However, similar extensions are conceivable in other species of market games as well\(^3\). The next section introduces a bare bones model that captures the issue at hand. In section three we develop an example that establishes the possibility of multiple prices for a commodity. Finally, some discussion and concluding remarks follow in sections four and five.

2 The model

In this section we present here an example of a multiple posts version of the model in [7]. Let \(H = \{K, L, M\}\) be a set of agents. There are two commodity types in the economy and the consumption set of each agent is identified with \(\mathbb{R}^2_+\). Each individual \(h \in H\) is characterized by an initial endowment \(e_h \in \mathbb{R}^2_+\) and a \(C^2\), strictly concave and strictly monotonic utility function \(u_h : \mathbb{R}^2_+ \rightarrow \mathbb{R}\). We postulate two trading posts labeled \(r\) and \(s\). Each consumer can offer \((b^r_h, b^s_h)\) units of commodity 1 and/or offer \((q^r_h, q^s_h)\) units of commodity 2 in the two posts. The strategy set of each agent \(h \in H\)

\(^2\)The robustness of our example will be evident in the sequel.

\(^3\)For a similar extension of the model appearing in [5] and in [4], the reader is directed to [3].
is thus $S_h = \{ (b^r_h, b^s_h, q^r_h, q^s_h) \in \mathbb{R}^4_+ : b^r_h + b^s_h \leq e^1_h, q^r_h + q^s_h \leq e^2_h \}$. Given a strategy profile, define for $j = r, s : B^j = \sum_{h \in H} b^j_h$, $Q^j = \sum_{h \in H} q^j_h$, $B^j_h = \sum_{k \neq h} b^j_k$ and $Q^j_{-h} = \sum_{k \neq h} q^j_k$. Transactions in each trading post clear through the price $p^j = B^j / Q^j$.

Each consumer receives $x^1_h = e^1_h + q^r_h \cdot p^r + q^s_h \cdot p^s - b^r_h - b^s_h$ units of commodity 1 and $x^2_h = e^2_h - q^r_h - q^s_h + b^r_h / p^r + b^s_h / p^s$ units of commodity 2, where divisions over zero are taken to equal zero. Consumers are viewed as solving the following problem:

$$\begin{align*}
\max_{(b^r_h, q^r_h) \in S_h} & \ U(x_h) \\
\text{s.t.} & \quad x^1_h = e^1_h + q^r_h \cdot \frac{B^r}{Q^r} + q^s_h \cdot \frac{B^s}{Q^s} - b^r_h - b^s_h \\
& \quad x^2_h = e^2_h - q^r_h - q^s_h + b^r_h \cdot \frac{Q^r}{B^r} + b^s_h \cdot \frac{Q^s}{B^s}
\end{align*}$$

(1)

An equilibrium is defined simply as a Nash equilibrium in bid-offer strategies. An equilibrium of the market game is termed interior if each agent is solving (1) in the interior of $S_h$. For such equilibria we have the following elementary facts:

**Fact 2.1** In an interior equilibrium, the prices in the two trading posts $s$ and $r$, must satisfy the following (non-arbitrage) condition:

$$(p^s)^2 = \frac{Q^r_{-h}}{B^r_{-h}} \cdot \frac{Q^s_{-h}}{B^s_{-h}} \cdot (p^r)^2, \quad \forall h \in H$$

(2)

**Proof:**

The first order conditions of (1) at an interior equilibrium are:

$$\begin{align*}
-\frac{Q^r_{-h}}{Q^r} \cdot \frac{\partial u_h}{\partial x^1_h} + \frac{B^r_{-h}Q^r}{(B^r)^2} \cdot \frac{\partial u_h}{\partial x^2_h} & = 0 \\
-\frac{Q^s_{-h}}{Q^s} \cdot \frac{\partial u_h}{\partial x^1_h} + \frac{B^s_{-h}Q^s}{(B^s)^2} \cdot \frac{\partial u_h}{\partial x^2_h} & = 0 \\
\frac{B^rQ^r_{-h}}{(Q^r)^2} \cdot \frac{\partial u_h}{\partial x^1_h} - \frac{B^r_{-h}}{B^r} \cdot \frac{\partial u_h}{\partial x^2_h} & = 0 \\
\frac{B^sQ^s_{-h}}{(Q^s)^2} \cdot \frac{\partial u_h}{\partial x^1_h} - \frac{B^s_{-h}}{B^s} \cdot \frac{\partial u_h}{\partial x^2_h} & = 0
\end{align*}$$

(3) (4) (5) (6)

Notice that (5) and (6) are the same as (3) and (4) respectively. Furthermore, the system of (3) and (4) can be written in matrix form as:

$$\begin{pmatrix}
-\frac{Q^r_{-h}}{Q^r} & -\frac{Q^s_{-h}}{Q^s} \\
\frac{B^r_{-h}Q^r}{(B^r)^2} & \frac{B^s_{-h}Q^s}{(B^s)^2}
\end{pmatrix}
\begin{pmatrix}
\frac{\partial u_h}{\partial x^1_h} \\
\frac{\partial u_h}{\partial x^2_h}
\end{pmatrix}
= \begin{pmatrix} 0 \\ 0 \end{pmatrix}
$$

Thus a solution exists if and only if:

$$\begin{pmatrix}
-\frac{Q^r_{-h}}{Q^r} & -\frac{Q^s_{-h}}{Q^s} \\
\frac{B^r_{-h}Q^r}{(B^r)^2} & \frac{B^s_{-h}Q^s}{(B^s)^2}
\end{pmatrix}
= 0$$

which implies that (2) is satisfied □
Fact 2.2 Under the assumptions made on preferences the sufficient second order conditions of (1) are always satisfied at an interior equilibrium.

Proof:
From equations (3)-(6) it is obvious that the rank of the matrix of second order derivatives of the objective function in (1) is equal to 2. Define:

\[ A = \frac{\partial^2 u_h}{(\partial x_h^2)} - 2 \cdot \frac{\partial^2 u_h}{\partial x_h^1 \partial x_h^2} \cdot \frac{B_{-h}^e}{Q_{-h}^e} \cdot \frac{(B^e)^2}{(Q^e)^2} + \frac{\partial^2 u_h}{(\partial x_h^2)^2} \cdot \left( \frac{B_{-h}^e}{Q_{-h}^e} \cdot \frac{(B^e)^2}{(Q^e)^2} \right)^2 \]

Notice that \( A < 0 \). Furthermore, given (2)-(4), we have that the determinants of the principal minors of the matrix of second order derivatives of the utility function are:

\[ |H_1| = \frac{\partial^2 u_h}{(\partial b_{h}^e)^2} = \left( \frac{Q_r^e}{Q_r^e} \right)^2 \cdot A - 2 \cdot \frac{\partial u_h}{\partial x_h^2} \cdot \frac{B_{-h}^e Q_r^e}{(B^e)^2} < 0 \] (7)

\[ |H_2| = \frac{\partial^2 u_h}{(\partial b_{h}^e)^2} \cdot \frac{\partial^2 u_h}{(\partial b_{h}^e)^2} - \left( \frac{\partial^2 u_h}{\partial b_{h}^e \partial b_{h}^e} \right)^2 = \left( \frac{Q_r^e}{Q_r^e} \right)^2 \cdot A - 2 \cdot \frac{\partial u_h}{\partial x_h^2} \cdot \frac{B_{-h}^e Q_r^e}{(B^e)^2} \cdot \frac{Q_{-h}^e Q_r^e}{Q_r^e Q_r^e} \cdot A - 2 \cdot \frac{\partial u_h}{\partial x_h^2} \cdot \frac{B_{-h}^e Q_r^e}{(B^e)^2} \cdot \frac{Q_{-h}^e Q_r^e}{Q_r^e Q_r^e} > 0 \] (8)

\[ |H_3| = |H_4| = 0. \]

Thus, at any point where the first order conditions are satisfied the matrix of second order derivatives of the objective function is negative semidefinite.

The crucial observation is that, according to the above facts, unequal prices may occur in equilibrium in as much as it is algebraically possible to find a strategy profile so that:

\[ \frac{B_{-h}^e}{Q_{-h}^e} \cdot \frac{Q_{-h}^e}{B_{-h}^e} = \frac{B_{-h}^e}{Q_{-h}^e} \cdot \frac{Q_{-h}^e}{B_{-h}^e} = \frac{B_{-h}^e}{Q_{-h}^e} \cdot \frac{Q_{-h}^e}{B_{-h}^e} \neq 1 \]

This observation has inspired the example that follows in the next section.

3 Equilibria with multiple prices

3.1 An Example

The example that follows features an equilibrium with two distinct positive prices for a commodity. This establishes the failure of the 'law of one price' when one allows for multiple trading posts. In constructing our example we proceed as follows: We construct a profile of strategies that satisfies (2) and then look for endowments and utility functions for which this profile of strategies is indeed a Nash equilibrium. To this end, consider the following profile of strategies:

\[ (b_{K}^e, b_{L}^e, b_{M}^e) = \left( \left( \frac{10}{9} \right)^2 \cdot \frac{5}{9}, \left( \frac{10}{9} \right)^2 \cdot \frac{3}{9}, \left( \frac{10}{9} \right)^2 \cdot \frac{1}{9} \right) \]
\[(b_K^s, b_K^l, b_M^s) = \left(\frac{7}{9}, \frac{1}{9}, \frac{2}{9}\right)\]
\[(q_K^r, q_L^r, q_M^r) = (1, 3, 1)\]
\[(q_K^s, q_L^s, q_M^s) = (2, 2, 1)\]

In this way we have:
\[p^r = \left(\frac{10}{9}\right)^2 \cdot \frac{1}{5} \quad \text{and} \quad p^s = \frac{2}{9} \quad \text{so that} \quad \frac{p^r}{p^s} = \frac{10}{9} \neq 1\]

**Remark 3.1** It is worthwhile noticing that the aggregate quantities submitted to each post are equal: \(Q^r = 5 = Q^s\).

We now turn to look for endowments and utility functions that validate the above profile as a Nash equilibrium. We begin by specifying endowments as follows:
\[(e_K^1, e_K^2) = \left(\frac{55}{18}, \frac{\left(\frac{10}{9}\right)^2}{18}\right)\]
\[(e_L^1, e_L^2) = \left(\frac{13}{12}, \frac{\left(\frac{10}{9}\right)^2}{6}\right)\]
\[(e_M^1, e_M^2) = \left(8 - \frac{4}{45}, \frac{\left(\frac{10}{9}\right)^2}{9}\right)\]

**Remark 3.2** Observe that for this specification of endowments the liquidity constraints are non-binding for the proposed strategy profile.

In this way, for the profile of strategies under consideration the allocation of commodities across agents turns out as follows:
\[(x_K^1, x_K^2) = (3, 8)\]
\[(x_L^1, x_L^2) = (2, 4)\]
\[(x_M^1, x_M^2) = (8, 4)\]

We now look for utility functions that will serve our purpose. We should find a utility function \(C^2\), strictly concave, monotonic) for each individual, such that equation (3) is satisfied when evaluated at the proposed profile. In view of the fact that the strategy profile in question satisfies (2) by construction, equations (4)-(6) will automatically be satisfied. Moreover, by virtue of fact 2.2 this will suffice to ensure that the proposed profile is indeed a Nash equilibrium one. Thus, we look for a solution to the following equations:
\[\frac{\partial u_K}{\partial x_K^1}(3, 8) = \frac{9}{4} \cdot \frac{\partial u_K}{\partial x_K^2}(3, 8)\]  \(9\)
\[\frac{\partial u_L}{\partial x_L^1}(2, 4) = \frac{27}{4} \cdot \frac{\partial u_L}{\partial x_L^2}(2, 4)\]  \(10\)
\[\frac{\partial u_M}{\partial x_M^1}(8, 4) = \frac{9}{2} \cdot \frac{\partial u_M}{\partial x_M^2}(8, 4)\]  \(11\)
If we look for a solution in the class of functions: $u_h(x_1^h, x_2^h) = (x_1^h)^{a_h} \cdot (x_2^h)^{1-a_h}$ for $h = K, L, M$, we find that equations (9)-(11) are satisfied for $a_K = 27/59$, $a_L = 27/35$ and $a_M = 9/10$.

**Remark 3.3** The constructive process that we followed makes it clear that our example is robust with respect to the choices of utility functions and endowments.

Finally, we show that the equilibrium allocation calculated above can not be achieved in the single trading post model. Let $z^i_h = x^i_h - e^i_h$ where $i = 1, 2$ and $h = K, L, M$ be the net trades of each agent. Notice that the net trades which are achievable with a single trading post, must satisfy $z^1_h = -p \cdot z^2_h$ for some clearing price $p$. In other words in order to be feasible in the single trading post model, a collection of pairs of (nonzero) net trades must satisfy:

$$\frac{z^1_K}{z^2_K} = \frac{z^1_L}{z^2_L} = \frac{z^1_M}{z^2_M}.$$

It can be readily checked that the net trades in the equilibrium allocation calculated above do not satisfy this set of equations, so it is can not be achieved via a single trading post.

### 3.2 Discussion of the example

For sure, the type of equilibria demonstrated in the example above, validate our suspicion that the structure of trading posts matters. On the other hand such equilibria seem odd, because they assign different values to the same commodity, according to which trading post it is traded. In the absence of some friction, conventional wisdom would suggest that if prices were unequal then an individual would find it profitable to shift purchases (sales) from a more (less) expensive post to a less (more) expensive one. In this way prices in the two posts would change accordingly so that in equilibrium prices would be equal. However, in an imperfectly competitive setup this conventional wisdom fails! The reason is that in the imperfectly competitive context, shifts of orders across posts involve a ‘price effect’. In this way infinitesimal ‘shifts’ of bids and offers across posts may change prices in a way that they are unfavorable to a consumer. Indeed prices do change in the ‘right’ direction to close the gap but, as the movement of prices is simultaneous, the relative magnitude of those changes may be such that it renders unfavorable a shift of bids (offers) from a more (less) expensive post towards a less (more) expensive one. The intuition of the example can be easily grasped in this way: if an agent reallocated some of the bid in the expensive post to the cheaper one, the price in the expensive post would fall and at the same time the price in the cheaper one would rise. In this way a higher price would have to be paid not only for the marginal unit but for all the units purchased from the cheaper post. Thus, the direct effect of shifting a part the bid is followed by an adverse indirect effect due to the price change. If this price effect is severe enough the net effect can be unfavorable. It is the absence of this price effect that validates the preceding argument in the competitive case. Finally, it is worthwhile mentioning that unlike the results in [1], the appearance of such equilibria here is not due to liquidity constraints.
4 Conclusion

We hope that the arguments in this paper have convinced the reader that the structure of trading posts in market games is essential. In view of this, the central message of this paper is that the concept of trading posts needs further clarification. The Arrow-Debreu description of an economy, being directed to the study of competitive equilibria where market institutions are irrelevant, does not suggest any particular structure of trading posts, so it is hard to decide how many posts should be allowed at the outset. Ultimately, a model where trading posts evolve, rather than being postulated, would resolve the issues raised here. Apparently, this will require some new insight as one would have to model more primitive concepts that would spark some life into trading posts. In lack of such an insight we propose a model with an infinite number of posts per commodity. In this way the number of (active) trading posts would probably vary across equilibria, but such a framework would certainly capture the equilibria of all possible configurations of trading posts.

On the other hand the possibility of a 'price distribution' for a commodity, as opposed to one price per commodity, seems interesting for a number of reasons. For one, it demonstrates that the lack of perfect competition can endogenously explain price disparities across markets or for equilibrium with arbitrage. An immediate conclusion that can be drawn from our example is that the law of one price entails perfectly competitive conditions of exchange. Furthermore, it adds a new dimension to the asymptotic convergence to competitive equilibria: in order to obtain equivalence with competitive equilibria via a model with multiple posts per commodity, it would be necessary that possible disparities of commodity prices in different trading posts tend to zero. Finally, given the possibility of equilibria with non-uniform prices for commodities, the introduction of multiple trading posts for commodities seems to provide an attractive context for the use of Nash equilibrium refinements.
References


